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Riesz transforms for Dunkl transform

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Abstract

In this paper we obtain the L^p-boundedness of Riesz transforms for the Dunkl transform for all $1 < p < \infty$.

Transformées de Riesz associés à la transformée de Dunkl

Résumé

Dans cet article, nous étudions la bornitude des transformées de Riesz associées à la transformée de Dunkl sur les espaces L^p, $1 < p < \infty$.

1. Introduction

On the Euclidean space $\mathbb{R}^N$, $N \geq 1$, the ordinary Riesz transform R_j, $j = 1, ..., N$ is defined as the multiplier operator

$$\widehat{R_j(f)}(\xi) = -i\frac{\xi_j}{\|\xi\|}\widehat{f}(\xi). \tag{1.1}$$

It can also be defined by the principal value of the singular integral

$$R_j(f)(x) = d_0 \lim_{\varepsilon\to 0}\int_{\|x-y\|>\varepsilon} \frac{x_j - y_j}{\|x-y\|} f(y)dy$$

where $d_0 = 2^{\frac{N}{2}}\dfrac{\Gamma(\frac{N+1}{2})}{\sqrt{\pi}}$. It follows from the general theory of singular integrals that Riesz transforms are bounded on $L^p(\mathbb{R}^N, dx)$ for all $1 < p < \infty$. What is done in this paper is to extend this result to the context of Dunkl theory where a similar operator is already defined.

Dunkl theory generalizes classical Fourier analysis on $\mathbb{R}^N$. It started twenty years ago with Dunkl's seminal work [3] and was further developed by several mathematicians. See for instance the surveys [5, 6, 7, 9] and the

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references cited therein. The study of the L^p-boundedness of Riesz transforms for Dunkl transform on $\mathbb{R}^N$ goes back to the work of S. Thangavelu and Y. Xu [10] where they established boundedness result only in a very special case of $N = 1$. It has been noted in [10] that the difficulty arises in the application of the classical L^p- theory of Caldéron-Zygmund, since Riesz transforms are singular integral operators. In this paper we describe how this theory can be adapted in Dunkl setting and gives an L^p-result for Riesz transforms for all $1 < p < \infty$. More precisely, through the fundamental result of M. Rösler [6] for the Dunkl translation of radial functions, we reformulate a Hörmander type condition for singular integral operators. The Riesz kernel is given by acting Dunkl operator on Dunkl translation of radial function.

This paper is organized as follows. In Section 2 we present some definitions and fundamental results from Dunkl's analysis. The Section 3 is devoted to proving L^p-boundedness of Riesz transforms. As applications, we will prove a generalized Riesz and Sobolev inequalities. Throughout this paper C denotes a constant which can vary from line to line.

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2. Preliminaries

In this section we collect notations and definitions and recall some basic facts. We refer to [5, 3, 6, 7, 9].

Let $G \subset \mathrm{O}(\mathbb{R}^N)$ be a finite reflection group associated to a reduced root system R and $k : R \to [0, +\infty)$ be a G-invariant function (called multiplicity function). Let R_+ be a positive root subsystem. We shall assume that R is normalized in the sense that $\|\alpha\|^2 = \langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R$, where $\langle\ ,\ \rangle$ is the standard Euclidean scalar product on $\mathbb{R}^N$.

The Dunkl operators T_ξ, $\xi \in \mathbb{R}^N$ are the following k–deformations of directional derivatives ∂_ξ by difference operators:

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha)\, \langle \alpha, \xi \rangle\, \frac{f(x) - f(\sigma_\alpha . x)}{\langle \alpha,\, x \rangle}, \quad x \in \mathbb{R}^N$$

where σ_α denotes the reflection with respect to the hyperplane orthogonal to α. For the standard basis vectors of $\mathbb{R}^N$, we simply write $T_j = T_{e_j}$.

The operators ∂_ξ and T_ξ are intertwined by a Laplace–type operator

$$V_k f(x) \; = \int_{\mathbb{R}^N} f(y)\, d\mu_x(y),$$

associated to a family of compactly supported probability measures

$$\{\, \mu_x \,|\, x \in \mathbb{R}^N \}\,.$$

Specifically, μ_x is supported in the the convex hull $co(G.x)$.

For every $\lambda \in \mathbb{C}^N$, the simultaneous eigenfunction problem,

$$T_\xi f = \langle \lambda, \xi \rangle\, f, \qquad \xi \in \mathbb{R}^N$$

has a unique solution $f(x) = E_k(\lambda, x)$ such that $E_k(\lambda, 0) = 1$, which is given by

$$E_k(\lambda, x) \; = \; V_k(e^{\langle \lambda, \,.\, \rangle})(x) \; = \int_{\mathbb{R}^N} e^{\langle \lambda, y \rangle}\, d\mu_x(y), \qquad x \in \mathbb{R}^N.$$

Furthermore $\lambda \mapsto E_k(\lambda, x)$ extends to a holomorphic function on $\mathbb{C}^N$.

Let m_k be the measure on $\mathbb{R}^N$, given by

$$dm_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)} dx.$$

For $f \in L^1(m_k)$ (the Lebesgue space with respect to the measure m_k) the Dunkl transform is defined by

$$\mathcal{F}_k(f)(\xi) = \frac{1}{c_k} \int_{\mathbb{R}^N} f(x)\, E_k(-i\,\xi, x) dm_k(x), \quad c_k \; = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{2}}\, dm_k(x).$$

This new transform shares many analogous properties of the Fourier transform.

(i) The Dunkl transform is a topological automorphism of $\mathcal{S}(\mathbb{R}^N)$ (*Schwartz space*).

(ii) (*Plancherel Theorem*) The Dunkl transform extends to an isometric automorphism of $L^2(m_k)$.

(iii) (*Inversion formula*) For every $f \in L^1(m_k)$ such that $\mathcal{F}_k f \in L^1(m_k)$, we have

$$f(x) = \mathcal{F}_k^2 f(-x), \qquad x \in \mathbb{R}^N.$$

(iv) For all $\xi \in \mathbb{R}^N$ and $f \in \mathcal{S}(\mathbb{R}^N)$

$$\mathcal{F}_k(T_\xi(f))(x) =< i\xi, x > \mathcal{F}_k(f)(x), \qquad x\in\mathbb{R}^N. \tag{2.1}$$

Let $x \in \mathbb{R}^N$, the Dunkl translation operator τ_x is defined on $L^2(m_k)$ by,

$$\mathcal{F}_k(\tau_x(f))(y) = E_k(ix, y)\mathcal{F}_k f(y), \quad y \in \mathbb{R}^N. \tag{2.2}$$

If f is a continuous radial function in $L^2(m_k)$ with $f(y) = \widetilde{f}(\|y\|)$, then

$$\tau_x(f)(y) = \int_{\mathbb{R}^N} \widetilde{f}\Big(\sqrt{\|x\|^2 + \|y\|^2 + 2 < y, \eta >} \Big) d\mu_x(\eta). \tag{2.3}$$

This formula is first proved by M. Rösler [6] for $f \in \mathcal{S}(\mathbb{R}^N)$ and recently is extended to continuous functions by F. and H. Dai Wang [2].

We collect below some useful facts :

(i) For all $x, y \in \mathbb{R}^N$,

$$\tau_x(f)(y) = \tau_y(f)(x). \tag{2.4}$$

(ii) For all $x, \xi \in \mathbb{R}^N$ and $f \in \mathcal{S}(\mathbb{R}^N)$,

$$T_\xi \tau_x(f) = \tau_x T_\xi(f). \tag{2.5}$$

(iii) For all $x \in \mathbb{R}^N$ and $f,\ g \in L^2(m_k)$,

$$\int_{\mathbb{R}^N} \tau_x(f)(-y)g(y)dm_k(y) = \int_{\mathbb{R}^N} f(y)\tau_x g(-y)dm_k(y). \tag{2.6}$$

(iv) For all $x \in \mathbb{R}^N$ and $1 \le p \le 2$, the operator τ_x can be extended to all radial functions f in $L^p(m_k)$ and the following holds

$$\|\tau_x(f)\|_{p,k} \le \|f\|_{p,k}. \tag{2.7}$$

$\|.\|_{p,k}$ is the usual norm of $L^p(m_k)$.

3. Riesz transforms for the Dunkl transform.

In Dunkl setting the Riesz transforms (see [10]) are the operators $\mathcal{R}_j$, $j = 1...N$ defined on $L^2(m_k)$ by

$$\mathcal{R}_j(f)(x) = d_k \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} \tau_x(f)(-y)\frac{y_j}{\|y\|^{p_k}} dm_k(y), \quad x \in \mathbb{R}^N$$

where

$$d_k = 2^{\frac{p_k-1}{2}} \frac{\Gamma(\frac{p_k}{2})}{\sqrt{\pi}}; \quad p_k = 2\gamma_k + N + 1 \quad \text{and} \quad \gamma_k = \sum_{\alpha \in R+} k(\alpha).$$

It has been proved by S. Thangavelu and Y. Xu [10], that $\mathcal{R}_j$ is a multiplier operator given by

$$\mathcal{F}_k(\mathcal{R}_j(f))(\xi) = -i\frac{\xi_j}{\|\xi\|}\mathcal{F}_k(f)(\xi), \quad f \in \mathcal{S}(\mathbb{R}^N),\ \xi \in \mathbb{R}^N, \tag{3.1}$$

The authors state that if $N = 1$ and $2\gamma_k \in \mathbb{N}$ the operator $\mathcal{R}_j$ is bounded on $L^p(m_k)$, $1 < p < \infty$. In [1] this result is improved by removing $2\gamma_k \in \mathbb{N}$, where Riesz transform is called Hilbert transform. If $\gamma_k = 0$ ($k = 0$), this operator coincides with the usual Riesz transform R_j given by (1.1). Our interest is to prove the boundedness of this operator for $N \geq 2$ and $k \geq 0$. To do this, we invoke the theory of singular integrals. Our basic is the following,

Theorem 3.1. *Let $\mathcal{K}$ be a measurable function on $\mathbb{R}^N \times \mathbb{R}^N \setminus \{(x, g.x);\ x \in \mathbb{R}^N,\ g \in G\}$ and S be a bounded operator from $L^2(m_k)$ into itself, associated with a kernel $\mathcal{K}$ in the sense that*

$$S(f)(x) = \int_{\mathbb{R}^N} \mathcal{K}(x, y)f(y)dm_k(y), \tag{3.2}$$

for all compactly supported function f in $L^2(m_k)$ and for a.e $x \in \mathbb{R}^N$ satisfying $g.x \notin supp(f)$, for all $g \in G$. If $\mathcal{K}$ satisfies

$$\int_{\min_{g\in G}\|g.x-y\|>2\|y-y_0\|} |\mathcal{K}(x, y) - \mathcal{K}(x, y_0)|dm_k(x) \leq C, \quad y, y_0 \in \mathbb{R}^N, \tag{3.3}$$

then S extends to a bounded operator from $L^p(m_k)$ into itself for all $1 < p \leq 2$.

Proof. We first note that $(\mathbb{R}^N, m_k)$ is a space of homogenous type, that is, there is a fixed constant $C > 0$ such that

$$m_k(B(x, 2r)) \leq Cm_k(B(x, r)), \quad \forall\, x \in \mathbb{R}^N,\ r > 0 \tag{3.4}$$

where $B(x, r)$ is the closed ball of radius r centered at x (see [8], Ch 1). Then we can adapt to our context the classical technic which consist to show that S is weak type (1,1) and conclude by Marcinkiewicz interpolation theorem.

In fact, the Calderón-Zygmund decomposition says that for all $f \in L^1(m_k) \cap L^2(m_k)$ and $\lambda > 0$, there exist a decomposition of f, $f = h + b$ with $b = \sum_j b_j$ and a sequence of balls $(B(y_j, r_j))_j = (B_j)_j$ such that for some constant C, depending only on the multiplicity function k

(i) $\|h\|_\infty \leq C\lambda$;

(ii) $supp(b_j) \subset B_j$;

(iii) $\displaystyle\int_{B_j} b_j(x) dm_k(x) = 0$;

(iv) $\|b_j\|_{1,k} \leq C\, \lambda\, m_k(B_j)$;

(v) $\displaystyle\sum_j m_k(B_j) \leq C\, \frac{\|f\|_{1,k}}{\lambda}$.

The proof consists in showing the following inequality hold for $w = h$ and $w = b$:

$$\rho_\lambda(S(w)) = m_k\Big(\{x \in \mathbb{R}^N;\ |S(w)(x)| > \frac{\lambda}{2}\}\Big) \leq C\, \frac{\|f\|_{1,k}}{\lambda}. \tag{3.5}$$

By using the L^2-boundedness of S we get

$$\rho_\lambda(S(h)) \leq \frac{4}{\lambda^2} \int_{\mathbb{R}^N} |S(h)(x)|^2 dm_k(x) \leq \frac{C}{\lambda^2} \int_{\mathbb{R}^N} |h(x)|^2 dm_k(x). \tag{3.6}$$

From (i) and (v),

$$\int_{\cup B_j} |h(x)|^2 dm_k(x) \leq C\lambda^2 \mu_k(\cup B_j) \leq C\lambda \|f\|_{1,k}. \tag{3.7}$$

Since on $(\cup B_j)^c$, $f(x) = h(x)$, then

$$\int_{(\cup B_j)^c} |h(x)|^2 dm_k(x) \leq C\lambda \|f|_{1,k}. \tag{3.8}$$

From (3.6), (3.7) and (3.8), the inequality (3.5) is satisfied for h.

Next we turn to the inequality (3.5) for the function b. Consider

$$B_j^* = B(y_j, 2r_j\ ); \quad \text{and} \quad Q_j^* = \bigcup_{g \in G} g.B_j^*.$$

Then

$$\rho_\lambda(S(b)) \leq m_k\Big(\bigcup_j Q_j^*\Big) + m_k\{x \in \Big(\bigcup_j Q_j^*\Big)^c; |S(b)(x)| > \frac{\lambda}{2}\}.$$

Now by (3.4) and *(v)*

$$m_k\Big(\bigcup_j Q_j^*\Big) \leq |G| \sum_j m_k(B_j^*) \leq C \sum_j m_k(B_j) \leq C\ \frac{\|f\|_{1,k}}{\lambda}.$$

Furthermore if $x \notin Q_j^*$, we have

$$\min_{g\in G} \|g.x - y_j\| > 2\|y - y_j\|, \quad y \in B_j.$$

Thus, from (3.2),*(iii)* ,*(ii)*, (3.3), *(iv)* and *(v)*

$$\begin{aligned}
&\int_{(\cup Q_j^*)^c} |S(b)(x)| dm_k(x)\\
&\leq \sum_j \int_{(Q_j^*)^c} |S(b_j)(x)| dm_k(x)\\
&= \sum_j \int_{(Q_j^*)^c} \Big| \int_{\mathbb{R}^N} \mathcal{K}(x,y) b_j(y) dm_k(y)\Big| dm_k(x)\\
&= \sum_j \int_{(Q_j^*)^c} \Big| \int_{\mathbb{R}^N} b_j(y)\Big(\mathcal{K}(x,y) - \mathcal{K}(x,y_j)\Big) dm_k(y)\Big| dm_k(x)\\
&\leq \sum_j \int_{\mathbb{R}^N} |b_j(y)| \int_{(Q_j^*)^c} |\mathcal{K}(x,y) - \mathcal{K}(x,y_j)| dm_k(x) dm_k(y)\\
&\leq \sum_j \int_{\mathbb{R}^N} |b_j(y)| \int_{\min\limits_{g\in G}\|g.x - y_j\| > 2\|y-y_j\|} |\mathcal{K}(x,y) - \mathcal{K}(x,y_j)| dm_k(x) dm_k(y)\\
&\leq C \sum_j \|b_j\|_{1,k}\\
&\leq C\ \|f\|_{1,k}.
\end{aligned}$$

Therefore,

$$m_k\{x \in \Big(\bigcup_j Q_j^*\Big)^c; |S(b)(x)| > \frac{\lambda}{2}\}$$
$$\leq \frac{2}{\lambda} \int_{(\cup Q_j^*)^c} |S(b)(x)| dm_k(x) \leq C\ \frac{\|f\|_{1,k}}{\lambda}.$$

This achieves the proof of (3.5) for b. $\square$

Now, we will give an integral representation for the Riesz transform $\mathcal{R}_j$. For this end, we put for $x, y \in \mathbb{R}^N$ and $\eta \in co(G.x)$

$$A(x,y,\eta) = \sqrt{\|x\|^2 + \|y\|^2 - 2 < y, \eta >} = \sqrt{\|y-\eta\|^2 + \|x\|^2 - \|\eta\|^2}.$$

It is easy to check that

$$\min_{g\in G} \|g.x - y\| \leq A(x,y,\eta) \leq \max_{g\in G} \|g.x - y\|. \tag{3.9}$$

The following inequalities are clear

$$\Big|\frac{\partial A^\ell}{\partial y_r}(x,y,\eta)\Big| \leq CA^{\ell-1}(x,y,\eta),$$
$$\Big|\frac{\partial^2 A^\ell}{\partial y_r \partial y_s}(x,y,\eta)\Big| \leq CA^{\ell-2}(x,y,\eta) \tag{3.10}$$

and for $\alpha \in R_+$,

$$\Big|\frac{\partial A^\ell}{\partial y_r}(x,\sigma_\alpha.y,\eta)\Big| \leq CA^{\ell-1}(x,\sigma_\alpha.y,\eta),$$
$$\Big|\frac{\partial^2 A^\ell}{\partial y_r \partial y_s}(x,\sigma_\alpha.y,\eta)\Big| \leq CA^{\ell-2}(x,\sigma_\alpha.y,\eta), \tag{3.11}$$

where $r, s = 1, ...N$ and $\ell \in \mathbb{R}$.

Let us set

$$\begin{aligned}
\mathcal{K}_j^{(1)}(x,y) &= \int_{\mathbb{R}^N} \frac{\eta_j - y_j}{A^{p_k}(x,y,\eta)} d\mu_x(\eta) \\
\mathcal{K}_j^{(\alpha)}(x,y) &= \frac{1}{< y, \alpha >} \int_{\mathbb{R}^N} \Big[\frac{1}{A^{p_k-2}(x,y,\eta)} - \frac{1}{A^{p_k-2}(x,\sigma_\alpha.y,\eta)}\Big] d\mu_x(\eta), \\
\mathcal{K}_j(x,y) &= d_k\Big\{\mathcal{K}_j^{(1)}(x,y) + \sum_{\alpha\in R_+} \frac{k(\alpha)\alpha_j}{p_k - 2} \mathcal{K}_j^{(\alpha)}(x,y)\Big\},
\end{aligned}$$

where $\alpha \in R_+$.

Proposition 3.2. *If $f \in L^2(m_k)$ with compact support, then for all $x \in \mathbb{R}^N$ satisfying $g.x \notin supp(f)$ for all $g \in G$, we have*

$$\mathcal{R}_j(f)(x) = \int_{\mathbb{R}^N} \mathcal{K}_j(x,y) f(y) dm_k(y).$$

Proof. Let $f \in L^2(m_k)$ be a compact supported function and $x \in \mathbb{R}^N$, such that $g.x \notin supp(f)$ for all $g \in G$. For $0 < \varepsilon < \min_{g \in G} \min_{y \in supp(f)} |g.x - y|$ and $n \in \mathbb{N}$, we consider $\widetilde{\varphi}_{n,\varepsilon}$ a C^∞-function on $\mathbb{R}$, such that:

- $\widetilde{\varphi}_{n,\varepsilon}$ is odd .
- $\widetilde{\varphi}_{n,\varepsilon}$ is supported in $\{t \in \mathbb{R};\ \varepsilon \le |t| \le n+1\}$.
- $\widetilde{\varphi}_{n,\varepsilon} = 1$ in $\{t \in \mathbb{R};\ \varepsilon + \frac{1}{n} \le t \le n\}$.
- $|\widetilde{\varphi}_{n,\varepsilon}| \le 1$.

Let

$$\widetilde{\phi}_{n,\varepsilon}(t) = \int_{-\infty}^{t} \frac{\widetilde{\varphi}_{n,\varepsilon}(u)}{|u|^{p_k - 1}}\, du \quad \text{and} \quad \phi_{n,\varepsilon}(y) = \widetilde{\phi}_{n,\varepsilon}(\|y\|),$$

for $t \in \mathbb{R}$ and $y \in \mathbb{R}^N$. Clearly, $\phi_{n,\varepsilon}$ is a C^∞ radial function supported in the ball $B(0, n+1)$ and

$$\lim_{n \to +\infty} \widetilde{\varphi}_{n,\varepsilon}(\|y\|) = 1, \quad \forall\, y \in \mathbb{R}^N,\ \|y\| > \varepsilon.$$

The dominated convergence theorem, (2.5) and (2.6) yield

$$\begin{aligned}
&\int_{\|y\| > \varepsilon} \tau_x(f)(-y) \frac{y_j}{\|y\|^{p_k}} dm_k(y) \\
&\qquad = \lim_{n \to \infty} \int_{\mathbb{R}^N} \tau_x(f)(-y) \frac{y_j}{\|y\|^{p_k}} \widetilde{\varphi}_{n,\varepsilon}(\|y\|) dm_k(y) \\
&\qquad = \lim_{n \to \infty} \int_{\mathbb{R}^N} \tau_x(f)(-y) T_j(\phi_{n,\varepsilon})(y) dm_k(y) \\
&\qquad = \lim_{n \to \infty} \int_{\mathbb{R}^N} f(y) T_j \tau_x(\phi_{n,\varepsilon})(-y) dm_k(y).
\end{aligned}$$

Now we have

$$\begin{aligned}
&T_j \tau_x(\phi_{n,\varepsilon})(-y) \\
&\quad = \int_{\mathbb{R}^N} \frac{(\eta_j - y_j)\widetilde{\varphi}_{n,\varepsilon}(A(x,y,\eta))}{A^{p_k}(x,y,\eta)} d\mu_x(\eta) \\
&\quad + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \int_{\mathbb{R}^N} \frac{\widetilde{\phi}_{n,\varepsilon}(A(x,\sigma_\alpha.y,\eta)) - \widetilde{\phi}_{n,\varepsilon}(A(x,y,\eta))}{< y, \alpha >} d\mu_x(\eta),
\end{aligned}$$

where from (3.9)

$$\varepsilon < A(x, y, \eta) \;; \quad \varepsilon < A(x, \sigma_\alpha.y, \eta), \quad y \in supp(f),\ \eta \in co(G.x).$$

Then with the aid of dominated convergence theorem

$$\lim_{n\to\infty} T_j \tau_x(\phi_{n,\varepsilon})(-y) = \frac{1}{d_k}\mathcal{K}_j(x, y),$$

and

$$d_k \int_{\|y\|\geq\varepsilon} \tau_x(f)(-y)\frac{y_j}{\|y\|^{p_k}} dm_k(y) = \int_{\mathbb{R}^N} \mathcal{K}_j(x, y) f(y) dm_k(y).$$

Letting $\varepsilon \to 0$, it follows that

$$\mathcal{R}_j(f)(x) = \int_{\mathbb{R}^N} \mathcal{K}_j(x, y) f(y) dm_k(y),$$

which proves the result. □

Now, we are able to state our main result.

Theorem 3.3. *The Riesz transform $\mathcal{R}_j$, $j = 1...N$, is a bounded operator from $L^p(m_k)$ into itself, for all $1 < p < \infty$.*

Proof. Clearly, from (3.1) and Plancherel's theorem $\mathcal{R}_j$ is bounded from $L^2(m_k)$ into itself, with adjoint operator $\mathcal{R}_j^* = -\mathcal{R}_j$. Thus, via duality it's enough to consider the range $1 < p \leq 2$ and apply Theorem 3.1. In view of Proposition 3.2 it only remains to show that $\mathcal{K}_j$ satisfies condition (3.3).

Let $y, y_0 \in \mathbb{R}^N$, $y \neq y_0$ and $x \in \mathbb{R}^N$, such that

$$\min_{g\in G} \|g.x - y\| > 2\|y - y_0\|. \tag{3.12}$$

By mean value theorem,

$$
\begin{aligned}
|\mathcal{K}_j^{(1)}(x,y)-\mathcal{K}_j^{(1)}(x,y_0)| &= \Big|\sum_{i=0}^{N}(y_i-(y_0)_i)\int_0^1 \frac{\partial \mathcal{K}_j^{(1)}}{\partial y_i}(x,y_t)\,dt\Big| \\
&= \Big|\sum_{i=0}^{N}(y_i-(y_0)_i)\int_0^1\int_{\mathbb{R}^N}\Big(\frac{\delta_{i,j}}{A^{p_k}(x,y_t,\eta)} \\
&\qquad +\frac{p_k((y_t)_i-\eta_i)(\eta_j-(y_t)_j)}{A^{p_k+2}(x,y_t,\eta)}\Big)\,d\mu_x(\eta)\Big| \\
&\leq C\|y-y_0\|\int_0^1\int_{\mathbb{R}^N}\frac{1}{A^{p_k}(x,y_t,\eta)}\,d\mu_x(\eta)dt.
\end{aligned}
$$

where $y_t = y_0 + t(y-y_0)$ and $\delta_{i,j}$ is the Kronecker symbol.
In view of (3.9) and (3.12), we obtain

$$
\|y-y_0\| < A(x,y_t,\eta), \quad \eta\in co(G.x).
$$

Therefore,

$$
\begin{aligned}
&\Big|\mathcal{K}_j^{(1)}(x,y)-\mathcal{K}_j^{(1)}(x,y_0)\Big| \\
&\leq C\|y-y_0\|\int_0^1\int_{\mathbb{R}^N}\frac{1}{\Big(\|y-y_0\|^2+A^2(x,y_t,\eta)\Big)^{\frac{p_k}{2}}}\,d\mu_x(\eta)dt. \\
&\leq C\|y-y_0\|\int_0^1\tau_x(\psi)(y_t)dt
\end{aligned}
$$

where ψ is the function defined by

$$
\psi(z)=\frac{1}{(\|y-y_0\|^2+\|z\|^2)^{\frac{p_k}{2}}}, \quad z\in\mathbb{R}^N.
$$

Using Fubini's theorem, (2.4) and (2.7), we get

$$
\begin{aligned}
&\int_{\min_{g\in G}\|g.x-y\|>2\|y-y_0\|}|\mathcal{K}_j^{(1)}(x,y)-\mathcal{K}_j^{(1)}(x,y_0)|dm_k(x) \\
&\leq C\|y-y_0\|\int_0^1\int_{\mathbb{R}^N}\tau_{-y_t}(\psi)(x)dm_k(x)\,dt \\
&\leq C|y-y_0|\int_{\mathbb{R}^N}\psi(z)dm_k(z) = C\int_{\mathbb{R}^N}\frac{du}{(1+u^2)^{\frac{p_k}{2}}} = C'.
\end{aligned}
$$

This established the condition (3.3) for $\mathcal{K}_j^{(1)}$.

To deal with $\mathcal{K}_j^{(\alpha)}$, $\alpha \in R_+$, we put for $x, y \in \mathbb{R}^N$, $\eta \in co(G.x)$ and $t \in [0,1]$

$$\begin{aligned} U(x,y,\eta) &= A^{2p_k-4}(x,y,\eta), \\ V_\alpha(x,y,\eta) &= A^{p_k-2}A_\alpha^{p_k-2}(A^{p_k-2}+A_\alpha^{p_k-2}), \\ h_{\alpha,t}(y) &= y+t(\sigma_\alpha.y-y) = y-t<y,\alpha>\alpha, \end{aligned}$$

By mean value theorem we have

$$\begin{aligned} \mathcal{K}_j^{(\alpha)}(x,y) &= \int_{\mathbb{R}^N} \frac{1}{<y,\alpha>} \frac{U(x,\sigma_\alpha.y,\eta)-U(x,y,\eta)}{V_\alpha(x,y,\eta)} d\mu_x(\eta) \\ &= -\int_{\mathbb{R}^N}\int_0^1 \frac{\partial_\alpha U(x,h_{\alpha,t}(y),\eta)}{V_\alpha(x,y,\eta)}\, dt\, d\mu_x(\eta) \end{aligned}$$

and

$$\begin{aligned} &\mathcal{K}_j^{(\alpha)}(x,y)-\mathcal{K}_j^{(\alpha)}(x,y_0) \\ &\quad = \int_{\mathbb{R}^N}\int_0^1\int_0^1 \partial_{y-y_0}\Big(\frac{\partial_\alpha U(x,h_{\alpha,t}(.),\eta)}{V_\alpha(x,.,\eta)}\Big)(y_\theta)\, d\theta\, dt\, d\mu_x(\eta). \end{aligned} \tag{3.13}$$

Here the derivations are taken with respect to the variable y.

To simplify, let us denote by

$$A = A(x,y_\theta,\eta); \quad A_\alpha = A(x,\sigma_\alpha.y_\theta,\eta)$$

Then using (3.10) and the fact

$$\|\eta - h_{\alpha,t}(y_\theta)\| \le \max(\ \|\eta-y_\theta\|, \|\eta-\sigma_\alpha(y_\theta)\|\ ),$$

we obtain

$$\begin{aligned} \Big|\frac{\partial U}{\partial y_r}(x,h_{\alpha,t}(y_\theta),\eta)\Big| &\le C\Big(A^{2p_k-5}+A_\alpha^{2p_k-5}\Big) \\ \Big|\frac{\partial^2 U}{\partial y_r\partial y_s}(x,h_{\alpha,t}(y_\theta),\eta)\Big| &\le C\Big(A^{2p_k-6}+A_\alpha^{2p_k-6}\Big), \quad r,s=1,...,N. \end{aligned}$$

This gives us the following estimates

$$\Big|\partial_\alpha U(x,h_{\alpha,t}(y_\theta),\eta)\Big| \le C\Big(A^{2p_k-5}+A_\alpha^{2p_k-5}\Big), \tag{3.14}$$

$$\Big|\partial_{y-y_0}\Big(\partial_\alpha U(x,h_{\alpha,t}(.),\eta)\Big)(y_\theta)\Big| \le C\|y-y_0\|\Big(A^{2p_k-6}+A_\alpha^{2p_k-6}\Big). \quad (3.15)$$

By (3.10) and (3.11), we also have

$$\Big|\frac{\partial V_\alpha}{\partial y_r}((x,y_\theta,\eta))\Big| \le CA^{p_k-3}A_\alpha^{p_k-3}(A^{p_k-2}+A_\alpha^{p_k-2})(A+A_\alpha).$$

The elementary inequality $\dfrac{u+v}{u^\ell+v^\ell} \le \dfrac{3}{u^{\ell-1}+v^{\ell-1}}, \quad u,v>0,\ \ell\ge 1$, leads to

$$\begin{aligned}
&\frac{\Big|\partial_{y-y_0}V_\alpha(x,y_\theta,\eta)\Big|}{V_\alpha^2(x,y_\theta,\eta)}\\
&\qquad \le C\|y-y_0\|\ \frac{A_\alpha+A}{A^{p_k-1}A_\alpha^{p_k-1}(A^{p_k-2}+A_\alpha^{p_k-2})}\\
&\qquad \le C\|y-y_0\|\ \frac{1}{A^{p_k-1}A_\alpha^{p_k-1}(A^{p_k-3}+A_\alpha^{p_k-3})}. \qquad (3.16)
\end{aligned}$$

Now (3.14), (3.15) and (3.16) yield

$$\begin{aligned}
&\Big|\partial_{y-y_0}\Big(\frac{\partial_\alpha U(x,h_{\alpha,t}(.),\eta)}{V_\alpha(x,.,\eta)}\Big)(y_\theta)\Big|\\
&\qquad \le C\|y-y_0\|\,\frac{A^{2p_k-6}+A_\alpha^{2p_k-6}}{A^{p_k-2}A_\alpha^{p_k-2}(A^{p_k-2}+A_\alpha^{p_k-2})}\\
&\qquad + C\|y-y_0\|\,\frac{A^{2p_k-5}+A_\alpha^{2p_k-5}}{A^{p_k-1}A_\alpha^{p_k-1}(A^{p_k-3}+A_\alpha^{p_k-3})}\\
&\qquad \le C\|y-y_0\|\Big(\frac{1}{A^2A_\alpha^{p_k-2}}+\frac{1}{A^{p_k-2}A_\alpha^2}\Big)\\
&\qquad + C\|y-y_0\|\Big(\frac{1}{AA_\alpha^{p_k-1}}+\frac{1}{A^{p_k-1}A_\alpha}\Big)\\
&\qquad \le C\|y-y_0\|\Big(\frac{1}{A^{p_k}}+\frac{1}{A_\alpha^{p_k}}\Big)
\end{aligned}$$

where in the last equality we have used the fact that $\dfrac{1}{uv^{\ell-1}} \le \dfrac{1}{u^\ell}+\dfrac{1}{v^\ell}$, $u,v>0$ and $\ell\ge 1$.

Thus, in view of (3.13),

$$\left|\mathcal{K}_j^{(\alpha)}(x,y) - \mathcal{K}_j^{(\alpha)}(x,y_0)\right|$$

$$\leq C\|y-y_0\| \int_0^1 \int_{\mathbb{R}^N} \Big[\frac{1}{A^{p_k}(x,y_\theta,\eta)} + \frac{1}{A^{p_k}(x,\sigma_\alpha y_\theta,\eta)}\Big]\, d\mu_x(\eta)\, d\theta.$$

Then by same argument as for $\mathcal{K}_j^{(1)}$ we obtain

$$\int_{\min_{g\in G}\|g.x-y\|>2\|y-y_0\|} |\mathcal{K}_j^{(2)}(x,y) - \mathcal{K}_j^{(2)}(x,y_0)| dm_k(x) \leq C,$$

which established the condition (3.3) for the kernel $\mathcal{K}_j^{(\alpha)}$ and furnishes the proof. □

As applications, we will prove a generalized Riesz and Sobolev inequalities

Corollary 3.4 (Generalized Riesz inequalities)**.** *For all* $1 < p < \infty$ *there exists a constant* C_p *such that*

$$||T_rT_s(f)||_{k,p} \leq C_p||\Delta_k f||_{k,p}, \quad \textit{for all } \ f \in \mathcal{S}(\mathbb{R}^N), \tag{3.17}$$

where Δ_k *is the Dunkl laplacian:* $\Delta_k f = \sum_{r=1}^{N} T_r^2(f)$

Proof. From (2.1) and (3.1) one can see that

$$T_rT_s(f) = \mathcal{R}_r\mathcal{R}_s(-\Delta_k)(f), \quad r,\ s = 1...N,\ f \in \mathcal{S}(\mathbb{R}^N).$$

Then (3.17) is concluded by Theorem 3.3. □

Corollary 3.5 (Generalized Sobolev inequality)**.** *For all* $1 < p \leq q < 2\gamma(k) + N$ *with* $\frac{1}{q} = \frac{1}{p} - \frac{1}{2\gamma(k)+N}$, *we have*

$$\|f\|_{q,k} \leq C_{p,q}\|\nabla_k f\|_{p,k} \tag{3.18}$$

for all $f \in \mathcal{S}(\mathbb{R}^N)$. *Here* $\nabla_k f = (T_1 f, ..., T_N f)$ *and* $|\nabla_k f| = (\sum_{r=1}^{N} |T_r f|^2)^{\frac{1}{2}}$.

Proof. For all $f \in \mathcal{S}(\mathbb{R}^N)$, we write

$$\begin{aligned}\mathcal{F}_k(f)(\xi) &= \frac{1}{\|\xi\|}\sum_{r=1}^{N}\frac{-i\xi_r}{\|\xi\|}\Big(i\xi_r\mathcal{F}_k(f)(\xi)\Big)\\ &= \frac{1}{\|\xi\|}\sum_{r=1}^{d}\frac{-i\xi_r}{\|\xi\|}\Big(\mathcal{F}_k(T_rf)(\xi)\Big).\end{aligned}$$

This yields to the following identity

$$f = I_k^1\Big(\sum_{j=1}^{N}\mathcal{R}_j(T_jf)\Big),$$

where

$$I_k^\beta(f)(x) = (d_k^\beta)^{-1}\int_{\mathbb{R}^N}\frac{\tau_y f(x)}{\|y\|^{2\gamma(k)+N-\beta}}dm_k(y),$$

here

$$d_k^\beta = 2^{-\gamma(k)-N/2+\beta}\frac{\Gamma(\frac{\beta}{2})}{\Gamma(\gamma(k)+\frac{N-\beta}{2})}.$$

Theorem 1.1 of [4] asserts that I_k^β a bounded operator from $L^p(m_k)$ to $L^q(m_k)$. Then (3.18) follows from Theorem 3.3. □

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