STUDIA MATHEMATICA, T. XLIV. (1972)

Riesz-Bochner summability of Fourier integrals and Fourier series

by

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To A. Zygmund on the 50 th anniversary of his first publication.

Abstract. The two problems of Riesz–Bochner summability in k-dimensions for Fourier series and Fourier integrals are considered. In the case of Fourier integrals, solutions to the summability problem are obtained for values below the critical index ($\lambda_0 = (k-1)/2$), in particular for values $\lambda > \lambda_0 - 1$. In both cases, summability is obtained everywhere except on exceptional sets whose dimension is not greater than k-αp, for functions in the Lebesgue classes $\mathscr{L}_\alpha^p$. The main facts, which are proved to make these results possible, are that functions in $\mathscr{L}_\alpha^p$ may be redefined (if necessary) on a "small" set so that the set of Lebesgue points is thick, and that for every point x except points in a small exceptional set, integration over the surface of the sphere centered at x with radius $r > 0$ is possible for all r. In the second of these results, the integrals are uniformly bounded in r, for each x possible; but one must assume $\alpha > 1$.

Introduction. The purpose of this note is to reformulate a problem investigated by H. Federer in [3] so that some additional results are obtained. To explain this purpose we introduce some notation. The setting will be two fold, sometimes we will be considering functions $f\colon \boldsymbol{R}^k \to \boldsymbol{R}$ (the setting for Fourier integrals) and other times we will be considering functions $f\colon T^k \to \boldsymbol{R}$ (the setting for Fourier series). We will use the notation

$$f^{\wedge}(x) = \frac{1}{(2\pi)^k} \int_Y f(y) e^{-i(x\cdot y)} dy$$

where we understand that x is in $\boldsymbol{R}^k$ when $Y = \boldsymbol{R}^k$ or $x = n = (n_1, \ldots, n_k)$, a point in the k-dimensional lattice plane, when $Y = T^k$, the k-dimensional torus. For two points $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ we write $(x\cdot y) = x_1 y_1 + \ldots + x_k y_k$ and $|x|^2 = x_1^2 + \ldots + x_k^2$. For the above definitions we must assume $f \in L^1(Y)$. We also write $\lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda y_k)$.

The Riesz–Bochner means of index $\lambda \geqslant 0$ are

$$\sum_{|n|<R} \left(1 - \frac{|n|^2}{R^2}\right)^{\lambda} f^{\wedge}(n) e^{in\cdot x} = S_R^{\lambda}(f, x) \tag{1.1}$$

and

$$\int_{|y|<R} \left(1-\frac{|y|^2}{R^2}\right)^\lambda f\hat{\ }(y) e^{iy\cdot x} dy = \sigma_R^\lambda(f, x). \tag{1.2}$$

The basic convergence properties for these means are contained in [1], [7] and [8], for classes of functions which are characterized by integrability conditions. In all of these cases, the critical index $\lambda_0 = (k-1)/2$ plays a primary role with the effect that in general summability fails for $\lambda < \lambda_0$. In [3] summability for Fourier integrals is obtained for $\lambda > \lambda_0 - 1$, for functions having distribution derivatives which are measures of bounded total variation.

The point of view here will be to consider the summability of classes of functions belonging to $\mathscr{L}_\alpha^p = \{f: f = g_\alpha * f_0, f_0 \in L^p\}$ (see for example [2] or [9]) where g_α is the kernel of the Bessel potential, which we will describe shortly. The effect of requiring that f belong to the class $\mathscr{L}_\alpha^p$ is to ask, in a sense, that f have fractional derivatives of order α in L^p. It is here that we make a substitution for the class of functions considered in [3]. $\mathscr{L}_\alpha^p$ is normed by $\|f\|_{p,\alpha} = \|f_0\|_p$ where $f = f_0 * g_\alpha$.

One of the basic features of the result in [3] is that the complement of the Lebesgue set of the class of functions considered there is of Hausdorff $(k-1)$-dimension zero. We will instead consider a capacity introduced by Meyer in [5] and show that the complement of the Lebesgue set of a function in $\mathscr{L}_\alpha^p$ is of capacity zero. This capacity has a similarity to the Bessel capacity in [5] and compares closely to Hausdorff measure of dimension $k-\alpha p$. We will obtain summability for points in the Lebesgue set of a function in $\mathscr{L}_\alpha^p(\boldsymbol{R}^k)$ which satisfy an additional property for any index $\lambda > \lambda_0 - 1$. The complement of this set of summability is also of capacity zero.

§1. The capacity $C_{k,p}$. Here we give an outline of a theory of capacity. A more complete and more general development is given in [5].

Let k be a positive lower semi-continuous kernel on Y, which will be either $\boldsymbol{R}^k$ or T^k. Let $\mathfrak{M}$ be the space of Radon measures and $\mathscr{L}_1^+$ the space of all non-negative measures in with finite total variation. For ν in $\mathscr{L}_1^+$ we write $\|\nu\|_1$ for its total variation. By $k(\nu, x)$ we mean $k(\nu, x) = (k*\nu)(x) = \int k(x-y)d\nu(y)$. By L_+^p we mean the non-negative functions in L^p and we write $k(x, f) = (k*f)(x) = \int k(x-y)f(y)dy$. In order that these definitions make sense, we assume $k \in L_{\text{loc}}^1 \cap L^{p'}$ and $|k(x)| \leqslant M < \infty$ for $|x| > 1$. It will be necessary to assume $p > 1$ for much of the work, and this condition is assumed to hold unless otherwise stated. We will consistently write $p/(p-1) = p'$.

Two capacities are defined. First, we define for an arbitrary point set A,

$$C_{k,p}(A) = \inf_f \|f\|_p^p$$

where the infimum is taken over all f satisfying $k(x, f) \geqslant 1$ on A and $f \in L_+^p$. If no such f exists we write $C_{k,p}(A) = +\infty$. Secondly, we define for an arbitrary point set $A \subset \mathfrak{F}_1$ ($\mathfrak{F}_1$ denotes the σ-field of all sets which are ν-measurable for all $\nu \in \mathscr{L}_1^+$)

$$c_{k,p}(A) = \sup_\nu \|\nu\|_1$$

where the supremum is taken over the set of $\nu \in \mathscr{L}_1^+$ for which $\|k(\nu, \cdot)\|_{p'} \leqslant 1$.

The functions and measures satisfying the conditions in the above definitions are called *test functions* and *test measures* respectively. Let A be any set such that $C_{k,p}(A) < \infty$ and suppose there is a solution to the problem: $\min_f \|f\|_p^p$ where the minimum is obtained in the class of test functions for A. Such a solution is called a *$C_{k,p}$-capacitary distribution for A*, and $k(x, f)$ is called a *$C_{k,p}$-capacitary potential for A*. If $A \subset \mathfrak{F}_1$; then ν is called a *$c_{k,p}$-capacitary distribution*, and $k(\nu, x)$ is called a *$C_{k,p}$-capacitary potential for A* if ν is a test measure satisfying

$$\|\nu\|_1 = c_{k,p}(A).$$

The following facts concerning these two capacities hold for analytic sets A:

(i) $c_{k,p}(A) = [C_{k,p}(A)]^{1/p}$,

(ii) with $k(\nu, f) = \iint k(x-y)f(x)d\nu(y)dx$, $(c_{k,p}(A))^{-1} = \inf_\nu \sup_f k(\nu f) = \sup_f \inf_\nu k(\nu, f)$, where $f \in L_+^p$ with $\|f\|_p = 1$ and $\nu \in \mathscr{L}_1^+$, is concentrated on A, with $\|\nu\|_1 = 1$ and $k(\nu, y) \in L^{p'}$.

(iii) A has a $C_{k,p}$-capacitary distribution different from zero if and only if the functional $k(\nu, f)$ defined for ν and f as in (ii) has a saddle point (ν', f') (i.e. $k(\nu', f') \leqslant k(\nu, f')$ and $k(\nu', f') \geqslant k(\nu', f)$) where $k(\nu', f') > 0$. In this case,

$$(c_{k,p}(A))^{-1} = k(\nu', f'), \quad \nu = c_{k,p}(A)\nu' \quad \text{and} \quad f = c_{k,p}(A)f',$$

where ν and f are the respective capacitary distributions.

(iv) If A has capacitary distributions f and ν, then $(f(y))^{p-1} = (c_{k,p}(A))^{p-1}k(\nu, y)$ almost everywhere with respect to Lebesgue measure. Furthermore, ν is concentrated on a set $B = A \cap \{x: k(x, f) = 1\}$ and $c_{k,p}(B) = c_{k,p}(A)$.

§2. A capacitary weak-type inequality and the Lebesgue set. In this section we describe more fully the classes $\mathscr{L}_\alpha^p$ we will be interested in and obtain a weak-type inequality for capacity.

For $\mathbf{R}^k$ we will consider the kernel $g_a(x)$ where

$$g_a\hat{}(\xi) = (2\pi)^{-k}(1+|\xi|^2)^{-a/2}. \tag{2.1}$$

In fact,

$$g_a(x) = b_a|x|^{\frac{a-k}{2}} K_{\frac{k-a}{2}}(|x|), \quad a > 0 \tag{2.2}$$

where $b_a = 2^{\frac{2-a}{2}}(\Gamma(a/2))^{-1}$. The behavior of g_a is as follows

$$g_a(x) = 2^{\frac{k-a}{2}}\Gamma(k/2-a)(\Gamma(a/2))^{-1}|x|^{a-k}+0(|x|^{a-k}) \quad \text{as } x\to 0 \quad \text{with } 0<a<k, \tag{2.3}$$

$$g_a(x) \cong (2\pi)^{1/2}2^{-a}(\Gamma(a/2))^{-1}|x|^{1/2(a-k-1)}e^{-|x|} \quad \text{as } |x|\to\infty, \tag{2.4}$$

$$\frac{\partial}{\partial x_i} g_a(x) = bx_i\cdot|x|^{1/2(a-k)-1}\cdot K_{\frac{k-a}{2}+1}(|x|). \tag{2.5}$$

In general we will ignore the precise values of constants unless otherwise stated. We will use B or b to indicate constants, sometimes with subscripts, but not always the same at different appearances.

We introduce a *secondary kernel* at this time which we denote by k_a and define by

$$k_a(x) = \sup_{r>0}\frac{1}{r^k}\int_{|y|<r}|g_a(x-y)|\,dy.$$

The following lemma will be useful.

LEMMA 1. *$k_a(x) < b\bar{k}_a(x)$ where*

$$\bar{k}_a(x) = \min(|x|^{a-k}, |x|^{-k}).$$

Proof. There exists numbers $0<r_1<r_2<\infty$ and constants B_1, B_2 and B_3 so that

$$|g_a(x)| < B_1e^{-3/4|x|} \quad \text{if } |x|>r_1, \tag{2.6}$$

$$|g_a(x)| < B_2|x|^{a-k} \quad \text{if } |x|<r_2, \tag{2.7}$$

$$|g_a(x-y)| \leqslant B_3|x|^{-k} \quad \text{if } |x|>r_1 \text{ and } |y|<\frac{|x|}{2}. \tag{2.8}$$

All three of these inequalities are evident consequences of properties (2.3) and (2.4) for $g_a(x)$.

We consider $r^{-k}\int_{|y|<r}|g_a(x-y)|\,dy$ in two cases.

Case 1. ($|x|>r_2$). If $|x|>r_2$ and $r>|x|/2$ we write

$$\int_{|y|<r}|g_a(x-y)|\,dy = \int_{\substack{|y|<r\\|x-y|<r_1}} + \int_{\substack{|y|<r\\|x-y|>r_1}} = I_1+I_2.$$

Then $I_1 \leqslant \int_{|x-y|<r_1}|g_a(x-y)|\,dy \leqslant B_2\int_{|u|<r_1}|u|^{a-k}du = Br_1^a$. Hence $r^{-k}I_1 \leqslant B2^k|x|^{a-k}$.

For I_2 we have

$$I_2 \leqslant B_1\int_{|x-y|>r_1}e^{-3/4|x-y|}dy \leqslant B_1e^{-3/4r_1} \leqslant Br_1^a.$$

Again $r^{-k}I_2 \leqslant B2^k|x|^{a-k}$.

Continuing case 1, we now consider $|x|>r_2$ and $r<|x|/2$. Using (2.8), $|y|<r<|x|/2$, we have

$$r^{-k}\int_{|y|<r}|g_a(x-y)|\,dy \leqslant B_2r^{-k}|x|^{-k}\int_{|y|<r}dy = B|x|^{-k}.$$

Case 2. ($|x|<r_2$). If $|x|<r_2$ and $r>|x|/2$ we write

$$\int_{|y|<r}|g_a(x-y)|\,dy = \int_{\substack{|y|<r\\|x-y|<\frac{|x|}{2}}} + \int_{\substack{|y|<r\\\frac{|x|}{2}<|x-y|}} = J_1+J_2.$$

$$J_1 \leqslant \int_{|x-y|<|x|/2}|g_a(x-y)|\,dy \leqslant B_2\int_{|u|<|x|/2}|u|^{a-k}du = B|x|^a.$$

Hence $r^{-k}J_1 \leqslant B|x|^{a-k}$. Since $|x-y|>|x|/2$ it is easily seen that $r^{-k}J_2 \leqslant r^{-k}B_2|x|^{a-k}\int_{|y|<r}dy = B|x|^{a-k}$.

Finishing case 2, we consider $|x|<r_2$ and $r<|x|/2$. Then

$$\int_{|y|<r}|g_a(x-y)|\,dy \leqslant B_2\int_{|y|<r}|x-y|^{a-k}dy.$$

Since $|x-y| \geqslant |x|-|y| \geqslant |x|-r > |x|/2$ the integral is less than or equal $B|x|^{a-k}r^k$.

Combining case 1 and case 2 completes the Lemma.

We remark that $\bar{k}_a$ is integrable to the p'-th for $ap>k$.

Analogous to k_a we introduce a second kernel k'_a which we define by

$$k'_a(x) = \sup_{r>0}\frac{1}{r^{k-1}}\int_{|y|<r}|\nabla g_a(x-y)|\,dy.$$

For this kernel one can prove in a manner similar to the proof of Lemma 1 the following lemma.

LEMMA 2. *$k'_a(x) < b\bar{k}_a(x)$; where $1<a<k$.*

We now recall the maximal operator of Hardy–Littlewood. Suppose that f is at least locally integrable. Then Mf is defined at x by

$$Mf(x) = \sup_{r>0}\frac{1}{w_kr^k}\int_{|y|<r}|f(x+y)|\,dy$$

where w_k is the volume of the unit sphere in k-dimensions.

We consider the following set $E(f,\lambda) = E_\lambda = \{x\colon Mf(x) > \lambda\}$ and we have the following theorem.

THEOREM 1. *For f in $\mathcal{L}_a^p$ and $k = \bar{k}_a$ we have*

$$c_{k,p}(E_\lambda) \leqslant b\cdot\frac{\|f\|_{p,a}}{\lambda}$$

where the constant b does not depend on f or λ.

Proof. Since E_λ is a Borel set, it is analytic. We may assume $c_{k,p}(E_\lambda) < \infty$. If not, we would truncate in a sphere of large radius and pass to the limit. Let ν be the $c_{k,p}$-capacitary distribution for E_λ. We then have, using the saddle point property of (iii) with f' also the function in (iii) that $1 = k(\nu, f')$. We also have with $f\in\mathcal{L}_a^p$, writing $f = f_0 * g_a$

$$Mf(x) = \sup_{r>0}\frac{1}{w_k r^k}\int_{|y|\leqslant r}|(f_0 * g_a)(x+y)|\,dy$$

$$\leqslant \sup_{r<0}\frac{1}{w_k r^k}\int_{|y|\leqslant r}\int |g_a(x+y-z)|\,|f_0(z)|\,dz\,dy \leqslant b\int \bar{k}_a(x-z)\,|f_0(z)|\,dz.$$

Using this we have

$$c_{k,p}(E_\lambda) = \int_{E_\lambda} d\nu \leqslant \frac{1}{\lambda}\int_{E_\lambda} Mf(x)\,d\nu(x) \leqslant \frac{b}{\lambda}\int\int \bar{k}_a(x-z)\,|f_0(z)|\,dz\,d\nu(x)$$

$$= \frac{b}{\lambda}\bar{k}_a(\nu, |f_0|) = \frac{b}{\lambda}\bar{k}_a\left(\nu, \frac{|f_0|}{\|f_0\|_p}\right)\|f_0\|_p$$

$$\leqslant \frac{b}{\lambda}\bar{k}_a(\nu, f')\,\|f_0\|_p = \frac{b}{\lambda}\|f_0\|_p = \frac{b}{\lambda}\|f\|_{p,a}.$$

Theorem 1 has an immediate corollary concerning the Lebesgue sets of functions in the class $\mathcal{L}_a^p$. We write

$$\bar{D}(f)(x) = \limsup_{r\to 0}\frac{1}{w_k r^k}\int_{|y|<r} f(x+y)\,dy$$

and

$$\underline{D}f(x) = \liminf_{r\to 0}\frac{1}{w_k r^k}\int_{|y|<r} f(x+y)\,dy.$$

Clearly $\bar{D}f(x) \geqslant \underline{D}f(x)$. We call the set where $\underline{D}f(x) = \bar{D}f(x)$ the *Lebesgue set of* f and write for it $\Lambda(f)$. For its complement we write $\Lambda'(f)$.

THEOREM 2. *If $f\in\mathcal{L}_a^p$ then $\Lambda'(f)$ has $c_{k,p}$-capacity zero.*

Proof. As usual the proof of this result relies on approximation by functions g for which $\Lambda'(g)$ is empty. If $f = f_0 * f_a$ then we can approximate f_0 in L^p by a continuous g_0 in L^p. Then $g = g_0 * k_a$ approximates f in $\mathcal{L}_a^p$ as g_0 approximates f_0. Since g is continuous, $\Lambda'(g) = \varphi$. We may write $f = g + h$ where g is continuous and $\|h\|_{p,a} < \varepsilon$. Then if $F_\lambda(f) = \{x\colon \bar{D}f(x) - \underline{D}f(x) > \lambda\}$ we have $F_\lambda(f) = F_\lambda(h) \subset \{x\colon |\bar{D}h(x)| > \lambda/2\} \cup \{x\colon |\underline{D}h(x)| > \lambda/2\}$. We call these two sets $\bar{F}_\lambda(h)$ and $\underline{F}_\lambda(h)$ respectively. Now $\bar{F}_\lambda(h) \subset E_{\lambda/2}(h)$ and so $c_{k,p}(\bar{F}_\lambda(h)) \leqslant c_{k,p}(E_{\lambda/2}(h)) \leqslant \frac{2b}{\lambda}\|h\|_{p,a} < \frac{2\varepsilon b}{\lambda}$. Similarly $c_{k,p}(\underline{F}_\lambda(h)) \leqslant \frac{2\varepsilon b}{\lambda}$. Using the subadditivity of capacities, we obtain $c_{k,p}(F_\lambda(f)) = 0$, since ε is arbitrary. Again noting that $\Lambda'(f) = \bigcup_{\lambda>0} F_\lambda(f)$ and using the subadditivity of the capacity we obtain the result.

Hence if $f\in\mathcal{L}_a^p$, f is equivalent to a function $\bar{f}$ with $\Lambda'(\bar{f})$ of $c_{k,p}$-capacity zero. From this point on, we will use $\bar{f}$ to mean the function satisfying

$$\bar{f}(x) = \lim_{r\to 0}\frac{1}{w_k r^k}\int_{|y|<r} f(x+y)\,dy, \tag{2.9}$$

where the limit exists. An easy argument shows

$$\lim_{r\to 0} r^{-k}\int_{|y|<r}|\bar{f}(x+y) - \bar{f}(x)|\,dy = 0, \quad c_{k,p} \text{ almost everywhere.} \tag{2.10}$$

We now need to prove a result whose proof resembles that of Theorems 1 and 2 almost in entirety. To do this we recall some facts concerning the spaces $\mathcal{L}_a^p$. A good source for this material is [9] chapter V. The Sobolov spaces $L_n^p(\boldsymbol{R}^k)$, are the space of functions belonging to $L^p(\boldsymbol{R}^k)$ having distribution derivatives of orders $\leqslant n$ which are also functions belonging to $L^p(\boldsymbol{R}^k)$. If $a = n$, a non-negative integer, and $1 < p < \infty$, then $\mathcal{L}_n^p(\boldsymbol{R}^k) = L_n^p(\boldsymbol{R}^k)$. For $a < \beta$ we have $\mathcal{L}_a^p \subset \mathcal{L}_\beta^p$. Eventually we will need the Sobolev inequality which states that if $f\in\mathcal{L}_a^p(\boldsymbol{R}^k)$ then $f\in L^q$ if $0 < \frac{1}{q} = \frac{1}{p} - \frac{a}{k}$, with $\|f\|_q \leqslant A_p\|f\|_{p,a}$, A_p depends on p and a only.

We consider $k > a > 1$, and $p > 1$. Consequently, if $f\in\mathcal{L}_a^p$, f has distribution derivatives which are in L^1_{loc} and we can discuss $\nabla f = \left\{\frac{\partial f}{\partial x_i}\right\}_{i=1}^k$. If $f = f_0 * g_a$ it is evident that $\nabla f = f_0 * \nabla g_a$. Since we consider $a > 1$ we have $\nabla f\in L^1_{\mathrm{loc}}$. We fix a point $x_0\in\boldsymbol{R}^k$ and consider $\tilde{f}_{x_0}(r) = \int_{y'\in\Sigma} f(x_0 + ry')\,dy'$ where $\Sigma = \{y\colon |y| = 1\}$. Since $f\in L^1_{\mathrm{loc}}$, we have that $\tilde{f}_{x_0}(r)$ exists for almost every $r > 0$. We can now apply ([4], Theorem 4.5.9; also see [3] page 139) to see that for $s > r > 0$ we have

$$|\tilde{f}_{x_0}(s) - \tilde{f}_{x_0}(r)| \leqslant r^{1-k}\int_{r\leqslant|x-x_0|\leqslant s}|\nabla f(x)|\,dx$$

and that

$$V_r^s(\tilde{f}_{x_0}) \leqslant r^{1-k}\int_{r\leqslant|x-x_0|\leqslant s}|\nabla f(x)|\,dx$$

and

$$\int_\sigma^\tau r^{k-1} d_r [V_\sigma^r(\tilde{f_{x_0}})] \leqslant \int_{\sigma<|x-x_0|<\tau} |\nabla f(x)|\, dx.$$

Having established this notation we state

THEOREM 3. *For $c_{k,p}$-almost all points x_0, and $0<\sigma<\tau<\infty$ there is a constant M_{x_0}, depending on x_0 only, such that*

$$\tau^{1-k}\int_\sigma^\tau r^{k-1} d_r\big(V_\sigma^r(\tilde{f_{x_0}})\big) \leqslant M_{x_0} < \infty. \tag{2.11}$$

Proof. We simply notice that

$$\tau^{1-k}\int_\sigma^\tau r^{k-1} d_r\big(V_\sigma^r(\tilde{f_{x_0}})\big) \leqslant \tau^{1-k}\int_{|x-x_0|<\tau} |\nabla f(x)|\, dx$$
$$\leqslant \tau^{1-k}\int_{|x-x_0|<\tau} |f_0| * |\nabla g_a|(x)\, dx \leqslant (|f_0| * k_a')(x_0) \leqslant b(|f_0| * \bar{k}_a)(x).$$

One can now obtain a weak-type estimate for capacity $c_{k,p}$ as in Theorem 1 for

$$M'f(x) = \sup_{\tau>0} \tau^{1-k}\int_0^\tau r^{k-1} d_r\big(V_\sigma^r(\tilde{f_{x_0}})\big),$$

that is

$$c_{k,p}\{x:\ M'f(x) > \lambda\} \leqslant \frac{b}{\lambda}\|f\|_{a,p}.$$

From this it is clear that $M'f(x) = \infty$ only on a set of $c_{k,p}$-capacity zero, and (2.11) follows.

§ 3. Convergence of Riesz-Bochner summability methods.

We are now ready to prove convergence for Riesz–Bochner summability methods. For the first of these theorems we have:

THEOREM 4. *Suppose $f \in \mathscr{L}_a^p(\mathbf{R}^k) \cap L^1(\mathbf{R}^k)$. For a point where $\bar{f}(x)$ is defined as in (2.10) we have*

$$\sigma_R^\lambda(f, x) \to \bar{f}(x) \quad \text{as } R \to \infty \tag{3.1}$$

where $\lambda > \lambda_0$ and $a > 0$. Hence (3.1) holds $c_{k,p}$-almost everywhere, with the kernel $k = \bar{k}_a$.

THEOREM 5. *Suppose $f \in \mathscr{L}_a^p(T^k)$. For a point where $\bar{f}(x)$ is defined as in (2.10) we have*

$$S_R^\lambda(f, x) \to \bar{f}(x) \quad \text{as } R \to \infty \tag{3.2}$$

where $\lambda > \lambda_0$ and $a > 0$, and hence $c_{k,p}$-almost everywhere.

The proof of either theorem depends basically on the existence of Lebesgue points. We do not reproduce the proof here since it can be found in many sources; for example, in [6].

In Theorem 5 one may interchange $\mathscr{L}_a^p$ for other spaces. One natural choice is the space of Riesz potentials; that is,

$$L_a^p = \{f\colon \hat{f}(n) = \hat{f_0}(n)\cdot|n|^{-a}\colon f \in L^p\}.$$

With this change none of the essential character of the theorem is lost. See [11] or [9] for details on the spaces L_a^p.

We now apply both Theorem 2 and Theorem 3 to obtain

THEOREM 6. *For $c_{k,p}$ almost all points x_0,*

$$\sigma_R^\lambda(f, x_0) \to \bar{f}(x_0), \quad k-ap>0,\ a>1,\ p>1, \quad \lambda > \lambda_0 - 1.$$

Proof. The argument of [3] goes over in total (by using (2.10) and (2.11) in place of (2) and (3) on page 139 of [3]) and hence we have $\sigma_R^\lambda(f, x_0) \to \bar{f}(x_0)$ where ever (2.10) and (2.11) hold.

At this point, we will make some remarks which help to clarify the role of the $c_{k,p}$-capacity and allow an insight into the case $p = 1$, $a > 1$. We use notation from [5]. Let $\sigma_\varrho(x_0) = \{x\colon |x-x_0| < \varrho\}$, $\sigma_\varrho(0) = \sigma_\varrho$ and $\varphi(\varrho) = c_{k,p}(\sigma_\varrho)$. Then φ is a positive increasing function of $\varrho > 0$; in fact, there exists constants b_1 and b_2 such that

$$b_1\varrho^{k-ap} \leqslant \varphi(\varrho) \leqslant b_2\varrho^{k-ap}, \quad k-ap>0$$

and

$$b_1(\log\varrho^{-1})^{1-p} \leqslant \varphi(\varrho) \leqslant b_2(\log\varrho^{-1})^{1-p}, \quad k = ap;\ 0<\varrho<1.$$

For an arbitrary set A, the Hausdorff φ-measure of A is $H_\varphi(A) = \lim_{r\to 0}\big(\inf\sum_{i=1}^\infty \varphi(\varrho_i)\big)$ where the infimum is taken over all countable coverings of A by spheres $\sigma_{\varrho_i}(x_i)$ such that $\varrho_i \leqslant r$. When $\varphi(\varrho) = \varrho^s$ we write $H_\varphi(A) = H_s(A)$, and when $\varphi(\varrho) = (\log\varrho^{-1})^{1-p}$ $(0<\varrho<1)$ we write $H_\varphi(A) = H_{(\log)^{1-p}}(A)$. We restate the following propositions from [5] (given there for Bessel-capacities) for $c_{k,p}$ where $k = \bar{k}_a$.

PROPOSITION 1. *If A is a set for which $H_\varphi(A) < \infty$ then $c_{k,p}(A) = 0$. If $k-ap>0$ then H_φ can be replaced by H_{k-ap} and if $k-ap = 0$ then H_φ can be replaced by $H_{(\log)^{1-p}}$.*

PROPOSITION 2. *If $H_{k-ap+\varepsilon}(A) > 0$ for $\varepsilon > 0$ then $c_{k,p}(A) > 0$.*

The supremum of numbers s, such that $H_s(A) > 0$, is called the Hausdorff dimesion of the set A. For $k-ap>0$ these two propositions indicate the relationship of $k-ap$ to the Hausdorff dimension.

Suppose that $1 < a = a_1 + a_2$ where $a_2 > 0$ and $a_1 > 1$. If $f \in \mathscr{L}_a^1$ we may write $f = f_0 * g_a$ where $f_0 \in L^1$. By (2.1) it is easy to see that $f = (f_0 * g_{a_2}) * g_{a_1}$. By Sobolev's inequality we have $f_0 * g_{a_2} \in L^q$ with $\frac{1}{q} = 1 -$

$-\frac{a_2}{k}$ (see [2], page 36). Hence $f \in \mathscr{L}^q_{a_1}$ and Theorem 4 applies and we have

$$\sigma_R^\beta(f, x) \to \bar{f}(x) \quad c_{k,p} \quad \text{a.e.}, \ \beta > \lambda_0 - 1. \tag{3.10}$$

The "dimension" of this capacity is $\leqslant k - a_1 q = k(k-a)/(k-a_2)$. We let $a_2 \to 0$ and see that the "dimension" of the set where (3.10) holds is not larger than $k-a$.

Remark. The problem of Riesz–Bochner summability for Fourier series below the critical index for exceptional sets remains open. For almost everywhere results in Lebesgue measure see [7] and [8].

Remark. In Theorem 6 it is not necessary to assume $f \in L^1$, as in Theorem 4, provided one uses as the definition of $\sigma_R^\lambda(f, x)$, the Bochner integral representation (see [6]) instead of (1.2).

Acknowledgement. The author wishes to thank the many members of the Purdue University Division of Mathematical Sciences for providing a stimulating environment during the year 1970–71 for which time he was a visitor there. Particular thanks go to the analysis section, including the other visitors during this same year.

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Received July 7, 1971 (358)

STUDIA MATHEMATICA, T. XLIV. (1972)

Some remarks on interpolation of operators and Fourier coefficients

by

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To my teacher, Antoni Zygmund

Abstract. The weak interpolation theory is applied in this note to problems on Fourier coefficients of some special function classes which, in general, do not form linear spaces.

Introduction. The connection between the weak interpolation theory and theorems on Fourier coefficients is well known. In fact the theory of $L(p, q)$ spaces, the cornerstone of the weak interpolation theory, was motivated by the classical theorems of Hardy, Littlewood and Paley on Fourier coefficients.

Recently, we have shown [7], that the theory of weak interpolation can be generalized in a way which permits application also to problems on Fourier coefficients of special classes of functions.

In the first part of this note some results on interpolation are presented in brief, to make the exposition reasonably complete. We then present some applications of the theory to problems on Fourier coefficients. The use of interpolation and $L(p, q)$ notation make the statements and proofs of the theorems more conceptual. In most cases the theorems are also strengthened. We have therefore included theorems proved elsewhere, but by a different technique.

I. Interpolation.

DEFINITION 1. Let $\mathscr{A}$ be a vector space. A subset Q of $\mathscr{A}$ is called a *quasi-cone* (QC) iff $Q+Q \subset Q$. It is a *cone* if also $\lambda Q \subset Q$ for all $0 < \lambda$.

DEFINITION 2. Let $\mathscr{A}$ be a vector space. A quasi-norm on $\mathscr{A}$ is a function $\| \ \|: \mathscr{A} \to R^+$ satisfying:

(a) $\|a\| = 0$ iff $a = 0$.

(b) For all $\lambda \in C$, $a \in \mathscr{A}$, $\|\lambda a\| = |\lambda| \, \|a\|$.

(c) A number $k = k(\mathscr{A})$ exists so that

$$\|a_1 + a_2\| \leqslant k(\|a_1\| + \|a_2\|) \quad \text{all } a_1, a_2 \in \mathscr{A}.$$