ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 6, Number 3, Summer 1976

RIGHT-ORDERABLE DECK TRANSFORMATION GROUPS

F. THOMAS FARRELL

0. **Introduction.** Let $p : E \to B$ be a regular covering space such that E is path connected, and B is a Hausdorff, paracompact space with a countable fundamental group. Also let $\mathbf{R}$ denote the real line, and $q : B \times \mathbf{R} \to B$ be projection onto the first factor.

QUESTION. Does there exist an embedding $f: E \to B \times \mathbf{R}$ such that the composite of f with q is p?

We show that the answer to this question is yes, if and only if $\pi_1 B/p_\# \pi_1 E$ is a right-orderable group.

In addition, if B happens to be a manifold and $\pi_1 B/p_\# \pi_1 E$ is right orderable, then we show that $B \times \mathbf{R}$ can be foliated so that at least one of its leaves is a one-to-one continuous image of E, and the remaining leaves are one-to-one continuous images of intermediate covering spaces of B.

Rubin [10] had previously answered an important case of this Question. Namely he considered the universal cover of any space homotopically equivalent to a countable wedge of circles. Rubin's covering space result played a key role in the proof by R. D. Edwards and R. T. Miller [3] that cell-like closed-0-dimensional decompositions of R^3 are R^4 factors. Also Edwards and Miller extended Rubin's result to answer the above Question when $\pi_1 B/p_\# \pi_1 E$ is a countable free group.

1. **Preliminary facts about right-orderable groups.**

1.1. DEFINITION. A right-ordered group is a pair $(G, >)$ where G is a group, and $>$ is a total order on G, such that for all x, y, and z in G, $x > y$ implies that $xz > yz$. A group G is right-orderable, if there exists an order $>$ such that $(G, >)$ is a right-ordered group.

The following basic facts about right-orderable groups can be found in [1] and [4].

1.2. *Right-orderable groups are torsion-free.*

1.3. *Any free group is right-orderable. Also any free abelian group is right-orderable.*

Received by the editors on August 23, 1974, and in revised form on May 1, 1975.

1.4. *Any extension of a right-orderable group by a right-orderable group is right-orderable.*

1.5. Example. By 1.3 and 1.4, the fundamental group of any closed 2-dimensional manifold, other than the sphere and the projective plane, is a right-orderable group.

We will need the next result in § 2.

1.6. Lemma. *If S is a countable, totally ordered set, then there is an order-preserving injection $f: S \to \mathbf{R}$, such that the image of f is a discrete subspace of* $\mathbf{R}$.

Proof. By adjoining two extra elements $\pm\infty$ to S, we can form a totally ordered set S', with maximal element $+\infty$ and minimal element $-\infty$, into which S order-preservingly injects. Hence we may as well assume that S has both a maximal and a minimal element.

Let $x_0, x_1, x_2, \cdots$ be an enumeration of the elements of S, such that x_0 is its minimal element, and x_1 is its maximal element; let $S_n = \{x_i \mid i \leqq n\}$. For each integer $n \geqq 2$, let x_n^- be the largest element in S_n which is smaller than x_n, and let x_n^+ be the smallest element in S_n which is larger than x_n.

Denote the Cantor middle-third set by C. Then $[0, 1] - C$ is the disjoint union of a collection $\mathcal{I}$ of open intervals. If I is in $\mathcal{I}$, we denote its length by $|I|$ and its midpoint by b_I. For each integer $n \geqq 2$, let

$$A_n = \{I \in \mathcal{I} \mid |I| = 3^{1-n}\} \text{ and } B_n = \{b_I \mid I \in A_n\}.$$

In addition, define

$$B_0 = \{-2\}, B_1 = \{2\}, \text{and } B = \bigcup_{n \geqq 0} B_n.$$

Note that B is a discrete subset of $\mathbf{R}$. Also notice that if $n > 1$, and

$$x, y \in \bigcup_{n > i} B_i \text{ with } x > y,$$

then there exists an element z in B_n such that $x > z > y$.

We now inductively define an order-preserving function $f: S \to B$ such that $f(x_n) \in B_n$. Start by putting $f(x_0) = -2$ and $f(x_1) = 2$. If $f(x_0), f(x_1), \cdots, f(x_{n-1})$ have already been defined, then let $f(x_n)$ be the smallest element b in B_n such that $f(x_n^+) > b > f(x_n^-)$.

The remainder of this section will be used only in § 3.

1.7. Definition. An order $>$ on a set S is said to be fine if, for each pair of points $x > y$, there exists a third point z such that $x > z > y$, and S contains neither a first nor a last point.

1.8. Lemma. *If S is a countable set and* $>$ *is a fine total order on S, then there exists an order-preserving injection* $f: S \to \mathbf{R}$ *whose image is dense in* $\mathbf{R}$.

The proof of this fact is left to the reader. It is similar to and easier than the proof of Lemma 1.6.

1.9. Lemma. *If* $(G, >)$ *is a countable, right-ordered group, then there exists an order-preserving monomorphism* $f: G \to H$ *where* $(H, >)$ *is a countable right-ordered group and* $>$ *is a fine order.*

Proof. Let $\mathbf{Q}$ be the additive group of rational numbers, then H can be chosen to be the direct sum $G \oplus \mathbf{Q}$ lexicographically ordered, i.e., $(a, b) > (c, d)$ if and only if either $a > c$, or $a = c$ and $b > d$. And we can take $f(x)$ to be $(x, 0)$.

In the next lemma, G has the discrete topology.

1.10. Lemma. *If G is a countable, right-orderable group, then* $\mathbf{R}$ has *a right G-space structure with at least one of its isotropy subgroups trivial.*

Proof. Let $>$ be an order on G so that $(G, >)$ is right-ordered. By Lemma 1.9, it suffices to consider the case when $>$ is fine. Also, by Lemma 1.8, we can identify G with a dense subset of $\mathbf{R}$. We proceed to define, for each x in G, a homeomorphism $f(\ ,x): \mathbf{R} \to \mathbf{R}$. For r in G, define $f(r, x)$ to equal rx. Since $f(\ ,x): G \to G$ is an order-preserving bijection, and G is dense in $\mathbf{R}$, $f(\ ,x)$ has a unique extension to a homeomorphism of $\mathbf{R}$. And it is easy to check that $f: \mathbf{R} \times G \to \mathbf{R}$ is a G-space.

2. **The main result.** We begin by fixing some notation and assumptions to be used throughout this section. Let E be a path connected space with base point e_0, and $p: E \to B$ a regular covering space; i.e., a principal bundle with discrete structure group. (See [11], page 70, for this definition.) Assume that B is a Hausdorff paracompact space with a countable fundamental group and base point $b_0 = p(e_0)$. Use G to denote $\pi_1(B, b_0)/p_\# \pi_1(E, e_0)$. Then we identify G with the group of deck transformations as follows. Let T be a deck transformation and choose a path α from e_0 to $T(e_0)$. Then p composed with α is some closed curve γ in B based at b_0. The map which sends T to the equivalence class represented by γ in G is our posited isomorphism. Finally, if $x \in G$ and S is either a subset or a point of E, then xS denotes the image of S under the action of x.

2.1. LEMMA. *If there exists a continuous function* $h : E \to \mathbf{R}$ *such that the map* $f: E \to B \times \mathbf{R}$ *defined by* $f(a) = (p(a), h(a))$ *is an injection, then G is right-orderable. If, in addition, the image of f is a closed subset of* $B \times \mathbf{R}$, *then G is either trivial or infinite cyclic.*

PROOF. Define a total order on G as follows: $x > y$, if and only if $h(xe_0) > h(ye_0)$. We proceed, via proof by contradiction, to show that $(G, >)$ is a right-ordered group. Thus assume that x, y, and z are elements in G such that $x > y$, but $yz > xz$.

Let γ be a loop in B based at b_0 whose equivalence class in $\pi_1 (B, b_0)/p_{\#} \pi_1 (E, e_0)$ is z. Lift γ to paths γ_1 and γ_2 in E such that $\gamma_1(0)$ is xe_0 and $\gamma_2(0)$ is ye_0; then $\gamma_1(1)$ is xze_0 and $\gamma_2(1)$ is yze_0.

Consider the function $\ell : [0,1] \to R$ defined by $\ell(t) = h\gamma_2(t) - h\gamma_1(t)$. Since $\ell(1) > 0 > \ell(0)$, there exists a real number t_0 such that $\ell(t_0)$ is zero. Therefore $f\gamma_1(t_0)$ equals $f\gamma_2(t_0)$; hence, $\gamma_1(t_0)$ equals $\gamma_2(t_0)$. But two liftings of γ which agree at one point must agree everywhere. This implies that x equals y, which is the desired contradiction.

Now we continue under the added assumption that the image of f is closed. Therefore $\varphi : G \to \mathbf{R}$ defined by $\varphi(x) = h(xe_0)$ is an order-preserving bijection onto a closed subset S of $\mathbf{R}$. Hence, either $>$ is fine, or the positive elements of G form a well-ordered set. The second possibility can occur only when G is either trivial or infinite cyclic. On the other hand, the first possibility implies that S is perfect. And S cannot be perfect since it is a countable set.

2.2 LEMMA. *If B is a locally finite simplicial complex and G is a right-orderable group, then there exists a continuous function* $h : E \to \mathbf{R}$ *such that the map* $f: E \to B \times \mathbf{R}$ *defined by* $f(a) = (p(a), h(a))$ *is an embedding.*

PROOF. Put on E the simplicial structure induced from the one of B via p, i.e., the simplexes are the liftings to E of the simplexes in B, and p becomes a simplicial map. For each vertex v of B, choose a point in $p^{-1}(v)$ and denote it by v'.

Let $>$ be an order on G such that $(G, >)$ is right-ordered. Then, by Lemma 1.6, there is an order-preserving injection $\varphi : G \to \mathbf{R}$ whose image is a discrete subset of $\mathbf{R}$.

We define h on the vertices of E as follows. For each vertex v of B and each element x in G, let $h(xv')$ be $\varphi(x)$. Then we linearly extend h to the rest of E. To be specific, consider the barycentric representation of a point c in E, i.e.,

$$c = t_0x_0v_0' + t_1x_1v_1' + \cdots + t_nx_nv_n',$$

where $t_i \in [0, 1]$, $x_i \in G$, and the v_i are vertices in B, such that $t_0 + t_1 + \cdots + t_n = 1$ and $x_0 v_0', x_1 v_1', \cdots, x_n v_n'$ *are the vertices of* a simplex in E containing c. Then define $h(c)$ to equal

$$t_0\varphi(x_0) + t_1\varphi(x_1) + \cdots + t_n\varphi(x_n).$$

We will show that f is an injection. (And leave the reader to check that f is an embedding.) To do this, suppose that c and d are distinct elements in E such that $p(c) = p(d)$. And let σ be a simplex of B containing $p(c)$. Then denote by σ_1 and σ_2 the simplexes in E such that $c \in \sigma_1, d \in \sigma_2$, and $p(\sigma_1) = p(\sigma_2) = \sigma$. Thus there exists an element x in G such that $x\sigma_1 = \sigma_2$ and $x^{-1}\sigma_2 = \sigma_1$. Let e denote the identity element of G, then either $x > e$ or $x^{-1} > e$; hence, by symmetry, we may assume that $x > e$.

Let $v_0, v_1, \cdots, v_n$ be the vertices of σ, then there exists elements x_i in G such that the points $x_i v_i'$ are the verticies of σ_1; since $x\sigma_1 = \sigma_2$, the vertices of σ_2 are the points $xx_i v_i'$ where $i = 0, 1 \cdots, n$. Thus c can be written in barycentric co-ordinates as

$$c = t_0 x_0 v_0' + t_1 x_1 v_1' + \cdots + t_n x_n v_n',$$

where each $t_i \in [0, 1]$ and $t_0 + t_1 + \cdots + t_n = 1$; consequently,

$$d = t_0 x x_0 v_0' + t_1 x x_1 v_1' + \cdots + t_n x x_n v_n',$$

since $p(c) = p(d)$. Therefore,

$$h(c) = t_0\varphi(x_0) + t_1\varphi(x_1) + \cdots + t_n\varphi(x_n),$$

while

$$h(d) = t_0\varphi(xx_0) + t_1\varphi(xx_1) + \cdots + t_n\varphi(xx_n).$$

But $x > e$ implies that $xx_i > x_i$, and $\varphi(xx_i) > \varphi(x_i)$ for $i = 0, \cdots, n$; hence $h(d) > h(c)$. And this shows that f is an injection.

2.3. Theorem. *There exists a continuous function* $h : E \to \mathbf{R}$ *such that the map* $f : E \to B \times \mathbf{R}$ *defined by* $f(a) = (p(a), h(a))$ *is an embedding, if and only if G is a right-orderable group.*

Proof. There exists a connected, locally finite simplicial complex X whose universal covering space $p' : X' \to X$ classifies principal G-bundles over Hausdorff, paracompact base spaces. (To see this, use [8, Th. 5.1] together with [2, Th. 7.5] and [9, Th. 1].) Thus there exist continuous functions $\ell : B \to X$ and $\ell' : E \to X'$, such that $p'\ell' = \ell p$, and such that the function $k : E \to B \times X'$ defined by $k(a) = (p(a), \ell'(a))$ is an embedding onto a closed subset of $B \times X'$.

Assume that G is right-orderable, then Lemma 2.2 is applicable to $p' : X' \to X$. Thus we obtain a continuous function $h' : X' \to \mathbf{R}$ such that the map $f' : X' \to X \times \mathbf{R}$ defined by $f'(a) = (p'(a), h'(a))$ is an embedding. Let h be the composite of $\mathfrak{L}'$ with h', then it is easily verified that f is an embedding.

The other half of Theorem 2.3 is an immediate consequence of Lemma 2.1.

2.4. Corollary. *There exists a continuous function $h : E \to \mathbf{R}$, such that the map $f: E \to B \times \mathbf{R}$ defined by $f(a) = (p(a),\ h(a))$ is a homeomorphism onto a closed subset of $B \times \mathbf{R}$, if and only if G is either trivial or infinite cyclic.*

Proof. If G is trivial, then h can be chosen to be identically zero. If G is infinite cyclic, then the space X used in the proof of Theorem 2.3 can be taken to be the circle S^1. In which case, X' is $\mathbf{R}$, and we can choose h to be $\mathfrak{L}'$.

The other half of Corollary 2.4 is a consequence of Lemma 2.1.

3. **A foliation of $M \times \mathbf{R}$.** We recall the definition of a codimension one foliation from [7]. (Lawson's paper is a good general reference on foliations.)

3.1. Definition. By a topological codimension one foliation of an m-dimension manifold W we mean a decomposition of W into a union of disjoint connected subsets $\{\mathcal{L}_i\}_{i \in I}$, called the leaves of the foliation, with the following property: Every point in W has a neighborhood U and a system of local coordinates $x = (x^1, \cdots, x^m) : U \to R^m$ such that for each leaf $\mathcal{L}_i$, each component of $U \cap \mathcal{L}_i$ is described by an equation of the form $x^m =$ constant.

Let $p : E \to M$ be a regular covering space, where M is a manifold, E is connected, and $G = \pi_1 M/p\#\pi_1 E$ is right-orderable. Also let $q : M \times \mathbf{R} \to M$ denote projection onto the first factor.

3.2. Theorem. *There is a topological codimension one folidation of $M \times \mathbf{R}$ whose leaves $\mathcal{L}_i$ are indexed by some set I such that, to each $i \in I$, there corresponds an intermediate covering space $p_i : E_i \to M$, and a continuous bijection $f_i : E_i \to \mathcal{L}_i$ with $p_i = qf_i$; furthermore, there is at least one index i with $p_i : E_i \to M$ equal to $p : E \to M$.*

Proof. Put on $\mathbf{R}$ the right G-space structure posited in Lemma 1.10. (Since M is a connected manifold, G is countable; hence, Lemma 1.10 is applicable.) Then form the bundle $p' : E' \to M$ with fibre $\mathbf{R}$ associated to the principal G-bundle $p : E \to M$. Note that E' is the quotient space of $\mathbf{R} \times E$ via the identifications $(rx, a) = (r, xa)$, where

$r \in \mathbf{R}$, $x \in G$, and $a \in E$. (Here we have reversed, for convenience, the customary procedure in which the group of a principal bundle acts on the right side of its total space and on the left side of its associated fibre. See Chapter 4 of [6] for basic material on principal bundles.) It is easily seen that E' has a foliation possessing the properties described in Theorem 3.2. (The leaves of this foliation are in 1-1 correspondence with the orbits of the action of G on $\mathbf{R}$.)

Let Top $\mathbf{R}$, Top$(0, 1)$, and Top$[0, 1]$ denote the order-preserving homeomorphisms of $\mathbf{R}$, $(0, 1)$, and $[0, 1]$ respectively. Put on Top$[0, 1]$ the topology of uniform convergence and topologize Top$(0, 1)$ by the natural identification of Top$(0, 1)$ with Top$[0, 1]$. Fix a homeomorphism $f: \mathbf{R} \rightarrow (0, 1)$; identify Top $\mathbf{R}$ to Top$(0, 1)$ via conjugation with f; and thus induce a topology on Top $\mathbf{R}$. (This topology is independent of f.) Since G acts on $\mathbf{R}$ via elements from Top $\mathbf{R}$, we can *enlarge* the structure group of $p' : E' \rightarrow M$ from G to Top $\mathbf{R}$. But by Theorem 1.1.1 of [5] Top $\mathbf{R}$ is contractible, hence $p' : E' \rightarrow M$ is topologically trivial; i.e., there exists a homeomorphism $k : E' \rightarrow M \times \mathbf{R}$ with $p' = qk$. Thus the foliation of E' induces, via k, the desired foliation of $M \times \mathbf{R}$.

References

1. P. Conrad, *Right-ordered groups*, Michigan Math. J. 6(1959), 267–275.

2. A. Dold, *Partitions of unity in the theory of fibrations*, Annals of Math 78 (1963), 223–255.

3. R. D. Edwards and R. T. Miller, *Cell-like closed-0-dimensional decompositions of* $\mathbf{R}^3$ *are* $\mathbf{R}^4$ *factors*, (to appear).

4. L. Fuchs, *Teilweise geordnete algebraische Strukturen*, Vandenhoeck + Ruprecht in Göttingen, 1966.

5. M-E. Hamstrom, *Homotopy in homeomorphism spaces, Top and PL*, Bull. of Amer. Math. Soc. 80(1974), 207–230.

6. D. Husemoller, *Fibre Bundles*, McGraw-Hill, New York, 1966.

7. H. B. Lawson, *Foliations*, Bull. of Amer. Math. Soc. 80(1974), 369–418.

8. J. Milnor, *Construction of universal bundles*, II, Annals of Math. 63(1956), 430–436.

9. J. Milnor, *On spaces having the homotopy type of a CW-complex*, Trans. of Amer. Math. Soc. 90(1959), 272–280.

10. L. R. Rubin, *A general class of factors of* E^4, Trans. of Amer. Math. Soc. 166(1972), 215–224.

11. N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, Princeton, 1951.

Pennsylvania State University, University Park, Pennsylvania 16802