Filomat 34:3 (2020), 931–944
https://doi.org/10.2298/FIL2003931S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Riesz Potential, Marcinkiewicz Integral and Their Commutators on Mixed Morrey Spaces

Andrea Scapellato[a]

[a]*Università degli Studi di Catania, Dipartimento di Matematica e Informatica, Viale Andrea Doria 6, 95125 Catania, Italy*

Abstract. This paper deals with the boundedness of integral operators and their commutators in the framework of mixed Morrey spaces. Precisely, we study the mixed boundedness of the commutator $[b, I_\alpha]$, where I_α denotes the fractional integral operator of order α and b belongs to a suitable homogeneous Lipschitz class. Some results related to the higher order commutator $[b, I_\alpha]^k$ are also shown. Furthermore, we examine some boundedness properties of the Marcinkiewicz-type integral μ_Ω and the commutator $[b, \mu_\Omega]$ when b belongs to the *BMO* class.

1. Introduction

In the last decades a lot of studies on integral operators and partial differential equations have been carried out. Many authors studied several areas in harmonic analysis, emphasizing real-variable methods, and leading to the study of prosperous areas of research including the Calderón-Zygmund theory of singular integral operators and commutators, the Muckenhoupt theory of A_p weight, the Fefferman-Stein theory of H^p spaces. See for instance the classical book [21] where the author, among other useful contents, discusses about the Calderón-Zygmund decomposition of locally integrable functions, fractional integration, the John-Nirenberg class of functions having bounded mean oscillation and develops the essentials of the Calderón-Zygmund theory of singular integral operators. In the above mentioned book, the author also deals with the Coifman-McIntosh-Meyer real variable approach to Calderón's commutator theorem. As an application of several real-variable methods, in [21], it is treated in detail the problem of the solution to the Dirichlet and Neumann problems on a C^1 domain by means of the layer potential methods.

For a deeper discussion of Calderón-Zygmund theory and weighted norm inequalities, we refer the reader to [11].

A deep study of the theory of fractional integration is contained in [23], where the authors studied the fractional integrations and some topics related to mean oscillation properties of functions, including the classed of Hölder continuous functions and the space of functions having bounded mean oscillation. It is interesting to point out that the motivation for studying fractional integration is provided by a sub-representation formula, which in higher dimensions plays a role roughly similar to the one played by the fundamental theorem of integral calculus in one dimension. The norm estimates for fractional integral operators derived in [23] are applied to obtain local and global first-order Poincaré-Sobolev inequalities,

2010 *Mathematics Subject Classification*. Primary 31B10, 42B35; Secondary 26B10
Keywords. Mixed Morrey spaces, integral operators, commutators
Received: 31 December 2019; Accepted: 07 January 2020
Communicated by Maria Alessandra Ragusa
Email address: scapellato@dmi.unict.it (Andrea Scapellato)

including endpoint cases. In this context it is useful to emphasize that the authors also extended the above subrepresentation formula for smooth functions to functions with a weak gradient.

Let us fix $T > 0$ and, for $t \in [0, T]$ and $0 < \alpha < n$, let us consider the Riesz fractional integral operator of order α (Riesz potential) defined by

$$I_\alpha f(x,t) = \int_{\mathbb{R}^n} \frac{f(y,t)}{|x-y|^{n-\alpha}}\, \mathrm{d}y, \qquad \text{a.e. in } \mathbb{R}^n.$$

For a locally integrable function b, the commutator is defined by

$$[b, I_\alpha] = b(x,t) I_\alpha f(x,t) - I_\alpha(bf)(x,t), \qquad b \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times [0,T]),\ b(x,t) = b(x).$$

Adams ([1]) proved that the fractional integral operator is bounded from the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$. Later, Chiarenza and Frasca ([5]) gave another proof of this boundedness result.

Di Fazio and Ragusa ([9]) showed that if b is in the class $BMO(\mathbb{R}^n)$ of functions having bounded mean oscillation, then for suitable p, q, λ, the commutator $[b, I_\alpha]$ is bounded from the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, and conversely, under some restriction on α, if the commutator $[b, I_\alpha]$ is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, then $b \in BMO(\mathbb{R}^n)$.

Continuing this study of commutators, in this paper we prove some new results dealing with the boundedness of the Marcinkiewicz integral and the boundedness of the commutator associated to such integral and a function b having bounded mean oscillation.

Let us define these operators, denoting by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$, $n \geq 2$, equipped with the normalized Lebesgue measure $\mathrm{d}\sigma$. Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $1 < q \leq \infty$ be homogeneous of degree zero and satisfy the cancellation property

$$\int_{\mathbb{S}^{n-1}} \Omega(x')\, \mathrm{d}x' = 0$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$.

The Marcinkiewicz integral of higher dimension μ_Ω is defined by

$$\mu_\Omega(f)(x,t) = \left(\int_0^\infty |F_{\Omega,s}(x,t)|^2 \frac{\mathrm{d}s}{s^3} \right)^{\frac{1}{2}},$$

where

$$F_{\Omega,s}(x,t) = \int_{|x-y|\leq s} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y,t)\, \mathrm{d}y.$$

In the sequel we consider the commutator $[b, \mu_\Omega]$ defined as follows:

$$[b, \mu_\Omega] f(x,t) = \left(\int_0^\infty |F^b_{\Omega,s}(x,t)|^2 \frac{\mathrm{d}s}{s^3} \right)^{\frac{1}{2}},$$

where

$$F^b_{\Omega,s}(x,t) = \int_{|x-y|\leq s} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)] f(y,t)\, \mathrm{d}y.$$

Let $\mathbb{S}^{n-1}$ stand for the unit sphere in $\mathbb{R}^n$, with $n \geq 2$, equipped with the normalized Lebesgue measure $\mathrm{d}\sigma$.

Stein ([19]) proved that, if $\Omega \in \mathrm{Lip}_\alpha(\mathbb{S}^{n-1})$ (i.e., $|\Omega(x) - \Omega(y)| \leq |x-y|^\alpha$), with $0 < \alpha \leq 1$, then μ_Ω is of type (p,p), for $1 < p \leq 2$ and of weak type $(1,1)$.

The *weighted* boundedness of Marcinkiewicz integral was firstly studied by Torchinsky and Wang in [22]. They proved that if $\Omega \in \mathrm{Lip}_{\alpha}(\mathbb{S}^{n-1})$, for $0 < \alpha \leq 1$, and $w \in A_p$, for $1 < p < \infty$, then μ_Ω is bounded on $L^p(w)$. In the same paper, Torchinsky and Wang also proved the L^p-boundedness of the commutator $[\mu_\Omega, b]$ for $1 < p < \infty$, if $\Omega \in \mathrm{Lip}_{\alpha}(\mathbb{S}^{n-1})$, for $0 < \alpha \leq 1$. Ding, Lu and Yabuta in [10] proved the L^p-boundedness of the above commutator if $\Omega \in L^q(\mathbb{S}^{n-1})$, for $1 < q \leq \infty$.

Following the Stein's point of view, in his book [20] on singular integrals, the Marcinkiewicz integral is the key to the L^p boundedness of the operators. Nowadays, although there are many other approaches to singular integral theory, many authors studied boundedness properties of the Marcinkiewicz integral. For recent developments of this area, we refer the reader, for example, to the papers [2, 3, 6–8].

In line with the contents of the paper [18], we investigate the boundedness of the operator μ_Ω, the commutators $[b, \mu_\Omega]$ and higher order commutators on mixed Morrey spaces.

2. Mathematical background

Let us assume that Ω is a bounded open set of $\mathbb{R}^n$ such that there exists $A > 0$ such that $|Q(x, \rho) \cap \Omega| \geq A\rho^n$ for every $x \in \Omega$ and $\rho \in [0, \mathrm{diam}(\Omega)]$, being $Q(x, \rho)$ a cube centered in x, with edges parallel to the coordinate axes and length 2ρ.

First of all we recall the definition of classical Morrey space ([14]).

Let $p \in]1, \infty[$, $\lambda \in]0, n[$ and f be a real measurable function defined in $\Omega \subset \mathbb{R}^n$. If $|f|^p$ is locally summable in Ω and the set described by the quantity

$$\frac{1}{\rho^\lambda} \int\limits_{\Omega \cap B_\rho(x)} |f(y)|^p \, \mathrm{d}y,$$

when ρ varies in $]0, \mathrm{diam}\,\Omega[$ and x varies in Ω, has an upper bound, then we say that f belongs to the *Morrey Space* $L^{p,\lambda}(\Omega)$.

If $f \in L^{p,\lambda}(\Omega)$, we define

$$\|f\|^p_{L^{p,\lambda}(\Omega)} := \sup_{\substack{x \in \Omega \\ 0<\rho<\mathrm{diam}\,\Omega}} \frac{1}{\rho^\lambda} \int\limits_{\Omega \cap B_\rho(x)} |f(y)|^p \, \mathrm{d}y \tag{1}$$

and the vector space naturally associated to the set of functions in $L^p(\Omega)$ such that (1) is finite, endowed with the norm (1), is a Banach space.

The exponent λ can take values that are not belonging only to $]0, n[$ but the unique cases of real interest are those for which $\lambda \in]0, n[$.

Similarly we can define the Morrey space in $L^{p,\lambda}(\mathbb{R}^n)$ as the space of functions such that is finite:

$$\|f\|^p_{L^{p,\lambda}(\mathbb{R}^n)} := \sup_{\substack{x \in \mathbb{R}^n \\ \rho > 0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f(y)|^p \, \mathrm{d}y.$$

The above defined space is used in the theory of regularity of solutions to nonlinear partial differential equations and for the study of local behavior of solutions to nonlinear equations and systems (see, for instance, [15, 16]).

The following definition appears in the recent paper [18].

Definition 2.1. *Let $1 < p, q < +\infty$, $0 < \lambda < n$, $0 < \mu < 1$. We define the set $L^{q,\mu}(0, T, L^{p,\lambda}(\Omega))$ as the class of functions $f : \Omega \times [0, T] \to \mathbb{R}$ such that the quantity*

$$\|f\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\Omega))} := \left(\sup_{\substack{t_0 \in [0,T] \\ \rho>0}} \frac{1}{\rho^\mu} \int_{[0,T]\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x \in \Omega \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{\Omega \cap B_\rho(x)} |f(y,t)|^p \, \mathrm{d}y \right)^{\frac{q}{p}} \mathrm{d}t \right)^{\frac{1}{q}}, \tag{2}$$

is finite. The same definition holds if $\Omega = \mathbb{R}^n$.

It can be shown that the linear space naturally of functions f such that (2) is finite endowed with the norm (2), is a Banach space.

For further details and recent results dealing with the Morrey spaces with mixed norm, we refer the reader to [4].

Definition 2.2 ([12]). *Let f be a locally integrable function defined on $\mathbb{R}^n$. We say that f is in the space $BMO(\mathbb{R}^n)$ of functions having bounded mean oscillation if*

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |f(y)-f_B|dy < \infty$$

where B runs over the class of all balls in $\mathbb{R}^n$ and $f_B = \frac{1}{|B|}\int_B f(y)dy$.

Let $f \in BMO(\mathbb{R}^n)$ and $r > 0$. We define the function

$$\eta(r) = \sup_{\rho\le r} \frac{1}{|B_\rho|}\int_{B_\rho} |f(y)-f_{B_\rho}|dy$$

where B_ρ is a ball with radius ρ, $\rho \le r$.

BMO is a Banach space with the norm $\|f\|_* = \sup_{r>0} \eta(r)$.

The following theorem holds.

Theorem 2.3. *Let $b \in BMO(\mathbb{R}^n)$. Then, for any $1 \le p < \infty$, we have*

$$\sup_B \left(\frac{1}{|B|}\int_B |b(x)-b_B|^p \, dx \right)^{\frac{1}{p}} \le c\|b\|_*.$$

Definition 2.4 ([17]). *We say that a function $f \in BMO$ is in the Sarason class $VMO(\mathbb{R}^n)$ of functions with vanishing mean oscillation if*

$$\lim_{r\to 0^+} \eta(r) = 0.$$

The function η is said to be the VMO modulus of f.

Definition 2.5 (Lipschitz space). *We define the homogeneous Lipschitz space of order β, $0 < \beta < 1$, by*

$$\dot{\Lambda}_\beta(\mathbb{R}^n) = \{f : |f(x)-f(y)| \le C|x-y|^\beta\}.$$

The smallest constant $C > 0$ is the Lipschitz norm $\|\cdot\|_{\dot{\Lambda}_\beta}$.

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we recall the following Hardy-Littlewood maximal function

$$Mf(x) = \sup_{\rho>0} \frac{1}{|B_\rho(x)|}\int_{B_\rho(x)} |f(y)|\, dy,$$

where $B_\rho(x)$ is a ball centered at x and with radius ρ.

Given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ the sharp maximal function is defined by

$$f^\sharp(x) = \sup_{B\supset\{x\}} \frac{1}{|B|}\int_B |f(y) - f_B|\, dy,$$

for a.e. $x \in \mathbb{R}^n$, where B is a generic ball in $\mathbb{R}^n$.

Set $t \in [0, T]$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times [0, T])$ and $0 < \eta < 1$. Let us consider the fractional maximal function

$$(M_\eta f)(x) = \sup_{B \supset \{x\}} \frac{1}{|B|^{1-\eta}} \int_B |f(y,t) - f_B| \, dy,$$

for a.e. $x \in \mathbb{R}^n$, where B is a generic ball in $\mathbb{R}^n$.

Throughout the paper, we write $A \lesssim B$ to mean that there exists a constant $C > 0$ such that $A \leq CB$. Moreover, we write $A \sim B$ if there exists a constant $C > 1$ such that $\frac{1}{C} \leq \frac{A}{B} \leq C$.

3. Boundedness of the commutator $[b, I_\alpha]$

In order to prove our theorems we need some technical results. A useful tool is a pointwise inequality that connect the sharp maximal function and the fractional integral operators. The classical L^p case is discussed in [21] where the reader can find a proof due to Strömberg of a result of Coifman, Rochberg and Weiss.

In [9] Di Fazio and Ragusa obtain a similar result in the framework of classical Morrey spaces.

Lemma 3.1 ([9])**.** *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$, $1 < r, s < \min\left(p, \frac{n}{\alpha}\left(1 - \frac{\lambda}{n}\frac{1}{p}\right)\right)$ and $b \in BMO(\mathbb{R}^n)$. Then, there exists a constant $C > 0$, independent of b and f, such that*

$$([b, I_\alpha](f))^\sharp(x) \leq C\|b\|_* \left[(M|I_\alpha f|^r)^{\frac{1}{r}}(x) + (M_{\frac{\alpha s}{n}}|f|^s)^{\frac{1}{s}}(x) \right]$$

for almost all $x \in \mathbb{R}^n$ and every $f \in L^{p,\lambda}(\mathbb{R}^n)$.

We can naturally extend the previous result to the case $f \in L^{q,\mu}(0, T, L^{p,\lambda}(\mathbb{R}^n))$, with $0 < \mu < 1$, $1 < q < \infty$. The next results are contained in [18].

Theorem 3.2. *Let $1 < p < +\infty$, $0 < \lambda < n$, $1 < q' < +\infty$, $0 < \mu < 1$ and $f \in L^{q',\mu}(0, T, L^{p,\lambda}(\mathbb{R}^n))$. Then, there exists a positive constant C, independent of f, such that*

$$\|Mf\|_{L^{q',\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \leq C \, \|f\|_{L^{q',\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))}.$$

Theorem 3.3. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$, $1 < q' < +\infty$, $0 < \mu' < 1$ and $f \in L^{q',\mu'}(0, T, L^{p,\lambda}(\mathbb{R})^n)$. Then, there exists a positive constant C, independent of f, such that*

$$\|I_\alpha f\|_{L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))} \leq C\|f\|_{L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))}.$$

Theorem 3.4. *Let $1 < p, q < \infty$, $0 < \lambda < n$, $0 < \mu < 1$ and $f \in L^{q,\mu}(0, T, L^{p,\lambda}(\mathbb{R}^n))$. Then, there exists a positive constant C, independent of f, such that*

$$\|Mf\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))} \leq C \, \|f^\sharp\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\mathbb{R}^n))}.$$

Theorem 3.5. *Let $1 < p, q, q_1 < \infty$, $0 < \lambda < n$, $0 < \mu_1 < 1$ and $f \in L^{q_1,\mu_1}(0, T, L^{p,\lambda}(\mathbb{R}^n))$. Then, for every $\eta \in]0, (1 - \frac{\lambda}{n})\frac{1}{p}[$, there exists a positive constant C, independent of f, such that*

$$\|M_\eta f\|_{L^{q_1,\mu_1}(0,T,L^{q,\lambda}(\mathbb{R}^n))} \leq C \, \|f\|_{L^{q_1,\mu_1}(0,T,L^{p,\lambda}(\mathbb{R}^n))}$$

where $\frac{1}{q} = \frac{1}{p} - \frac{n\eta}{n-\lambda}$.

The next results deal with the boundedness of commutator in two different cases: while in the first we assume that the multiplication function b has bounded mean oscillation, in the second one we take b in a homogeneous Lipschitz space.

Theorem 3.6. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$, $1 < q' < +\infty$, $0 < \mu' < 1$, $f \in L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ *and* $b \in BMO(\mathbb{R}^n \times [0,T])$, $b(x,t) = b(x)$. *Then,* $[b, I_\alpha]$ *is bounded from* $L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ *to* $L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))$.

Proof. Let us fix

$$1 < r, s < \min\left(p, \frac{n}{\alpha}\left(1 - \frac{\lambda}{n}\right)\frac{1}{p}\right).$$

Using Lemma 3.1, Theorems 3.2-3.5, we obtain:

$$\begin{aligned}
\|[b,I_\alpha]f\|_{L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))} &\le \|M([b,I_\alpha])(f)\|_{L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))} \\
&\le C\|([b,I_\alpha](f))^\sharp\|_{L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))} \\
&\le C\|b\|_* \left[\|(M|I_\alpha f|^r)^{\frac{1}{r}}\|_{L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))} + \|(M_{\frac{\alpha s}{n}}|f|^s)^{\frac{1}{s}}\|_{L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))}\right] \\
&\le C\|b\|_* \left[\|M|I_\alpha f|^r\|^{\frac{1}{r}}_{L^{q',\mu'}(0,T,L^{\frac{q}{r},\lambda}(\mathbb{R}^n))} + \|M_{\frac{\alpha s}{n}}|f|^s\|^{\frac{1}{s}}_{L^{q',\mu'}(0,T,L^{\frac{q}{s},\lambda}(\mathbb{R}^n))}\right] \\
&\le C\|b\|_* \left[\||I_\alpha f|^r\|^{\frac{1}{r}}_{L^{q',\mu'}(0,T,L^{\frac{q}{r},\lambda}(\mathbb{R}^n))} + \||f|^s\|^{\frac{1}{s}}_{L^{q',\mu'}(0,T,L^{\frac{p}{s},\lambda}(\mathbb{R}^n))}\right] \\
&\le C\|f\|_{L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))},
\end{aligned}$$

and this completes the proof of the theorem. □

Theorem 3.7. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$, $1 < q' < +\infty$, $0 < \mu' < 1$, $f \in L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$, $0 < \beta < 1$, $0 < \alpha + \beta < n$ *and* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. *Then,* $[b, I_\alpha]$ *is bounded from* $L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ *to* $L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))$.

Proof. We begin by proving a pointwise inequality. Precisely, from the definition of the function space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ it follows that:

$$\begin{aligned}
|[b,I_\alpha]f(x,t)| &= \left|\int_{\mathbb{R}^n} \frac{(b(x)-b(y))f(y,t)}{|x-y|^{n-\alpha}}\,\mathrm{d}y\right| \\
&\le \int_{\mathbb{R}^n} \frac{|b(x)-b(y)|\cdot|f(y,t)|}{|x-y|^{n-\alpha}}\,\mathrm{d}y \\
&\le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|f(y,t)|}{|x-y|^{n-(\alpha+\beta)}}\,\mathrm{d}y \\
&= C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} I_{\alpha+\beta}(|f|)(x),
\end{aligned}$$

for a.e. $(x,t) \in \mathbb{R}^n \times [0,T]$.

Then, using Theorem 3.3, we obtain

$$\begin{aligned}
\|[b,I_\alpha]f\|_{L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))} &\le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|I_{\alpha+\beta}(|f|)\|_{L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))} \\
&\le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|f\|_{L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))}
\end{aligned}$$

and the proof is complete. □

Let $T > 0$ and for $t \in]0,T[$, we consider a higher order commutator operator defined as follows:

$$[b,I_\alpha]^k f(x,t) := \int_{\mathbb{R}^n} \frac{\Delta_\xi^k b(x) f(\xi,t)}{|\xi|^{n-\alpha}}\,\mathrm{d}\xi,$$

where

$$\Delta_\xi^1 b(x) = \Delta_\xi b(x) = b(x+\xi) - b(x), \quad \Delta_\xi^{k+1} b(x) = \Delta_\xi^k b(x) - \Delta_\xi^k b(y), \ k \geq 1.$$

Let $0 < \beta < k \leq n$, k an integer and n be the dimension of the whole space. For $\beta > 0$, we say that b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ if

$$\|b\|_{\dot{\Lambda}_\beta} = \sup_{\substack{x,\xi \in \mathbb{R}^n \\ x \neq h}} \frac{|\Delta_\xi^k b(x)|}{|\xi|^\beta} < \infty, \qquad k \geq 1.$$

Using the same argument as in Theorem 3.7, it is possible to prove the following result.

Theorem 3.8. *Under the same assumptions as Theorem 3.7, if $b = b_1 + P$, where $b_1 \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and P is a polynomial of degree less than k, then $[b, I_\alpha]^k$ is bounded from $L^{q',\mu'}(0,T,L^{p,\lambda}(\mathbb{R}^n))$ to $L^{q',\mu'}(0,T,L^{q,\lambda}(\mathbb{R}^n))$.*

4. Estimates for the Marcinkiewicz integral and its commutator

The main goal of this section is to prove two boundedness results; the first one concerns μ_Ω, the second one deals with the commutator $[b, \mu_\Omega]$.

Theorem 4.1. *Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $1 < q \leq \infty$. Then, for every $1 < p < \infty$, $0 < \lambda < n$, $0 < \mu < 1$, there exists a positive constant C, independent of f, such that*

$$\|\mu_\Omega(f)\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\Omega))} \leq C\|f\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\Omega))}.$$

Proof. The proof is divided in two steps. In the first step we obtain a classical Morrey-type inequality; in the second step, integrating and taking the supremum, we achieve the mixed Morrey norm.

First step.

Let us fix a ball $B = B(x_0, r) \subseteq \mathbb{R}^n$ and let $kB = B(x_0, kr)$, for any $k > 0$. Let $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$, $f_2 = f\chi_{(2B)^c}$ and χ_{2B} denotes the characteristic function of $B(x_0, 2r)$. Then, using Minkowski inequality, we have

$$\frac{1}{r^{\frac{\lambda}{p}}}\left(\int_B |\mu_\Omega f(x,t)|^p \, dx\right)^{\frac{1}{p}} \leq \frac{1}{r^{\frac{\lambda}{p}}}\left(\int_B |\mu_\Omega f_1(x,t)|^p \, dx\right)^{\frac{1}{p}} + \frac{1}{r^{\frac{\lambda}{p}}}\left(\int_B |\mu_\Omega f_2(x,t)|^p \, dx\right)^{\frac{1}{p}} = K_1(t) + K_2(t).$$

The classical L^p boundedness of Marcinkiewicz integral with rough kernel (see, e.g., [13]) implies that

$$\begin{aligned}
K_1(t) &\lesssim \frac{1}{|B|^{\frac{\lambda}{np}}}\left(\int_{2B} |\mu_\Omega f(x,t)|^p \, dx\right)^{\frac{1}{p}} \\
&\lesssim \frac{(2r)^{\frac{\lambda}{p}}}{r^{\frac{\lambda}{p}}} \sup_{B \subseteq \mathbb{R}^n} \frac{1}{(2r)^{\frac{\lambda}{p}}}\left(\int_{2B} |\mu_\Omega f(x,t)|^p \, dx\right)^{\frac{1}{p}} \\
&\lesssim \left(\sup_{\substack{x \in \mathbb{R}^n \\ \rho > 0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f(y,t)|^p \, dy\right)^{\frac{1}{p}}
\end{aligned}$$

In order to estimate K_2, we observe that, if $x \in B$ and $y \in 2^{j+1}B \setminus 2^j B$, $j \geq 1$, then

$$|x-y| \geq |y-x_0| - |x-x_0| \geq 2^{j-1}r.$$

Therefore

$$
\begin{aligned}
K_2(t) &\le \left(\int_0^\infty \left| \int_{(2B)^c \cap \{y: |x-y| \le s\}} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y,t)\,\mathrm{d}y \right|^2 \frac{\mathrm{d}s}{s^3} \right)^{\frac{1}{2}} \\
&\le \sum_{j=1}^{\infty} \left(\int_{2^{j+1}B \setminus 2^j B} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y,t)|\,\mathrm{d}y \right) \cdot \left(\int_{2^{j-1}r}^{\infty} \frac{\mathrm{d}s}{s^3} \right)^{\frac{1}{2}} \\
&\lesssim \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{\frac{1}{n}}} \int_{2^{j+1}B \setminus 2^j B} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y,t)|\,\mathrm{d}y.
\end{aligned}
$$

Let $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Using Hölder inequality we obtain the following estimate:

$$
\begin{aligned}
|\mu_\Omega(f_2)(x,t)| &\le C\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \sum_{j=1}^{+\infty} \frac{1}{|2^{j+1}B|^{\frac{1}{n}}} \cdot \frac{1}{|2^{j+1}B|^{\frac{n-1}{n}}} \int_{2^{j+1}B} |f(y,t)|\,\mathrm{d}y \\
&\lesssim \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(y,t)1^{\frac{1}{p}}|1^{-\frac{1}{p}}\,\mathrm{d}y \\
&\lesssim \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \frac{1}{(2^{j+1}r)^{\frac{\lambda}{p}}} \left(\int_{2^{j+1}B} |f(y,t)|^p 1\,\mathrm{d}y \right)^{\frac{1}{p}} (2^{j+1}r)^{\frac{\lambda}{p}} |2^{j+1}B|^{\frac{1}{p'}} \\
&\lesssim \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \frac{1}{(2^{j+1}r)^{\frac{\lambda}{p}}} \left(\int_{2^{j+1}B} |f(y,t)|^p 1\,\mathrm{d}y \right)^{\frac{1}{p}} (2^{j+1}r)^{\frac{\lambda}{p}} \frac{|2^{j+1}B|}{|2^{j+1}B|^{\frac{1}{p}}} \\
&\lesssim \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \frac{1}{(2^{j+1}r)^{\frac{\lambda}{p}}} \left(\int_{2^{j+1}B} |f(y,t)|^p 1\,\mathrm{d}y \right)^{\frac{1}{p}} (2^{j+1}r)^{\frac{\lambda}{p}} \frac{|2^{j+1}B|}{|2^{j+1}r|^{\frac{n}{p}}} \\
&\lesssim \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^\lambda} \int_{B_\rho(x)} |f(y,t)|^p\,\mathrm{d}y \right)^{\frac{1}{p}} \sum_{j=1}^{\infty} |2^{j+1}r|^{\frac{\lambda-n}{p}}.
\end{aligned}
$$

Now, let $\Omega \in L^q(\mathbb{S}^{n-1})$, $1 < q < \infty$. From Hölder inequality we get

$$
\begin{aligned}
|\mu_\Omega(f_2)(x,t)| &\lesssim \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{\frac{1}{n}}} \int_{2^{j+1}B \setminus 2^j B} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y,t)|\,\mathrm{d}y \\
&\lesssim \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{\frac{1}{n}}} \left(\int_{2^{j+1}B \setminus 2^j B} |\Omega(x-y)|^q\,\mathrm{d}y \right)^{\frac{1}{q}} \left(\int_{2^{j+1}B \setminus 2^j B} \frac{|f(y,t)|^{q'}}{|x-y|^{nq'}}\,\mathrm{d}y \right)^{\frac{1}{q'}}.
\end{aligned}
$$

For $x \in B$ and $y \in 2^{j+1}B \setminus 2^jB$, a direct calculation shows that $2^{j-1}r \le |y-x| < 2^{j+2}r$. Hence

$$\begin{aligned}\left(\int\limits_{2^{j+1}B\setminus 2^jB} |\Omega(x-y)|^q \,\mathrm{d}y\right)^{\frac{1}{q}} &= \left(\int\limits_{2^{j-1}r\le|z|\le 2^{j+1}r} |\Omega(z)|^q \,\mathrm{d}z\right)^{\frac{1}{q}} \\ &= \left(\int\limits_{2^{j-1}r}^{2^{j+1}r}\int\limits_{\mathbb{S}^{n-1}} |\Omega(z)|^q \rho^{n-1}\,\mathrm{d}\sigma(z)\,\mathrm{d}\rho\right)^{\frac{1}{q}} \\ &\le \|\Omega\|_{L^q(\mathbb{S}^{n-1})}\left(\int\limits_{2^{j-1}r}^{2^{j+1}r} \rho^{n-1}\,\mathrm{d}\rho\right)^{\frac{1}{q}} \\ &\lesssim \|\Omega\|_{L^q(\mathbb{S}^{n-1})}|2^{j+1}B|^{\frac{1}{q}}.\end{aligned} \tag{3}$$

We also note that if $x \in B$, $y \in (2B)^c$, then $|y-x| \sim |y-x_0|$. Consequently we have

$$\left(\int\limits_{2^{j+1}B\setminus 2^jB} \frac{|f(y,t)|^{q'}}{|x-y|^{nq'}}\,\mathrm{d}y\right)^{\frac{1}{q'}} \le \frac{1}{|2^{j+1}B|}\left(\int\limits_{2^{j+1}B} |f(y,t)|^{q'}\,\mathrm{d}y\right)^{\frac{1}{q'}}.$$

So we have

$$|\mu_\Omega(f_2)(x,t)| \lesssim \|\Omega\|_{L^q(\mathbb{S}^{n-1})}\sum_{j=1}^{\infty}\left(\frac{1}{|2^{j+1}B|}\int\limits_{2^{j+1}B} |f(y,t)|^{q'}\,\mathrm{d}y\right)^{\frac{1}{q'}}.$$

Let us now obtain another useful inequality. If $p = q'$, we have

$$\begin{aligned}\mu_\Omega(f_2)(x,t) &\lesssim \|\Omega\|_{L^q(\mathbb{S}^{n-1})}\sum_{j=1}^{\infty}\left(\frac{1}{|2^{j+1}B|}\int\limits_{2^{j+1}B} |f(y,t)|^{q'}\,\mathrm{d}y\right)^{\frac{1}{q'}} \\ &\lesssim \|\Omega\|_{L^q(\mathbb{S}^{n-1})}\sum_{j=1}^{\infty}\left(\frac{1}{|2^{j+1}B|}\int\limits_{2^{j+1}B} |f(y,t)|^{p}\,\mathrm{d}y\right)^{\frac{1}{p}} \\ &\le \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p\,\mathrm{d}y\right)^{\frac{1}{p}} \sum_{j=1}^{\infty} |2^{j+1}r|^{\frac{\lambda-n}{p}}.\end{aligned}$$

Thanks to a straightforward calculation, similar to that one in the case $\Omega \in L^\infty(\mathbb{S}^{n-1})$, we have

$$|\mu_\Omega(f_2)(x,t)| \lesssim \|\Omega\|_{L^q(\mathbb{S}^{n-1})}\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p\,\mathrm{d}y\right)^{\frac{1}{p}} \sum_{j=1}^{\infty} |2^{j+1}r|^{\frac{\lambda-n}{p}}.$$

Hence, for $1 < q \le \infty$, $1 < p < \infty$, taking into account the estimations above, we have that

$$K_2 \lesssim \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p\,\mathrm{d}y\right)^{\frac{1}{p}}.$$

Combining the inequalities for K_1 and K_2 and taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we get the $L^{p,\lambda}-$estimates.

Second step. From the previous step, we have the classical Morrey inequality

$$\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |\mu_\Omega f(y,t)|^p\,\mathrm{d}y\right)^{\frac{1}{p}} \lesssim \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p\,\mathrm{d}y\right)^{\frac{1}{p}}.$$

Elevating to q, integrating over $[0,T]\cap(t_0-\rho,t_0+\rho)$, it follows that

$$\int_{[0,T]\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |\mu_\Omega f(y,t)|^p\,\mathrm{d}y\right)^{\frac{q}{p}}\mathrm{d}t \lesssim \int_{[0,T]\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p\,\mathrm{d}y\right)^{\frac{q}{p}}\mathrm{d}t.$$

Multiplying the inequality above to $\frac{1}{\rho^\mu}$ and taking the supremum of both sides and, finally, elevating both sides to $\frac{1}{q}$, we obtain

$$\left(\sup_{\substack{t_0\in[0,T]\\ \rho>0}} \frac{1}{\rho^\mu}\int_{[0,T]\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |\mu_\Omega f(y,t)|^p\,\mathrm{d}y\right)^{\frac{q}{p}}\mathrm{d}t\right)^{\frac{1}{q}} \lesssim$$

$$\lesssim \left(\sup_{\substack{t_0\in[0,T]\\ \rho>0}} \frac{1}{\rho^\mu}\int_{[0,T]\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p\,\mathrm{d}y\right)^{\frac{q}{p}}\mathrm{d}t\right)^{\frac{1}{q}}.$$

and the proof is complete. □

Theorem 4.2. *Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $1 < q \leq \infty$ and $b \in BMO(\mathbb{R}^n \times [0,T])$, $b(x,t) = b(x)$. Then, for every $1 < p < \infty$, $0 < \lambda < n$, $0 < \mu < 1$, there exists a positive constant C, independent of f, such that*

$$\|[b,\mu_\Omega](f)\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\Omega))} \leq C\|f\|_{L^{q,\mu}(0,T,L^{p,\lambda}(\Omega))}.$$

Proof. Let us fix a ball $B = B(x_0,r) \subseteq \mathbb{R}^n$ and let $kB = B(x_0,kr)$ for any $k>0$. Let $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ being χ_{2B} the characteristic function of $B(x_0,2r)$. Then, we have

$$\begin{aligned}\frac{1}{r^{\frac{\lambda}{p}}}\left(\int_B |[b,\mu_\Omega]f(x,t)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} &\leq \frac{1}{r^{\frac{\lambda}{p}}}\left(\int_B |[b,\mu_\Omega]f_1(x,t)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} + \frac{1}{r^{\frac{\lambda}{p}}}\left(\int_B |[b,\mu_\Omega]f_2(x,t)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\ &= K_1'(t) + K_2'(t), \quad \text{for } t\in[0,T].\end{aligned}$$

Using the weighted L^p-estimate (for $w \equiv 1$) stated in [22], we obtain

$$K_1'(t) \lesssim \|b\|_* \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p\,\mathrm{d}y\right)^{\frac{1}{p}}, \quad \text{for } t\in[0,T].$$

Now, we deal with the term $K_2'(t)$. For any fixed $x \in B$, we have

$$
\begin{aligned}
|[b,\mu_\Omega]f_2(x,t)| &\leq |b(x)-b_B|\left(\int_0^\infty \left|\int_{(2B)^c\cap\{y:|x-y|\leq s\}} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y,t)\,\mathrm{d}y\right|^2 \frac{\mathrm{d}s}{s^3}\right)^{\frac{1}{2}} + \\
&\quad + \left(\int_0^\infty \left|\int_{(2B)^c\cap\{y:|x-y|\leq s\}} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(y)-b_B] f(y,t)\,\mathrm{d}y\right|^2 \frac{\mathrm{d}s}{s^3}\right)^{\frac{1}{2}} \\
&:= I + II.
\end{aligned}
$$

In the proof of Theorem 4.1 we have already proved that

$$
\begin{aligned}
I &\lesssim |b(x)-b_B|\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p\,\mathrm{d}y\right)^{\frac{1}{p}} \sum_{j=1}^{n} |2^{j+1}B|^{\frac{\frac{\lambda}{n}-1}{p}} \\
&\lesssim |b(x)-b_B|\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p\,\mathrm{d}y\right)^{\frac{1}{p}} \sum_{j=1}^{n} |2^{j+1}r|^{\frac{\lambda-n}{p}},
\end{aligned}
$$

consequently

$$
\frac{1}{r^{\frac{\lambda}{p}}}\left(\int_B I^p \mathrm{d}x\right)^{\frac{1}{p}} \lesssim \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p\,\mathrm{d}y\right)^{\frac{1}{p}} \frac{1}{r^{\frac{\lambda}{p}}} \sum_{j=1}^{\infty} \frac{1}{(2^{j+1}r)^{\frac{n-\lambda}{p}}} \left(\int_B |b(x)-b_B|^p \mathrm{d}x\right)^{\frac{1}{p}}
$$

and then, using Theorem 2.3

$$
\begin{aligned}
\frac{1}{r^{\frac{\lambda}{p}}}\left(\int_B I^p \mathrm{d}x\right)^{\frac{1}{p}} &\lesssim \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p\,\mathrm{d}y\right)^{\frac{1}{p}} \left(\frac{1}{|B|}\int_B |b(x)-b_B|^p \mathrm{d}x\right)^{\frac{1}{p}} \\
&\lesssim \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p\,\mathrm{d}y\right)^{\frac{1}{p}} \|b\|_*.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
II &\lesssim \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{\frac{1}{n}}} \int_{2^{j+1}B\setminus 2^jB} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |b(y)-b_B||f(y,t)|\mathrm{d}y \\
&\lesssim \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{\frac{1}{n}}} \int_{2^{j+1}B\setminus 2^jB} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |b(y)-b_{2^{j+1}B}||f(y,t)|\mathrm{d}y \\
&\quad + \sum_{j=1}^{\infty} \frac{|b_{2^{j+1}B}-b_B|}{|2^{j+1}B|^{\frac{1}{n}}} \int_{2^{j+1}B\setminus 2^jB} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y,t)|\mathrm{d}y \\
&= III + IV, \quad \text{for } t\in[0,T].
\end{aligned}
$$

If $\Omega \in L^{\infty}(\mathbb{S}^{n-1})$, using Hölder inequality and Theorem 2.3, we attain

$$\begin{aligned} III &\lesssim \|\Omega\|_{L^{\infty}(\mathbb{S}^{n-1})} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \int\limits_{2^{j+1}B} |b(y) - b_{2^{j+1}B}||f(y,t)| \mathrm{d}y \\ &\lesssim \|\Omega\|_{L^{\infty}(\mathbb{S}^{n-1})} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} \left(\int\limits_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{p'} 1^{-\frac{p'}{p}} \mathrm{d}y \right)^{\frac{1}{p'}} \cdot \left(\int\limits_{2^{j+1}B} |f(y,t)|^{p} \cdot 1 \mathrm{d}y \right)^{\frac{1}{p}} \\ &\lesssim \|\Omega\|_{L^{\infty}(\mathbb{S}^{n-1})} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^{\lambda}} \int_{B_{\rho}(x)} |f(y,t)|^{p} \, \mathrm{d}y \right)^{\frac{1}{p}} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} (2^{j+1}r)^{\frac{\lambda}{p}} \cdot \left(\int\limits_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{p'} 1^{-\frac{p'}{p}} \mathrm{d}y \right)^{\frac{1}{p'}} \\ &\lesssim \|\Omega\|_{L^{\infty}(\mathbb{S}^{n-1})} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^{\lambda}} \int_{B_{\rho}(x)} |f(y,t)|^{p} \, \mathrm{d}y \right)^{\frac{1}{p}} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} (2^{j+1}r)^{\frac{\lambda}{p}} (2^{j+1}r)^{\frac{n}{p'}} \\ &\lesssim \|\Omega\|_{L^{\infty}(\mathbb{S}^{n-1})} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^{\lambda}} \int_{B_{\rho}(x)} |f(y,t)|^{p} \, \mathrm{d}y \right)^{\frac{1}{p}} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|} (2^{j+1}r)^{\frac{\lambda}{p}} (2^{j+1}r)^{n-\frac{n}{p}} \\ &\lesssim \|\Omega\|_{L^{\infty}(\mathbb{S}^{n-1})} \|b\|_{*} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^{\lambda}} \int_{B_{\rho}(x)} |f(y,t)|^{p} \, \mathrm{d}y \right)^{\frac{1}{p}} \sum_{j=1}^{\infty} (2^{j+1}r)^{\frac{\lambda-n}{p}}. \end{aligned}$$

If $\Omega \in L^{q}(\mathbb{S}^{n-1})$, using Hölder's inequality and (3), we gain

$$\begin{aligned} III &\lesssim \|\Omega\|_{L^{\infty}(\mathbb{S}^{n-1})} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{\frac{1}{q'}}} \left(\int\limits_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{q'} |f(y,t)|^{q'} \mathrm{d}y \right)^{\frac{1}{q'}} \\ &\lesssim \|\Omega\|_{L^{\infty}(\mathbb{S}^{n-1})} \|b\|_{*} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^{\lambda}} \int_{B_{\rho}(x)} |f(y,t)|^{p} \, \mathrm{d}y \right)^{\frac{1}{p}} \sum_{j=1}^{\infty} (2^{j+1}r)^{\frac{\lambda-n}{p}}. \end{aligned}$$

Hence, for $1 < q \leq \infty$, we achieve

$$\frac{1}{r^{\frac{\lambda}{p}}} \left(\int\limits_{B} III^{p} \mathrm{d}x \right)^{\frac{1}{p}} \lesssim \|b\|_{*} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^{\lambda}} \int_{B_{\rho}(x)} |f(y,t)|^{p} \, \mathrm{d}y \right)^{\frac{1}{p}}.$$

In the proof of Theorem 4.1 we reached

$$\frac{1}{|2^{j+1}B|^{\frac{1}{n}}} \int\limits_{2^{j+1}B\setminus 2^{j}B} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y,t)| \mathrm{d}y \lesssim \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^{\lambda}} \int_{B_{\rho}(x)} |f(y,t)|^{p} \, \mathrm{d}y \right)^{\frac{1}{p}} (2^{j+1}r)^{\frac{\lambda-n}{p}}.$$

Hence, by an easy calculation, it can be shown that if $b \in BMO(\mathbb{R}^n)$, then $|b_{2^{j+1}B} - b_{B}| \leq Cj\|b\|_{*}$. Then,

$$IV \lesssim \|b\|_{*} \left(\sup_{\substack{x\in\mathbb{R}^n \\ \rho>0}} \frac{1}{\rho^{\lambda}} \int_{B_{\rho}(x)} |f(y,t)|^{p} \, \mathrm{d}y \right)^{\frac{1}{p}} \sum_{j=1}^{\infty} j(2^{j+1}r)^{\frac{\lambda-n}{p}}.$$

From the last inequality it follows that

$$\begin{aligned}\frac{1}{r^{\frac{\lambda}{p}}}\left(\int_B IV^p \mathrm{d}x\right)^{\frac{1}{p}} &\lesssim \|b\|_* \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p \,\mathrm{d}y\right)^{\frac{1}{p}} \sum_{j=1}^{\infty} j\cdot \frac{r^{\frac{n-\lambda}{p}}}{(2^{j+1}r)^{\frac{n-\lambda}{p}}}\\ &\lesssim \|b\|_* \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p \,\mathrm{d}y\right)^{\frac{1}{p}} \sum_{j=1}^{\infty} \frac{j}{(2^j)^{\frac{n-\lambda}{p}}}\\ &\lesssim \|b\|_* \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p \,\mathrm{d}y\right)^{\frac{1}{p}}.\end{aligned}$$

Combining the above estimates and taking the supremum over all balls $B \subseteq \mathbb{R}^n$ we obtain the classical Morrey estimate:

$$\left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |[b,\mu_\Omega]f(y,t)|^p \,\mathrm{d}y\right)^{\frac{1}{p}} \lesssim \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p \,\mathrm{d}y\right)^{\frac{1}{p}}, \quad \text{for } t\in[0,T]. \tag{4}$$

Finally, we derive the desired mixed-Morrey estimate. Elevating (4) to q, integrating over $[0,T]\cap(t_0-\rho, t_0+\rho)$, it follows that

$$\int_{[0,T]\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |[b,\mu_\Omega]f(y,t)|^p \,\mathrm{d}y\right)^{\frac{q}{p}} \mathrm{d}t \lesssim \int_{[0,T]\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p \,\mathrm{d}y\right)^{\frac{q}{p}} \mathrm{d}t.$$

Multiplying the inequality above to $\frac{1}{\rho^\mu}$, taking the supremum of both sides and elevating both sides to $\frac{1}{q}$ we get

$$\left(\sup_{\substack{t_0\in[0,T]\\ \rho>0}} \frac{1}{\rho^\mu}\int_{[0,T]\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |[b,\mu_\Omega]f(y,t)|^p \,\mathrm{d}y\right)^{\frac{q}{p}} \mathrm{d}t\right)^{\frac{1}{q}} \lesssim$$

$$\left(\sup_{\substack{t_0\in[0,T]\\ \rho>0}} \frac{1}{\rho^\mu}\int_{[0,T]\cap(t_0-\rho,t_0+\rho)} \left(\sup_{\substack{x\in\mathbb{R}^n\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B_\rho(x)} |f(y,t)|^p \,\mathrm{d}y\right)^{\frac{q}{p}} \mathrm{d}t\right)^{\frac{1}{q}}.$$

and the theorem is completed. □

References

[1] D.R. Adams, A note on Riesz potential, Duke Math. J. **42**, 765-778, (1975).

[2] A. Akbulut, V.S. Guliyev, M.N. Omarova, Marcinkiewicz integrals associated with Schrödinger operator and their commutators on vanishing generalized Morrey spaces, Bound. Value Probl. (2017) 2017: 121. https://doi.org/10.1186/s13661-017-0851-4

[3] S.S. Aliyev, V.S. Guliyev, Boundedness of parametric Marcinkiewicz integral operator and their commutators on generalized Morrey spaces, Georgian Math. J. 19 (2012), 195-208.

[4] F. Anceschi, C.S. Goodrich, A. Scapellato, Operators with Gaussian Kernel Bounds on Mixed Morrey Spaces, Filomat 33:16 (2019), 5219-5230, https://doi.org/10.2298/FIL1916219A

[5] F. Chiarenza, M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend. Mat. Appl. **7**, 273-279, (1987).

[6] F. Deringoz, Commutators of parametric Marcinkiewicz integrals on generalized Orlicz-Morrey spaces, Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat., Volume 66, Number 1, Pages 115-123 (2017) DOI: 10.1501/Commua1_000000078, ISSN 1303-5991.

[7] F. Deringoz, Parametric Marcinkiewicz integral operator and higher order commutators on generalized weighted Morrey spaces, Transactions of NAS of Azerbaijan, Issue Mathematics Series of Physical-Technical and Mathematical Sciences, in press.

[8] F. Deringoz, S.G. Hasanov, Parametric Marcinkiewicz integral operator on generalized Orlicz-Morrey spaces, Transactions of NAS of Azerbaijan, Issue Mathematics, 36 (4), 70-76 (2016). Series of Physical-Technical and Mathematical Sciences.

[9] G. Di Fazio, M.A. Ragusa, Commutators and Morrey spaces, Bollettino U.M.I., **7**, 323-332, (1991).

[10] Y. Ding, S.Z. Lu, K. Yabuta, On commutators of Marcinkiewicz integrals with rough kernel, J. Math. Anal. and Appl., 275(2002), 60-68.

[11] J.García-Cuerva, J.L. Rubio De Francia, Weighted Norm Inequalities and Related Topics, North-Holland Mathematics Studies 116, North-Holland Publishing Co., Amsterdam, 1985.

[12] F. John, L. Nirenberg, On functions of bounded mean oscillation, Commun. Pure Appl. Math., **14** (1961), 415–426.

[13] S. Lu, Y. Ding, D. Yan, Singular Integrals and Related Topics, World Scientific, 2007.

[14] C.B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. (43) (1938), 126-166.

[15] M.A. Ragusa, S. Polidoro, Hölder Regularity for Solutions of Ultraparabolic Equations in Divergence Form, Potential Anal. (2001) 14: 341. https://doi.org/10.1023/A:1011261019736.

[16] M.A. Ragusa, Cauchy-Dirichlet problem associated to divergence form parabolic equations, Commun. Contemp. Math., Vol. 6, No. 3, (2004), 377-393.

[17] D. Sarason, On functions of vanishing mean oscillation, Trans. Amer. Math. Soc., **207** (1975), 391–405.

[18] A. Scapellato, New perspectives in the theory of some function spaces and their applications, AIP Conference Proceedings 1978, 140002 (2018); https://doi.org/10.1063/1.5043782

[19] E.M. Stein, On the function of Littlewood-Paley, Lusin and Marcinkiewicz, Trans. Amer. Math. Soc. 88 (1958), 430-466.

[20] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, 1970.

[21] A. Torchinsky, Real-variable Mathods in Harmonic Analysis, Academic Press, 1986.

[22] A. Torchinsky, S. Wang, A note on the Marcinkiewicz integral, Collect. Math., 61-62(1990), 235-243.

[23] R.L. Wheeden, A. Zygmund, Measure and Integral: An Introduction to Real Analysis, Second Edition, CRC Press (2015).