arXiv:0909.0293v1 [math.QA] 1 Sep 2009

RIGHT COIDEAL SUBALGEBRAS OF NICHOLS ALGEBRAS AND THE DUFLO ORDER ON THE WEYL GROUPOID

I. HECKENBERGER AND H.-J. SCHNEIDER

Abstract. We study graded right coideal subalgebras of Nichols algebras of semisimple Yetter-Drinfeld modules. Assuming that the Yetter-Drinfeld module admits all reflections and the Nichols algebra is decomposable, we construct an injective order preserving and order reflecting map between morphisms of the Weyl groupoid and graded right coideal subalgebras of the Nichols algebra. Here morphisms are ordered with respect to right Duflo order and right coideal subalgebras are ordered with respect to inclusion. If the Weyl groupoid is finite, then we prove that the Nichols algebra is decomposable and the above map is bijective. In the special case of the Borel part of quantized enveloping algebras our result implies a conjecture of Kharchenko.

Introduction

It is well-known that quantum groups do not have "enough" Hopf subalgebras. Instead the larger class of right (or left) coideal subalgebras should be studied. A right coideal subalgebra $E\subset A$ of a Hopf algebra A with comultiplication Δ is a subalgebra of A with $\Delta(E)\subset E\otimes A$.

1. Right coideal subalgebras of quantized enveloping algebras $U^{\ge 0}$. Let $\mathfrak{g}$ be a semisimple complex Lie algebra, Π a basis of its root system with respect to a fixed Cartan subalgebra, and $U=U_q(\mathfrak{g})$ the quantized enveloping algebra of $\mathfrak{g}$ in the sense of [Jan96, Ch. 4]. We assume that q is not a root of unity. Let U^+ and U^0 be the subalgebras of U generated by the sets $\{E_\alpha\,|\,\alpha\in\Pi\}$ and $\{K_\alpha,K_\alpha^{-1}\,|\,\alpha\in\Pi\}$, respectively, and let $U^{\ge 0}=U^+U^0$. For any element w of the Weyl group W of $\mathfrak{g}$ let $U^+[w]\subset U^+$ be the subspace defined in [Jan96, 8.24] in terms of root vectors constructed via Lusztig's automorphisms. We prove in Thm. 7.3, see also Cor. 6.13, the following:

The map $w\mapsto U^+[w]U^0$ defines an order preserving bijection between W and the set of all right coideal subalgebras of $U^{\ge 0}$ containing U^0, where right coideal subalgebras are ordered by inclusion and W is ordered by the Duflo order. If

2000 *Mathematics Subject Classification.* 17B37,16W30;20F55.

Key words and phrases. Hopf algebra, quantum group, root system, Weyl group.

The work of I.H. is supported by DFG within a Heisenberg fellowship.

$E_1 \subset E_2$ are right coideal subalgebras of $U^{\geq 0}$ containing U^0, then E_2 is free over E_1 as a right module.

Recall that if w_1, w_2 are elements in W, then $w_1 \leq_D w_2$ in the (right) Duflo order if and only if any reduced expression of w_1 can be extended to a reduced expression of w_2 beginning with w_1.

In particular, the number of right coideal subalgebras of $U^{\geq 0}$ containing U^0 is equal to the order of the Weyl group W. This last statement was conjectured by Kharchenko in [Kha09] for simple Lie algebras $\mathfrak{g}$. The conjecture was proven for $\mathfrak{g}$ of type A_n [KS08], B_n [Kha09] and G_2 [Pog09] by combinatorial calculations using Lyndon words. In these papers right coideal subalgebras are classified in terms of certain subsets of positive roots.

The subspaces $U^+[w] \subset U^+$ are familiar objects in quantum groups. Among others, they are used by Lusztig [Lus93] to establish a PBW basis for U^+, by De Concini, Kac and Procesi [CKP95] to introduce quantum Schubert calculus, and are identified by Yakimov [Yak09] as quotients of quantized Bruhat cell translates [Jos95, Gor00]. It was essentially well-known that $U^+[w]U^0$ is a right coideal subalgebra of $U^{\geq 0}$: proofs and indications in this direction can be found in [LS90], [CKP95, 2.2], [CP92, 9.3], [AJS94]. The arguments often use case by case considerations and reduction to the rank two case, and sometimes they work only in the h-adic setting. The algebras $U^+[w]$ are known to depend only on w and not on the chosen reduced expression of w, see e. g. [Lus93, 40.2.1] and [Jan96, 8.21]. With our systematic approach to graded right coideal subalgebras we offer a new way to study U^+ without the usual case by case considerations, and intrinsically characterize the algebras $U^+[w]$ and their ordering with respect to inclusion. With the necessary modifications, our results also apply to the small quantum groups of semisimple Lie algebras where q is a root of unity, and to multiparameter versions of U, see Cor. 6.17 and Rem. 7.4.

2. Right coideal subalgebras of Nichols algebras. The paper is written in the very general context of Nichols algebras $\mathcal{B}(M)$ of semisimple Yetter-Drinfeld modules $M \in {}^H_H\mathcal{YD}$, where H is an arbitrary Hopf algebra with bijective antipode. Nichols algebras, also called quantum symmetric algebras, see [Ros98], appear as fundamental objects in the classification theory of Hopf algebras [AS98, AS02, AS05], in particular of Hopf algebras which are generated by group-like and skew-primitive elements. For example, in the setting of quantized enveloping algebras, $M = \oplus_{\alpha\in\Pi} \Bbbk E_\alpha$ is a Yetter-Drinfeld module over U^0, and $\mathcal{B}(M) = U^+$. Finite-dimensional Nichols algebras of diagonal type are classified in [Hec09]. Recently, much progress in the understanding of finiteness properties of Nichols algebras of nonabelian group type has been achieved, see e. g. [AFGV09b, AFGV09a], [HS08] and references therein.

In rather general situations (if M admits all reflections, see Sect. 6) one can associate a Weyl groupoid $\mathcal{W}(M)$ to M, see [AHS08], [HS08]. In case of the Borel part of a quantized Kac-Moody algebra $\mathfrak{g}$, $\mathcal{W}(M)$ is essentially the Weyl group of $\mathfrak{g}$. Under the assumption that $\mathcal{W}(M)$ is finite, we prove in Cor. 6.13 a PBW-theorem for the Nichols algebra $\mathcal{B}(M)$ and its right coideal subalgebras, where the subalgebra generated by a root vector in the quantum group case is replaced by the Nichols algebra of a finite-dimensional irreducible Yetter-Drinfeld module. As a consequence we can show that the real roots associated with $\mathcal{W}(M)$ satisfy the axioms of a root system in the sense of [HY08b], see also [HS08]. In Thm. 6.14 we provide generalizations of results of Levendorskii and Soibelman [LS90, LS91] on coproducts and commutators of root vectors. Our proofs are new even for U^+, since they are free of case by case calculations, and do not use the braid relations for Lusztig's automorphisms.

We note that a PBW-theorem for right coideal subalgebras of character Hopf algebras (where the braiding is diagonal) is obtained by Kharchenko in terms of Lyndon words, see [Kha08]. For Nichols algebras of diagonal type a PBW theorem in the spirit of Lusztig was proven by the first author and Yamane [HY08a].

The main results in this paper rely on the crucial coproduct formula in Thm. 4.2. For quantum groups this formula amounts to an explicit computation of the braided coproduct of U^+ in the image of $T_\alpha(U^+)$ as a subalgebra of U. Our formula has the advantage to involve only algebra maps, and hence it is well-suited to study coideal subalgebras.

To provide more details, let $\theta \in \mathbb{N}$, let $M_1, \dots, M_\theta$ be finite-dimensional irreducible objects in ${}^H_H\mathcal{YD}$, and $M = (M_1, \dots, M_\theta)$. The goal is to understand the Nichols algebra

$$\mathcal{B}(M) = \mathcal{B}(M_1 \oplus \cdots \oplus M_\theta)$$

as a Hopf algebra in the braided category ${}^H_H\mathcal{YD}$. Let $\mathbb{Z}^\theta$ be the free abelian group of rank θ with standard basis $\alpha_1, \dots, \alpha_\theta$. The Nichols algebra $\mathcal{B}(M)$ is $\mathbb{Z}^\theta$-graded where $\deg(M_i) = \alpha_i$ for all $1 \le i \le \theta$.

First we define reflection operators R_i, $1 \le i \le \theta$. Assume that for all $j \neq i$,

$$a^M_{ij} = -\max\{m \,|\, (\mathrm{ad}\, M_i)^m(M_j) \neq 0\} < \infty.$$

Define $a^M_{ii} = 2$. Then $(a^M_{ij})_{i,j\in\{1,\dots,\theta\}}$ is a generalized Cartan matrix. Let $s^M_i \in \mathrm{Aut}(\mathbb{Z}^\theta)$ be the corresponding reflection. Define $R_i(M)_i = M_i^*$, and

$$R_i(M)_j = (\mathrm{ad}\, M_i)^{-a^M_{ij}}(M_j) \text{ for all } j \neq i,$$

and let $R_i(M) = (R_i(M)_1, \dots, R_i(M)_\theta)$. Finally let $K_i^M = \mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}$, where the coinvariant elements are defined with respect to the projection of $\mathcal{B}(M)$ onto $\mathcal{B}(M_i)$. By [AHS08, Thm. 3.12] there is an algebra isomorphism

$$\Omega_i^M : K_i^M \# \mathcal{B}(M_i^*) \to \mathcal{B}(R_i(M))$$

which is the identity on all $R_j(M)_j \subset K_i^M \# \mathcal{B}(M_i^*)$. By the coproduct formula in Thm. 4.2, Ω_i^M becomes an isomorphism of $\mathbb{Z}^\theta$-graded braided Hopf algebras.

In Prop. 7.1 we show that in the quantum group case the inverse of Ω_i^M can be identified with Lusztig's automorphism T_{α_i} restricted to U^+.

Assume that all iterations of the construction $M \mapsto R_j(M)$ are well-defined. In [AHS08], [HS08, Thm. 6.10] the Weyl groupoid $\mathcal{W}(M)$ of M is defined. The objects of $\mathcal{W}(M)$ are sequences of isomorphism classes $[N] = ([N_1], \dots, [N_\theta])$ where the sequence of Yetter-Drinfeld modules $(N_1, \dots, N_\theta)$ is obtained from M by iterating the operations R_j. The morphisms are generated by elementary reflections $s_i^N : R_i(N) \to N$. Then our main result on right coideal subalgebras in the general case, see Thm. 6.15 and Cor. 6.17, says the following.

Assume that the Weyl groupoid of M is finite. Then there exists an order preserving bijection $\varkappa^M$ between the set of morphisms of $\mathcal{W}(M)$ with target $[M]$ and the set of $\mathbb{N}_0^\theta$-graded right coideal subalgebras of $\mathcal{B}(M)\#H$ containing H, where right coideal subalgebras are ordered with respect to inclusion and the morphisms are ordered by the Duflo order.

The map $\varkappa^M$ also exists for non-finite $\mathcal{W}(M)$, if we assume that $\mathcal{B}(M)$ is decomposable, see Def. 6.8. Then $\varkappa^M$ is always injective, order preserving and order reflecting by Thm. 6.12.

Acknowledgement. The first author would like to thank S. Kolb for interesting discussions on coideal subalgebras of $U_q(\mathfrak{g})$.

1. Weyl groupoids and the Duflo order

Recall the definition of the Weyl groupoid of a root system from [CH09, Sect. 2], see also [HS08, Sect. 5].

Let I be a non-empty finite set and $(\alpha_i)_{i\in I}$ the standard basis of $\mathbb{Z}^I$. Let $\mathcal{X}$ be a non-empty set, and for all $i \in I$ and $X \in \mathcal{X}$ let $r_i : \mathcal{X} \to \mathcal{X}$ be a map and $A^X = (a_{jk}^X)_{j,k\in I}$ a generalized Cartan matrix. The quadruple

$$\mathcal{C} = \mathcal{C}(I, \mathcal{X}, (r_i)_{i\in I}, (A^X)_{X\in\mathcal{X}}),$$

is called a *Cartan scheme* if

(C1) $r_i^2 = \mathrm{id}$ for all $i \in I$,

(C2) $a_{ij}^X = a_{ij}^{r_i(X)}$ for all $X \in \mathcal{X}$ and $i, j \in I$.

Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, (r_i)_{i\in I}, (A^X)_{X\in\mathcal{X}})$ be a Cartan scheme. For all $i \in I$, $X \in \mathcal{X}$ define $s_i^X \in \mathrm{Aut}(\mathbb{Z}^I)$ by

$$s_i^X(\alpha_j) = \alpha_j - a_{ij}^X \alpha_i \qquad \text{for all } j \in I.$$

Recall that a groupoid is a category where all morphisms are isomorphisms. The *Weyl groupoid of* $\mathcal{C}$ is the groupoid $\mathcal{W}(\mathcal{C})$ with $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = \mathcal{X}$, where the morphisms are generated by all s_i^X (considered as morphism in $\mathrm{Hom}(X, r_i(X))$ with $i \in I$, $X \in \mathcal{X}$. Then $s_i^{r_i(X)} s_i^X = \mathrm{id}_X$ in $\mathrm{Hom}(X, X)$. We will write s_i instead of s_i^X if X is uniquely determined by the context.

For any groupoid $\mathcal{G}$ and any $X \in \mathrm{Ob}(\mathcal{G})$ let

$$\mathrm{Hom}(\mathcal{G}, X) = \bigcup_{Y\in\mathrm{Ob}(\mathcal{G})} \mathrm{Hom}(Y, X) \quad \text{(disjoint union)}.$$

Let $\mathcal{C}$ be a Cartan scheme and let $X \in \mathcal{X}$. Following [Kac90, §5.1] we say that

$$\boldsymbol{\Delta}^{X\,\mathrm{re}} = \{w(\alpha_i) \,|\, i \in I, w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), X)\} \tag{1.1}$$

is the set of *real roots* (of X), where $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), X)$ is interpreted as an element in $\mathrm{Aut}(\mathbb{Z}^I)$. A real root $\alpha \in \boldsymbol{\Delta}^{X\,\mathrm{re}}$ is called *positive*, if $\alpha \in \mathbb{N}_0^I$. The set of positive real roots is denoted by $\boldsymbol{\Delta}_+^{X\,\mathrm{re}}$.

Remark 1.1. Weyl groupoids associated to Nichols algebras satisfy additional properties which do not follow from the axioms of Cartan schemes, see Thm. 6.15. For example, cf. [CH09, Pf. of Thm. 6.1], let $\mathcal{X} = \{X_1, X_2, X_3\}$, $I = \{1, 2\}$, $r_1(X_i) = X_{\sigma(i)}$, $r_2(X_i) = X_{\tau(i)}$, where $\sigma = (1\,2)$, $\tau = (2\,3)$. Let

$$A^{X_1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad A^{X_2} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}, \quad A^{X_3} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

Then $\mathcal{C}(I, \mathcal{X}, (r_i)_{i\in I}, (A^X)_{X\in\mathcal{X}})$ is a Cartan scheme with finitely many real roots

$$\begin{aligned}
\boldsymbol{\Delta}^{X_1\,\mathrm{re}} =& \{\pm 1, \pm 2, \pm 12, \pm 12^2, \pm 12^3, \pm 1^2 2^3,\\
& \pm 1^3 2^4, \pm 1^3 2^5, \pm 1^4 2^5, \pm 1^4 2^7, \pm 1^5 2^7, \pm 1^5 2^8\},\\
\boldsymbol{\Delta}^{X_2\,\mathrm{re}} =& \{\pm 1, \pm 2, \pm 12, \pm 12^2, \pm 12^3, \pm 1^2 2^3,\\
& \pm 12^4, \pm 12^5, \pm 1^2 2^5, \pm 1^2 2^7, \pm 1^3 2^7, \pm 1^3 2^8\},\\
\boldsymbol{\Delta}^{X_3\,\mathrm{re}} =& \{\pm 12^{-1}, \pm 1, \pm 2, \pm 12, \pm 1^2 2, \pm 12^2,\\
& \pm 12^3, \pm 1^2 2^3, \pm 12^4, \pm 1^3 2^4, \pm 1^2 2^5, \pm 1^3 2^5\},
\end{aligned}$$

where $k\alpha_1 + l\alpha_2$ is abbreviated by $1^k 2^l$ for all $k, l \in \mathbb{Z}$. Observe that $\boldsymbol{\Delta}^{X_3\,\mathrm{re}}$ contains the real root $\alpha_1 - \alpha_2$, and hence $\mathcal{C}(I, \mathcal{X}, (r_i)_{i\in I}, (A^X)_{X\in\mathcal{X}})$ does not admit a root system in the sense of the following definition.

We say that

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (\boldsymbol{\Delta}^X)_{X\in\mathcal{X}})$$

is a *root system of type* $\mathcal{C}$ if $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, (r_i)_{i\in I}, (A^X)_{X\in\mathcal{X}})$ is a Cartan scheme and $\boldsymbol{\Delta}^X \subset \mathbb{Z}^I$, where $X \in \mathcal{X}$, are subsets such that

(R1) $\boldsymbol{\Delta}^X = (\boldsymbol{\Delta}^X \cap \mathbb{N}_0^I) \cup -(\boldsymbol{\Delta}^X \cap \mathbb{N}_0^I)$ for all $X \in \mathcal{X}$,
(R2) $\boldsymbol{\Delta}^X \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$ for all $i \in I$, $X \in \mathcal{X}$,
(R3) $s_i^X(\boldsymbol{\Delta}^X) = \boldsymbol{\Delta}^{r_i(X)}$ for all $i \in I$, $X \in \mathcal{X}$,
(R4) $(r_ir_j)^{m_{i,j}^X}(X) = X$ for all $i, j \in I$ and $X \in \mathcal{X}$ such that $i \neq j$ and $m_{i,j}^X := \#(\boldsymbol{\Delta}^X \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j))$ is finite.

If $\mathcal{R}(\mathcal{C}, (\boldsymbol{\Delta}^X)_{X\in\mathcal{X}})$ is a root system of type $\mathcal{C}$, then $\mathcal{W}(\mathcal{R}) := \mathcal{W}(\mathcal{C})$ is called the *Weyl groupoid of* $\mathcal{R}$. The elements of $\boldsymbol{\Delta}_+^X := \boldsymbol{\Delta}^X \cap \mathbb{N}_0^I$ and $\boldsymbol{\Delta}_-^X := -\boldsymbol{\Delta}_+^X$ are called *positive* and *negative roots*, respectively. Note that (R3) implies that $w(\boldsymbol{\Delta}^Y) = \boldsymbol{\Delta}^X$ for all $X, Y \in \mathcal{X}$ and $w \in \mathrm{Hom}(Y, X)$.

Recall that a groupoid $\mathcal{G}$ is *connected*, if for all $X, Y \in \mathrm{Ob}(\mathcal{G})$ the set $\mathrm{Hom}(Y, X)$ is non-empty. It is *finite*, if $\mathrm{Hom}(\mathcal{G})$ is finite.

The following claim was proven in [CH09, Lemma 2.11].

Lemma 1.2. *Let $\mathcal{C}$ be a Cartan scheme and $\mathcal{R}$ a root system of type $\mathcal{C}$. Assume that $\mathcal{W}(\mathcal{R})$ is connected. Then the following are equivalent.*

(1) $\boldsymbol{\Delta}^X$ *is finite for at least one* $X \in \mathrm{Ob}(\mathcal{W}(\mathcal{R}))$.
(2) $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ *is finite for at least one* $X \in \mathrm{Ob}(\mathcal{W}(\mathcal{R}))$.
(3) $\mathrm{Hom}(\mathcal{W}(\mathcal{R}), X)$ *is finite for at least one* $X \in \mathrm{Ob}(\mathcal{W}(\mathcal{R}))$.
(4) *The groupoid* $\mathcal{W}(\mathcal{R})$ *is finite.*

Further, (1)–(3) hold for one $X \in \mathrm{Ob}(\mathcal{W}(\mathcal{R}))$ *if and only if they hold for all* $X \in \mathrm{Ob}(\mathcal{W}(\mathcal{R}))$.

Remark 1.3. The equivalence of (1), (2), and (4) was stated and proven in [CH09, Lemma 2.11]. Clearly, (4) implies (3). For the proof of the implication (4)⇒(1) in [CH09, Lemma 2.11] one needs only to assume (3), and hence all claims of Lemma 1.2 are equivalent.

Let $\mathcal{C}$ be a Cartan scheme and $\mathcal{R}$ a root system of type $\mathcal{C}$. Then $\mathcal{R}$ is called *finite*, see [CH09, Def. 2.20], if $\boldsymbol{\Delta}^X$ is finite for all $X \in \mathrm{Ob}(\mathcal{W}(\mathcal{R}))$. If $\mathcal{W}(\mathcal{R})$ is connected, then this is equivalent to the conditions in Lemma 1.2.

Let ℓ denote the length function on Weyl groupoids of root systems: for each $X \in \mathcal{X}$ and each $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{R}), X)$ let

$$\ell(w) = \min\{m \in \mathbb{N}_0 \,|\, \text{there exist } i_1, \ldots, i_m \in I \text{ with } w = \mathrm{id}_X s_{i_1} \cdots s_{i_m}\}.$$

Proposition 1.4. [CH09, Prop. 2.12] *Let $\mathcal{C}$ be a Cartan scheme and $\mathcal{R}$ a root system of type $\mathcal{C}$. Let $X \in \mathcal{X}$, $m \in \mathbb{N}_0$, and $i_1, \ldots, i_m \in I$ such that $\ell(w) = m$ for $w = s_{i_1} \cdots s_{i_m} \in \mathrm{Hom}(\mathcal{W}(\mathcal{R}), X)$. Then the roots*

$$\beta_k = \mathrm{id}_X s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \qquad k \in \{1, 2, \ldots, m\},$$

are positive and pairwise distinct. If $\mathcal{R}$ is finite and $w \in \mathrm{Hom}(\mathcal{W}(\mathcal{R}), X)$ is the unique longest element, then $\{\beta_k \,|\, 1 \le k \le \ell(w)\} = \boldsymbol{\Delta}^X_+$.

Prop. 1.4 implies in particular that the set of roots of a finite root system is uniquely determined by its Cartan scheme and coincides with the set of real roots.

Definition 1.5. Let $\mathcal{X}$ and I be non-empty sets, $(r_i)_{i\in I}$ a family of maps $r_i : \mathcal{X} \to \mathcal{X}$, and $(m^X_{i,j})_{i,j\in I, X\in\mathcal{X}}$ a family of numbers in $\mathbb{N} \cup \{\infty\}$ such that $m^X_{i,i} = 1$ and $(r_i r_j)^{m^X_{i,j}}(X) = X$ for all $X \in \mathcal{X}$ and $i, j \in I$ with $m^X_{i,j} < \infty$. [1] Let $\mathcal{G}$ be a groupoid with $\mathrm{Ob}(\mathcal{G}) = \mathcal{X}$, and let $(s^X_i)_{i\in I, X\in\mathcal{X}}$ be a family of morphisms $s^X_i \in \mathrm{Hom}(X, r_i(X))$. We say that $(\mathcal{G}, (s^X_i)_{i\in I, X\in\mathcal{X}})$ *satisfies the Coxeter relations* if

$$\underbrace{s_i^{r_j(r_ir_j)^{m^X_{i,j}-1}(X)} s_j^{(r_ir_j)^{m^X_{i,j}-1}(X)} \ldots s_j^{r_ir_j(X)} s_i^{r_j(X)} s_j^X}_{2m^X_{i,j} \text{ factors}} = \mathrm{id}_X \tag{1.2}$$

for all $X \in \mathcal{X}$ and $i, j \in I$ with $m^X_{i,j} < \infty$. In particular, Eq. (1.2) means for $i = j$ that $s_i^{r_i(X)} s_i^X = \mathrm{id}_X$ for all $X \in \mathcal{X}$ and $i \in I$.

Let $\mathcal{W}$ be a groupoid and $(s^X_i)_{i\in I, X\in\mathcal{X}}$ a family of morphisms as above. We say that $(\mathcal{W}, (s^X_i)_{i\in I, X\in\mathcal{X}})$ is a *Coxeter groupoid*, if

(1) $(\mathcal{W}, (s^X_i)_{i\in I, X\in\mathcal{X}})$ satisfies the Coxeter relations, and
(2) for each pair $(\mathcal{G}, (t^X_i)_{i\in I, X\in\mathcal{X}})$ satisfying the Coxeter relations (with the same $\mathcal{X}$, I, (r_i) and $(m^X_{i,j})$ as for $\mathcal{W}$) there is a unique functor $F : \mathcal{W} \to \mathcal{G}$ such that F is the identity on $\mathcal{X} = \mathrm{Ob}(\mathcal{W}) = \mathrm{Ob}(\mathcal{G})$ and $F(s^X_i) = t^X_i$ for all $i \in I$, $X \in \mathcal{X}$.

The universal property of a Coxeter groupoid $(\mathcal{W}, (s^X_i)_{i\in I, X\in\mathcal{X}})$ implies that $\mathrm{Hom}(\mathcal{W})$ is generated by the morphisms $s^X_i \in \mathrm{Hom}(X, r_i(X))$, where $i \in I$ and $X \in \mathcal{X}$.

For the rest of this section let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, (r_i)_{i\in I}, (A^X)_{X\in\mathcal{X}})$ be a Cartan scheme and let $\mathcal{R} = \mathcal{R}(\mathcal{C}, (\boldsymbol{\Delta}^X)_{X\in\mathcal{X}})$ be a root system of type $\mathcal{C}$.

Theorem 1.6. [HY08b, Thm. 1] *For all $i, j \in I$ and $X \in \mathcal{X}$ let*

$$m^X_{i,j} = \#(\boldsymbol{\Delta}^X \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)).$$

[1] This slight extension of notation is compatible with (R4) and (C1).

Then $(\mathcal{W}(\mathcal{R}),(s_i^X)_{i\in I,X\in\mathcal{X}})$ *is a Coxeter groupoid with respect to* $(m_{i,j}^X)$.

Definition 1.7. For all $X\in\mathcal{X}$, $m\in\mathbb{N}$ and $(i_1,\dots,i_m)\in I^m$ let $\Lambda_+^X()=\emptyset$ and

$$\begin{aligned}&\beta_k=s_{i_1}^{r_{i_1}(X)}s_{i_2}^{r_{i_2}r_{i_1}(X)}\cdots s_{i_{k-1}}^{r_{i_{k-1}}\cdots r_{i_2}r_{i_1}(X)}(\alpha_{i_k})\in\boldsymbol{\Delta}^X,\quad k\in\{1,2,\dots,m\},\\&\Lambda^X(i_1,\dots,i_m)=(\beta_k)_{k\in\{1,2,\dots,m\}},\\&\Lambda_+^X(i_1,\dots,i_m)=\big\{\lambda\in\boldsymbol{\Delta}_+^X\,|\,\#\{k\in\mathbb{Z}\,|\,1\le k\le m,\lambda=\pm\beta_k\}\text{ is odd}\big\}.\end{aligned}$$

Lemma 1.8. *Let* $m\in\mathbb{N}$, $i_1,\dots,i_m\in I$, $X\in\mathcal{X}$, *and* $Y=r_{i_1}(X)$. *Then*

$$\Lambda_+^X(i_1,\dots,i_m)=\begin{cases}s_{i_1}^Y(\Lambda_+^Y(i_2,\dots,i_m))\cup\{\alpha_{i_1}\} & \text{if } \alpha_{i_1}\notin\Lambda_+^Y(i_2,\dots,i_m),\\ s_{i_1}^Y\big(\Lambda_+^Y(i_2,\dots,i_m)\setminus\{\alpha_{i_1}\}\big) & \text{if } \alpha_{i_1}\in\Lambda_+^Y(i_2,\dots,i_m).\end{cases}$$

Proof. The claim follows from Def. 1.7 and basic properties of the map $s_{i_1}^Y$. □

The sets $\Lambda_+^X(i_1,\dots,i_m)$ ultimately describe the elements of $\mathrm{Hom}(\mathcal{W}(\mathcal{R}),X)$.

Proposition 1.9. *Let* $X\in\mathcal{X}$, $m,m'\in\mathbb{N}_0$ *and* $i_1,\dots,i_m,i'_1,\dots,i'_{m'}\in I$. *The following are equivalent.*

(1) $\Lambda_+^X(i_1,\dots,i_m)=\Lambda_+^X(i'_1,\dots,{i'}_{m'})$,
(2) $s_{i_1}\cdots s_{i_m}=s_{i'_1}\cdots s_{i'_{m'}}$ *in* $\mathrm{Hom}(\mathcal{W}(\mathcal{R}),X)$.

Moreover, $\#\Lambda_+^X(i_1,\dots,i_m)=\ell(\mathrm{id}_X s_{i_1}\cdots s_{i_m})$.

Proof. Let $\mathcal{V}$ denote the category with $\mathrm{Ob}(\mathcal{V})=\mathcal{X}$ and morphisms

$$\begin{aligned}\mathrm{Hom}(Y,Z)=\big\{&\big(\Lambda_+^Z(j_1,\dots,j_n),\mathrm{id}_Z s_{j_1}\cdots s_{j_n}\big)\,|\\&n\in\mathbb{N}_0,\,j_1,\dots,j_n\in I,r_{j_1}\cdots r_{j_n}(Y)=Z\big\}\end{aligned}$$

for all $Y,Z\in\mathcal{X}$. Composition of morphisms is defined via concatenation:

$$\begin{aligned}&\big(\Lambda_+^Z(j_1,\dots,j_n),\mathrm{id}_Z s_{j_1}\cdots s_{j_n}\big)\circ\big(\Lambda_+^Y(k_1,\dots,k_p),\mathrm{id}_Y s_{k_1}\cdots s_{k_p}\big)\\&\qquad=\big(\Lambda_+^Z(j_1,\dots,j_n,k_1,\dots,k_p),\mathrm{id}_Z s_{j_1}\cdots s_{j_n}s_{k_1}\cdots s_{k_p}\big)\end{aligned}$$

for all $Y,Z\in\mathcal{X}$, $n,p\in\mathbb{N}_0$ and $j_1,\dots,j_n,k_1,\dots,k_p\in I$ with $Z=r_{j_1}\cdots r_{j_n}(Y)$. First we prove that $\mathcal{V}$ is indeed a category.

Let $n,n',p,p'\in\mathbb{N}_0$, $j_1,\dots,j_n$, $j'_1,\dots,j'_{n'}$, $k_1,\dots,k_p$, $k'_1,\dots,k'_{p'}\in I$ and $Z\in\mathcal{X}$ such that

$$\Lambda_+^Z(j_1,\dots,j_n)=\Lambda_+^Z(j'_1,\dots,j'_{n'}),\quad\Lambda_+^Y(k_1,\dots,k_p)=\Lambda_+^Y(k'_1,\dots,k'_{p'}),$$

and $\mathrm{id}_Z s_{j_1}\cdots s_{j_n}=\mathrm{id}_Z s_{j'_1}\cdots s_{j'_{n'}}$, where $Y=r_{j_n}\cdots r_{j_1}(Z)$. The definition of Λ_+^Z implies that

$$\Lambda_+^Z(j_1,\dots,j_n,k_1,\dots,k_p)=\Lambda_+^Z(j_1,\dots,j_n,k'_1,\dots,k'_{p'}),\tag{1.3}$$

$$\Lambda_+^Z(j_1,\dots,j_n,k_1,\dots,k_p)=\Lambda_+^Z(j'_1,\dots,j'_{n'},k_1,\dots,k_p),\tag{1.4}$$

and hence the composition is independent of the choice of representatives. Clearly, the composition is associative, and $(\Lambda^Z_+(), \mathrm{id}_Z)$ is the identity for any $Z \in \mathcal{X}$, and hence $\mathcal{V}$ is a category. Moreover, $\mathrm{Hom}(\mathcal{V})$ is generated by the morphisms $(\Lambda^Z_+(i), s_i^{r_i(Z)})$ with $i \in I$ and $Z \in \mathcal{X}$, which are invertible since $\Lambda^Z_+(i,i) = \emptyset$. Thus $\mathcal{V}$ is a groupoid.

Let $Z \in \mathcal{X}$ and $i, j \in I$. Assume that $m^Z_{i,j} < \infty$. Then

$$\Lambda^Z_+(\underbrace{i,j,i,j,\dots,i,j}_{2m^Z_{i,j}\text{ entries}}) = \emptyset, \tag{1.5}$$

since the entries β_k of $\Lambda^Z(i,j,i,j,\dots,i,j)$ are the elements of $(\mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j) \cap \mathbf{\Delta}^Z$, each appearing with multiplicity one, see [HY08b, Lemma 6]. In the special case, where $i = j$, we have $\Lambda^Z(i,i) = (\alpha_i, -\alpha_i)$. Hence the pair

$$(\mathcal{V}, ((\Lambda^Z_+(i), s_i^{r_i(Z)}))_{i\in I, Z\in\mathcal{X}})$$

satisfies the Coxeter relations in the sense of Def. 1.5. By Thm. 1.6, there is a functor from $\mathcal{W}(\mathcal{R})$ to $\mathcal{V}$ which maps $s_i^{r_i(Z)}$ to $(\Lambda^Z_+(i), s_i^{r_i(Z)})$ for each $Z \in \mathcal{X}$ and $i \in I$. This proves the implication (2)⇒(1) in the claim of the lemma.

We prove (1)⇒(2). Assume that $\Lambda^X_+(i_1,\dots,i_m) = \Lambda^X_+(i'_1,\dots,{i'}_{m'})$. Using that

$$\Lambda^Z_+(i,i,j_1,\dots,j_n) = \Lambda^Z_+(j_1,\dots,j_n), \quad \mathrm{id}_Z s_i s_i s_{j_1}\cdots s_{j_n} = \mathrm{id}_Z s_{j_1}\cdots s_{j_n}$$

for all $Z \in \mathcal{X}$ and $n \in \mathbb{N}_0$, $i, j_1, \dots, j_n \in I$, we may assume that $m' = 0$. By the first part of the lemma we may restrict to the case when $\mathrm{id}_X s_{i_1}\cdots s_{i_m}$ is a reduced expression, that is, $\ell(\mathrm{id}_X s_{i_1}\cdots s_{i_m}) = m$. Then we have to show that $m = 0$. The roots

$$s_{i_1}^{r_{i_1}(X)} \cdots s_{i_{k-1}}^{r_{i_{k-1}}\cdots r_{i_1}(X)}(\alpha_{i_k}) \in \mathbf{\Delta}^X,$$

where $1 \le k \le m$, are pairwise distinct and positive by Prop. 1.4. Hence the assumption $\Lambda^X_+(i_1,\dots,i_m) = \emptyset$ implies the desired claim $m = 0$. The last assertion of the lemma follows from the first one and from Prop. 1.4. □

Let $X, Y \in \mathcal{X}$, $w \in \mathrm{Hom}(Y, X)$, $m \in \mathbb{N}_0$, and $i_1, \dots, i_m \in I$ such that $w = s_{i_1}\cdots s_{i_m}$. Let $\Lambda^X_+(w) = \Lambda^X_+(i_1,\dots,i_m)$. This notation is justified by Prop. 1.9.

Corollary 1.10. *Let $X, Y \in \mathcal{X}$, $w \in \mathrm{Hom}(Y, X)$, and $i \in I$. Then*

$$\ell(s_i^X w) = \begin{cases} \ell(w) + 1 & \text{if } \alpha_i \notin \Lambda^X_+(w), \\ \ell(w) - 1 & \text{if } \alpha_i \in \Lambda^X_+(w). \end{cases}$$

Proof. By Prop. 1.9, $\ell(s_i^X w) = \ell(w)+1$ if and only if $\#\Lambda^{r_i(X)}_+(s_i^X w) = \#\Lambda^X_+(w)+1$. Then the claim holds by Lemma 1.8. □

Definition 1.11. Let $X \in \mathcal{X}$. For all $Y, Z \in \mathcal{X}$ and $x \in \mathrm{Hom}(Y, X)$, $y \in \mathrm{Hom}(Z, Y)$ we write $x \le_D xy$ (in $\mathrm{Hom}(\mathcal{W}(\mathcal{R}), X)$) if and only if $\ell(xy) = \ell(x) + \ell(y)$. Then $\le_D$ is a partial order on $\mathrm{Hom}(\mathcal{W}(\mathcal{R}), X)$ which is called the *(right) Duflo order.*

The definition stems from the corresponding notion for Weyl groups of semisimple Lie algebras, see [Mel04], [Jos95, A 1.2]. The Duflo order is also known as the *weak order* [BB05].

Remark 1.12. As for the right Duflo order for Weyl groups, $\le_D$ is the weakest partial order on $\mathcal{W}(\mathcal{R})$ such that $x \le_D xs_i$ for all $x \in \mathrm{Hom}(\mathcal{W}(\mathcal{R}))$ and $i \in I$ with $\ell(x) < \ell(xs_i)$.

The following theorem gives a characterization of the right Duflo order.

Theorem 1.13. *Let $X \in \mathcal{X}$ and let $w_1, w_2 \in \mathrm{Hom}(\mathcal{W}(\mathcal{R}), X)$. Then $w_1 \le_D w_2$ if and only if $\Lambda^X_+(w_1) \subset \Lambda^X_+(w_2)$,*

Proof. We proceed by induction on $\ell(w_1)$. If $w_1 = \mathrm{id}_X$, then $\Lambda^X_+(w_1) = \emptyset$ and $w_1 \le_D w_2$, and hence the claim holds. If $\ell(w_1) = 1$, then $w_1 = s_i^{r_i(X)}$ for some $i \in I$, and hence $\Lambda^X_+(w_1) = \alpha_i$. By definition, $w_1 \le_D w_2$ if and only if $\ell(w_2) = 1 + \ell(s_i^X w_2)$. Hence the claim holds by Cor. 1.10.

Assume now that $\ell(w_1) > 1$. Let $i \in I$ with $\ell(w_1) = \ell(w) + 1$ for $w = s_i^X w_1$. Then

$$\alpha_i \in \Lambda^X_+(w_1) \tag{1.6}$$

by Cor. 1.10. Thus Lemma 1.8 implies that $\Lambda^X_+(w_1) \subset \Lambda^X_+(w_2)$ if and only if

$$\alpha_i \in \Lambda^X_+(w_2) \quad \text{and} \quad \Lambda^{r_i(X)}_+(s_i^X w_1) \subset \Lambda^{r_i(X)}_+(s_i^X w_2). \tag{1.7}$$

Induction hypothesis, (1.6) and Cor. 1.10 imply that the relations in (1.7) are equivalent to

$$\ell(s_i^X w_2) = \ell(w_2) - 1, \quad \ell(s_i^X w_2) = \ell(s_i^X w_1) + \ell(w_1^{-1} w_2). \tag{1.8}$$

Since (1.8) implies that $w_1 \le_D w_2$, the if part of the claim holds. Further, $w_1 \le_D w_2$ implies that

$$\ell(s_i^X w_2) \le \ell(s_i^X w_1) + \ell(w_1^{-1} w_2) = \ell(w_1) - 1 + \ell(w_1^{-1} w_2) = \ell(w_2) - 1,$$

that is, that (1.8) holds. Therefore the only if part of the claim holds as well. □

2. Braided Hopf algebras and Nichols algebras

Let $\Bbbk$ be a field and let H be a Hopf algebra over $\Bbbk$ with bijective antipode. Let ${}^H_H\mathcal{YD}$ denote the category of Yetter-Drinfeld modules over H.

Let R be a bialgebra in ${}^H_H\mathcal{YD}$. We use the Sweedler notation for the coaction $\delta : R \to H \otimes R$ and the coproduct $\Delta_R : R \to R \otimes R$ in the following form: $\delta(r) = r_{(-1)} \otimes r_{(0)}$, $\Delta_R(r) = r^{(1)} \otimes r^{(2)}$ for all $r \in R$.

Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$ with antipode S_R. Let $R\#H$ be the bosonization of R, see e. g. [AHS08, Sect. 1.4]. Recall that $R\#H$ is a Hopf algebra with projection $\pi_H : R\#H \to H$, and

$$(r\#h)(r'\#h') = r(h_{(1)} \cdot r')\#h_{(2)}h', \tag{2.1}$$

$$r_{(-1)} \otimes r_{(0)} = \pi_H(r_{(1)}) \otimes r_{(2)}, \quad r_{(1)} \otimes r_{(2)} = r^{(1)}{r^{(2)}}_{(-1)} \otimes {r^{(2)}}_{(0)} \tag{2.2}$$

for all $r, r' \in R$, $h, h' \in H$, where $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for all $a \in R\#H$.

Let S denote the antipode of the Hopf algebra $R\#H$. Then

$$S_R(r) = r_{(-1)}S(r_{(0)}), \quad S(r) = S(r_{(-1)})S_R(r_{(0)}) \quad \text{for all } r \in R, \tag{2.3}$$

and $S_R \in \mathrm{End}(R)$ is a morphism in ${}^H_H\mathcal{YD}$ satisfying

$$S_R(rs) = S_R(r_{(-1)} \cdot s)S_R(r_{(0)}), \tag{2.4}$$

$$\Delta_R(S_R(r)) = S_R({r^{(1)}}_{(-1)} \cdot r^{(2)}) \otimes S_R({r^{(1)}}_{(0)}) \tag{2.5}$$

for all $r, s \in R$. If S is bijective, then the map $S_R^{-1} : R \to R$,

$$S_R^{-1}(r) = S^{-1}(r_{(0)})r_{(-1)} \quad \text{for all } r \in R \tag{2.6}$$

is a morphism in ${}^H_H\mathcal{YD}$ and is inverse to S_R. Moreover,

$$S^{-1}(r) = S_R^{-1}(r_{(0)})S^{-1}(r_{(-1)}) \quad \text{for all } r \in R. \tag{2.7}$$

In this case, Eqs. (2.4), (2.5) are equivalent to

$$S_R^{-1}(rs) = S_R^{-1}(s_{(0)})S_R^{-1}(S^{-1}(s_{(-1)}) \cdot r), \tag{2.8}$$

$$\Delta_R(S_R^{-1}(r)) = S_R^{-1}({r^{(2)}}_{(0)}) \otimes S_R^{-1}(S^{-1}({r^{(2)}}_{(-1)}) \cdot r^{(1)}) \tag{2.9}$$

for all $r, s \in R$.

Remark 2.1. (i) Let A be a Hopf algebra. Then A^{op} is a Hopf algebra if and only if the antipode S of A is bijective. In this case S^{-1} is the antipode of A^{op}.

(ii) Let B be a bialgebra in ${}^H_H\mathcal{YD}$ with a coalgebra filtration

$$\Bbbk 1 = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B, \quad \cup_{n=0}^{\infty} B_n = B.$$

Then B is a Hopf algebra in ${}^H_H\mathcal{YD}$, and the antipodes of B and $B\#H$ are bijective. Indeed,

$$H \subset B_1\#H \subset B_2\#H \subset \cdots \subset B\#H$$

is a coalgebra filtration of $B\#H$ and

$$H^{\mathrm{op}} \subset (B_1\#H)^{\mathrm{op}} \subset (B_2\#H)^{\mathrm{op}} \subset \cdots \subset (B\#H)^{\mathrm{op}}$$

is a coalgebra filtration of $(B\#H)^{\mathrm{op}}$. Since H and H^{op} are Hopf algebras, $B\#H$ and $(B\#H)^{\mathrm{op}}$ are Hopf algebras by [Mon93, Lemma 5.2.10]. By Part (i) the antipode of $B\#H$ is bijective. Hence S_B is bijective with inverse given in Eq. (2.6) (for $R = B$). □

Let $E \subset R$ be a subspace. We say that E is a *right coideal subalgebra of R in ${}^H_H\mathcal{YD}$* if $E \subset R$ is a subobject in ${}^H_H\mathcal{YD}$ and a subalgebra (containing 1) with $\Delta_R(E) \subset E \otimes R$. If H is the trivial 1-dimensional Hopf algebra, we follow the traditional terminology and call E a right coideal subalgebra of R.

Let G be a group. We say that a right coideal subalgebra E of a (braided) Hopf algebra R is *G-graded*, if $R = \oplus_{g\in G} R_g$ is a G-graded algebra and $E = \oplus_{g\in G}(E \cap R_g)$. For any G-graded algebras A, B, and any algebra map $f : A \to B$ we say that f is a *homomorphism of G-graded algebras*, if $f(A_g) \subset B_g$ for all $g \in G$.

Lemma 2.2. *Let $B \subset R$ be a Hopf subalgebra in ${}^H_H\mathcal{YD}$. Assume that there exists a morphism $\pi : R \to B$ of Hopf algebras in ${}^H_H\mathcal{YD}$ such that $\pi|_B = \mathrm{id}_B$. Let $R^{\mathrm{co}\,B} = \{r \in R \,|\, r^{(1)} \otimes \pi(r^{(2)}) = r \otimes 1\}$. Let $E \subset R$ be a right coideal subalgebra in ${}^H_H\mathcal{YD}$ such that $B \subset E$. Then the multiplication map $(R^{\mathrm{co}\,B} \cap E) \otimes B \to E$ is an isomorphism.*

Proof. The inverse of the multiplication map is given by

$$E \to (R^{\mathrm{co}\,B} \cap E) \otimes B, \quad r \mapsto r^{(1)} S_R(\pi(r^{(2)})) \otimes \pi(r^{(3)})$$

for all $r \in E$. □

Proposition 2.3. *Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$.*

(i) Let $E \subset R$. If E is a right coideal subalgebra of R in ${}^H_H\mathcal{YD}$, then $E\#H$ is a right coideal subalgebra of $R\#H$.

(ii) Let E' be a right coideal subalgebra of $R\#H$ with $H \subset E'$. Then $E = {E'}^{\mathrm{co}\,H}$ is a right coideal subalgebra of R in ${}^H_H\mathcal{YD}$ with $E' = E\#H$.

(iii) Let G be a group, and assume that the algebra $R = \oplus_{g\in G} R_g$ is G-graded and $R_g \in {}^H_H\mathcal{YD}$ for all $g \in G$. Then $R\#H = \oplus_{g\in G}(R\#H)_g$ is G-graded with $(R\#H)_g = R_g\#H$. Let $E \subset R$ be a subobject in ${}^H_H\mathcal{YD}$. Then E is a G-graded subalgebra of R if and only if $E\#H$ is a G-graded subalgebra of $R\#H$.

Proof. (i) and (iii) follow from Eqs. (2.1), (2.2). (ii) is a special case of Lemma 2.2 with $H = \Bbbk 1$ and $B = H$. □

Let $V \in {}^H_H\mathcal{YD}$. Assume that $\dim_\Bbbk V < \infty$. Then $V^* \in {}^H_H\mathcal{YD}$ with the following properties:

$$\langle h \cdot f, v\rangle = \langle f, S(h)\cdot v\rangle, \tag{2.10}$$

$$f_{(-1)}\langle f_{(0)}, v\rangle = S^{-1}(v_{(-1)})\langle f, v_{(0)}\rangle, \tag{2.11}$$

for all $h \in H$, $v \in V$, $f \in V^*$, see e.g. [AHS08, Sect. 1.2]. Let $\mathcal{B}(V)$ and $\mathcal{B}(V^*)$ denote the Nichols algebra of V and V^*, respectively. These are $\mathbb{N}_0$-graded braided Hopf algebras in ${}^H_H\mathcal{YD}$ with degree 1 parts $\mathcal{B}^1(V) \simeq V$, $\mathcal{B}^1(V^*) \simeq V^*$ and with $\Bbbk$ as degree 0 part. Since the antipode of H is bijective, the antipodes of $\mathcal{B}(V)$, $\mathcal{B}(V^*)$, $\mathcal{B}(V)\#H$ and $\mathcal{B}(V^*)\#H$ are bijective by Remark 2.1(ii).

The evaluation map between V^* and V induces a bilinear form

$$\langle\cdot,\cdot\rangle : \mathcal{B}(V^*) \times \mathcal{B}(V) \to \Bbbk, \tag{2.12}$$

see [AHS08, Sect. 1.5] for the origins. This pairing is non-degenerate, and it satisfies the equations

$$\langle 1, 1\rangle = 1, \quad \langle f, v\rangle = 0 \quad \text{for all } f \in \mathcal{B}^k(V^*),\ v \in \mathcal{B}^l(V),\ k \neq l, \text{ and} \tag{2.13}$$

$$\langle h \cdot f, v\rangle = \langle f, S(h)\cdot v\rangle, \tag{2.14}$$

$$f_{(-1)}\langle f_{(0)}, v\rangle = S^{-1}(v_{(-1)})\langle f, v_{(0)}\rangle, \tag{2.15}$$

$$\langle f, vw\rangle = \langle f^{(1)}, w\rangle\langle f^{(2)}, v\rangle, \tag{2.16}$$

$$\langle fg, v\rangle = \langle g, v^{(2)}\rangle\langle f, v^{(1)}\rangle \tag{2.17}$$

for all $f, g \in \mathcal{B}(V^*)$, $v, w \in \mathcal{B}(V)$, $h \in H$.

Let $\{b^\alpha\}$ and $\{b_\alpha\}$ be $\mathbb{N}_0$-graded dual bases of $\mathcal{B}(V^*)$ and $\mathcal{B}(V)$, respectively. Eqs. (2.14)-(2.17) imply the following for all $h \in H$, $v \in \mathcal{B}(V)$, $f \in \mathcal{B}(V^*)$.

$$\varepsilon(h)\sum_\alpha b_\alpha \otimes b^\alpha = \sum_\alpha h_{(1)}\cdot b_\alpha \otimes h_{(2)}\cdot b^\alpha, \tag{2.18}$$

$$\sum_\alpha S(h)\cdot b_\alpha \otimes b^\alpha = \sum_\alpha b_\alpha \otimes h\cdot b^\alpha, \tag{2.19}$$

$$\sum_\alpha 1\otimes b_\alpha \otimes b^\alpha = \sum_\alpha b_{\alpha(-1)}{b^\alpha}_{(-1)} \otimes b_{\alpha(0)} \otimes {b^\alpha}_{(0)}, \tag{2.20}$$

$$\sum_\alpha b_{\alpha(-1)} \otimes b_{\alpha(0)} \otimes b^\alpha = \sum_\alpha S({b^\alpha}_{(-1)}) \otimes b_\alpha \otimes {b^\alpha}_{(0)}, \tag{2.21}$$

$$\sum_\alpha v b_\alpha \otimes b^\alpha = \sum_\alpha b_\alpha \otimes \langle b^{\alpha(2)}, v\rangle b^{\alpha(1)}, \tag{2.22}$$

$$\sum_\alpha b_\alpha v \otimes b^\alpha = \sum_\alpha b_\alpha \otimes \langle b^{\alpha(1)}, v\rangle b^{\alpha(2)}, \tag{2.23}$$

$$\sum_\alpha b_\alpha \otimes b^\alpha f = \sum_\alpha \langle f, {b_\alpha}^{(1)}\rangle {b_\alpha}^{(2)} \otimes b^\alpha, \tag{2.24}$$

$$\sum_\alpha b_\alpha \otimes f b^\alpha = \sum_\alpha \langle f, {b_\alpha}^{(2)}\rangle {b_\alpha}^{(1)} \otimes b^\alpha, \tag{2.25}$$

$$\sum_{\alpha,\beta} b_\alpha \otimes b_\beta \otimes b^\beta b^\alpha = \sum_\gamma \Delta_{\mathcal{B}(V)} b_\gamma \otimes b^\gamma. \tag{2.26}$$

Let $K \in {}^{\mathcal{B}(V)\#H}_{\mathcal{B}(V)\#H}\mathcal{YD}$. We write

$$\delta_{\mathcal{B}(V)\#H}(x) = x_{[-1]} \otimes x_{[0]} \tag{2.27}$$

for the left coaction of $\mathcal{B}(V)\#H$ on $x \in K$. Then $K \in {}^H_H\mathcal{YD}$, where the action is the restriction of the action of $\mathcal{B}(V)\#H$ to H, and the coaction is

$$\delta = (\pi_H \otimes \mathrm{id})\delta_{\mathcal{B}(V)\#H},$$

and $\pi_H : \mathcal{B}(V)\#H \to H$ is the canonical projection. Let $\delta_{\mathcal{B}(V)} : K \to \mathcal{B}(V) \otimes K$,

$$\delta_{\mathcal{B}(V)}(x) = x_{[-1]} S(x_{[0](-1)}) \otimes x_{[0](0)} \quad \text{for all } x \in K. \tag{2.28}$$

We use modified Sweedler notation for $\delta_{\mathcal{B}(V)}$ in the form

$$\delta_{\mathcal{B}(V)}(x) = x^{(-1)} \otimes x^{(0)} \quad \text{for all } x \in K. \tag{2.29}$$

The map $\delta_{\mathcal{B}(V)}$ is H-linear and H-colinear via diagonal action and coaction. We are going to study the right action of $\mathcal{B}(V)\#H$ on K defined by

$$x \triangleleft v = S^{-1}(v) \cdot x \quad \text{for all } v \in \mathcal{B}(V)\#H,\ x \in K. \tag{2.30}$$

Lemma 2.4. *Let $K \in {}^{\mathcal{B}(V)\#H}_{\mathcal{B}(V)\#H}\mathcal{YD}$.*

(i) Let $x \in K$, $v, w \in \mathcal{B}(V)$ and $h \in H$. Then

$$h \cdot (x \triangleleft v) = (h_{(1)} \cdot x) \triangleleft (h_{(2)} \cdot v), \tag{2.31}$$

$$\delta(x \triangleleft v) = x_{(-1)} v_{(-1)} \otimes x_{(0)} \triangleleft v_{(0)}, \tag{2.32}$$

$$(x \triangleleft v) \triangleleft w = x \triangleleft vw. \tag{2.33}$$

(ii) For all $f \in V^*$ and $x \in K$ let $\partial^L_f(x) = \langle f, x^{(-1)}\rangle x^{(0)}$. Then $\partial^L_f(x) \in K$ and

$$\partial^L_f(x \triangleleft v) = \partial^L_f(x) \triangleleft v + \langle S^{-1}(x_{(-1)}) \cdot f, v^{(1)}\rangle x_{(0)} \triangleleft v^{(2)} - \langle f, v_{(2)}\rangle x \triangleleft v_{(1)} \tag{2.34}$$

for all $f \in V^*$, $x \in K$, $v \in \mathcal{B}(V)$.

(iii) Suppose that K is a $\mathcal{B}(V)\#H$-module algebra. Then

$$(xy) \triangleleft v = (x \triangleleft {v^{(2)}}_{(0)})(S^{-1}({v^{(2)}}_{(-1)}) \cdot (y \triangleleft v^{(1)})) \tag{2.35}$$

for all $v \in \mathcal{B}(V)$ and $x, y \in K$.

Proof. (i) By the relations of $\mathcal{B}(V)\#H$,

$$\begin{aligned} h\cdot(x\triangleleft v) =& hS^{-1}(v)\cdot x = S^{-1}(vS(h))\cdot x = S^{-1}(S(h_{(1)})(h_{(2)}\cdot v))\cdot x\\ =& S^{-1}(h_{(2)}\cdot v)\cdot(h_{(1)}\cdot x) = (h_{(1)}\cdot x)\triangleleft(h_{(2)}\cdot v). \end{aligned}$$

Using the Yetter-Drinfeld structure of K we obtain that

$$\begin{aligned} \delta(x\triangleleft v) =& \delta(S^{-1}(v)\cdot x) = \pi_H(S^{-1}(v)_{(1)}x_{[-1]}S(S^{-1}(v)_{(3)}))\otimes S^{-1}(v)_{(2)}\cdot x_{[0]}\\ =& \pi_H(S^{-1}(v_{(3)})x_{[-1]}v_{(1)})\otimes S^{-1}(v_{(2)})\cdot x_{[0]}\\ =& \pi_H(x_{[-1]}v_{(1)})\otimes S^{-1}(v_{(2)})\cdot x_{[0]} = x_{(-1)}v_{(-1)}\otimes x_{(0)}\triangleleft v_{(0)}, \end{aligned}$$

where the fourth relation follows from $v\in\mathcal{B}(V)$, and the last one from the definitions of δ and $\triangleleft$. Eq. (2.33) holds since S^{-1} is an algebra antiautomorphism of $\mathcal{B}(V)\#H$.

(ii) Let $f\in V^*$, $v\in\mathcal{B}(V)$, and $x\in K$. Let $\vartheta:\mathcal{B}(V^*)\#H\to\mathcal{B}(V^*)$ be the linear map with $\vartheta(w\#h)=w\varepsilon(h)$ for all $w\in\mathcal{B}(V^*)$, $h\in H$. It is well-known that $\vartheta(w)=w_{(1)}\pi_H(S(w_{(2)}))$ for all $w\in\mathcal{B}(V^*)\#H$. We get

$$\begin{aligned} \partial^L_f(x\triangleleft v) =& \langle f,(x\triangleleft v)^{(-1)}\rangle(x\triangleleft v)^{(0)} \overset{(2.28)}{=} \langle f,\vartheta(S^{-1}(v_{(3)})x_{[-1]}v_{(1)})\rangle x_{[0]}\triangleleft v_{(2)}\\ =& \langle f,\vartheta(S^{-1}(v_{(3)})\pi_H(x_{[-1]}v_{(1)}))\rangle x_{[0]}\triangleleft v_{(2)}\\ &+\langle f,\vartheta(\pi_H(S^{-1}(v_{(3)}))x_{[-1]}\pi_H(v_{(1)}))\rangle x_{[0]}\triangleleft v_{(2)}\\ &+\langle f,\vartheta(\pi_H(S^{-1}(v_{(3)})x_{[-1]})v_{(1)})\rangle x_{[0]}\triangleleft v_{(2)}\\ =& \langle f,\vartheta(S^{-1}(v_{(2)}))\rangle x\triangleleft v_{(1)}+\langle f,\vartheta(x_{[-1]})\rangle x_{[0]}\triangleleft v+\langle f,\vartheta(x_{(-1)}v_{(1)})\rangle x_{(0)}\triangleleft v_{(2)}\\ =& \langle f,S^{-1}(v_{(3)})\pi_H(v_{(2)})\rangle x\triangleleft v_{(1)}+\partial^L_f(x)\triangleleft v+\langle f,x_{(-1)}\cdot\vartheta(v_{(1)})\rangle x_{(0)}\triangleleft v_{(2)}\\ \overset{(2.6)}{=}& \langle f,S^{-1}_{\mathcal{B}(V)}(v_{(2)})\rangle x\triangleleft v_{(1)}+\partial^L_f(x)\triangleleft v+\langle f,x_{(-1)}\cdot v^{(1)}\rangle x_{(0)}\triangleleft v^{(2)}\\ =& -\langle f,v_{(2)}\rangle x\triangleleft v_{(1)}+\partial^L_f(x)\triangleleft v+\langle S^{-1}(x_{(-1)})\cdot f,v^{(1)}\rangle x_{(0)}\triangleleft v^{(2)} \end{aligned}$$

which proves (ii). In the latter transformations, the third equation follows from the fact that f is of degree 1 in $\mathcal{B}(V^*)$, and hence the pairing of a homogeneous product with f vanishes if the sum of the degrees of the factors is not 1. Further, the fourth equation is obtained from $(\mathrm{id}\otimes\pi_H)\Delta(v)=v\otimes 1$ and from the definition of ϑ. In the fifth equation, the definition of ϑ and of $\partial^L_f(x)$ is used. The last equation comes from Eq. (2.14) and the facts that f has degree 1 and each element of V is primitive.

(iii) Let $v \in \mathcal{B}(V)$ and $x, y \in K$. Then

$$\begin{aligned}(xy) \triangleleft v =& S^{-1}(v) \cdot (xy) = (S^{-1}(v_{(2)}) \cdot x)(S^{-1}(v_{(1)}) \cdot y)\\ =&(S^{-1}({v^{(2)}}_{(0)}) \cdot x)(S^{-1}(v^{(1)}{v^{(2)}}_{(-1)}) \cdot y)\\ =&(x \triangleleft {v^{(2)}}_{(0)})(S^{-1}({v^{(2)}}_{(-1)}) \cdot (y \triangleleft v^{(1)}))\end{aligned}$$

since K is a $\mathcal{B}(V)\#H$-module algebra. □

We extend the definition of ∂_f^L given in Lemma 2.4. Note that $\mathcal{B}(V)$ is a coalgebra. For any left $\mathcal{B}(V)$-comodule L and any $f \in V^*$ let $\partial_f^L : L \to L$ be the map defined by

$$\partial_f^L(x) = \langle f, x^{(-1)} \rangle x^{(0)} \quad \text{for all } x \in L, \tag{2.36}$$

where $\delta_{\mathcal{B}(V)}(x) = x^{(-1)} \otimes x^{(0)}$.

Lemma 2.5. *Let L be a left $\mathcal{B}(V)$-comodule. Let $L_0 \subset L$ and let $\langle L_0 \rangle$ be the smallest left $\mathcal{B}(V)$-subcomodule of L containing L_0. Then $\langle L_0 \rangle$ is the smallest subspace of L which contains L_0 and is stable under the maps ∂_f^L, $f \in V^*$.*

Proof. By assumption,

$$\langle L_0 \rangle = \operatorname{span}_{\Bbbk}\{g(x^{(-1)})x^{(0)} \mid x \in L_0, g \in \mathcal{B}(V)^*\}.$$

The non-degeneracy of the pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{B}(V^*)$ and $\mathcal{B}(V)$, see (2.12), implies that

$$\langle L_0 \rangle = \operatorname{span}_{\Bbbk}\{\langle g, x^{(-1)} \rangle x^{(0)} \mid x \in L_0, g \in \mathcal{B}(V^*)\}.$$

Eq. (2.17) further implies that

$$\begin{aligned}\langle L_0 \rangle =& \operatorname{span}_{\Bbbk}\{\langle f_1 \cdots f_k, x^{(-1)} \rangle x^{(0)} \mid x \in L_0, k \in \mathbb{N}_0, f_1, \dots, f_k \in V^*\}\\ =& \operatorname{span}_{\Bbbk}\{\partial_{f_1}^L \cdots \partial_{f_k}^L(x) \mid x \in L_0, k \in \mathbb{N}_0, f_1, \dots, f_k \in V^*\}.\end{aligned}$$

This proves the lemma. □

3. The algebra map $\mathfrak{R}$

Let $V, V' \in {}^H_H\mathcal{YD}$ and $W = V \oplus V'$ with $\dim_{\Bbbk} W < \infty$. Let $\pi : \mathcal{B}(W)\#H \to \mathcal{B}(V)\#H$ denote the Hopf algebra projection corresponding to the decomposition $W = V \oplus V'$, and let

$$K = \mathcal{B}(W)^{\operatorname{co} \mathcal{B}(V)\#H} = \{x \in \mathcal{B}(W) \mid (\operatorname{id} \otimes \pi)\Delta_{\mathcal{B}(W)\#H}(x) = x \otimes 1\} \tag{3.1}$$

be the algebra of right coinvariants with respect to the right coaction of $\mathcal{B}(V)\#H$. Then K is a braided Hopf algebra in ${}^{\mathcal{B}(V)\#H}_{\mathcal{B}(V)\#H}\mathcal{YD}$, and hence the results in the previous section apply.

Recall from [AHS08, Def. 2.5, Remark 2.7] that $K\#\mathcal{B}(V^*)$ is an algebra in ${}^H_H\mathcal{YD}$ such that the multiplication map $K\otimes\mathcal{B}(V^*)\to K\#\mathcal{B}(V^*)$ is an isomorphism of left K-modules and right $\mathcal{B}(V^*)$-modules, and

$$(1\#f)(x\#1)=\langle f^{(2)},\pi(x^{(1)})\rangle x^{(2)}{}_{(0)}\#S^{-1}(x^{(2)}{}_{(-1)})\cdot f^{(1)} \tag{3.2}$$

for all $f\in\mathcal{B}(V^*)$, $x\in K$. For brevity we will often write xf instead of $x\#f$ for all $x\in K$ and $f\in\mathcal{B}(V^*)$. Note that if $f\in V^*$ and $x\in K$ then

$$\begin{aligned}(1\#f)(x\#1)=&\langle f,x^{(1)}\rangle x^{(2)}\#1+x_{(0)}\#S^{-1}(x_{(-1)})\cdot f\\ =&\partial^L_f(x)\#1+x_{(0)}\#S^{-1}(x_{(-1)})\cdot f\end{aligned} \tag{3.3}$$

by Eqs. (3.2), (2.36).

Let ad denote the left adjoint action of $\mathcal{B}(W)\#H$ on itself. Assume that $\dim(\operatorname{ad}\mathcal{B}(V)\#H)(x)<\infty$ for all $x\in V'$. Since K is generated as an algebra by $(\operatorname{ad}\mathcal{B}(V)\#H)(V')$, see [AHS08, Prop. 3.6], the left adjoint action of $\mathcal{B}(V)\#H$ on K is locally finite.

Let $\mathfrak{R}:K\otimes K\to K\otimes(K\#\mathcal{B}(V^*))$ be the linear map with

$$\mathfrak{R}(x\otimes y)=\sum_\alpha x\triangleleft b_\alpha\otimes b^\alpha y \tag{3.4}$$

for all $x,y\in K$, where $\{b^\alpha\}$ and $\{b_\alpha\}$ are dual bases of the $\mathbb{N}_0$-graded algebras $\mathcal{B}(V^*)$ and $\mathcal{B}(V)$, respectively, and $\triangleleft$ is defined in Eq. (2.30). By the local finiteness of the left adjoint action of $\mathcal{B}(V)\#H$ on K, the sum in Eq. (3.4) is finite for all $x,y\in K$. Since $1\triangleleft v=\varepsilon(v)1$ for all $v\in\mathcal{B}(V)\#H$, we conclude that

$$\mathfrak{R}(1\otimes y)=1\otimes y\quad\text{for all } y\in K. \tag{3.5}$$

Let $\overline{\mathfrak{R}}:K\otimes K\#\mathcal{B}(V^*)\to K\otimes K\#\mathcal{B}(V^*)$ be the linear map with

$$\overline{\mathfrak{R}}(x\otimes y)=\sum_\alpha x\triangleleft S_{\mathcal{B}(W)}(b_\alpha)\otimes b^\alpha y \tag{3.6}$$

for all $x\in K$ and $y\in K\#\mathcal{B}(V^*)$. Then

$$\overline{\mathfrak{R}}\mathfrak{R}(x\otimes y)=x\otimes y\quad\text{for all } x,y\in K. \tag{3.7}$$

Indeed,

$$\begin{aligned}\overline{\mathfrak{R}}\,\mathfrak{R}(x\otimes y)=&\overline{\mathfrak{R}}\Big(\sum_\alpha x\triangleleft b_\alpha\otimes b^\alpha y\Big)=\sum_{\alpha,\beta}x\triangleleft b_\alpha S_{\mathcal{B}(W)}(b_\beta)\otimes b^\beta b^\alpha y\\ =&\sum_\gamma x\triangleleft b_\gamma^{(1)}S_{\mathcal{B}(W)}(b_\gamma^{(2)})\otimes b^\gamma y=\sum_\gamma x\otimes\varepsilon(b_\gamma)b^\gamma y=x\otimes y,\end{aligned}$$

where the third equation follows from Eq. (2.26).

Lemma 3.1. *For all $x, y \in K$ and $h \in H$,*

$$h \cdot \mathfrak{R}(x \otimes y) = \mathfrak{R}(h_{(1)} \cdot x \otimes h_{(2)} \cdot y), \tag{3.8}$$

$$\delta\mathfrak{R}(x \otimes y) = x_{(-1)}y_{(-1)} \otimes \mathfrak{R}(x_{(0)} \otimes y_{(0)}). \tag{3.9}$$

Proof. By Eqs. (3.4), (2.18) and (2.31),

$$\begin{aligned} h \cdot \mathfrak{R}(x \otimes y) &= \sum_\alpha h_{(1)} \cdot (x \triangleleft b_\alpha) \otimes (h_{(2)} \cdot b^\alpha)(h_{(3)} \cdot y) \\ &= \sum_\alpha (h_{(1)} \cdot x) \triangleleft (h_{(2)} \cdot b_\alpha) \otimes (h_{(3)} \cdot b^\alpha)(h_{(4)} \cdot y) \\ &= \sum_\alpha (h_{(1)} \cdot x) \triangleleft b_\alpha \otimes b^\alpha (h_{(2)} \cdot y) = \mathfrak{R}(h_{(1)} \cdot x \otimes h_{(2)} \cdot y) \end{aligned}$$

for all $h \in H$ and $x, y \in K$. Similarly, Eqs. (3.4), (2.32) and (2.20) imply that

$$\begin{aligned} \delta\mathfrak{R}(x \otimes y) &= \delta\Big(\sum_\alpha x \triangleleft b_\alpha \otimes b^\alpha y\Big) \\ &= \sum_\alpha x_{(-1)} b_{\alpha(-1)} {b^\alpha}_{(-1)} y_{(-1)} \otimes x_{(0)} \triangleleft b_{\alpha(0)} \otimes {b^\alpha}_{(0)} y_{(0)} \\ &= \sum_\alpha x_{(-1)} y_{(-1)} \otimes x_{(0)} \triangleleft b_\alpha \otimes b^\alpha y_{(0)} = x_{(-1)} y_{(-1)} \otimes \mathfrak{R}(x_{(0)} \otimes y_{(0)}) \end{aligned}$$

for all $x, y \in K$. □

The vector space $K \otimes K$ is an algebra in ${}^H_H\mathcal{YD}$ with product

$$(x \otimes y)(z \otimes w) = x(\operatorname{ad}\pi(y_{(1)}))(z) \otimes y_{(2)} w \tag{3.10}$$

for all $x, y, z, w \in K$. Note that this is the usual product of $K \otimes K$ in ${}^{\mathcal{B}(V)\#H}_{\mathcal{B}(V)\#H}\mathcal{YD}$. Similarly, $K\#B(V^*) \otimes K\#\mathcal{B}(V^*)$ is an algebra in ${}^H_H\mathcal{YD}$ with product

$$(x \otimes y)(z \otimes w) = x(y_{(-1)} \cdot z) \otimes y_{(0)} w \tag{3.11}$$

for all $x, y, z, w \in K\#\mathcal{B}(V^*)$.

Theorem 3.2. *The map $\mathfrak{R} : K \otimes K \to K\#\mathcal{B}(V^*) \otimes K\#\mathcal{B}(V^*)$ is an algebra map in ${}^H_H\mathcal{YD}$.*

Proof. The map $\mathfrak{R}$ is a morphism in ${}^H_H\mathcal{YD}$ by Lemma 3.1. Further,

$$\begin{aligned} &\mathfrak{R}((x \otimes y)(1 \otimes z)) = \mathfrak{R}(x \otimes yz) \\ &\quad = \sum_\alpha x \triangleleft b_\alpha \otimes b^\alpha yz = \mathfrak{R}(x \otimes y)(1 \otimes z) = \mathfrak{R}(x \otimes y)\mathfrak{R}(1 \otimes z) \end{aligned}$$

for all $x, y, z \in K$ by Eq. (3.5). Hence it suffices to show that

$$\mathfrak{R}((x\otimes 1)(y\otimes 1)) = \mathfrak{R}(x\otimes 1)\mathfrak{R}(y\otimes 1), \tag{3.12}$$
$$\mathfrak{R}((1\otimes x)(y\otimes 1)) = \mathfrak{R}(1\otimes x)\mathfrak{R}(y\otimes 1) \tag{3.13}$$

for all $x, y \in K$. Let $x, y \in K$. Then

$$\begin{aligned}
\mathfrak{R}(x\otimes 1)\mathfrak{R}(y\otimes 1) &\overset{(3.4)}{=} \Big(\sum_\alpha x\triangleleft b_\alpha\otimes b^\alpha\Big)\Big(\sum_\beta y\triangleleft b_\beta\otimes b^\beta\Big)\\
&\overset{(3.11)}{=} \sum_{\alpha,\beta}(x\triangleleft b_\alpha)({b^\alpha}_{(-1)}\cdot(y\triangleleft b_\beta))\otimes {b^\alpha}_{(0)}b^\beta\\
&\overset{(2.21)}{=} \sum_{\alpha,\beta}(x\triangleleft b_{\alpha(0)})(S^{-1}(b_{\alpha(-1)})\cdot(y\triangleleft b_\beta))\otimes b^\alpha b^\beta\\
&\overset{(2.26)}{=} \sum_\gamma(x\triangleleft {b_\gamma^{(2)}}_{(0)})(S^{-1}({b_\gamma^{(2)}}_{(-1)})\cdot(y\triangleleft b_\gamma^{(1)}))\otimes b^\gamma\\
&\overset{(2.35)}{=} \sum_\gamma (xy)\triangleleft b_\gamma\otimes b^\gamma \overset{(3.4)}{=} \mathfrak{R}(xy\otimes 1).
\end{aligned}$$

This proves Eq. (3.12). Further,

$$\begin{aligned}
\mathfrak{R}((1\otimes x)(y\otimes 1)) &\overset{(3.10)}{=} \mathfrak{R}((x_{[-1]}\cdot y)\otimes x_{[0]})\\
&\overset{(3.4)}{=} \sum_\alpha (x_{[-1]}\cdot y)\triangleleft b_\alpha\otimes b^\alpha x_{[0]} \overset{(2.30)}{=} \sum_\alpha y\triangleleft(S(x_{[-1]})b_\alpha)\otimes b^\alpha x_{[0]}\\
&\overset{(3.2)}{=} \sum_\alpha y\triangleleft(S(x_{[-1]})b_\alpha)\otimes\langle b^{\alpha(2)},\pi({x_{[0]}}^{(1)})\rangle {{x_{[0]}}^{(2)}}_{(0)}(S^{-1}({{x_{[0]}}^{(2)}}_{(-1)})\cdot b^{\alpha(1)})\\
&\overset{(2.22)}{=} \sum_\alpha y\triangleleft(S(x_{[-1]})\pi({x_{[0]}}^{(1)})b_\alpha)\otimes {{x_{[0]}}^{(2)}}_{(0)}(S^{-1}({{x_{[0]}}^{(2)}}_{(-1)})\cdot b^\alpha)\\
&= \sum_\alpha y\triangleleft(S\pi(x_{(1)})\pi(x_{(2)})S\pi_H(x_{(3)})b_\alpha)\otimes x_{(5)}(S^{-1}\pi_H(x_{(4)})\cdot b^\alpha)\\
&\overset{(2.19)}{=} \sum_\alpha y\triangleleft(S\pi_H(x_{(1)})(\pi_H(x_{(2)})\cdot b_\alpha))\otimes x_{(3)}b^\alpha\\
&= \sum_\alpha y\triangleleft(b_\alpha S(x_{(-1)}))\otimes x_{(0)}b^\alpha \overset{(2.30)}{=} \sum_\alpha x_{(-1)}\cdot(y\triangleleft b_\alpha)\otimes x_{(0)}b^\alpha\\
&\overset{(3.11)}{=} (1\otimes x)\Big(\sum_\alpha y\triangleleft b_\alpha\otimes b^\alpha\Big) \overset{(3.4),(3.5)}{=} \mathfrak{R}(1\otimes x)\mathfrak{R}(y\otimes 1).
\end{aligned}$$

This proves Eq. (3.13), and the proof of the theorem is completed. □

4. Reflections of Nichols algebras

Let $\theta \in \mathbb{N}$ and $\mathbb{I} = \{1, 2, \dots \theta\}$. Let $\mathcal{F}_\theta$ denote the set of θ-tuples of finite-dimensional irreducible objects in ${}^H_H\mathcal{YD}$, and let $\mathcal{X}_\theta$ denote the set of θ-tuples of isomorphism classes of finite-dimensional irreducible objects in ${}^H_H\mathcal{YD}$. For each $M = (M_1, \dots, M_\theta) \in \mathcal{F}_\theta$ let $[M] = ([M_1], \dots, [M_\theta]) \in \mathcal{X}_\theta$ denote the corresponding θ-tuple of isomorphism classes.

Let $\{\alpha_1, \dots, \alpha_\theta\}$ be the standard basis of $\mathbb{Z}^\theta$. For all $M = (M_1, \dots, M_\theta) \in \mathcal{F}_\theta$ define an algebra grading by $\mathbb{N}_0^\theta$ on the Nichols algebra $\mathcal{B}(M) := \mathcal{B}(M_1 \oplus \cdots \oplus M_\theta)$ by $\deg M_j = \alpha_j$ for all $j \in \mathbb{I}$, see [AHS08, Rem. 2.8]. We call this the *standard* $\mathbb{N}_0^\theta$*-grading of* $\mathcal{B}(M)$.

Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta$. Let

$$K_i^M = \mathcal{B}(M)^{\mathrm{co}\, \mathcal{B}(M_i) \# H}. \tag{4.1}$$

Clearly, $M_j \subset K_i^M$ for all $j \in \mathbb{I} \setminus \{i\}$. As in [HS08, Def. 6.4] we say that M is *i-finite* if $(\mathrm{ad}\, \mathcal{B}(M_i) \# H)(M_j)$ is finite-dimensional for all $j \in \mathbb{I} \setminus \{i\}$. Note that if $N \in \mathcal{F}_\theta$ with $[M] = [N]$ then M is i-finite if and only if N is i-finite.

Proposition 4.1. [AHS08, Prop. 3.6] *Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta$. Assume that M is i-finite.*

(i) The algebra K_i^M is generated by $\oplus_{j \in \mathbb{I} \setminus \{i\}} (\mathrm{ad}\, \mathcal{B}(M_i))(M_j)$.

(ii) The left adjoint action of $\mathcal{B}(M_i)$ on K_i^M is locally finite.

Let us recall some crucial definitions introduced in [AHS08, Sect. 3.4]. Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta$. If M is not i-finite, let $R_i(M) = M$. Otherwise let $a_{ij}^M \in \mathbb{Z}$, where $j \in \mathbb{I}$, and $M' = (M'_1, \dots, M'_\theta) \in ({}^H_H\mathcal{YD})^\theta$ be given by

$$a_{ij}^M = \begin{cases} 2 & \text{if } j = i, \\ -\max\{m \mid (\mathrm{ad}\, M_i)^m(M_j) \neq 0\} & \text{if } j \neq i, \end{cases} \tag{4.2}$$

$$M'_i = M_i^*, \quad M'_j = (\mathrm{ad}\, M_i)^{-a_{ij}^M}(M_j) \text{ for all } j \in \mathbb{I} \setminus \{i\}. \tag{4.3}$$

and let

$$s_i^M \in \mathrm{Aut}(\mathbb{Z}^\theta), \quad s_i^M(\alpha_j) = \alpha_j - a_{ij}^M \alpha_i \text{ for all } j \in \mathbb{I}. \tag{4.4}$$

By [AHS08, Thm. 3.8], M'_j is finite-dimensional and irreducible for all $j \in \mathbb{I}$, and hence $M' \in \mathcal{F}_\theta$. Let $R_i(M) = M'$ in this case. Note that $[R_i(M)] = [R_i(N)]$ in $\mathcal{X}_\theta$ for all $N \in \mathcal{F}_\theta$ with $[N] = [M]$. Thus we may define

$$r_i([M]) = [R_i(M)],$$

and these definitions provide us with maps $R_j : \mathcal{F}_\theta \to \mathcal{F}_\theta$, $r_j : \mathcal{X}_\theta \to \mathcal{X}_\theta$ for all $j \in \mathbb{I}$. Further, if $N \in \mathcal{F}_\theta$ with $[M] = [N]$ and M is i-finite, then $a_{ij}^M = a_{ij}^N$ for all

$j \in \mathbb{I}$, and $s_i^M = s_i^N$. Thus we may write $a_{ij}^{[M]}$ and $s_i^{[M]}$ instead of a_{ij}^M and s_i^M if needed.

Let $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta$. Assume that M is i-finite. Let

$$\Omega_i^M : K_i^M \# \mathcal{B}(M_i^*) \to \mathcal{B}(R_i(M)) \tag{4.5}$$

be the unique algebra map which is the identity on all $M'_j \subset K_i^M \# \mathcal{B}(M_i^*)$, where $j \in \mathbb{I}$ — see [AHS08, Prop. 3.14]. Then Ω_i^M is a map in ${}^H_H\mathcal{YD}$ and

$$\Omega_i^M(x_\alpha) \in \mathcal{B}(R_i(M))_{s_i^M(\alpha)} \tag{4.6}$$

for all $x_\alpha \in K_i^M \# \mathcal{B}(M_i^*)$ of degree $\alpha \in \mathbb{Z}^\theta$, where $K_i^M \subset \mathcal{B}(M)$ is graded by the standard grading of $\mathcal{B}(M)$ and $\deg M_i^* = -\alpha_i$. Further, Ω_i^M is bijective and

$$S_{\mathcal{B}(R_i(M))}\Omega_i^M(K_i^M) = K_i^{R_i(M)} \tag{4.7}$$

by the last paragraph of the proof of [AHS08, Thm. 3.12]. By [AHS08, Eq. (3.37)] the restriction of $S_{\mathcal{B}(R_i(M))}\Omega_i^M$ to M_j defines an isomorphism

$$\varphi_{ij}^M : M_j \to R_i^2(M)_j, \quad j \in \mathbb{I} \setminus \{j\},$$

in ${}^H_H\mathcal{YD}$, and there is a canonical isomorphism $\varphi_{ii}^M : M_i \to R_i^2(M)_i = M_i^{**}$ in ${}^H_H\mathcal{YD}$, see [AHS08, Rem. 1.4]. Let

$$\varphi_i^M = (\varphi_{ij}^M)_{j \in \mathbb{I}} : M \to R_i^2(M) \tag{4.8}$$

be the family of these isomorphisms.

The following property of Ω_i^M will be one of the main ingredients to characterize right coideal subalgebras of Nichols algebras.

Theorem 4.2. *Let $M \in \mathcal{F}_\theta$ and $i \in \mathbb{I}$. Assume that M is i-finite. Then the following diagram is commutative:*

$$\begin{array}{ccccc} K_i^M & \xrightarrow{\Delta_{K_i^M}} & K_i^M \otimes K_i^M & \xrightarrow{\mathfrak{R}} & K_i^M \otimes (K_i^M \# \mathcal{B}(M_i^*)) \\ \Omega_i^M \downarrow & & & & \downarrow \Omega_i^M \otimes \Omega_i^M \\ \mathcal{B}(R_i(M)) & \xrightarrow{\Delta_{\mathcal{B}(R_i(M))}} & & & \mathcal{B}(R_i(M)) \otimes \mathcal{B}(R_i(M)) \end{array} \tag{4.9}$$

that is,

$$\Delta_{\mathcal{B}(R_i(M))}\Omega_i^M(x) = (\Omega_i^M \otimes \Omega_i^M)\mathfrak{R}\Delta_{K_i^M}(x) \tag{4.10}$$

for all $x \in K_i^M$.

Proof. For all $j \in \mathbb{I} \setminus \{i\}$ and $k \in \mathbb{N}$ let

$$L_j = (\operatorname{ad} \mathcal{B}(M_i))(M_j), \quad L_j^k = (\operatorname{ad} M_i)^{k-1}(M_j). \tag{4.11}$$

Then

$$\mathit{\Delta}_{\mathcal{B}(M)}(x) - x \otimes 1 \in \mathcal{B}(M_i) \otimes L_j \tag{4.12}$$

for all $j \in \mathbb{I} \setminus \{i\}$ and $x \in L_j$ by [AHS08, Eq. (3.11)].

By [AHS08, Prop. 3.14], $\Omega_i^M : K_i^M \# \mathcal{B}(M_i^*) \to \mathcal{B}(R_i(M))$ is an algebra map. By Thm. 3.2, $\mathfrak{R}$ is an algebra map. By Prop. 4.1(i), K_i^M is generated as an algebra by $\cup_{j \neq i} L_j$. Hence it suffices to prove that Eq. (4.10) holds for all $x \in L_j$, $j \in \mathbb{I} \setminus \{i\}$. Let $j \in \mathbb{I} \setminus \{i\}$ and let $x \in L_j$. Eq. (4.12) implies that $\mathit{\Delta}_{K_i^M}(x) = 1 \otimes x + x \otimes 1$. Hence

$$(\Omega_i^M \otimes \Omega_i^M)\mathfrak{R}\mathit{\Delta}_{K_i^M}(x) = 1 \otimes \Omega_i^M(x) + \sum_\alpha \Omega_i^M(x \triangleleft b_\alpha) \otimes b^\alpha \tag{4.13}$$

by Eqs. (3.4), (3.5) and since $\Omega_i^M|_{\mathcal{B}(M_i^*)} = \mathrm{id}$.

Since M is i-finite, Prop. 4.1(ii) tells that the left adjoint action of $\mathcal{B}(M_i)$ on K_i^M is locally finite. Then [AHS08, Thm. 3.8] can be applied, that is, $L_j^{1-a_{ij}^M}$ generates L_j as a $\mathcal{B}(M_i)$-comodule. By Lemma 2.5 it suffices to show that the set of solutions of Eq. (4.10) contains $L_j^{1-a_{ij}^M}$ and is stable under the maps ∂_f^L for all $f \in M_i^*$.

Suppose first that $x \in L_j^{1-a_{ij}^M}$. Then

$$(\Omega_i^M \otimes \Omega_i^M)\mathfrak{R}\mathit{\Delta}_{K_i^M}(x) = 1 \otimes \Omega_i^M(x) + \Omega_i^M(x) \otimes 1 = \mathit{\Delta}_{\mathcal{B}(R_i(M))}(\Omega_i^M(x))$$

by Eq. (4.13) and since $\Omega_i^M(x) \in R_i(M)_j$. Thus the set of solutions of Eq. (4.10) contains $L_j^{1-a_{ij}^M}$.

Let $k \in \mathbb{N}$, $k \leq 1 - a_{ij}^M$. Assume that Eq. (4.10) holds for all $x \in L_j^k$. That is,

$$\mathit{\Delta}_{\mathcal{B}(R_i(M))}\Omega_i^M(x) = 1 \otimes \Omega_i^M(x) + \sum_\alpha \Omega_i^M(x \triangleleft b_\alpha) \otimes b^\alpha \quad \text{for all } x \in L_j^k. \tag{4.14}$$

Let $y \in L_j^k$ and $f \in M_i^*$. We have to prove that Eq. (4.14) holds for $x = \partial_f^L(y)$. Since $\partial_f^L(y) = fy - y_{(0)}(S^{-1}(y_{(-1)}) \cdot f)$ in $K_i^M \# \mathcal{B}(M_i^*)$ by Eq. (3.3), and since Ω_i^M is an algebra map in ${}_H^H\mathcal{YD}$ with $\Omega_i^M|_{\mathcal{B}(M_i^*)} = \mathrm{id}$, we obtain that

$$\begin{aligned}
\mathit{\Delta}_{\mathcal{B}(R_i(M))}&\Omega_i^M(\partial_f^L(y)) = \mathit{\Delta}_{\mathcal{B}(R_i(M))}(f\Omega_i^M(y) - \Omega_i^M(y_{(0)})(S^{-1}(y_{(-1)}) \cdot f)) \\
&= (f \otimes 1 + 1 \otimes f)\Big(1 \otimes \Omega_i^M(y) + \sum_\alpha \Omega_i^M(y \triangleleft b_\alpha) \otimes b^\alpha\Big) - \Big(1 \otimes \Omega_i^M(y_{(0)}) \\
&\quad + \sum_\alpha \Omega_i^M(y_{(0)} \triangleleft b_\alpha) \otimes b^\alpha\Big)(1 \otimes S^{-1}(y_{(-1)}) \cdot f + S^{-1}(y_{(-1)}) \cdot f \otimes 1).
\end{aligned}$$

By applying the product rule in $\mathcal{B}(R_i(M))\otimes\mathcal{B}(R_i(M))$ this becomes

$$\begin{aligned}\varDelta_{\mathcal{B}(R_i(M))}\Omega_i^M(\partial_f^L(y)) &= 1\otimes(f\Omega_i^M(y)-\Omega_i^M(y_{(0)})(S^{-1}(y_{(-1)})\cdot f))\\ &+f\otimes\Omega_i^M(y)-y_{(-1)}S^{-1}(y_{(-2)})\cdot f\otimes\Omega_i^M(y_{(0)})\\ &+\sum_\alpha(f\Omega_i^M(y\triangleleft b_\alpha)\otimes b^\alpha+f_{(-1)}\cdot\Omega_i^M(y\triangleleft b_\alpha)\otimes f_{(0)}b^\alpha)\\ &-\sum_\alpha\Omega_i^M(y_{(0)}\triangleleft b_\alpha)\otimes b^\alpha(S^{-1}(y_{(-1)})\cdot f)\\ &-\sum_\alpha\Omega_i^M(y_{(0)}\triangleleft b_\alpha)(b^\alpha{}_{(-1)}S^{-1}(y_{(-1)})\cdot f)\otimes b^\alpha{}_{(0)}.\end{aligned}$$

In the last expression, the first line equals $1\otimes\Omega_i^M(\partial_f^L(y))$ and the second is zero. We rewrite all other terms such that the second tensor factors contain only b^α.

Eqs. (2.25) and (2.15) and the definition of $\triangleleft$ yield that

$$\begin{aligned}\sum_\alpha f_{(-1)}\cdot\Omega_i^M(y\triangleleft b_\alpha)\otimes f_{(0)}b^\alpha &= \sum_\alpha f_{(-1)}\cdot(\langle f_{(0)},b_\alpha{}^{(2)}\rangle\Omega_i^M(y\triangleleft b_\alpha{}^{(1)}))\otimes b^\alpha\\ &=\sum_\alpha\langle f,b_\alpha{}^{(2)}{}_{(0)}\rangle S^{-1}(b_\alpha{}^{(2)}{}_{(-1)})\cdot\Omega_i^M(y\triangleleft b_\alpha{}^{(1)})\otimes b^\alpha\\ &=\sum_\alpha\langle f,b_\alpha{}^{(2)}{}_{(0)}\rangle\Omega_i^M(y\triangleleft b_\alpha{}^{(1)}b_\alpha{}^{(2)}{}_{(-1)})\otimes b^\alpha=\sum_\alpha\langle f,b_{\alpha(2)}\rangle\Omega_i^M(y\triangleleft b_{\alpha(1)})\otimes b^\alpha,\end{aligned}$$

Eq. (2.24) implies that

$$\sum_\alpha\Omega_i^M(y_{(0)}\triangleleft b_\alpha)\otimes b^\alpha(S^{-1}(y_{(-1)})\cdot f)=\sum_\alpha\langle S^{-1}(y_{(-1)})\cdot f,b_\alpha{}^{(1)}\rangle\Omega_i^M(y_{(0)}\triangleleft b_\alpha{}^{(2)})\otimes b^\alpha,$$

and from Eq. (2.21) we conclude that

$$\begin{aligned}&\sum_\alpha\Omega_i^M(y_{(0)}\triangleleft b_\alpha)(b^\alpha{}_{(-1)}S^{-1}(y_{(-1)})\cdot f)\otimes b^\alpha{}_{(0)}\\ &\qquad=\sum_\alpha\Omega_i^M(y_{(0)}\triangleleft b_{\alpha(0)})(S^{-1}(y_{(-1)}b_{\alpha(-1)})\cdot f)\otimes b^\alpha.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\Omega_i^M\otimes\Omega_i^M)\mathfrak{R}\varDelta_{K_i^M}(\partial_f^L(y)) &= (\Omega_i^M\otimes\Omega_i^M)\mathfrak{R}(1\otimes\partial_f^L(y)+\partial_f^L(y)\otimes 1)\\ &=1\otimes\Omega_i^M(\partial_f^L(y))+\sum_\alpha\Omega_i^M(\partial_f^L(y)\triangleleft b_\alpha)\otimes b^\alpha.\end{aligned}$$

Comparing coefficients in front of b^α we conclude that Eq. (4.14) holds for $x=\partial_f^L(y)$ if equation

$$\begin{aligned}f(y\triangleleft b)+\langle f,b_{(2)}\rangle(y\triangleleft b_{(1)})-\langle S^{-1}(y_{(-1)})\cdot f,b^{(1)}\rangle y_{(0)}\triangleleft b^{(2)}&\\ -(y_{(0)}\triangleleft b_{(0)})(S^{-1}(y_{(-1)}b_{(-1)})\cdot f)&=\partial_f^L(y)\triangleleft b\end{aligned}\tag{4.15}$$

holds in $K_i^M\#\mathcal{B}(M_i^*)$ for all $b\in\mathcal{B}(M_i)$. Using Eq. (3.2) with $f\in M_i^*$, Eq. (4.15) becomes equivalent to

$$\partial_f^L(y\triangleleft b)=-\langle f,b_{(2)}\rangle y\triangleleft b_{(1)}+\langle S^{-1}(y_{(-1)})\cdot f,b^{(1)}\rangle y_{(0)}\triangleleft b^{(2)}+\partial_f^L(y)\triangleleft b.$$

The latter is true by Lemma 2.4(ii). Thus Eq. (4.14) (and hence Eq. (4.10)) holds for $x=\partial_f^L(y)$ and hence for all $x\in L_j$. This finishes the proof of the theorem. $\square$

5. Right coideal subalgebras of Nichols algebras

Let $\theta\in\mathbb{N}$ and $M=(M_1,\dots,M_\theta)\in\mathcal{F}_\theta$. Let $\mathcal{K}(M)$ denote the set of all $\mathbb{N}_0^\theta$-graded right coideal subalgebras of $\mathcal{B}(M)$ in ${}^H_H\mathcal{YD}$, where $\mathcal{B}(M)$ is graded by the standard $\mathbb{N}_0^\theta$-grading, see Sect. 4.

For all $\alpha\in\mathbb{Z}^\theta$ let $t^\alpha=t_1^{n_1}\cdots t_\theta^{n_\theta}\in\mathbb{N}_0[t_1^{\pm1},\dots,t_\theta^{\pm1}]$, where $\alpha=\sum_{i\in\mathbb{I}}n_i\alpha_i$. For any $N\in\mathcal{F}_\theta$ and any $\mathbb{N}_0^\theta$-graded object $X=\oplus_{\alpha\in\mathbb{N}_0^\theta}X_\alpha\subset\mathcal{B}(N)$ in ${}^H_H\mathcal{YD}$ let

$$\mathcal{H}_X(t)=\sum_{\alpha\in\mathbb{N}_0^\theta}(\dim X_\alpha)t^\alpha\in\mathbb{N}_0[[t_1,\dots,t_\theta]] \tag{5.1}$$

be the *(multivariate) Hilbert series* of X.

There is a $\mathbb{Z}$-linear action of $\mathrm{GL}(\theta,\mathbb{Z})$ on $\mathbb{Z}[t_1^{\pm1},\dots,t_\theta^{\pm1}]$ via $gt^\alpha=t^{g(\alpha)}$ for all $g\in\mathrm{GL}(\theta,\mathbb{Z})$, $\alpha\in\mathbb{Z}^\theta$. This extends to a partially defined $\mathbb{Z}$-linear action of $\mathrm{GL}(\theta,\mathbb{Z})$ on $\mathbb{Z}[[t_1,\dots,t_\theta]]$: the action of each $g\in\mathrm{GL}(\theta,\mathbb{Z})$ is well-defined on the subring of $\mathbb{Z}[[t_1,\dots,t_\theta]]$ consisting of those formal power series $\sum_{\alpha\in\mathbb{N}_0^\theta}a_\alpha t^\alpha$, where $a_\alpha\in\mathbb{Z}$ for all α and $a_\alpha=0$ if $g(\alpha)\notin\mathbb{N}_0^\theta$.

We start our considerations of right coideal subalgebras with general lemmata.

Lemma 5.1. *Let $M\in\mathcal{F}_\theta$ and let E be an $\mathbb{N}_0^\theta$-graded right coideal of $\mathcal{B}(M)$ in ${}^H_H\mathcal{YD}$. If $E\neq\Bbbk1$, then there exists $i\in\mathbb{I}$ such that $M_i\subset E$.*

Proof. Let $\mathrm{pr}_1:\mathcal{B}(M)\to M_1\oplus\cdots\oplus M_\theta$ be the $\mathbb{N}_0$-graded projection to the homogeneous elements of degree 1. Recall that the map

$$(\mathrm{pr}_1\otimes\mathrm{id})\Delta_{\mathcal{B}(M)}:\oplus_{n=1}^\infty\mathcal{B}^n(M)\to(M_1\oplus\cdots\oplus M_\theta)\otimes\mathcal{B}(M)$$

is injective. By assumption, $\Delta_{\mathcal{B}(M)}(E)\subset E\otimes\mathcal{B}(M)$, and $E\neq\Bbbk1$. Thus

$$0\neq(\mathrm{pr}_1\otimes\mathrm{id})\Delta_{\mathcal{B}(M)}(E)\subset\big(E\cap(M_1\oplus\cdots\oplus M_\theta)\big)\otimes\mathcal{B}(M)$$

and hence $E\cap(M_1\oplus\cdots\oplus M_\theta)\neq0$. Since E is $\mathbb{N}_0^\theta$-graded, there exists $i\in\mathbb{I}$ such that $E\cap M_i\neq0$. Since $E\in{}^H_H\mathcal{YD}$ and $M_i\in{}^H_H\mathcal{YD}$ is irreducible, this implies that $M_i\subset E$. $\square$

Lemma 5.2. *Let $M\in\mathcal{F}_\theta$, $i\in\mathbb{I}$, and $E\in\mathcal{K}(M)$. Then $M_i\not\subset E$ if and only if $S_{\mathcal{B}(M)}(E)\subset K_i^M$.*

Proof. The homogeneous subspace of K_i^M of degree α_i is 0, hence $S_{\mathcal{B}(M)}(E) \subset K_i^M$ implies that $M_i \not\subset E$. Conversely, assume that $M_i \not\subset E$ and let $\pi : \mathcal{B}(M) \to \mathcal{B}(M_i)$ be the canonical map. Then $M_i \not\subset \pi(E)$ since E is $\mathbb{N}_0^\theta$-homogeneous. Since $\pi(E)$ is a right coideal of $\mathcal{B}(M_i)$, we conclude that $\pi(E) = \Bbbk 1$ by Lemma 5.1 and hence

$$\pi(S_{\mathcal{B}(M)}(E)) = \Bbbk 1. \tag{5.2}$$

On the other hand, $S_{\mathcal{B}(M)}(E)$ is a left coideal subalgebra of $\mathcal{B}(M)$, and by Eq. (5.2) it is contained in $\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)} = K_i^M$ which proves the lemma. □

Corollary 5.3. *Let $M \in \mathcal{F}_\theta$, $i \in \mathbb{I}$, and $E \in \mathcal{K}(R_i(M))$. Assume that M is i-finite and $R_i(M)_i \not\subset E$. Then $(\Omega_i^M)^{-1}(E) \subset K_i^M$.*

Proof. Since M is i-finite the map Ω_i^M is bijective, see Sect. 4. Since $R_i(M)_i \not\subset E$, Lemma 5.2 yields that

$$S_{\mathcal{B}(R_i(M))}(E) \subset K_i^{R_i(M)} = S_{\mathcal{B}(R_i(M))}\Omega_i^M(K_i^M), \tag{5.3}$$

where the last equation holds by Eq. (4.7). The relation (5.3) gives the claim. □

Lemma 5.4. *Let $M \in \mathcal{F}_\theta$ and $i \in \mathbb{I}$. Suppose that M is not i-finite. Then there exist infinitely many $\mathbb{N}_0^\theta$-graded right coideal subalgebras of $\mathcal{B}(M)$ in ${}^H_H\mathcal{YD}$ which do not contain any M_j with $j \in \mathbb{I} \setminus \{i\}$.*

Proof. Let $k \in \mathbb{I}\setminus\{i\}$ such that $\dim(\mathrm{ad}\,\mathcal{B}(M_i))(M_k) = \infty$. For all $n \in \mathbb{N}$ let E_n be the subalgebra of $\mathcal{B}(M)$ generated by $(\mathrm{ad}\,\mathcal{B}^n(M_i))(M_k)$ and M_i. By assumption, $E_n \neq 0$ for all $n \in \mathbb{N}$. By construction, $E_n \subset E_m$ for all $m \le n$ and

$$(E_n)_{\alpha_k + m\alpha_i} = \begin{cases} 0 & \text{if } m < n, \\ (\mathrm{ad}\,\mathcal{B}^n(M_i))(M_k) & \text{if } m = n. \end{cases} \tag{5.4}$$

Hence $E_1 \supset E_2 \supset \cdots$ is a strictly decreasing sequence of nontrivial $\mathbb{N}_0^\theta$-graded subalgebras of $\mathcal{B}(M)$ in ${}^H_H\mathcal{YD}$ with $E_n \cap M = M_i$ for all $n \in \mathbb{N}$. It remains to prove that each E_n is a right coideal of $\mathcal{B}(M)$. But this is true since $\Delta_{\mathcal{B}(M)}(x) \in E_n \otimes \mathcal{B}(M)$ for each generator x of E_n by Eq. (4.12). □

Recall that K_i^M is a Hopf algebra in the braided category ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}$. Its comultiplication is denoted by $\Delta_{K_i^M}$. Assume that M is i-finite. Regard $K_i^M\#\mathcal{B}(M_i^*)$ as a Hopf algebra in ${}^H_H\mathcal{YD}$, such that the algebra map $\Omega_i^M : K_i^M\#\mathcal{B}(M_i^*) \to \mathcal{B}(R_i(M))$ is an isomorphism of Hopf algebras in ${}^H_H\mathcal{YD}$.

Lemma 5.5. *Let $M \in \mathcal{F}_\theta$ and $i \in \mathbb{I}$. Assume that M is i-finite. Let $F \subset K_i^M$ be a subalgebra in ${}^H_H\mathcal{YD}$. Then the following are equivalent:*

(1) *$F\mathcal{B}(M_i)$ is a right coideal subalgebra of $\mathcal{B}(M)$ in ${}^H_H\mathcal{YD}$.*

(2) *F is a right coideal subalgebra of K_i^M in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}$, and* $(\operatorname{ad} M_i)(F) \subset F$.
(3) *F is a right coideal subalgebra of $K_i^M \# \mathcal{B}(M_i^*)$ in ${}^H_H\mathcal{YD}$.*

Proof. Assume (1) and let $E = F\mathcal{B}(M_i)$. Then $F = E \cap K_i^M$, and $(\operatorname{ad} M_i)(F) \subset F$ since $M_i \subset E$. Let $\pi : \mathcal{B}(M) \to \mathcal{B}(M_i)$ be the canonical map. Since

$$\Delta_{K_i^M}(x) = x^{(1)} S_{\mathcal{B}(M_i)} \pi(x^{(2)}) \otimes x^{(3)} \quad \text{for all } x \in K_i^M,$$

we obtain that $\Delta_{K_i^M}(E \cap K_i^M) \subset E \otimes K_i^M$. This proves (2). Similarly, (2) implies (1).

Assume (2). By definition and Thm. 4.2 the restriction of the comultiplication of $K_i^M \# \mathcal{B}(M_i^*)$ to K_i^M is given by the map

$$K_i^M \xrightarrow{\Delta_{K_i^M}} K_i^M \otimes K_i^M \xrightarrow{\mathfrak{R}} K_i^M \otimes (K_i^M \# \mathcal{B}(M_i^*)).$$

Hence F is a right coideal subalgebra of $K_i^M \# \mathcal{B}(M_i^*)$ in ${}^H_H\mathcal{YD}$.

Conversely, assume (3). Let $\pi' = \varepsilon \otimes \operatorname{id}_{\mathcal{B}(M_i^*)} : K_i^M \# \mathcal{B}(M_i^*) \to \mathcal{B}(M_i^*)$. Then π' is an algebra map by Eq. (3.2). Let $x \in F$. Since F is a right coideal subalgebra of $K_i^M \# \mathcal{B}(M_i^*)$ and $(\operatorname{id} \otimes \pi')\Delta_{K_i^M}(x) = x \otimes 1$ for all $x \in K_i^M$, it follows that

$$F \otimes \mathcal{B}(M_i^*) \ni (\operatorname{id}_{K_i^M} \otimes \pi')\mathfrak{R}\Delta_{K_i^M}(x) = \sum_\alpha S^{-1}_{\mathcal{B}(M_i)\#H}(b_\alpha) \cdot x \otimes b^\alpha.$$

Since F is also an H-module, F is stable under the adjoint action $\cdot$ of $\mathcal{B}(M_i)$. Since $\mathfrak{R}\Delta_{K_i^M}(F) \subset F \otimes (K_i^M \# \mathcal{B}(M_i^*))$ by assumption, it follows from (3.7) that F is a right coideal subalgebra of K_i^M in ${}^{\mathcal{B}(M_i)\#H}_{\mathcal{B}(M_i)\#H}\mathcal{YD}$. □

Recall from Eq. (4.8) that for all $i \in \mathbb{I}$ and $M \in \mathcal{F}_\theta$, $\varphi_i^M : M \to R_i^2(M)$ is a family of isomorphisms of objects in ${}^H_H\mathcal{YD}$. Let $\mathcal{B}(\varphi_i^M) : \mathcal{B}(M) \to \mathcal{B}(R_i^2(M))$ be the induced isomorphism of braided Hopf algebras in ${}^H_H\mathcal{YD}$. The following theorem is the key result in the proof of Thms. 6.12, 6.15. When M is i-finite we will use the isomorphisms

$$K_i^M \# \mathcal{B}(M_i^*) \xrightarrow{\Omega_i^M} \mathcal{B}(R_i(M)),$$

$$K_i^{R_i(M)} \# \mathcal{B}(R_i(M)_i^*) \xrightarrow{\Omega_i^{R_i(M)}} \mathcal{B}(R_i^2(M)) \xleftarrow{\mathcal{B}(\varphi_i^M)} \mathcal{B}(M)$$

to define bijections between the right coideal subalgebras of $\mathcal{B}(M)$ and of $\mathcal{B}(R_i(M))$.

Theorem 5.6. *Let $M \in \mathcal{F}_\theta$ and $i \in \mathbb{I}$. Assume that M is i-finite. Then the maps $\sigma_i^M : \mathcal{K}(M) \to \mathcal{K}(R_i(M))$ defined for all $E \in \mathcal{K}(M)$ by*

$$\sigma_i^M(E) = \begin{cases} \Omega_i^M(E \cap K_i^M) & \text{if } M_i \subset E, \\ (\Omega_i^{R_i(M)})^{-1}(\mathcal{B}(\varphi_i^M)(E))\mathcal{B}(R_i(M)_i) & \text{if } M_i \not\subset E, \end{cases}$$

and $\bar\sigma_i^{R_i(M)} : \mathcal{K}(R_i(M)) \to \mathcal{K}(M)$ *defined for all* $E \in \mathcal{K}(R_i(M))$ *by*

$$\bar\sigma_i^{R_i(M)}(E) = \begin{cases} \mathcal{B}(\varphi_i^M)^{-1}(\Omega_i^{R_i(M)}(E \cap K_i^{R_i(M)})) & \text{if } R_i(M)_i \subset E, \\ (\Omega_i^M)^{-1}(E)\mathcal{B}(M_i) & \text{if } R_i(M)_i \not\subset E, \end{cases}$$

are bijective. More precisely, the following hold.

(1) For all $E \in \mathcal{K}(M)$*,* $M_i \subset E$ *if and only if* $R_i(M)_i \not\subset \sigma_i^M(E)$*. For all* $E \in \mathcal{K}(R_i(M))$*,* $R_i(M)_i \subset E$ *if and only if* $M_i \not\subset \bar\sigma_i^{R_i(M)}(E)$*.*

(2) $\bar\sigma_i^{R_i(M)}\sigma_i^M = \mathrm{id}_{\mathcal{K}(M)}$*,* $\sigma_i^M\bar\sigma_i^{R_i(M)} = \mathrm{id}_{\mathcal{K}(R_i(M))}$*.*

Proof. By Cor. 5.3 the maps σ_i^M and $\bar\sigma_i^{R_i(M)}$ are well-defined in the sense that $\sigma_i^M(E) \subset \mathcal{B}(R_i(M))$ for all $E \in \mathcal{K}(M)$ and $\bar\sigma_i^{R_i(M)}(E) \subset \mathcal{B}(M)$ for all $E \in \mathcal{K}(R_i(M))$. It remains to prove (1) and that σ_i^M maps $\mathcal{K}(M)$ to $\mathcal{K}(R_i(M))$ and $\bar\sigma_i^{R_i(M)}$ maps $\mathcal{K}(R_i(M))$ to $\mathcal{K}(M)$. Then the equations in (2) follow from Lemma 2.2.

We prove that $\sigma_i^M(E) \in \mathcal{K}(R_i(M))$ for all $E \in \mathcal{K}(M)$, and that the part of (1) regarding σ_i^M holds. The analogous claims for $\bar\sigma_i^{R_i(M)}$ can be shown similarly.

Let $E \in \mathcal{K}(M)$. Assume first that $M_i \subset E$, and let $F = E \cap K_i^M$. Since K_i^M is an $\mathbb{N}_0^\theta$-graded algebra in ${}^H_H\mathcal{YD}$, F is an $\mathbb{N}_0^\theta$-graded subalgebra of $\mathcal{B}(M)$ in ${}^H_H\mathcal{YD}$. Further, $E = F\mathcal{B}(M_i)$ by Lemma 2.2. By Lemma 5.5 (1)⇒(3), and since $\Omega_i^M : K_i^M\#\mathcal{B}(M_i^*) \to \mathcal{B}(R_i(M))$ is an isomorphism of $\mathbb{N}_0^\theta$-graded Hopf algebras in ${}^H_H\mathcal{YD}$, we conclude that $\Omega_i^M(F) \in \mathcal{K}(R_i(M))$. Further, $R_i(M)_i \not\subset \Omega_i^M(F)$ by (4.6) and since $(E \cap K_i^M)_{-\alpha_i} = 0$.

Assume now that $M_i \not\subset E$. Since $\mathcal{B}(\varphi_i^N) : \mathcal{B}(N) \to \mathcal{B}(R_i^2(N))$ is an isomorphism of $\mathbb{N}_0^\theta$-graded braided Hopf algebras, we conclude that $R_i^2(M)_i \not\subset E$ and $\mathcal{B}(\varphi_i^M)(E) \in \mathcal{K}(R_i^2(M))$. Further, $R_i^2(M)$ is i-finite. Let $F = (\Omega_i^{R_i(M)})^{-1}(\mathcal{B}(\varphi_i^M)(E))$. Then $F \subset K_i^{R_i(M)}$ by Cor. 5.3. Further, F is an $\mathbb{N}_0^\theta$-graded subalgebra of $K_i^{R_i(M)}$ and a right coideal subalgebra of $K_i^{R_i(M)}\#\mathcal{B}(R_i(M)_i^*)$ in ${}^H_H\mathcal{YD}$ by the definition of the braided Hopf algebra structure of $K_i^{R_i(M)}\#\mathcal{B}(R_i(M)_i^*)$. By Lemma 5.5 (3)⇒(1), $\sigma_i^M(E) = F\mathcal{B}(R_i(M)_i) \in \mathcal{K}(R_i(M))$. Clearly, $R_i(M)_i \subset \sigma_i^M(E)$, and hence we are done. □

Corollary 5.7. *Let* $M \in \mathcal{F}_\theta$*,* $i \in \mathbb{I}$*, and* $E_1, E_2 \in \mathcal{K}(M)$*. Assume that* M *is* i*-finite and that* $\mathcal{H}_{E_1} = \mathcal{H}_{E_2}$*. Then* $\mathcal{H}_{\sigma_i^M(E_1)} = \mathcal{H}_{\sigma_i^M(E_2)}$*.*

Proof. Note that $M_i = \mathcal{B}(M)_{\alpha_i}$ is irreducible in ${}^H_H\mathcal{YD}$. Hence $M_i \subset E_1$ if and only if $M_i \subset E_2$, since $\mathcal{H}_{E_1} = \mathcal{H}_{E_2}$. Assume first that $M_i \subset E_1$. By Lemma 2.2, $E_l \simeq (E_l \cap K_i^M) \otimes \mathcal{B}(M_i)$ as $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$ for $l \in \{1, 2\}$. Hence the claim follows from Thm. 5.6 and (4.6). The case $M_i \not\subset E_1$ is treated similarly. □

Corollary 5.8. *Let $M \in \mathcal{F}_\theta$ and let $i \in \mathbb{I}$. Then $\mathcal{K}(M)$ is finite if and only if $\mathcal{K}(R_i(M))$ is finite. In this case M is i-finite, and $\mathcal{K}(M)$ and $\mathcal{K}(R_i(M))$ have the same cardinality.*

Proof. If M is not i-finite, then $R_i(M) = M$, and hence $\mathcal{K}(M) = \mathcal{K}(R_i(M))$ is infinite by Lemma 5.4. If M is i-finite, then the claim follows from the bijectivity of σ_i^M in Thm. 5.6. □

6. Construction of right coideal subalgebras

Let $\theta \in \mathbb{N}$ and $M = (M_1, \dots, M_\theta) \in \mathcal{F}_\theta$. Let

$$\mathcal{F}_\theta(M) = \{R_{i_1} \cdots R_{i_n}(M) \mid n \in \mathbb{N}_0, i_1, \dots, i_n \in \mathbb{I}\},$$
$$\mathcal{X}_\theta(M) = \{r_{i_1} \cdots r_{i_n}([M]) \mid n \in \mathbb{N}_0, i_1, \dots, i_n \in \mathbb{I}\}$$

where R_i and r_i, $i \in \mathbb{I}$, are defined in Sect. 4. We say that M *admits all reflections*, if N is i-finite for all $N \in \mathcal{F}_\theta(M)$ and $i \in \mathbb{I}$. This is for example the case if $(M_1 \oplus \cdots \oplus M_\theta)^{\otimes m}$ is semisimple in ${}_H^H\mathcal{YD}$ for all $m \geq 1$ and the Gelfand-Kirillov dimension of $\mathcal{B}(M)$ is finite, see [HS08, Thms. I, III]. Also, Cor. 5.8 and the definition of $\mathcal{F}_\theta(M)$ yield the following.

Proposition 6.1. *Let $M \in \mathcal{F}_\theta$. Assume that $\mathcal{K}(M)$ is finite. Then M admits all reflections.*

Recall from [AHS08] the following crucial result.

Theorem 6.2. [AHS08], [HS08, Thm. 6.10] *Let $M \in \mathcal{F}_\theta$. If M admits all reflections, then $\mathcal{C}(M) = (\mathbb{I}, \mathcal{X}_\theta(M), (r_i|_{\mathcal{X}_\theta(M)})_{i\in\mathbb{I}}, (A^X)_{X\in\mathcal{X}_\theta(M)})$ is a Cartan scheme.*

Therefore, if $M \in \mathcal{F}_\theta(M)$ and M admits all reflections, then we may attach the Weyl groupoid $\mathcal{W}(M) := \mathcal{W}(\mathcal{C}(M))$ to M. Later on, for brevity we will write r_i instead of $r_i|_{\mathcal{X}_\theta(M)}$.

In this section we associate a right coideal subalgebra $E^N(w)$ of $\mathcal{B}(N)$ to any $N \in \mathcal{F}_\theta(M)$ and $w \in \mathrm{Hom}(\mathcal{W}(M), [N])$.

Recall that $\Bbbk 1 \in \mathcal{K}(N)$ for all $N \in \mathcal{F}_\theta$. By Thm. 5.6, $\bar{\sigma}_i^{R_i(N)} : \mathcal{K}(R_i(N)) \to \mathcal{K}(N)$ is a bijection for all $N \in \mathcal{F}_\theta$ and $i \in \mathbb{I}$, where N is i-finite.

Definition 6.3. Let $M \in \mathcal{F}_\theta$. Assume that M admits all reflections. For all $N \in \mathcal{F}_\theta(M)$, $m \in \mathbb{N}_0$, $i_1, \dots, i_m \in \mathbb{I}$, let $E^N() = \Bbbk 1$ and

$$E^N(i_1, \dots, i_m) = \bar{\sigma}_{i_1}^{R_{i_1}(N)} \bar{\sigma}_{i_2}^{R_{i_2}R_{i_1}(N)} \cdots \bar{\sigma}_{i_m}^{R_{i_m}\cdots R_{i_1}(N)}(\Bbbk 1) \in \mathcal{K}(N).$$

Lemma 6.4. *Let $M \in \mathcal{F}_\theta$. Assume that M admits all reflections. Let $N \in \mathcal{F}_\theta(M)$, $m \in \mathbb{N}_0$, and $i_1, \dots, i_m \in \mathbb{I}$. Then $E^N(i_1, \dots, i_m)$ is the unique element $E \in \mathcal{K}(N)$ with $\mathcal{H}_E = \mathcal{H}_{E^N(i_1,\dots,i_m)}$.*

Proof. By Thm. 5.6,

$$\begin{aligned}\sigma_{i_m}^{R_{i_{m-1}}\cdots R_{i_1}(N)}&\cdots\sigma_{i_1}^N(E^N(i_1,\dots,i_m))\\&=\sigma_{i_m}^{R_{i_{m-1}}\cdots R_{i_1}(N)}\cdots\sigma_{i_1}^N\bar\sigma_{i_1}^{R_{i_1}(N)}\cdots\bar\sigma_{i_m}^{R_{i_m}\cdots R_{i_1}(N)}(\Bbbk 1)=\Bbbk 1.\end{aligned}$$

Let $E\in\mathcal{K}(N)$ with $\mathcal{H}_E=\mathcal{H}_{E^N(i_1,\dots,i_m)}$, and let $E'=\sigma_{i_m}^{R_{i_{m-1}}\cdots R_{i_1}(N)}\cdots\sigma_{i_1}^N(E)$. By Cor. 5.7 the Hilbert series of E' and of $\sigma_{i_m}^{R_{i_{m-1}}\cdots R_{i_1}(N)}\cdots\sigma_{i_1}^N(E^N(i_1,\dots,i_m))$ coincide. Hence $\mathcal{H}_{E'}=1$, and therefore $E'=\Bbbk 1$. Thus

$$E=\bar\sigma_{i_1}^{R_{i_1}(N)}\cdots\bar\sigma_{i_m}^{R_{i_m}\cdots R_{i_1}(N)}(E')=\bar\sigma_{i_1}^{R_{i_1}(N)}\cdots\bar\sigma_{i_m}^{R_{i_m}\cdots R_{i_1}(N)}(\Bbbk 1)=E^N(i_1,\dots,i_m).$$

This proves the lemma. □

Definition 6.5. Let $M\in\mathcal{F}_\theta$. Assume that M admits all reflections. Let $N\in\mathcal{F}_\theta(M)$, $m\in\mathbb{N}_0$ and $i_1,\dots,i_m,j\in\mathbb{I}$. Let $T_j^{R_j(N)}:\mathcal{B}(R_j(N))\to\mathcal{B}(N)$ be the composition of the linear maps

$$\mathcal{B}(R_j(N))\xrightarrow{(\Omega_j^N)^{-1}}K_j^N\#\mathcal{B}(N_j^*)\xrightarrow{\mathrm{id}\otimes\varepsilon}K_j^N\subset\mathcal{B}(N).$$

For all $1\le l\le m$ define $\beta_l^{[N]}(i_1,\dots,i_m)\in\boldsymbol{\Delta}^{[N]\,\mathrm{re}}$ and $N_l(i_1,\dots,i_m)\in{}^H_H\mathcal{YD}$ by

$$\begin{aligned}\beta_l^{[N]}(i_1,\dots,i_m)&=s_{i_1}^{r_{i_1}([N])}s_{i_2}^{r_{i_2}r_{i_1}([N])}\cdots s_{i_{l-1}}^{r_{i_{l-1}}\cdots r_{i_1}([N])}(\alpha_{i_l}),\\ N_l(i_1,\dots,i_m)&=T_{i_1}^{R_{i_1}(N)}T_{i_2}^{R_{i_2}R_{i_1}(N)}\cdots T_{i_{l-1}}^{R_{i_{l-1}}\cdots R_{i_2}R_{i_1}(N)}(R_{i_{l-1}}\cdots R_{i_2}R_{i_1}(N)_{i_l})\end{aligned}$$

(where $\beta_1^N(i_1,\dots,i_m)=\alpha_{i_1}$, and $N_1(i_1,\dots,i_m)=N_{i_1}$).

We say that $(i_1,\dots,i_m)$ is N-*admissible* if for all $1\le k\le m-1$ and $1\le l\le m-k$,

$$\alpha_{i_k}\ne\beta_l^{r_{i_k}\cdots r_{i_2}r_{i_1}([N])}(i_{k+1},\dots,i_m).$$

Equivalently, $(i_1,\dots,i_m)$ is N-admissible if and only if

$$\beta_l^{[N]}(i_1,\dots,i_m)\ne-\beta_k^{[N]}(i_1,\dots,i_m)\quad\text{for all }1\le k<l\le m.\tag{6.1}$$

Lemma 6.6. *Let $M\in\mathcal{F}_\theta$, and assume that M admits all reflections. Let $m\in\mathbb{N}_0$, $i_1,\dots,i_m\in\mathbb{I}$ and $N\in\mathcal{F}_\theta(M)$. Assume that $(i_1,\dots,i_m)$ is N-admissible. For all $1\le l\le m$ let $\beta_l=\beta_l^{[N]}(i_1,\dots,i_m)$, and $N_{\beta_l}=N_l(i_1,\dots,i_m)$. Then*

1. *$\beta_1,\dots,\beta_m$ are pairwise distinct elements in $\mathbb{N}_0^\theta$.*
2. *For all $1\le l\le m$, $N_{\beta_l}\subset E^N(i_1,\dots,i_m)$ is a finite-dimensional irreducible subobject in ${}^H_H\mathcal{YD}$ of degree β_l, and $N_{\beta_l}\simeq R_{i_{l-1}}\cdots R_{i_2}R_{i_1}(N)_{i_l}$ in ${}^H_H\mathcal{YD}$.*
3. *For all $1\le l\le m$, the subalgebra $\Bbbk\langle N_{\beta_l}\rangle$ of $\mathcal{B}(N)$ generated by N_{β_l} is isomorphic to $\mathcal{B}(N_{\beta_l})$ as an algebra and as an $\mathbb{N}_0^\theta$-graded object in ${}^H_H\mathcal{YD}$, where N_{β_l} has degree β_l.*
4. *The multiplication map $\Bbbk\langle N_{\beta_m}\rangle\otimes\cdots\otimes\Bbbk\langle N_{\beta_1}\rangle\to E^N(i_1,\dots,i_m)$ is an isomorphism of $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$.*

Proof. The cases $m = 0, 1$ are clear since $N_{\alpha_{i_1}} = N_{i_1}$, and $E^N(i_1) = \mathcal{B}(N_{i_1})$ by Thm. 5.6. Let $m > 1$ and assume that $(i_1, \dots, i_m)$ is N-admissible. Then $(i_2. \dots, i_m)$ is $R_{i_1}(N)$-admissible. To prove the Lemma for $(i_1, \dots, i_m)$ we may assume by induction that (1)–(4) hold for $(i_2, \dots, i_m)$, that is, if we define

$$\gamma_l = s_{i_2}^{r_{i_2} r_{i_1}([N])} s_{i_3}^{r_{i_3} r_{i_2} r_{i_1}([N])} \cdots s_{i_l}^{r_{i_l} \cdots r_{i_2} r_{i_1}([N])}(\alpha_{i_{l+1}}),$$
$$R_{i_1}(N)_{\gamma_l} = T_{i_2}^{R_{i_2} R_{i_1}(N)} T_{i_3}^{R_{i_3} R_{i_2} R_{i_1}(N)} \cdots T_{i_l}^{R_{i_l} \cdots R_{i_2} R_{i_1}(N)}(R_{i_l} \cdots R_{i_2} R_{i_1}(N)_{i_{l+1}})$$

for all $1 \le l \le m-1$, then

(a) $\gamma_1, \dots, \gamma_{m-1}$ are pairwise distinct elements in $\mathbb{N}_0^\theta$.
(b) For all $1 \le l \le m-1$, $R_{i_1}(N)_{\gamma_l} \subset E^{R_{i_1}(N)}(i_2, \dots, i_m) \subset \mathcal{B}(R_{i_1}(N))$ is an irreducible finite-dimensional subobject in ${}_H^H\mathcal{YD}$ of degree γ_l.
(c) For all $1 \le l \le m-1$, the subalgebra $\Bbbk\langle R_{i_1}(N)_{\gamma_l}\rangle$ of $\mathcal{B}(R_{i_1}(N))$ generated by $R_{i_1}(N)_{\gamma_l}$ is isomorphic to $\mathcal{B}(R_{i_1}(N)_{\gamma_l})$ as an algebra and as an $\mathbb{N}_0^\theta$-graded object in ${}_H^H\mathcal{YD}$, where $R_{i_1}(N)_{\gamma_l}$ has degree γ_l.
(d) The multiplication map

$$\Bbbk\langle R_{i_1}(N)_{\gamma_{m-1}}\rangle \otimes \cdots \otimes \Bbbk\langle R_{i_1}(N)_{\gamma_1}\rangle \to E^{R_{i_1}(N)}(i_2, \dots, i_m)$$

is an isomorphism of $\mathbb{N}_0^\theta$-graded objects in ${}_H^H\mathcal{YD}$.

By Definition 6.3

$$\bar{\sigma}_{i_1}^{R_{i_1}(N)}(E^{R_{i_1}(N)}(i_2, \dots, i_m)) = E^N(i_1, \dots, i_m).$$

By assumption, $\alpha_{i_1} \ne \gamma_l$ for all $1 \le l \le m-1$. Hence by degree reasons it follows from (d) that $R_{i_1}(N)_{i_1} \not\subset E^{R_{i_1}(N)}(i_2 \dots, i_m)$, since by (b) $R_{i_1}(N)_{\gamma_l}$ has degree γ_l for all $1 \le l \le m-1$, and $\gamma_1, \dots, \gamma_{m-1} \in \mathbb{N}_0^\theta$ by (a). Then

$$\bar{\sigma}_{i_1}^{R_{i_1}(N)}(E^{R_{i_1}(N)}(i_2, \dots, i_m)) = (\Omega_{i_1}^N)^{-1}(E^{R_{i_1}(N)}(i_2, \dots, i_m))\mathcal{B}(N_{i_1}),$$

and $(\Omega_{i_1}^N)^{-1}(E^{R_{i_1}(N)}(i_2, \dots, i_m)) \subset K_{i_1}^N$ by Thm. 5.6. Thus the multiplication map

$$(\Omega_{i_1}^N)^{-1}(E^{R_{i_1}(N)}(i_2, \dots, i_m)) \otimes \mathcal{B}(N_{i_1}) \to E^N(i_1, \dots, i_m) \tag{6.2}$$

is bijective. Moreover the restriction of the map $T_{i_1}^{R_{i_1}(N)}$ to $E^{R_{i_1}(N)}(i_2, \dots, i_m)$ is the restriction of the algebra isomorphism $(\Omega_{i_1}^N)^{-1}$. Therefore we obtain from (d) that the multiplication map

$$\Bbbk\langle T_{i_1}^{R_{i_1}(N)}(R_{i_1}(N)_{\gamma_{m-1}})\rangle \otimes \cdots \otimes \Bbbk\langle T_{i_1}^{R_{i_1}(N)}(R_{i_1}(N)_{\gamma_1})\rangle \to T_{i_1}^{R_{i_1}(N)}(E^{R_{i_1}(N)}(i_2, \dots, i_m))$$

is bijective. Since $T_{i_1}^{R_{i_1}(N)}(R_{i_1}(N)_{\gamma_l}) = N_{\beta_{l+1}}$ for all $1 \le l \le m-1$, (4) follows from the bijectivity of the map in Eq. (6.2).

Note that $s_{i_1}^{r_{i_1}([N])}(\gamma_l) = \beta_{l+1}$ for all $1 \le l \le m-1$. By (4.6) the subspace $T_{i_1}^{R_{i_1}(N)}(R_{i_1}(N)_{\gamma_l}) = (\Omega_{i_1}^N)^{-1}(R_{i_1}(N)_{\gamma_l}) = N_{\beta_{l+1}}$ of $\mathcal{B}(N)$ has degree β_{l+1} for all $1 \le l \le m-1$. By definition $N_{\beta_1} = N_{i_1}$ has degree $\beta_1 = \alpha_{i_1}$. It now follows from (4) that $\beta_l \in \mathbb{N}_0^\theta$ for all $1 \le l \le m$. Thus (1) holds by the characterization of N-admissibility via Eq. (6.1). Finally, (2) and (3) follow from (b) and (c) since $(\Omega_{i_1}^N)^{-1}$ is an algebra isomorphism in ${}^H_H\mathcal{YD}$, and the change of grading is given by (4.6). □

Lemma 6.7. *Let $M \in \mathcal{F}_\theta$, and assume that M admits all reflections. Let $m \in \mathbb{N}_0$, $i_1, \dots, i_m \in \mathbb{I}$ and $N \in \mathcal{F}_\theta(M)$. For all $1 \le l \le m$ let $\beta_l = \beta_l^{[N]}(i_1, \dots, i_m)$.*

1. *Let $j \in \mathbb{I}$ and assume that $\alpha_j \notin \{\beta_1, \dots, \beta_m\}$ and that $(i_1, \dots, i_m)$ is N-admissible. Then $(j, i_1, \dots, i_m)$ is $R_j(N)$-admissible.*
2. *Assume that $\alpha_j \in \{\beta_1, \dots, \beta_m\}$ for all $j \in \mathbb{I}$ and that $(i_1, \dots, i_m)$ is N-admissible. Then $E^N(i_1, \dots, i_m) = \mathcal{B}(N)$.*
3. *Assume that $\mathcal{C}(M)$ is the Cartan scheme of a root system. Then $(i_1, \dots, i_m)$ is N-admissible if and only if $\mathrm{id}_{[N]} s_{i_1} \cdots s_{i_m}$ is a reduced expression.*

Proof. (1) holds by definition, and (2) follows from Lemma 6.6 (2) since $N_j \subset E^N(i_1, \dots, i_m)$ for all $j \in \mathbb{I}$ implies that $E^N(i_1, \dots, i_m) = \mathcal{B}(N)$.

Suppose in (3) that $(i_1, \dots, i_m)$ is N-admissible. Then $\mathrm{id}_{[N]} s_{i_1} \cdots s_{i_m}$ is a reduced expression by Lemma 6.6 (1) and Prop. 1.9. Conversely, suppose that $\mathrm{id}_{[N]} s_{i_1} \cdots s_{i_m}$ is a reduced expression. Then $\beta_1, \dots, \beta_m$ are pairwise distinct elements in $\mathbb{N}_0^\theta$ by Prop. 1.4. Hence $(i_1, \dots, i_m)$ is N-admissible. □

We recall a notion from [HS08].

Definition 6.8. Let $N \in \mathcal{F}_\theta$. Then the Nichols algebra $\mathcal{B}(N)$ of N is called *decomposable* if there exist a totally ordered index set $(L, \le)$ and a family $(W_l)_{l \in L}$ of finite-dimensional irreducible $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$ such that

$$\mathcal{B}(N) \simeq \otimes_{l \in L} \mathcal{B}(W_l) \tag{6.3}$$

as $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$, where $\deg N_i = \alpha_i$ for $1 \le i \le \theta$.

In such a decomposition the isomorphism classes of the Yetter-Drinfeld modules W_l and their degrees in $\mathbb{N}_0^\theta$ are uniquely determined by [HS08, Lemma 4.7], and we define the positive roots $\boldsymbol{\Delta}_+^{[N]}$ and the roots $\boldsymbol{\Delta}^{[N]}$ of $[N]$ by

$$\boldsymbol{\Delta}_+^{[N]} = \{\deg(W_l) \mid l \in L\},$$
$$\boldsymbol{\Delta}^{[N]} = \boldsymbol{\Delta}_+^{[N]} \cup -\boldsymbol{\Delta}_+^{[N]}.$$

In [HS08] we showed

Theorem 6.9. [HS08, Thm. 6.11] *Let $M \in \mathcal{F}_\theta$ and assume that M admits all reflections and that $\mathcal{B}(M)$ is decomposable. Then $\mathcal{B}(N)$ is decomposable for all $N \in \mathcal{F}_\theta(M)$, and $\mathcal{R}(M) = (\mathcal{C}(M), (\boldsymbol{\Delta}^X)_{X \in \mathcal{X}_\theta(M)})$ is a root system of type $\mathcal{C}(M)$.*

Corollary 6.10. *Let $M \in \mathcal{F}_\theta$ and assume that M admits all reflections and that $\mathcal{B}(M)$ is decomposable. Let $N \in \mathcal{F}_\theta(M)$ and $\lambda \in \boldsymbol{\Delta}^{[N]\,\mathrm{re}}_+$.*

(1) *There is exactly one $l(\lambda) \in L$ with $\lambda = \deg W_{l(\lambda)}$ in (6.3).*
(2) *Let $P \in \mathcal{F}_\theta(M)$, $w = \mathrm{id}_{[N]} s_{i_1} \cdots s_{i_m} \in \mathrm{Hom}([P],[N])$ be a reduced expression and $i \in \mathbb{I}$ such that $w(\alpha_i) = \lambda$. Let $N_\lambda = N_{m+1}(i_1, \dots, i_m, i) \subset \mathcal{B}(N)$. Then $\deg N_\lambda = \lambda$, $W_{l(\lambda)} \simeq P_i \simeq N_\lambda$ in ${}^H_H\mathcal{YD}$, and $\Bbbk\langle N_\lambda\rangle \cong \mathcal{B}(W_{l(\lambda)})$ as algebras and $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$.*
(3) *Let $j \in \mathbb{I}$, $\mu = s_j^N(\lambda)$, $Q = R_j(N)$, and assume that $\lambda \neq \alpha_j$. Similarly to N_λ in (2), define Q_μ using a reduced expression of an element $w' \in \mathrm{Hom}(\mathcal{W}(M),[Q])$. Then $N_\lambda \simeq Q_\mu$ in ${}^H_H\mathcal{YD}$.*

Proof. (1) is shown in [HS08, Lemma 7.1 (1)], and in (2), $W_{l(\lambda)} \simeq P_i$ by [HS08, Lemma 7.1 (2)]. Since $\mathrm{id}_{[N]} s_{i_1} \cdots s_{i_m}$ is a reduced expression, $(i_1, \dots, i_m)$ is N-admissible by Lemma 6.7 (3). Then $(i_1, \dots, i_m, i)$ is N-admissible: Indeed, $\lambda = \beta^{[N]}_{m+1}(i_1, \dots, i_m, i) \in \mathbb{N}_0^\theta$, and hence it differs from all $-\beta^{[N]}_l(i_1, \dots, i_m, i)$ with $1 \le l \le m$ by Lemma 6.6 (1). The remaining part of (2) follows from Lemma 6.6 since $P \simeq R_{i_m} \cdots R_{i_1}(N)$.

Now we prove (3). Since $\lambda \neq \alpha_j$, $\mu \in \boldsymbol{\Delta}^{[P]\,\mathrm{re}}_+$, and hence Q_μ can be defined. By (2), Q_μ is independent of the choice of w'. Hence we may choose $w' = s_j w$. Then (2) yields that $N_\lambda \simeq P_i$ and $Q_\mu \simeq P_i$ in ${}^H_H\mathcal{YD}$, which proves the claim. □

In the next lemma, which will be needed for Thm. 6.12, we follow the notation in Cor. 6.10. For each $\lambda \in \boldsymbol{\Delta}^{[N]\,\mathrm{re}}_+$ we choose $w_\lambda \in \mathrm{Hom}(\mathcal{W}(M),[N])$, a reduced expression $\mathrm{id}_{[N]} s_{j_1} \cdots s_{j_n}$ of w_λ, and $i \in \mathbb{I}$ such that $w_\lambda(\alpha_i) = \lambda$. Then we define

$$N_\lambda = N_{n+1}(j_1, \dots, j_n, i) \subset \mathcal{B}(N). \tag{6.4}$$

By Cor. 6.10, the isomorphism class of $N_\lambda \in {}^H_H\mathcal{YD}$ does not depend on w_λ and i.

Lemma 6.11. *Let $M \in \mathcal{F}_\theta$ and $N \in \mathcal{F}_\theta(M)$. Assume that M admits all reflections and that $\mathcal{B}(M)$ is decomposable. Let $m \in \mathbb{N}_0$, $i_1, \dots, i_m \in \mathbb{I}$. Then there exists an isomorphism*

$$E^N(i_1, \dots, i_m) \simeq \bigotimes_{\lambda \in \Lambda^{[N]}_+(i_1, \dots, i_m)} \mathcal{B}(N_\lambda)$$

of $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$.

Proof. We proceed by induction on m. Since $E^N() = \Bbbk 1$, the claim holds for $m = 0$.

Assume now that $m > 0$. Let $P = R_{i_1}(N)$, and assume that

$$E := E^P(i_2, \dots, i_m) \simeq \bigotimes_{\lambda \in \Lambda_+^{[P]}(i_2,\dots,i_m)} \mathcal{B}(P_\lambda). \tag{6.5}$$

Case 1: $\alpha_{i_1} \notin \Lambda_+^{[P]}(i_2, \dots, i_m)$. Then $P_{i_1} \not\subset E$ by degree reasons, and $(\Omega_{i_1}^P)^{-1}(E) \subset K_{i_1}^N$ by Cor. 5.3. Hence

$$E^N(i_1, \dots, i_n) = \bar{\sigma}_{i_1}^P(E) = (\Omega_{i_1}^N)^{-1}(E)\mathcal{B}(N_{i_1}) \simeq \Big(\bigotimes_{\lambda \in \Lambda_+^{[P]}(i_2,\dots,i_m)} \mathcal{B}(P_\lambda)\Big) \otimes \mathcal{B}(N_{i_1})$$

in ${}_H^H\mathcal{YD}$. Here the first two equations follow by definition, and the isomorphism is obtained from Lemma 2.2 and since $(\Omega_{i_1}^N)^{-1}$ is a morphism in ${}_H^H\mathcal{YD}$. Now, $\deg P_\lambda = s_{i_1}^P(\lambda)$ for all $\lambda \in \Lambda_+^{[P]}(i_2, \dots, i_m)$ by (4.6). Further, $N_{s_{i_1}^P(\lambda)} \simeq P_\lambda$ in ${}_H^H\mathcal{YD}$ by Cor. 6.10 (3). Thus the claim follows from Lemma 1.8.

Case 2: $\alpha_{i_1} \in \Lambda_+^{[P]}(i_2, \dots, i_m)$. Then $E \simeq (E \cap K_{i_1}^P) \otimes \mathcal{B}(P_{i_1})$ as $\mathbb{N}_0^\theta$-graded objects in ${}_H^H\mathcal{YD}$ by Lemma 2.2, and hence

$$E \cap K_{i_1}^P \simeq \bigotimes_{\lambda \in \Lambda_+^{[P]}(i_2,\dots,i_m) \setminus \{\alpha_{i_1}\}} \mathcal{B}(P_\lambda) \tag{6.6}$$

as $\mathbb{N}_0^\theta$-graded objects in ${}_H^H\mathcal{YD}$ by Eq. (6.5) and [HS08, Lemma 4.8]. Therefore

$$E^N(i_1, \dots, i_n) = \bar{\sigma}_{i_1}^P(E) = \mathcal{B}(\varphi_{i_1}^N)^{-1}\big(\Omega_{i_1}^P(E \cap K_{i_1}^P)\big) \simeq \bigotimes_{\lambda \in \Lambda_+^{[P]}(i_2,\dots,i_m) \setminus \{\alpha_{i_1}\}} \mathcal{B}(P_\lambda)$$

as $\mathbb{N}_0^\theta$-graded objects in ${}_H^H\mathcal{YD}$, where $\deg P_\lambda = s_{i_1}^P(\lambda)$, see (4.6). Indeed, the first two equations hold by definition, and the isomorphism follows from Eq. (6.6) and since $\Omega_{i_1}^P$ and $\mathcal{B}(\varphi_{i_1}^N)$ are morphisms in ${}_H^H\mathcal{YD}$. By Cor. 6.10 (3) we may replace P_λ by $N_{s_{i_1}^P(\lambda)}$, and then the claim follows from Lemma 1.8. □

Theorem 6.12. *Let $M \in \mathcal{F}_\theta$, and assume that M admits all reflections and that $\mathcal{B}(M)$ is decomposable. Let $N \in \mathcal{F}_\theta(M)$. Then for all $w \in \mathrm{Hom}(\mathcal{W}(M), [N])$ the right coideal subalgebra*

$$E^N(w) = E^N(i_1, \dots, i_m) \subset \mathcal{B}(N),$$

where $m \geq 0$ and $1 \leq i_1, \dots, i_m \leq \theta$ such that $w = \mathrm{id}_{[N]} s_{i_1} \cdots s_{i_m}$, is independent of the choice of $i_1, \dots, i_m$. The map

$$\varkappa^N : \mathrm{Hom}(\mathcal{W}(M), [N]) \to \mathcal{K}(N),\ w \mapsto E^N(w),$$

is injective, order preserving, and order reflecting, where the set of morphisms $\mathrm{Hom}(\mathcal{W}(M), [N])$ *is ordered by the right Duflo order and right coideal subalgebras are ordered with respect to inclusion.*

Proof. To prove that $\varkappa^N$ is a well-defined map, assume that $w = \mathrm{id}_{[N]}s_{i_1}\cdots s_{i_m} = \mathrm{id}_{[N]}s_{j_1}\cdots s_{j_n}$ in $\mathrm{Hom}(\mathcal{W}(M),[N])$, where $1 \le i_1,\dots,i_m,j_1,\dots j_n \le \theta$, $m,n \ge 0$. By Prop. 1.9, $\Lambda_+^{[N]}(i_1,\dots,i_m) = \Lambda_+^{[N]}(j_1,\dots,j_n)$. Hence by Lemma 6.11 the Hilbert series of $E^N(i_1,\dots,i_m)$ and of $E^N(j_1,\dots,j_n)$ coincide, and by Lemma 6.4 $E^N(i_1,\dots,i_m) = E^N(j_1,\dots,j_n)$.

Let $w,w' \in \mathrm{Hom}(\mathcal{W}(M),[N])$ with $E^N(w) = E^N(w')$. Then $\Lambda_+^{[N]}(w) = \Lambda_+^{[N]}(w')$ by Lemma 6.11 and [HS08, Lemma 4.7]. Therefore $w = w'$ by Prop. 1.9. Thus $\varkappa^N$ is injective.

By Thm. 1.13 $\varkappa^N$ is order preserving and order reflecting if and only if the following are equivalent for all $w_1,w_2 \in \mathrm{Hom}(\mathcal{W}(M),[N])$.

(1) $E^N(w_1) \subset E^N(w_2)$,
(2) $\Lambda_+^{[N]}(w_1) \subset \Lambda_+^{[N]}(w_2)$.

To prove the equivalence of (1) and (2) we proceed by induction on $\ell(w_1)$. If $w_1 = \mathrm{id}_{[N]}$, then $E^N(w_1) = \Bbbk 1$, $\Lambda_+^{[N]}(w_1) = \emptyset$ and hence (1) and (2) are both true. If $\ell(w_1) = 1$, then $w_1 = s_i^{r_i([N])}$ for some $i \in \mathbb{I}$. Then $\Lambda_+^{[N]}(w_1) = \alpha_i$ and $E^N(w_1) = \mathcal{B}(N_i)$. Hence (2) is equivalent to (1) by Lemma 6.11, since if $N_i \subset E^N(w_2)$, then $E^N(w_1) = \mathcal{B}(N_i) \subset E^N(w_2)$. Assume now that $\ell(w_1) > 1$. Let $i \in \mathbb{I}$ with $\ell(w_1) = \ell(w)+1$ for $w = s_i^N w_1$. Then

$$\alpha_i \in \Lambda_+^{[N]}(w_1) \tag{6.7}$$

by Cor. 1.10. Therefore Lemma 1.8 implies that (2) holds if and only if

$$\alpha_i \in \Lambda_+^{[N]}(w_2) \quad \text{and} \quad \Lambda_+^{r_i([N])}(s_i^N w_1) \subset \Lambda_+^{r_i([N])}(s_i^N w_2). \tag{6.8}$$

Since $\alpha_i = \Lambda_+^{[N]}(s_i^{r_i([N])})$, the induction hypothesis gives that the relations in (6.8) are equivalent to $N_i \subset E^N(w_2)$, $E^{R_i(N)}(s_i^N w_1) \subset E^{R_i(N)}(s_i^N w_2)$. Since $N_i \subset E^N(w_1)$ by Lemma 6.11 and by (6.7), the latter is equivalent to (1) by Thm. 5.6. □

Corollary 6.13. *Let $M \in \mathcal{F}_\theta$, $N \in \mathcal{F}_\theta(M)$, and assume that M admits all reflections and that $\mathcal{B}(M)$ is decomposable. Let $w_1,w_2 \in \mathrm{Hom}(\mathcal{W}(M),[N])$ with $E^N(w_1) \subset E^N(w_2)$. Then there are $m,n \in \mathbb{N}_0$, $m \le n$, and $i_1,\dots,i_n \in \mathbb{I}$ such that $w_1 = \mathrm{id}_{[N]}s_{i_1}\cdots s_{i_m}$ and $w_2 = \mathrm{id}_{[N]}s_{i_1}\cdots s_{i_n}$ are reduced expressions. Let*

$$\beta_l = \beta_l^{[N]}(i_1,\dots,i_n),\ N_{\beta_l} = N_l(i_1,\dots,i_n)$$

for all $1 \le l \le n$. Then

(1) *For all $1 \le l \le n$, $N_{\beta_l} \subset \mathcal{B}(N)$ is an irreducible finite-dimensional subobject in ${}_H^H\mathcal{YD}$ of degree β_l, and $\beta_k \ne \beta_l$ for all $k \ne l$.*

(2) *For all* $1 \le l \le n$*, the subalgebra* $\Bbbk\langle N_{\beta_l}\rangle$ *of* $\mathcal{B}(N)$ *generated by* N_{β_l} *is isomorphic to* $\mathcal{B}(N_{\beta_l})$ *as an algebra and as an* $\mathbb{N}_0^\theta$*-graded object in* ${}^H_H\mathcal{YD}$*, where* N_{β_l} *has degree* β_l*.*

(3) *The multiplication maps*

$$\Bbbk\langle N_{\beta_n}\rangle \otimes \cdots \otimes \Bbbk\langle N_{\beta_1}\rangle \to E^N(w_2),$$
$$\Bbbk\langle N_{\beta_m}\rangle \otimes \cdots \otimes \Bbbk\langle N_{\beta_1}\rangle \to E^N(w_1)$$

are bijective. In particular, $E^N(w_2)$ *is a free right module over* $E^N(w_1)$*.*

Proof. By Thm. 6.12 $w_1 \le_D w_2$. Hence by definition of the Duflo order, any reduced presentation $w_1 = \mathrm{id}_{[N]}s_{i_1}\cdots s_{i_m}$ of w_1 can be extended to a reduced presentation $w_2 = \mathrm{id}_{[N]}s_{i_1}\cdots s_{i_m}\cdots s_{i_n}$ of w_2. Then (1),(2) and (3) follow from Lemma 6.6 and Lemma 6.7 (3). □

The following results generalize properties of commutators and coproducts of PBW generators of quantized enveloping algebras.

Theorem 6.14. *Let* $M \in \mathcal{F}_\theta$*,* $N \in \mathcal{F}_\theta(M)$*, and assume that* M *admits all reflections and that* $\mathcal{B}(M)$ *is decomposable. Let* $n \in \mathbb{N}_0$*,* $i_1,\dots,i_n \in \mathbb{I}$*, and* $w = \mathrm{id}_{[N]}s_{i_1}\cdots s_{i_n} \in \mathrm{Hom}(\mathcal{W}(M),[N])$ *such that* $\ell(w) = n$*. For all* $1 \le l \le n$ *let* $\beta_l \in \boldsymbol{\Delta}^{[N]\,\mathrm{re}}$ *and* $N_{\beta_l} \subset \mathcal{B}(N)$ *as in Lemma 6.6. Then in* $E^N(w)$

$$xy - (x_{(-1)}\cdot y)x_{(0)} \in \Bbbk\langle N_{\beta_{l-1}}\rangle\Bbbk\langle N_{\beta_{l-2}}\rangle\cdots\Bbbk\langle N_{\beta_{k+1}}\rangle \tag{6.9}$$

for all $1 \le k < l \le n$*,* $x \in N_{\beta_k}$*,* $y \in N_{\beta_l}$*, and*

$$\Delta_{\mathcal{B}(N)}(x) - x \otimes 1 \in \Bbbk\langle N_{\beta_{l-1}}\rangle\Bbbk\langle N_{\beta_{l-2}}\rangle\cdots\Bbbk\langle N_{\beta_1}\rangle \otimes \mathcal{B}(N) \tag{6.10}$$

for all $1 \le l \le n$*,* $x \in N_{\beta_l}$*.*

Proof. By Cor. 6.13 and the definition of the N_{β_l}, for the proof of Eq. (6.9) it is enough to consider the case $k = 1$, $l = n$. In that case

$$xy - (x_{(-1)}\cdot y)x_{(0)} = (\mathrm{ad}x)(y) \in K^N_{i_1} \cap E^N(w) = \Bbbk\langle N_{\beta_n}\rangle\Bbbk\langle N_{\beta_{n-1}}\rangle\cdots\Bbbk\langle N_{\beta_2}\rangle,$$

since $y \in K^N_{i_1}$ and $x \in N_{i_1}$. Further, $\deg\,(\mathrm{ad}\,x)(y) = \beta_1 + \beta_n$. But $\beta_m \ne \beta_1 = \alpha_{i_1}$ for all $2 \le m \le n$, and hence $(\mathrm{ad}\,x)(y)$ has no summand with a factor in $\Bbbk\langle N_{\beta_n}\rangle$.

Now we prove Eq. (6.10). Since $E^N(w') \in \mathcal{K}(N)$ for all $w' \in \mathrm{Hom}(\mathcal{W}(M),[N])$, by Cor. 6.13 it suffices to consider the case $l = n$. Since $\Delta_{\mathcal{B}(N)}$ is $\mathbb{N}_0^\theta$-graded and $\mathcal{B}(N)$ is a connected coalgebra, (that is $z \in \mathcal{B}(N)$, $\deg z = 0$ implies that $z \in \Bbbk$,) the claim follows by degree reasons. □

Theorem 6.15. *Let* $M \in \mathcal{F}_\theta$*. Then the following are equivalent.*

(1) $\mathcal{K}(M)$ *is finite.*

(2) *M admits all reflections and the length of N-admissible sequences, where $N \in \mathcal{F}_\theta(M)$, is bounded.*
(3) *M admits all reflections and $\boldsymbol{\Delta}^{[M]\,\mathrm{re}}$ is finite.*

Assume the equivalent conditions (1) – (3). Then $\mathcal{B}(M)$ is decomposable, $\mathcal{R}(M) = (\mathcal{C}(M), (\boldsymbol{\Delta}^{X\,\mathrm{re}})_{X \in \mathcal{X}_\theta(M)})$ is a finite root system of type $\mathcal{C}(M)$, and for all $N \in \mathcal{F}_\theta(M)$, the map

$$\varkappa^N : \mathrm{Hom}(\mathcal{W}(M), [N]) \to \mathcal{K}(N)$$

is bijective.

Proof. Assume (2), and let $t \in \mathbb{N}$ such that $t \geq m$ for all $N \in \mathcal{F}_\theta(M)$ and all N-admissible sequences $(i_1, \dots, i_m)$. We prove (1), (3) and the second half of the theorem.

Suppose an m-tuple $(i_1, \dots, i_m)$ of elements in $\mathbb{I}$ is P-admissible for some $P \in \mathcal{F}_\theta(M)$. If there exists an element $j \in \mathbb{I}$ such that $\alpha_j \neq \beta_l^{[P]}(i_1, \dots, i_m)$ for all $1 \leq l \leq m$, then $(j, i_1, \dots, i_m)$ is $R_j(P)$-admissible by definition, and $t \geq m+1$.

Let $N \in \mathcal{F}_\theta(M)$. By the previous paragraph, there is a largest integer $m \geq 1$ such that there is a P-admissible sequence $(i_m, \dots, i_1)$ with $P = R_{i_m} \cdots R_{i_1}(N)$. Hence $E^P(i_m, \dots, i_1) = \mathcal{B}(P)$ by Lemma 6.7 (2), and by Lemma 6.6 there is an isomorphism of $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$

$$\mathcal{B}(P_{\gamma_m}) \otimes \cdots \otimes \mathcal{B}(P_{\gamma_1}) \cong \mathcal{B}(P), \tag{6.11}$$

where $\gamma_1, \dots, \gamma_m$ are pairwise distinct elements in $\mathbb{N}_0^\theta$. This means that the Nichols algebra of P is decomposable. Hence $\mathcal{B}(M)$ is decomposable by [HS08, Lemma 6.8], and the root system $\mathcal{R}(M)$ exists by Thm. 6.9. Moreover $\mathcal{R}(M)$ is finite, and for all objects $X \in \mathcal{X}_\theta(M)$, $2m = |\boldsymbol{\Delta}^{X\,\mathrm{re}}|$ and $m = |\boldsymbol{\Delta}_+^{X\,\mathrm{re}}|$. This proves (3).

We note that $\mathrm{id}_{[P]} s_{i_m} \cdots s_{i_1}$ is a reduced expression by Lemma 6.7 (3). Therefore the inverse $\mathrm{id}_{[N]} s_{i_1} \cdots s_{i_m}$ is a reduced expression. It cannot be extended to a reduced expression $\mathrm{id}_{[R_j(N)]} s_j s_{i_1} \cdots s_{i_m}$, $1 \leq j \leq \theta$, by Lemma 6.6 (1) since $m = |\boldsymbol{\Delta}_+^{R_j(N)\,\mathrm{re}}|$. Thus by Lemma 6.7 (1),(2), $E^N(w_0) = \mathcal{B}(N)$, where $w_0 = \mathrm{id}_{[N]} s_{i_1} \cdots s_{i_m}$.

By Thm. 6.12, $\varkappa^N$ is injective. To prove surjectivity of $\varkappa^N$, let $E \in \mathcal{K}(N)$. Let $w \in \mathrm{Hom}(\mathcal{W}(M), [N])$ be a shortest element such that $E \subset E^N(w)$. Such a w exists, since $\mathcal{R}(M)$ is finite, hence $\mathrm{Hom}(\mathcal{W}(M), [N])$ is finite by Lemma 1.2 and $E \subset E^N(w_0) = \mathcal{B}(N)$. We prove by induction on $\ell(w)$ that $E = E^N(w)$.

Assume first that $\ell(w) = 0$. Then $\Bbbk 1 \subset E \subset E^N(\mathrm{id}_{[N]}) = \Bbbk 1$ and hence $E = E^N(\mathrm{id}_{[N]})$.

Assume now that $\ell(w) > 0$. Then $E \neq \Bbbk 1$ by the minimality of w. By Lemma 5.1 there exists $i \in \mathbb{I}$ such that $N_i \subset E$. Then $N_i \subset E^N(w)$, and hence $w = s_i^{r_i([N])} w'$ by Cor. 6.13 with $w_1 = s_i^{r_i([N])}$ and $w_2 = w$, where $w' = s_i^N w$ and

$\ell(w) = \ell(w') + 1$. Further, $\sigma_i^N(E) \subset \sigma_i^N(E^N(w)) = E^{R_i(N)}(w')$ by Thm. 5.6. Hence

$$\sigma_i^N(E) = E^{R_i(N)}(w'')$$

for some $w'' \in \mathrm{Hom}(\mathcal{W}(M), r_i([N]))$ by induction hypothesis. Thus

$$E = \bar{\sigma}_i^{R_i(N)}(\sigma_i^N(E)) = \bar{\sigma}_i^{R_i(N)}(E^{R_i(N)}(w'')) = E^N(s_i w'').$$

Hence $\varkappa^N$ is bijective. In particular, $\varkappa^M$ is bijective. Therefore (1) holds since $\mathrm{Hom}(\mathcal{W}(M), [M])$ is finite by Lemma 1.2.

Finally we prove (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2).

Assume (1) and let $t = \#(\mathcal{K}(M))$. First, M admits all reflections by Prop. 6.1. Hence $\#(\mathcal{K}(M)) = \#(\mathcal{K}(N)) = t$ for all $N \in \mathcal{F}_\theta(M)$ by Cor. 5.8. It follows from Lemma 6.6 (4) that for all $N \in \mathcal{F}_\theta(M)$ the length of N-admissible sequences $(i_1, \dots, i_m)$ is bounded by t since $E^N(i_1, \dots, i_k) \neq E^N(i_1, \dots, i_l)$ for all k, l with $1 \le k < l \le m$.

Assume (3). Let $t = \#(\boldsymbol{\Delta}^{M\,\mathrm{re}})$. Since the Weyl groupoid is connected it follows that $\#(\boldsymbol{\Delta}^{M\,\mathrm{re}}) = \#(\boldsymbol{\Delta}^{N\,\mathrm{re}}) = t$ for all $N \in \mathcal{F}_\theta(M)$. By Lemma 6.6 the length of admissible sequences is bounded by t. □

Corollary 6.16. *Let $M \in \mathcal{F}_\theta$. Assume that M admits all reflections and that $\boldsymbol{\Delta}^{[M]\,\mathrm{re}}$ is finite. Then $\mathcal{B}(M)$ is decomposable and $\mathcal{R}(M) = (\mathcal{C}(M), (\boldsymbol{\Delta}^{X\,\mathrm{re}})_{X \in \mathcal{X}_\theta(M)})$ is a finite root system of type $\mathcal{C}(M)$. Let $w \in \mathrm{Hom}(\mathcal{W}(M), [M])$ be a longest element. Let $m = \ell(w)$ and let $w = \mathrm{id}_{[M]} s_{i_1} \cdots s_{i_m}$ be a reduced decomposition. For each $l \in \{1, \dots, m\}$ let $\beta_l \in \boldsymbol{\Delta}_+^{[M]}$ and $N_{\beta_l} \subset \mathcal{B}(M)$ as in Lemma 6.6. Then for each $l \in \{1, \dots, m\}$ the identity on N_{β_l} induces an isomorphism $\Bbbk\langle N_{\beta_l}\rangle \simeq \mathcal{B}(N_{\beta_l})$ of $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$, where N_{β_l} has degree β_l. Further, the multiplication map*

$$\Bbbk\langle N_{\beta_m}\rangle \otimes \cdots \otimes \Bbbk\langle N_{\beta_2}\rangle \otimes \Bbbk\langle N_{\beta_1}\rangle \to \mathcal{B}(M)$$

is an isomorphism of $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$.

Proof. The first claim is proven in Thm. 6.15. The rest follows from Lemma 6.6 and Lemma 6.7 (3). □

Corollary 6.17. *Let $M \in \mathcal{F}_\theta$. Assume that M admits all reflections and that $\boldsymbol{\Delta}^{[M]\,\mathrm{re}}$ is finite. Then there exist order preserving bijections between*

(1) *the set of $\mathbb{N}_0^\theta$-graded right coideal subalgebras of $\mathcal{B}(M)\#H$ containing H,*
(2) *the set of $\mathbb{N}_0^\theta$-graded right coideal subalgebras of $\mathcal{B}(M)$ in ${}^H_H\mathcal{YD}$,*
(3) $\mathrm{Hom}(\mathcal{W}(M), [M])$,

where right coideal subalgebras are ordered with respect to inclusion and the set $\mathrm{Hom}(\mathcal{W}(M), [M])$ is ordered by the right Duflo order.

Proof. See Thm. 6.15 for the bijection between (2) and (3) and Prop. 2.3 for the bijection between (1) and (2). □

Remark 6.18. Assume that $\mathcal{W}(M)$ is standard, that is, for each $N \in \mathcal{F}_\theta(M)$ we have $a_{ij}^N = a_{ij}^M$ for all $i, j \in \mathbb{I}$. Then $\mathrm{Hom}(\mathcal{W}(M), [M])$ can be identified with the Weyl group W of $\mathfrak{g}$, see [CH09, Thm. 3.3(1)].

7. Right coideal subalgebras of $U^{\geq 0}$

In this section we are going to establish a close relationship between the maps $T_j^{R_j(M)}$, see Def. 6.5, and Lusztig's automorphisms T_α of quantized enveloping algebras. Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra and let Π be a basis of the root system with respect to a fixed Cartan subalgebra. Let W be the Weyl group of $\mathfrak{g}$ and let $(\cdot, \cdot)$ be the invariant scalar product on the real vector space generated by Π such that $(\alpha, \alpha) = 2$ for all short roots in each component. For each $\alpha \in \Pi$ let $d_\alpha = (\alpha, \alpha)/2$. Let $U = U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$ in the sense of [Jan96, Ch. 4]. More precisely, let $\Bbbk$ be a field with $\mathrm{char}(\Bbbk) \neq 2$, and if $\mathfrak{g}$ has a component of type G_2, then assume additionally that $\mathrm{char}(\Bbbk) \neq 3$. Let $q \in \Bbbk$ with $q \neq 0$ and $q^n \neq 1$ for all $n \in \mathbb{N}$. As a unital associative algebra, U is defined over $\Bbbk$ with generators $K_\alpha, K_\alpha^{-1}, E_\alpha, F_\alpha$, where $\alpha \in \Pi$, and relations given in [Jan96, 4.3]. By [Jan96, Prop. 4.11] there is a unique Hopf algebra structure on U such that

$$\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, \qquad \varepsilon(E_\alpha) = 0, \tag{7.1}$$

$$\Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha, \qquad \varepsilon(F_\alpha) = 0, \tag{7.2}$$

$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \qquad \varepsilon(K_\alpha) = 1. \tag{7.3}$$

For all $m \in \mathbb{N}$, $\alpha \in \Pi$ let $q_\alpha = q^{d_\alpha}$, $[m]_\alpha = (q_\alpha^m - q_\alpha^{-m})/(q_\alpha - q_\alpha^{-1})$, $[m]_\alpha^! = \prod_{i=1}^m [i]_\alpha$ and $E_\alpha^{(m)} = E_\alpha^m/[m]_\alpha^!$, $F_\alpha^{(m)} = F_\alpha^m/[m]_\alpha^!$. By [Jan96, 8.14] there exist unique algebra automorphisms T_α, $\alpha \in \Pi$ of U such that

$$T_\alpha(K_\alpha) = K_\alpha^{-1}, \qquad T_\alpha(K_\beta) = K_\beta K_\alpha^{-a_{\alpha\beta}}, \tag{7.4}$$

$$T_\alpha(E_\alpha) = -F_\alpha K_\alpha, \qquad T_\alpha(F_\alpha) = -K_\alpha^{-1} E_\alpha, \tag{7.5}$$

$$T_\alpha(E_\beta) = \mathrm{ad}(E_\alpha^{(-a_{\alpha\beta})}) E_\beta, \qquad T_\alpha(F_\beta) = \sum_{i=0}^{-a_{\alpha\beta}} (-q_\alpha)^i F_\alpha^{(i)} F_\beta F_\alpha^{(-a_{\alpha\beta} - i)}, \tag{7.6}$$

$\alpha \neq \beta$, where ad denotes the usual left adjoint action of U on itself.

As in [Jan96, 4.6, 4.22], let U^+ and $U^{\geq 0}$ be the subalgebras of U generated by the sets $\{E_\alpha \,|\, \alpha \in \Pi\}$ and $\{K_\alpha, K_\alpha^{-1}, E_\alpha \,|\, \alpha \in \Pi\}$, respectively.

Recall that $U^+ \in {}_{U^0}^{U^0}\mathcal{YD}$ via the left action $\mathrm{ad}|_{U^0}$ and left coaction

$$\delta(E_{\alpha_1} \cdots E_{\alpha_k}) = K_{\alpha_1} \cdots K_{\alpha_k} \otimes E_{\alpha_1} \cdots E_{\alpha_k}, \quad k \in \mathbb{N}_0,\ \alpha_1, \dots, \alpha_k \in \Pi.$$

Identify now Π with $\mathbb{I}=\{1,\dots,\theta\}$, where $\theta=\#\Pi$ is the rank of $\mathfrak{g}$. Let $\alpha\in\Pi$. Following the notation in Sect. 4 we obtain that $M=(\Bbbk E_\beta)_{\beta\in\Pi}\in\mathcal{F}_\theta$ and that $R_\alpha(M)=(R_\alpha(M)_\beta)_{\beta\in\Pi}\in\mathcal{F}_\theta$, where

$$R_\alpha(M)_\beta=\begin{cases}\Bbbk\,\mathrm{ad}(E_\alpha^{(-a_{\alpha\beta})})E_\beta & \text{if } \beta\neq\alpha,\\ (\Bbbk E_\alpha)^* & \text{if } \beta=\alpha.\end{cases} \tag{7.7}$$

Let $\vartheta_\alpha: M_1\oplus\cdots\oplus M_\theta\to R_\alpha(M)_1\oplus\cdots\oplus R_\alpha(M)_\theta$,

$$\vartheta_\alpha(E_\beta)=\begin{cases}\mathrm{ad}(E_\alpha^{(-a_{\alpha\beta})})E_\beta & \text{if } \beta\neq\alpha,\\ (q_\alpha^{-3}-q_\alpha^{-1})^{-1}E_\alpha^* & \text{if } \beta=\alpha\end{cases} \tag{7.8}$$

for all $\beta\in\Pi$, where $E_\alpha^*\in(\Bbbk E_\alpha)^*$ such that $E_\alpha^*(E_\alpha)=1$. Note that

$$\delta(\vartheta_\alpha(E_\beta))=K_\beta K_\alpha^{-a_{\alpha\beta}}\otimes\vartheta_\alpha(E_\beta)\quad\text{for all } \alpha,\beta\in\Pi. \tag{7.9}$$

In particular, $[M]\neq[R_\alpha(M)]$ in $\mathcal{X}_\theta$. Nevertheless ϑ_α is an isomorphism of braided vector spaces. Indeed, the braiding c satisfies

$$\begin{aligned} c(E_\beta\otimes E_\gamma)=&\mathrm{ad}(K_\beta)E_\gamma\otimes E_\beta=q^{(\beta,\gamma)}E_\gamma\otimes E_\beta,\\ c(\vartheta_\alpha(E_\beta)\otimes\vartheta_\alpha(E_\gamma))=&\mathrm{ad}(K_\beta K_\alpha^{-a_{\alpha\beta}})\vartheta_\alpha(E_\gamma)\otimes\vartheta_\alpha(E_\beta)\\ =&q^{(\beta-a_{\alpha\beta}\alpha,\gamma-a_{\alpha\gamma}\alpha)}\vartheta_\alpha(E_\gamma)\otimes\vartheta_\alpha(E_\beta)=q^{(\beta,\gamma)}\vartheta_\alpha(E_\gamma)\otimes\vartheta_\alpha(E_\beta)\end{aligned}$$

for all $\beta,\gamma\in\Pi$ because of the W-invariance of $(\cdot,\cdot)$. Hence $\mathcal{B}(\vartheta_\alpha):\mathcal{B}(M)\to\mathcal{B}(R_\alpha(M))$ is an isomorphism of $\mathbb{N}_0^\theta$-graded algebras and coalgebras.

Proposition 7.1. *Let $\alpha\in\Pi$. Let $\iota_\alpha: K_\alpha^M\#\mathcal{B}(M_\alpha^*)\to U$ be the linear map with $\iota_\alpha(x\#(E_\alpha^*)^m)=(q_\alpha^{-1}-q_\alpha^{-3})^m x(F_\alpha K_\alpha)^m$ for all $x\in K_\alpha^M$, $m\in\mathbb{N}_0$. Then ι_α is an injective algebra map, and the following diagram is commutative.*

$$\begin{array}{ccc}\mathcal{B}(M)=U^+ & \xrightarrow{T_\alpha} & U\\ {\scriptstyle\mathcal{B}(\vartheta_\alpha)}\big\downarrow & & \big\uparrow{\scriptstyle\iota_\alpha}\\ \mathcal{B}(R_\alpha(M)) & \xrightarrow[(\Omega_\alpha^M)^{-1}]{} & K_\alpha^M\#\mathcal{B}(M_\alpha^*)\end{array} \tag{7.10}$$

Proof. We first prove that ι_α is an algebra map. By definition, $\iota_\alpha|_{K_\alpha^M\#1}$ and $\iota_\alpha|_{1\#\mathcal{B}(M_\alpha^*)}$ are algebra maps. By Prop. 4.1 (i), the algebra K_α^M is generated by the elements $\mathrm{ad}(E_\alpha^{(n)})E_\beta$, $\beta\in\Pi\setminus\{\alpha\}$, $0\le n\le -a_{\alpha\beta}$, and the algebra $\mathcal{B}(M_\alpha^*)$ is generated by E_α^*. Further,

$$\begin{aligned}\partial^L_{E_\alpha^*}(\mathrm{ad}(E_\alpha^{(n)})E_\beta)=&E_\alpha^*\mathrm{ad}(E_\alpha^{(n)})E_\beta-(\mathrm{ad}(E_\alpha^{(n)})E_\beta)(K_\alpha^{-n}K_\beta^{-1}\cdot E_\alpha^*)\\ =&E_\alpha^*\mathrm{ad}(E_\alpha^{(n)})E_\beta-q^{(n\alpha+\beta,\alpha)}(\mathrm{ad}(E_\alpha^{(n)})E_\beta)E_\alpha^*\end{aligned}$$

for all $\beta \in \Pi \setminus \{\alpha\}$ and $0 \le n \le -a_{\alpha\beta}$ by Eq. (3.3), where $\partial^L_{E^*_\alpha}(x) = \langle E^*_\alpha, x^{(1)}\rangle x^{(2)}$ for all $x \in K^M_\alpha$. By [Jan96, 8A.5(2)] we obtain that $\partial^L_{E^*_\alpha}(E_\beta) = 0$ and

$$\partial^L_{E^*_\alpha}(\mathrm{ad}(E^{(n)}_\alpha)E_\beta) = q^{n-1}_\alpha(1 - q_\alpha^{-2(-a_{\alpha\beta}-n+1)})\mathrm{ad}(E^{(n-1)}_\alpha)E_\beta$$

for all $\beta \in \Pi \setminus \{\alpha\}$ and $1 \le n \le -a_{\alpha\beta}$. Since $F_\alpha K_\alpha E_\beta = q^{(\alpha,\beta)}E_\beta F_\alpha K_\alpha$ for all $\beta \in \Pi \setminus \{\alpha\}$, it suffices to prove that the following relations hold in U.

$$\begin{aligned} q^{n-1}_\alpha&(1 - q_\alpha^{-2(-a_{\alpha\beta}-n+1)})\mathrm{ad}(E^{(n-1)}_\alpha)E_\beta \\ &= (q^{-1}_\alpha - q^{-3}_\alpha)(F_\alpha K_\alpha \mathrm{ad}(E^{(n)}_\alpha)E_\beta - q^{(n\alpha+\beta,\alpha)}(\mathrm{ad}(E^{(n)}_\alpha)E_\beta)F_\alpha K_\alpha), \end{aligned}$$

where $\beta \in \Pi \setminus \{\alpha\}$ and $1 \le n \le -a_{\alpha\beta}$. The latter equations follow from [Jan96, 8.9 (2)].

The injectivity of ι_α follows immediately from the triangular decomposition of U. Since all maps in the diagram (7.10) are algebra maps, it is enough to check that the diagram commutes on the algebra generators E_β, $\beta \in \Pi$, of U^+. This follows directly from the definitions of the maps involved. □

Remark 7.2. Prop. 7.1 implies that the PBW basis of U^+ given in [Jan96, Thm. 8.24] coincides with the PBW basis in Cor. 6.16. Let us indicate a proof.

First observe that $\mathcal{W}(M)$ is standard, since M is a Yetter-Drinfeld module of Cartan type [AHS08, Rem. 3.27]. This means that for each $N \in \mathcal{F}_\theta(M)$ we have $a^N_{ij} = a^M_{ij}$ for all $i, j \in \mathbb{I}$. Hence $\mathrm{Hom}(\mathcal{W}(M), [M])$ can be identified with the Weyl group W of $\mathfrak{g}$, see [CH09, Thm. 3.3(1)].

Let $\alpha, \beta, \gamma \in \Pi$ such that $\ell(s_\alpha s_\beta s_\gamma) = 3$. There exists a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{B}(M) & & & & & & \\
{\scriptstyle \mathcal{B}(\vartheta_\gamma)}\downarrow & & & & & & \\
\mathcal{B}(R_\gamma(M)) & \xrightarrow{T_\gamma^{R_\gamma(M)}} & \mathcal{B}(M) & & & & \\
\downarrow & & {\scriptstyle \mathcal{B}(\vartheta_\beta)}\downarrow & & & & \\
\mathcal{B}(R_\gamma R_\beta(M)) & \xrightarrow{T_\gamma^{R_\gamma R_\beta(M)}} & \mathcal{B}(R_\beta(M)) & \xrightarrow{T_\beta^{R_\beta(M)}} & \mathcal{B}(M) & & \\
\downarrow & & \downarrow & & {\scriptstyle \mathcal{B}(\vartheta_\alpha)}\downarrow & & \\
\mathcal{B}(R_\gamma R_\beta R_\alpha(M)) & \xrightarrow{T_\gamma^{R_\gamma R_\beta R_\alpha(M)}} & \mathcal{B}(R_\beta R_\alpha(M)) & \xrightarrow{T_\beta^{R_\beta R_\alpha(M)}} & \mathcal{B}(R_\alpha(M)) & \xrightarrow{T_\alpha^{R_\alpha(M)}} & \mathcal{B}(M)
\end{array}$$

such that the unlabelled vertical arrows are isomorphisms of $\mathbb{N}^\theta_0$-graded algebras and coalgebras. The existence of such maps can be concluded by considering $\mathcal{F}_\theta$ as a category, where morphisms between $M, N \in \mathcal{F}_\theta$ are bijective maps $f : M_1 \oplus \cdots \oplus M_\theta \to N_1 \oplus \cdots \oplus N_\theta$ preserving the braiding and satisfying $f(M_i) \subset N_i$

for each $i \in \mathbb{I}$. Then $R_i : \mathcal{F}_\theta \to \mathcal{F}_\theta$ becomes a functor, and for example the vertical arrow left to $\mathcal{B}(\vartheta_\beta)$ is just $\mathcal{B}(R_\gamma(\vartheta_\beta))$. The PBW generators of $\mathcal{B}(M)$ constructed in Cor. 6.16 arise as images (at the lower right corner) of appropriate generators of the Nichols algebras in the lower line. Similarly, the PBW generators of $\mathcal{B}(M)$ arise by applying the maps T_α, T_β, ... appropriately to the algebras $\mathcal{B}(M)$ at the diagonal. Then Prop. 7.1 gives that the images obtained this way coincide.

Let w be an element of the Weyl group W, let $m = \ell(w)$, and let $s_{\alpha_1} \cdots s_{\alpha_m}$ be a reduced decomposition of w. Recall from [Jan96, 8.24] that $U^+[w] \subset U^+$ is the linear span of the products

$$E_{\beta_m}^{a_m} \cdots E_{\beta_2}^{a_2} E_{\beta_1}^{a_1}, \quad a_1, \dots, a_m \in \mathbb{N}_0, \tag{7.11}$$

where $E_{\beta_l} = T_{\alpha_1} \cdots T_{\alpha_{l-1}}(E_{\alpha_l})$ for all $1 \le l \le m$.

Theorem 7.3. *The map $\varkappa$ from W to the set of right coideal subalgebras of $U^{\ge 0}$ containing U^0, given by $\varkappa(w) = U^+[w]U^0$, is an order preserving bijection.*

Proof. Subalgebras of $U^{\ge 0}$ containing U^0 are $\mathbb{N}_0^\theta$-graded by the non-degeneracy of $(\cdot,\cdot)$ and since q is not a root of 1. Thus the claim is a special case of Cor. 6.17, see also Rem. 6.18 for the interpretation of W and Rem. 7.2 for the equality of the PBW generators in (7.11) and in Cor. 6.16. □

Remark 7.4. In view of Cor. 6.17, the claim of Thm. 7.3 holds also for multiparameter deformations of $\mathfrak{g}$ if q_α is not a root of 1 for all $\alpha \in \Pi$. Similarly, if q_α is a root of 1 for all $\alpha \in \Pi$, then the claim of Thm. 7.3 holds for the (multiparameter version of) small quantum groups, if we restrict ourselves to $\mathbb{N}_0^\theta$-graded right coideal subalgebras.

References

[AFGV09a] N. Andruskiewitsch, F. Fantino, M. Graña, and V. Vendramin, *Pointed Hopf algebras associated with the symmetric groups ii*, Preprint arXiv:0904.3978 (2009).

[AFGV09b] ———, *Pointed Hopf algebras over some sporadic simple groups*, Preprint arXiv:0906.1352 (2009).

[AHS08] N. Andruskiewitsch, I. Heckenberger, and H.-J. Schneider, *The Nichols algebra of a semisimple Yetter-Drinfeld module*, Preprint arXiv:0803.2430 (2008), 52 pp.

[AJS94] H.H. Andersen, J. Jantzen, and W. Soergel, *Representations of quantum groups at p-th roots of unity and of semisimple groups in characteristic p: Independence of p*, Astérisque **220** (1994), 321 pp.

[AS98] N. Andruskiewitsch and H.-J. Schneider, *Lifting of quantum linear spaces and pointed Hopf algebras of order p^3*, J. Algebra **209** (1998), 658–691.

[AS02] ———, *Pointed Hopf algebras*, New Directions in Hopf Algebras, MSRI Publications, vol. 43, Cambridge University Press, 2002.

[AS05] ———, *On the classification of finite-dimensional pointed Hopf algebras*, Accepted for publication in Ann. Math., Preprint math.QA/0502157 (2005), 43 pp.

[BB05] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Text in Mathematics, vol. 231, Springer, 2005.

[CH09] M. Cuntz and I. Heckenberger, *Weyl groupoids with at most three objects*, J. Pure Appl. Algebra **213** (2009), 1112–1128.

[CKP95] C. De Concini, V. Kac, and C. Procesi, *Some quantum analogues of solvable Lie groups*, Proceedings of the International Colloquium on Geometry and Analysis, Tata Institute, Bombay (1992), Oxford Univ. Press, 1995, pp. 41–66.

[CP92] C. De Concini and C. Procesi, *Quantum groups*, Lecture Notes in Mathematics, vol. 1565, pp. 31–140, Springer-Verlag, 1992.

[Gor00] M. Gorelik, *The prime and the primitive spectra of a quantum Bruhat cell translate*, J. Algebra **227** (2000), 211–253.

[Hec09] I. Heckenberger, *Classification of arithmetic root systems*, Adv. Math. **220** (2009), 59–124.

[HS08] I. Heckenberger and H.-J. Schneider, *Root systems and Weyl groupoids for Nichols algebras*, Preprint arXiv:0807.0691 (2008), 40 pp.

[HY08a] I. Heckenberger and H. Yamane, *Drinfel'd doubles and Shapovalov determinants*, Preprint arXiv:0810.1621v1 (2008), 50 pp.

[HY08b] ———, *A generalization of Coxeter groups, root systems, and Matsumoto's theorem*, Math. Z. **259** (2008), 255–276.

[Jan96] J.C. Jantzen, *Lectures on quantum groups*, Graduate studies in mathematics, vol. 6, Amer. Math. Soc., Providence, RI, 1996.

[Jos95] A. Joseph, *Quantum groups and their primitive ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1995.

[Kac90] V.G. Kac, *Infinite dimensional Lie algebras*, Cambridge Univ. Press, 1990.

[Kha08] V. Kharchenko, *PBW bases of coideal subalgebras and a freeness theorem*, Trans. Amer. Math. Soc. **360** (2008), no. 10, 5121–5143.

[Kha09] ———, *Right coideal subalgebras in* $U_q^+(\mathfrak{so}_{2n+1})$, Preprint arXiv:0908.4235 (2009), 53 pp.

[KS08] V. Kharchenko and A.V.L. Sagahon, *Right coideal subalgebras in* $U_q(\mathfrak{sl}_{n+1})$, J. Algebra **319** (2008), 2571–2625.

[LS90] S.Z. Levendorskii and Ya.S. Soibelman, *Some applications of the quantum weyl groups*, J. Geom. Phys. **7** (1990), 241–254.

[LS91] ———, *Algebras of functions on compact quantum groups, Schubert cells and quantum tori*, Comm. Math. Phys. **139** (1991), 141–170.

[Lus93] G. Lusztig, *Introduction to quantum groups*, Birkhäuser, Boston, MA, 1993.

[Mel04] A. Melnikov, *On orbital variety closures in* $\mathfrak{sl}_n$. *I. Induced Duflo order*, J. Algebra **271** (2004), 179–233.

[Mon93] S. Montgomery, *Hopf algebras and their actions on rings*, Amer. Math. Soc., Providence, RI, 1993.

[Pog09] B. Pogorelsky, *Right coideal subalgebras of the quantum Borel subalgebra of type* G_2, J. Algebra **322** (2009), 2335–2354.

[Ros98] M. Rosso, *Quantum groups and quantum shuffles*, Invent. Math. **133** (1998), 399–416.

[Yak09] M. Yakimov, *Invariant prime ideals in quantizations of nilpotent Lie algebras*, Preprint arXiv:0905.0852 (2009), 25 pp.

István Heckenberger, Mathematisches Institut, Universität Köln, Weyertal 86-90, D-50931 Köln, Germany

E-mail address: i.heckenberger@googlemail.com

Hans-Jürgen Schneider, Mathematisches Institut, Universität München, Theresienstr. 39, D-80333 Munich, Germany

E-mail address: Hans-Juergen.Schneider@mathematik.uni-muenchen.de