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RIEMANNIAN METRICS WITH LARGE λ_1

B. COLBOIS AND J. DODZIUK

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ABSTRACT. We show that every compact smooth manifold of three or more dimensions carries a Riemannian metric of volume one and arbitrarily large first eigenvalue of the Laplacian.

Let (M^n, g) be a compact, connected Riemannian manifold of n dimensions. The Laplacian Δ_g acting on functions on M has discrete spectrum. Let $\lambda_1(g)$ denote the smallest positive eigenvalue of Δ_g. Hersch [5] proved that

$$\lambda_1(g)\operatorname{vol}(S^2, g) \leq 8\pi$$

for every Riemannian metric g on the 2-sphere S^2.

In connection with this result, Berger [2] asked whether there exists a constant $k(M)$ such that

$$\lambda_1(g)\operatorname{vol}(M^n, g)^{2/n} \leq k(M) \tag{1}$$

for any Riemannian metric on M. Yang and Yau [8] proved that the inequality above holds for a compact surface S of genus γ with $k(S) = 8\pi(\gamma + 1)$.

Subsequently, numerous examples of manifolds were constructed for which (1) is false (cf. [3] for a discussion and references). In particular, for every $n \geq 3$, the sphere S^n admits metrics of volume one with λ_1 arbitrarily large [3, 6]. Bleecker conjectured in [3] that such metrics exist on every manifold M^n if $n \geq 3$. In this note we give a very simple proof of Bleecker's conjecture using known examples and quite general principles. The same result has been proved independently by Xu [7] by a construction similar to ours. His argument, however, is much harder than our proof.

Theorem 1. *Every compact manifold M^n with $n \geq 3$ admits metrics g of volume one with arbitrarily large $\lambda_1(g)$.*

Proof. The idea of the proof is very simple. We take a metric g_0 on S^n with $\operatorname{vol}(S^n, g_0) = 1$ and $\lambda_1(g_0) \geq k + 1$, where k is a large constant. We excise from S^n a very small ball $B(p, \eta) = B_\eta$ and form the connected sum of S^n with M. The resulting manifold is diffeomorphic to M and has a submanifold Ω, with smooth boundary, naturally identified with $S^n \setminus B_\eta$. Let g_1 be an

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arbitrary metric on M whose restriction to Ω is equal to $g_0|\Omega$. We modify the metric g_1 making it very small on "most of" $M \setminus \Omega$ without altering it on Ω. With the new metric, M looks practically like (S^n, g_0) in the sense that all of the topology of M is contained in a part which is metrically very small. In particular, the smallest positive eigenvalue of this metric is very close to $\lambda_1(g_0)$.

To make this into a rigorous proof we use results of Colin de Verdière [4, Theorem III.1] and Anné [1]. Thus, by [1, Theorem 2], if η is chosen sufficiently small, the first positive eigenvalue μ_1 of the Laplacian of (Ω, g_0) for the Neumann boundary conditions is a very good approximation of $\lambda_1(g_0)$ so that $\mu_1 \geq k + \frac{1}{2}$. Let ε be a small positive number. Take a sequence of smooth functions $F_{i,\varepsilon}$ such that $F_{i,\varepsilon}|\Omega \equiv 1$, $1 \geq F_{i,\varepsilon} \geq \varepsilon$, and $\lim_{i\to\infty} F_{i,\varepsilon}(x) = \varepsilon$ for every $x \in M \setminus \Omega$, and consider metrics $g_{i,\varepsilon} = F_{i,\varepsilon} g_1$. Colin de Verdière showed in the course of proof of Theorem III.1 of [4] that for every positive integer J the eigenvalues μ_j, $j \leq J$, of the Neumann problem for Ω can be approximated to arbitrary accuracy by $\lambda_j(g_{i,\varepsilon})$ by first choosing ε sufficiently small and then i sufficiently large (condition $(*)$ appearing in [4, Theorem III.1] is satisfied for some choice of indices and constants since the spectrum of (Ω, g_0) is discrete). It follows that $\lambda_1(g_{i,\varepsilon}) \geq k + \frac{1}{4}$ for appropriate choices of η, ε, and i. Finally, we multiply the metric $g_{i,\varepsilon}$ by a constant to make the volume equal to one and call the resulting metric g. If the choices of η and ε were sufficiently small and i is sufficiently large then the rescaling factor is practically equal to one so that $\lambda_1(g) \geq k$. □

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Departement Mathematik, Eidgenössiche Technische Hochschule Zürich, ETH Zentrum, CH-8092 Zürich, Switzerland

E-mail address: `COLBOIS@MATH.ETHZ.CH`

Ph.D. Program in Mathematics, Graduate School and University Center (CUNY), New York, New York 10036-8099

E-mail address: `JZDQC@CUNYVM.CUNY.EDU`