



Right-Angled Coxeter Polytopes, Hyperbolic 6-manifolds, and a Problem of Siegel

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- Our solution for $n = 6$ will be described in this talk.

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- As $\chi(M) \in \mathbb{Z}$, we get minimum volume if $|\chi(M)| = 1$.
- A compact orientable M satisfies $\chi(M) \in 2\mathbb{Z}$, so the minimum volume is most likely achieved by a noncompact manifold.

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- For $n = 5$, the minimum known volume is $7\zeta(3)/4$ where ζ is the Riemann zeta function.

Hyperbolic n -Space

- We work in the **hyperboloid model** of hyperbolic n -space

$$H^n = \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\}.$$

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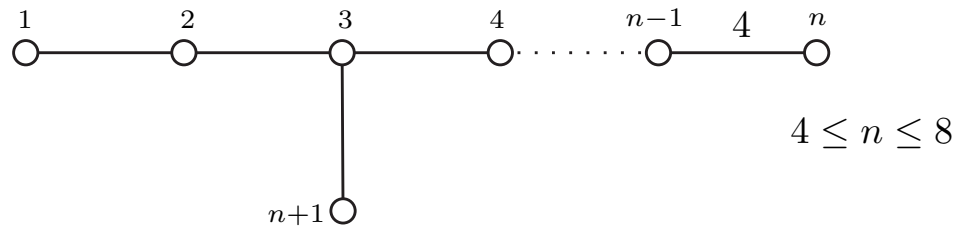
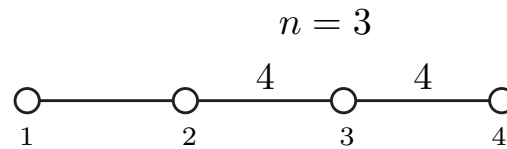
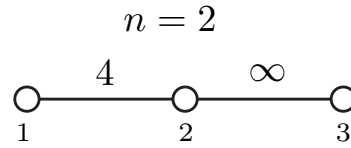
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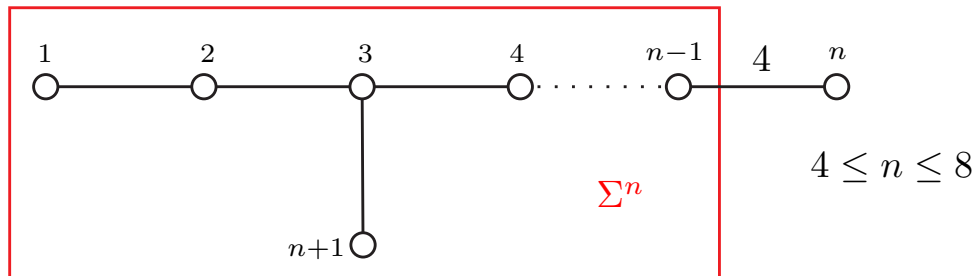
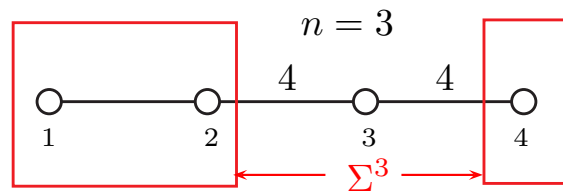
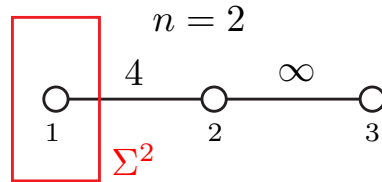
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- A Coxeter diagram for Δ^n is given on the next slide.

Coxeter diagrams



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The right-angled polytope P^n

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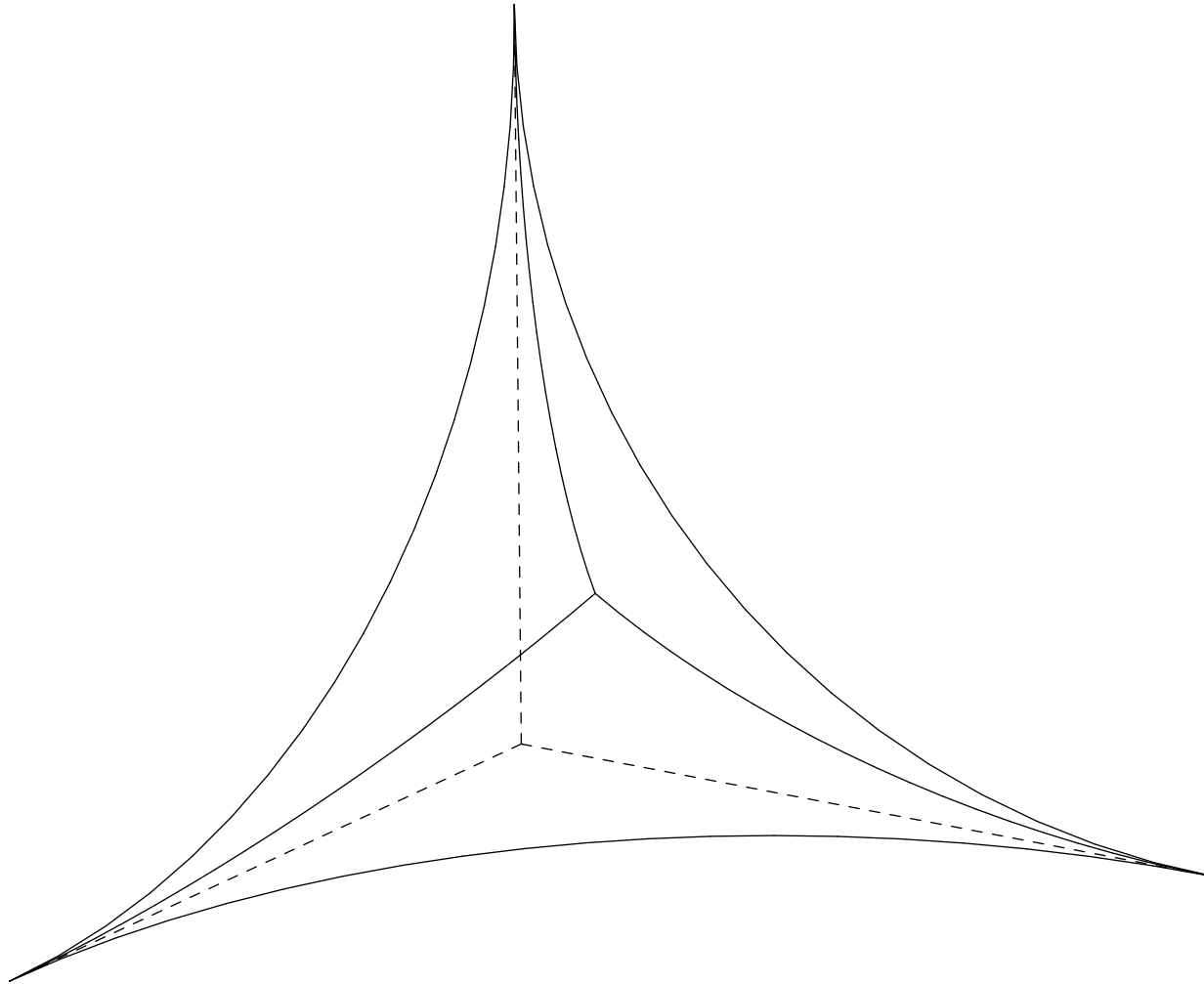
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- Then P^n is a convex polytope of finite volume with symmetry group Σ^n .
- The polytope P^n is right-angled for all n , and each side of P^n is congruent to P^{n-1} for all $n > 2$.

The right-angled polytope P^3



The Congruence Two Subgroup

- Let Γ_2^n be the congruence two subgroup of $\Gamma^n = PO_{n,1}\mathbb{Z}$, that is, the subgroup of all matrices in Γ^n that are congruent to the identity matrix modulo two.

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- The above matrix is in Γ_2^2 and represents the third Coxeter generator of Γ^2 .

Γ_2^n is a right-angled Coxeter group

- **Theorem:** For $n = 2, \dots, 7$, the congruence two subgroup Γ_2^n of Γ^n is a hyperbolic reflection group with Coxeter polytope the right-angled polytope P^n . Moreover, $\Gamma^n / \Gamma_2^n \cong \Sigma^n$.

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- **Theorem:** For $n = 2, \dots, 7$, the congruence two subgroup Γ_2^n of Γ^n is a hyperbolic reflection group with Coxeter polytope the right-angled polytope P^n . Moreover, $\Gamma^n / \Gamma_2^n \cong \Sigma^n$.
- **Corollary:** For $n = 2, \dots, 7$, every maximal finite subgroup of Γ_2^n is either elementary of order 2^n and conjugate to the stabilizer of an actual vertex of P^n or is elementary of order 2^{n-1} and conjugate to the stabilizer of a line edge of P^n .

The right-angled polytope P^6

- The polytope P^6 has 72 actual vertices and 27 ideal vertices, 432 ray edges, 216 line edges, 1089 P^2 -faces, 720 P^3 -faces, 216 P^4 -faces and 27 sides each congruent to P^5 .

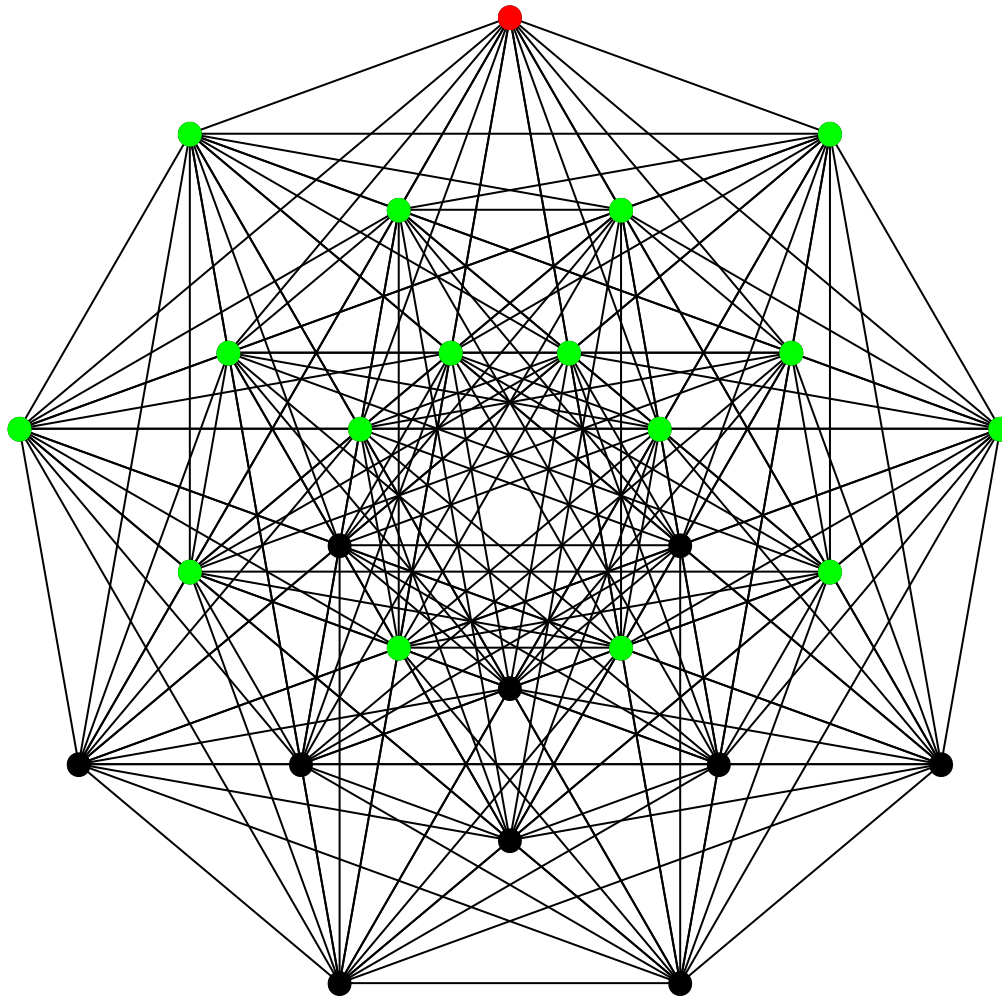
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- The Euclidean dual of P^6 is a semi-regular 6-dimensional polytope discovered by Gosset in 1900.
- The Gosset 6-polytope combinatorially parametrizes the arrangement of the 27 straight lines in a general cubic surface.

1-Skeleton of the Gosset 6-Polytope



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- By the Gauss-Bonnet theorem, P^6 has volume $\pi^3/15$ which is one-eighth the smallest volume possible for a hyperbolic 6-manifold.
- We constructed an orientable hyperbolic 6-manifold M of the smallest possible volume $8\pi^3/15$ and $\chi(M) = -1$ by gluing together eight copies of P^6 along their sides.

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- The 6-manifold M is isometric to the orbit space H^6/Γ of a **torsion-free** subgroup Γ of Γ^6 of index 414,720.

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- The group Γ is generated by $\Gamma \cap \Gamma_2^6$ and the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & -1 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \quad \text{with } \det(A) = 1.$$

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- The polytope Q^6 is a fundamental polytope for $\Gamma \cap \Gamma_2^6$.

- The group $\Gamma \cap \Gamma_2^6$ is generated by 252 elements of the form $s_i k_i$ where s_i is the reflection in the i th side of Q^6 and $k_i \in K^6$, with $\det(k_i) = -1$, and so $s_i k_i$ is orientation preserving for each i .

The group $\Gamma \cap \Gamma_2^6$ is torsion-free

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- This easily implies that $\Gamma \cap \Gamma_2^6$ is a normal subgroup of Γ_2^6 and $\Gamma_2^6 / (\Gamma \cap \Gamma_2^6) \cong K^6$ with r_i mapping to k_i , since $s_i k_i$ gets killed.

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- The group $\Gamma \cap \Gamma_2^6$ is **torsion-free** because each maximal finite subgroup of Γ_2^6 maps isomorphically into \mathbb{K}^6 under the isomorphism $\Gamma_2^6 / (\Gamma \cap \Gamma_2^6) \cong \mathbb{K}^6$.

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- The matrix A normalizes $\Gamma \cap \Gamma_2^6$, since A leaves the subgroup $(\Gamma \cap \Gamma_2^6) / [\Gamma_2^6, \Gamma_2^6]$ invariant.
- The quotient $\Gamma / (\Gamma \cap \Gamma_2^6)$ is a cyclic group of order 8, since A projects to a matrix \bar{A} in Σ^6 of order 8.

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- Now $A = r_2\bar{A}$ where $r_2 = \text{diag}(1, -1, 1, 1, 1, 1, 1)$.
- Rewriting the equation $1 = (hA^4)(hA^4)$ in terms of \bar{A} leads to a linear equation

$$(I + \bar{A}_*^4)(v) = v_9 + v_{11} + v_{12} + v_{14} + v_{20} + v_{21}.$$

in the $\mathbb{Z}/2$ -vector space $(\Gamma \cap \Gamma_2^6)/[\Gamma_2^6, \Gamma_2^6]$ with basis v_7, \dots, v_{27} which has no solution. Here v_i is the image of the Q^6 side-pairing map $r_i k_i$. Hence Γ is torsion-free.

H^6/Γ has minimum volume

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- The orientable hyperbolic manifold $M = H^6/\Gamma$ has the smallest possible volume $8\pi^3/15$ by the Gauss-Bonnet Theorem.
- The manifold M is noncompact with five cusps.

H^6/Γ has minimum volume

- As $\chi(\Gamma_2^6) = -1/8$ and $\Gamma_2^6/(\Gamma \cap \Gamma_2^6)$ has order 64, we have that

$$\chi(\Gamma \cap \Gamma_2^6) = -64/8 = -8.$$

- As $\Gamma/\Gamma \cap \Gamma_2^6$ has order 8, we have that

$$\chi(\Gamma) = -8/8 = -1.$$

- The orientable hyperbolic manifold $M = H^6/\Gamma$ has the smallest possible volume $8\pi^3/15$ by the Gauss-Bonnet Theorem.
- The manifold M is noncompact with five cusps.
- $H_1(M) \cong (\mathbb{Z}/2)^4 \oplus \mathbb{Z}/8$.

References

- The details for this talk are available in our preprint that can be downloaded from lanl.arXiv.org.

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- The preprint version of this talk will appear in *Mathematische Annalen*.