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# RIESZ GROUPS WITH A FINITE NUMBER OF DISJOINT ELEMENTS

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Let $G = (G, +, \leqq)$ be an ordered group (henceforth po-group). Two elements $a_1, a_2 \in G$ are disjoint if $a_1 > 0$, $a_2 > 0$, $a_1 \wedge a_2 = 0$, where $a_1 \wedge a_2$ denotes $\inf_G(a_1, a_2)$. $A = \{a_1, \ldots, a_n\}$ is called a disjoint subset of $G$ if $A \subseteqq G^+ \setminus \{0\}$ and any two elements $a_i, a_j \in A$, $i \neq j$ are disjoint.

P. CONRAD in [1] has studied the structure of a lattice-ordered group $G$ satisfying the following condition:

$(c_n)$ $G$ contains an $n$-element disjoint subset but does not contain an $(n + 1)$-element disjoint subset.

$l$-groups with the property $(c_2)$ had been studied by P. CONRAD and A. CLIFFORD in [2] and by F. ŠIK in [8].

Similarly J. JAKUBÍK in [4] has studied a po-group $G$ having the property:

$(q_2)$ There exist two $m$-disjoint elements $x, y \in G$ such that if $A \subseteqq G$ is an $m$-disjoint subset and card $A > 1$, then $A = \{x, y\}$.

($x, y \in G$ will be called $m$-disjoint if $0 \in x \wedge y$, where $x \wedge y$ is a multilattice operation in $G$.)

In this paper, Riesz groups with the property $(c_n)$ are investigated.

**0.** Let $G = (G, +, \leqq)$ be a po-group. $G$ will be called an *interpolation group* if to any $a_1, a_2, b_1, b_2 \in G$ satisfying $a_i \leqq b_j$ $(i = 1, 2; j = 1, 2)$, there exists $c \in G$ such that $a_i \leqq c \leqq b_j$ $(i = 1, 2; j = 1, 2)$ (i.e. the ordered set (po-set) $(G, \leqq)$ satisfies the interpolation property). A directed interpolation group is said to be a *Riesz group.* A po-set $S$ satisfying the interpolation property is said to be an *antilattice-ordered set* if it holds: If $a \wedge b[a \vee b]$ exists in $S$, then $a \wedge b = a$ or $a \wedge b = b[a \vee b = a$ or $a \vee b = b]$. A Riesz group $G = (G, +, \leqq)$ is said to be an *antilattice* if the po-set $(G, \leqq)$ is an antilattice-ordered set. A Riesz group $G$ is an antilattice if and only if it holds: If $a \wedge b = 0$ $(a, b \in G)$, then $a = 0$ or $b = 0$

(See [3, Lemma 7.1].) A po-group $G$ is said to be *regular* if the existence of $\inf_{G^+}(x, y)$ implies the existence of $\inf_G(x, y)$ for $x, y \in G^+$. ($G^+$ denotes the positive cone of $G$.) If $G$ is regular, then $c = \inf_{G^+}(x, y)$ implies $c = \inf_G(x, y)$.

If $\emptyset \neq A$ is a subset of a group $G$, then $\langle A \rangle$ will always denote the subgroup of $G$ that is generated by $A$.

**1.** *Any interpolation group is regular.* (See [6].)

**2.** *Let $G$ be a Riesz group satisfying the property* $(c_n)$ $(n \geqq 2)$ *and let* $\{a_1, \ldots, a_n\}$ *be an n-element disjoint subset of G. Then*

$$H_i = \{x \in G : x \wedge a_j = 0 \text{ for all } j \neq i\}$$

*is an antilattice-ordered convex subsemigroup with* 0 *of* $G^+$ *and* $G_i = \langle H_i \rangle$ *is an antilattice-ordered directed convex subgroup of G.*

Proof. a) Let $x, y \in H_i$, i.e. $x \wedge a_j = y \wedge a_j = 0$ for all $j \neq i$. Then, by [7, Hilfssatz 2], $(x + y) \wedge a_j = 0$ for all $j \neq i$, and hence $H_i$ is a subsemigroup with 0 of $G^+$. It is evident that $H_i$ contains with each element $x$ the whole interval $[0, x]$, therefore $H_i$ is convex.

b) By a) and by [5, Theorem 2.1], $G_i = \langle H_i \rangle$ is a directed convex subgroup of $G$ and $G_i^+ = H_i$. Since $G_i$ is convex and $G$ is an interpolation group, it follows that also $G_i$ is an interpolation group. Let us show that $G_i$ is antilattice-ordered. Let $0 \leqq x, y \in G_i$ (hence $x, y \in H_i$) and let $x \wedge y = 0$. Then $x = 0$ or $y = 0$, for otherwise $\{x, y, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n\}$ would be an $(n + 1)$-element disjoint subset of $G$.

c) From b) and from the regularity of $G$ it follows that $H_i$ is antilattice-ordered.

**3.** Let $G$ be a group, $H_1, \ldots, H_n$ subsemigroups of $G$, and let $A$ be the subsemigroup of $G$ that is generated by $H_1, \ldots, H_n$. Then $A = H_1 \oplus \ldots \oplus H_n$ (see also [1, p. 173]) will mean that

(1) $A = H_1 + \ldots + H_n$,

(2) $H_i \cap (H_1 + \ldots + H_{i-1} + H_{i+1} + \ldots + H_n) = \{0\}$ for all $i = 1, \ldots, n$,

(3) $x_i + x_j = x_j + x_i$ for all $x_i \in H_i$, $x_j \in H_j$, $i \neq j$.

**4.** *Let G be a Riesz group,* $H_1, \ldots, H_n$ $(n \geqq 2)$ *convex subsemigroups with* 0 *of* $G^+$ *such that* $H_i \cap H_j = \{0\}$ *for all* $i \neq j$, *and let A be the subsemigroup of G that is generated by* $H_1, \ldots, H_n$. *Then*

a) $A = H_1 \oplus \ldots \oplus H_n$;

b) *if* $x = x_1 + \ldots + x_n$ *where* $x_i \in H_i$ $(i = 1, \ldots, n)$, *then* $x = x_1 \vee \ldots \vee x_n$;

c) *A is convex.*

Proof. a) Let $x \in H_i$, $y \in H_j$, $i \neq j$. If $0 \leqq z \leqq x, y$, then the convexity of the subsemigroups $H_i, H_j$ implies $z \in H_i \cap H_j$, hence $z = 0$. Since (by 1) any Riesz group is regular, it is $x \wedge y = 0$. Hence by [7, Hilfssatz 2] it holds $x + y = x \vee y = y + x$, therefore $A = H_1 + \ldots + H_n$.

Let $x_i \in H_i \cap (H_1 + \ldots + H_{i-1} + H_{i+1} + \ldots + H_n)$. Then $x_i = x_1 + \ldots + x_{i-1} + x_{i+1} + \ldots + x_n$, where $x_k \in H_k$, $k \in \{1, \ldots, n\} \setminus \{i\}$. Thus the preceding part implies $x_i = x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots \vee x_n$.

Let further $x_i \in H_i$, $x_j \in H_j$, $i \neq j$. Then $x_j \in \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$ implies $0 = x_i \wedge x_j = (x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots \vee x_n) \wedge x_j = x_j$. Hence $0 = x_j$ for all $j \neq i$ and thus also $x_i = 0$. Therefore $A = H_1 \oplus \ldots \oplus H_n$.

b) The assertion b) is now evident.

c) Let $0 \leqq y \leqq x$, $x \in A$. Then $0 \leqq y \leqq x_1 + \ldots + x_n$, where $x_i \in H_i$, $i = 1, \ldots, n$. $G$ is a Riesz group, hence there exist $0 \leqq x_i' \leqq x_i$ $(i = 1, \ldots, n)$ such that $y = x_1' + \ldots + x_n'$. The subsemigroups $H_1, \ldots, H_n$ are convex, therefore $x_i' \in H_i$ $(i = 1, \ldots, n)$, i.e. $y \in A$.

If $G$ is a po-group, then $G = G_1 \boxplus \ldots \boxplus G_n$ means that $G$ is an $o$-direct sum of its $o$-ideals (i.e. normal directed convex subgroups) $G_i$.

**5.** *Let $G$ be a Riesz group satisfying the property $(c_n)$ $(n \geqq 2)$, $\{a_1, \ldots, a_n\}$ an $n$-element disjoint subset of $G$, $H_i = \{x \in G;\ x \wedge a_j = 0$ for all $j \neq i\}$ $(i = 1, \ldots, n)$, $A$ the subsemigroup of $G$ generated by $H_1, \ldots, H_n$. Then $\langle A \rangle = \langle H_1 \rangle \boxplus \ldots \boxplus \langle H_n \rangle$.*

Proof. First let us show that $\langle A \rangle$ is the direct sum $\langle H_1 \rangle \oplus \ldots \oplus \langle H_n \rangle$ of the subgroups $\langle H_1 \rangle, \ldots, \langle H_n \rangle$. Let us prove that for $i \neq j$ it is $H_i \cap H_j = \{0\}$. Let $x \in H_i \cap H_j$. But then $x \wedge a_k = 0$ for all $k = 1, \ldots, n$ and since $G$ has the property $(c_n)$, $x = 0$. Hence (by 4) it holds $A = H_1 \oplus \ldots \oplus H_n$ and $A$ is convex with 0. Therefore (by [5, Theorem 2.1]) $\langle A \rangle$ is a directed convex subgroup of $G$ and $\langle A \rangle^+ = A$.

Now let us show that $H_i$ $(i = 1, \ldots, n)$ is invariant in $A$. Let $y \in A$, $y = h_1 + \ldots + h_n$, $h_i \in H_i$ $(i = 1, \ldots, n)$, $x \in H_i$. Then

$$-y + x + y = -h_n - \ldots - h_1 + x + h_1 + \ldots + h_n,$$

hence by 4

$$-y + x + y = -h_i - h_n - \ldots - h_{i+1} - h_{i-1} - \ldots - h_1 + h_1 + \ldots$$
$$\ldots + h_{i-1} + h_{i+1} + \ldots + h_n + x + h_i = -h_i + x + h_i .$$

Let $j \neq i$. Then $0 = x \wedge a_j = -h_i + (x \wedge a_j) + h_i$, therefore by [7, Hilfssatz 2]

$$0 = (-h_i + x + h_i) \wedge (-h_i + a_j + h_i) =$$
$$= (-h_i + x + h_i) \wedge (-h_i + h_i + a_j) = (-h_i + x + h_i) \wedge a_j .$$

Hence $-h_i + x + h_i \in H_i$. This implies by [5, Theorem 3.1] that $\langle H_i \rangle$ $(i = 1, \dots, n)$ is a normal subgroup of $\langle A \rangle$.

Now let us prove that $\langle A \rangle = \langle H_1 \rangle + \dots + \langle H_n \rangle$. Let $z \in \langle A \rangle$. Then $z = x - y$, where $x, y \in A$, i.e. $x = h_1^{(x)} + \dots + h_n^{(x)}$, $y = h_1^{(y)} + \dots + h_n^{(y)}$, $h_i^{(x)}, h_i^{(y)} \in H_i$, $i = 1, \dots, n$. Thus $z = h_1^{(x)} + \dots + h_n^{(x)} - h_n^{(y)} - \dots - h_1^{(y)} \in \langle\langle H_1 \rangle, \dots, \langle H_n \rangle\rangle$. Since $\langle H_i \rangle = H_i - H_i$ $(i = 1, \dots, n)$ and since all elements of distinct subsemigroups $H_i, H_j$ commute, it holds also that all elements of $\langle H_i \rangle, \langle H_j \rangle$ commute. Hence $\langle\langle H_1 \rangle, \dots, \langle H_n \rangle\rangle = \langle H_1 \rangle + \dots + \langle H_n \rangle$, and so $\langle A \rangle \subseteqq \langle H_1 \rangle + \dots + \langle H_n \rangle$. The converse inclusion is evident.

Let now $x \in \langle H_1 \rangle + \dots + \langle H_{i-1} \rangle + \langle H_{i+1} \rangle + \dots + \langle H_n \rangle$. Then

$$x = h_1 - h_1' + \dots + h_{i-1} - h_{i-1}' + h_{i+1} - h_{i+1}' + \dots + h_n - h_n',$$

where $h_j, h_j' \in H_j$ $(j = 1, \dots, i-1, i+1, \dots, n)$, and thus

$$\begin{aligned} x = h_1 + .. + h_{i-1} + h_{i+1} + \dots + h_n - h_n' - \dots - h_{i+1}' - h_{i-1}' - \dots - h_1' = \\ = (h_1 + \dots + h_{i-1} + h_{i+1} + \dots + h_n) - \\ - (h_1' + \dots + h_{i-1}' + h_{i+1}' + \dots + h_n'). \end{aligned}$$

Hence $x \in \langle H_1 + \dots + H_{i-1} + H_{i+1} + \dots + H_n \rangle$. Therefore $\langle H_1 \rangle + \dots + \langle H_{i-1} \rangle + \langle H_{i+1} \rangle + .. + \langle H_n \rangle = \langle H_1 + \dots + H_{i-1} + H_{i+1} + \dots + H_n \rangle$.

It is clear that $B^{(i)} = H_1 + \dots + H_{i-1} + H_{i+1} + \dots + H_n$ is a subsemigroup with 0 of $G^+$. Indeed, all elements from any distinct summands commute. Let us show that $B^{(i)}$ is convex. Let $0 \leqq y \leqq h_1 + \dots + h_{i-1} + h_{i+1} + \dots + h_n$, $h_j \in H_j$, $j = 1, \dots, i-1, i+1, \dots, n$. Since $G$ is a Riesz group, $y = \bar{h}_1 + \dots + \bar{h}_{i-1} + \bar{h}_{i+1} + \dots + \bar{h}_n$ where $0 \leqq \bar{h}_j \leqq h_j$, $j = 1, \dots, i-1, i+1, \dots, n$. $H_j$ being convex implies $\bar{h}_j \in H_j$, and hence $y \in B^{(i)}$.

Now, since $G$ is a Riesz group it follows by [5, Theorems 2.1, 2.4, 3.1]

$$\begin{aligned} (\langle H_i \rangle \cap (\langle H_1 \rangle + \dots + \langle H_{i-1} \rangle + \langle H_{i+1} \rangle + \dots + \langle H_n \rangle))^+ = \\ = (\langle H_i \rangle \cap \langle H_1 + \dots + H_{i-1} + H_{i+1} + \dots + H_n \rangle)^+ = \\ = \langle H_i \rangle^+ \cap \langle H_1 + \dots + H_{i-1} + H_{i+1} + \dots + H_n \rangle^+ = \\ = H_i \cap (H_1 + \dots + H_{i-1} + H_{i+1} + \dots + H_n) = \{0\}. \end{aligned}$$

The subgroup $\langle H_i \rangle \cap (\langle H_1 \rangle + \dots + \langle H_{i-1} \rangle + \langle H_{i+1} \rangle + \dots + \langle H_n \rangle)$ is directed, thus also $\langle H_i \rangle \cap (\langle H_1 \rangle + \dots + \langle H_{i-1} \rangle + \langle H_{i+1} \rangle + \dots + \langle H_n \rangle) = \{0\}$. Therefore $\langle A \rangle = \langle H_1 \rangle \oplus \dots \oplus \langle H_n \rangle$.

Let now $0 \leqq x \in \langle A \rangle$, $x = x_1 + \dots + x_n$, $x_i \in \langle H_i \rangle$, $i = 1, \dots, n$. Since the subgroups $\langle H_i \rangle$ are directed, it holds

$$0 \leqq x_1 + \dots + x_n \leqq \bar{x}_1 + \dots + \bar{x}_n,$$

where $\bar{x}_i \in U(x_i, 0) \cap \langle H_i \rangle$, $i = 1, \dots, n$. ($U(x, y)$ means the set of all upper bounds of a subset $\{x, y\}$ in $G$.) And since $G$ is a Riesz group, there exist $0 \leqq u_i \leqq \bar{x}_i$ ($i = 1, \ldots, n$) such that

$$x_1 + \ldots + x_n = u_1 + \ldots + u_n\,.$$

$\langle H_i \rangle$ being convex, it is $u_i \in \langle H_i \rangle$, $i = 1, \ldots, n$. And since $\langle A \rangle$ is the direct sum of its subgroups $\langle H_i \rangle$, $0 \leqq x_i = u_i$, $i = 1, \ldots, n$. Therefore $\langle A \rangle = \langle H_1 \rangle \boxplus \ldots \boxplus \langle H_n \rangle$.

**6.** *Let $A$ be a Riesz group such that $A = A_1 \boxplus \ldots \boxplus A_n$, where $A_1, \ldots, A_n$ are antilattices, $A_i \neq \{0\}$ ($i = 1, \ldots, n$). Then $A$ satisfies the condition* ($c_n$).

Proof. Let $x_i \in A_i^+ \setminus \{0\}$, $i = 1, \ldots, n$. Then, by the proof 4a), $x_i \wedge x_j = 0$ for $i \neq j$. Thus $A$ contains an $n$-element disjoint subset. Let $Y = \{y_1, \ldots, y_n, y_{n+1}\}$ be an $(n+1)$-element disjoint subset in $A$, $y_j = y_{j1} + \ldots + y_{jn}$, $y_{ji} \in A_i$, $j = 1, \ldots, n, n+1$, $i = 1, \ldots, n$. But then for each $j \neq k$ and for each $i = 1, \ldots, n$ it is $y_{ji} \wedge y_{ki} = 0$. Since every $A_i$ is an antilattice, $y_{ji} = 0$ or $y_{ki} = 0$. Therefore it must hold that at most one of the $y_{1i}, \ldots, y_{ni}, y_{n+1,i}$ is strictly positive. But this means that some of the elements $y_1, \ldots, y_n, y_{n+1}$ is equal to 0, thus $Y$ is not a disjoint subset in $A$. Therefore $A$ has the property ($c_n$).

Throughout the following $G$ will denote a Riesz group with the property ($c_n$) ($n \geqq 2$), $\{a_1, \ldots, a_n\}$ an $n$-element disjoint subset in $G$, $H_i = \{x \in G;\ x \wedge a_j = 0$ for all $j \neq i\}$ ($i = 1, \ldots, n$), $A$ a subsemigroup of $G$ that is generated by the subsemigroups $H_1, \ldots, H_n$.

**7.** *Let $0 < b_i \in H_i$, $i = 1, \ldots, n$, and let $K_i = \{x \in G;\ x \wedge b_j = 0$ for all $j \neq i\}$. Then $H_i = K_i$, $i = 1, \ldots, n$.*

Proof. Let $x \in H_i$, $i \neq j$ and let $0 \leqq y \in G$ such that $y \leqq b_j, x$. Then the convexity of $H_i, H_j$ yields $y \in H_j \cap H_i$, hence $y = 0$. Therefore $x \wedge b_j = 0$ for all $j \neq i$, and so $x \in K_i$. This implies $H_i \subseteqq K_i$.

Similarly $K_i \subseteqq H_i$.

**8.** *If $\{b_1, \ldots, b_n\}$ is an $n$-element disjoint subset of $G$, then $\{b_1, \ldots, b_n\} \subseteqq A$. Moreover, there exists a permutation $\varphi$ on $\{1, \ldots, n\}$ such that $b_i \in H_{i\varphi}$ for all $i = 1, \ldots, n$.*

Proof. Let $i \neq j$ and let $\neg(b_k \wedge a_i = 0)$, $\neg(b_k \wedge a_j = 0)$. Since $G$ is a Riesz group, there exist $c_{ki}, c_{kj}$ such that $0 < c_{ki} \leqq b_k, a_i$; $0 < c_{kj} \leqq b_k, a_j$. But then $\{b_1, \ldots, b_{k-1}, c_{ki}, c_{kj}, b_{k+1}, \ldots, b_n\}$ is an $(n+1)$-element disjoint subset of $G$. This means that it holds $\neg(b_k \wedge a_i = 0)$ for at most one $i \in \{1, \ldots, n\}$, therefore $b_k \in H_i$ for some $i$. But since $H_i$ is antilattice-ordered, no two of the $b_k$'s can belong to the same $H_i$.

**9.** *$\langle A \rangle$ is a normal subgroup of $G$.*

Proof. Let $i \neq j$, $x, y \in G$, $y \leqq -x + a_i + x$, $y \leqq -x + a_j + x$. Then $x + y - x \leqq a_i, a_j$, hence $x + y - x \leqq 0$. This means $y \leqq -x + x = 0$. Therefore it holds $(-x + a_i + x) \wedge (-x + a_j + x) = 0$. Hence by 8, $0 < -x + a_i + x \in H_{i\varphi}$ for all $i$, where $\varphi$ is a permutation on $\{1, \ldots, n\}$. Thus by 7,

$$-x + A + x = -x + (H_1 \oplus \ldots \oplus H_n) + x \subseteqq H_{1\varphi} \oplus \ldots \oplus H_{n\varphi} = A .$$

Then, by [5, Theorem 3.1], $\langle A \rangle$ is normal in $G$.

## *References*


[1] *Conrad, P.:* The structure of a lattice-ordered group with a finite number of disjoint elements, Michigan Math. J., *7* (1960), 171—180.

[2] *Conrad, P.* and *Clifford, A. H.:* Lattice ordered groups having at most two disjoint elements, Proc. Glasgow Math. Assoc., *4* (1960), 111—113.

[3] *Fuchs, L.:* Riesz groups, Ann. Scuola Norm. Sup. Pisa, Ser. III, *19* (1965), 1—34.

[4] *Jakubík, J.:* Partially ordered groups with two disjoint elements, Colloq. Math., *21* (1970), 39—44.

[5] *Rachůnek, J.:* Directed convex subgroups of ordered groups, Acta Univ. Palack. Olomucensis, Fac. rer. nat., *41* (1973), 39—46.

[6] *Rachůnek, J.:* Prime subgroups of ordered groups, Czechoslovak Math. J., *24* (*99*) (1974), 541—551.

[7] *Šik, F.:* Zum Disjunktivitätsproblem auf geordneten Gruppen, Math. Nachr., *25* (1963), 85—93.

[8] *Šik, F.:* Struktur und Realisierungen von Verbandsgruppen III, Mem. Fac. Cie. Univ. Habana, vol. *1*, no. *4* (1966), 1—20.



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