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Right Regular and Left Regular Elements of E-Order-Preserving Transformation Semigroups

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Abstract

Let $(X, \leq)$ be a totally ordered set, E an arbitrary equivalence on X and $T(X)$ the full transformation semigroup on X. We consider a subsemigroup of $T(X)$ defined by

$$EOP(X) = \{\alpha \in T(X) : (a,b) \in E \text{ and } a \leq b \Rightarrow (a\alpha, b\alpha) \in E \text{ and } a\alpha \leq b\alpha\}$$

which is called the *E-order-preserving transformation semigroup* on X. The purpose of this paper is to characterize when elements of $EOP(X)$ are regular, left regular and right regular.

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1 Introduction and preliminaries

For a semigroup S, an element x of S is said to be *regular* if $x = xyx$ for some $y \in S$, *left* [*right*] *regular* if $x = yx^2$ [$x = x^2y$] for some $y \in S$. If all its elements are regular we called S a *regular semigroup*.

Let $T(X)$ be the full transformation semigroup on a set X, that is, the semigroup under usual composition of all maps from X into itself. It is well known that $T(X)$ is a regular semigroup. Over the last decades, notions of regularity of subsemigroups of $T(X)$ have been widely considered see [4], [5],

[2], [1], [3] and [6]. In 2010, Ma et al. [1] have introduced a subsemigroup of $T(X)$ defined by

$$EOP(X) = \{\alpha \in T(X) : (a,b) \in E \text{ and } a \leq b \Rightarrow (a\alpha, b\alpha) \in E \text{ and } a\alpha \leq b\alpha\}$$

where $(X, \leq)$ is a totally ordered set and E is an equivalence on X. $EOP(X)$ is called the *E-order-preserving transformation semigroup* on X. They also have investigated regularity and Green's relations of $EOP(X)$ where X is a finite set.

In this paper, regularity of elements for $EOP(X)$ are discussed. This characterization is a generalization of regularity in [1]. We determine left regular and right regular elements of $EOP(X)$.

Firstly, we introduce some notations and proposition that will be used in this paper. For a set X and $\alpha \in T(X)$, we denote by $\pi(\alpha)$ the partition of X induced by α, namely,

$$\pi(\alpha) = \{y\alpha^{-1} : y \in X\alpha\}$$

and α_* the natural bijection corresponding to α from $\pi(\alpha)$ onto $X\alpha$ defined by

$$P\alpha_* = x\alpha \quad \text{for all } P \in \pi(\alpha) \text{ and } x \in P.$$

Let A be a nonempty subset of X, we denote

$$\pi_A(\alpha) = \{P \in \pi(\alpha) : P \cap A \neq \emptyset\}.$$

For each nonempty subset A of X and $\alpha \in T(X)$, we denote a partition of A induce by α,

$$\pi(A, \alpha) = \{P' \cap A : P' \in \pi_A(\alpha)\}.$$

Let π be a collection of nonempty subsets of a partially ordered set X. We define a relation $\preceq$ on π by

$$P \preceq Q \quad \text{if and only if} \quad P = Q \text{ or } x \leq y \text{ for all } x \in P, y \in Q.$$

Then $(\pi, \preceq)$ is a partially ordered set.

Throughout of this paper, we assume that $(X, \leq)$ is a totally ordered set and E an equivalence on X.

Proposition 1.1. *For $A \in X/E$ and $\alpha \in EOP(X)$, $(\pi(A, \alpha), \preceq)$ is a totally ordered set.*

Proof. It is clearly seen that $\pi(A,\alpha)$ is a partition of A and $(\pi(A,\alpha),\preceq)$ is a partially ordered set. To show that $(\pi(A,\alpha),\preceq)$ is a totally ordered set, let $P,Q\in\pi(A,\alpha)$ be such that $Q\not\preceq P$. Then there exist $p\in P$ and $q\in Q$ such that $q\not\leq p$. Since $(X,\leq)$ is a totally ordered set, $p<q$. By the definition of $\pi(A,\alpha)$, we have that $P=P'\cap A$ and $Q=Q'\cap A$ for some $P',Q'\in\pi_A(\alpha)$. Since $P\neq Q$, $P'\neq Q'$ and hence $P'\alpha_*\neq Q'\alpha_*$. Claim that $P\preceq Q$. Suppose that $y<x$ for some $x\in P$ and $y\in Q$. We note here that $p,q,x,y\in A$. Since $p\leq q$, $(p,q)\in E$ and $\alpha\in EOP(X)$, we deduce that $p\alpha\leq q\alpha$. Since $P'\alpha_*\neq Q'\alpha_*$, $p\in P'$ and $q\in Q'$, $P'\alpha_*=p\alpha<q\alpha=Q'\alpha_*$. Similarly, we have that $Q'\alpha_*=y\alpha\leq x\alpha=P'\alpha_*$. This implies that

$$P'\alpha_*<Q'\alpha_*\leq P'\alpha_*$$

which is a contradiction. Hence $x\leq y$ for all $x\in P$ and $y\in Q$. So we have the claim. Therefore $(\pi(A,\alpha),\preceq)$ is a totally ordered set as desired. □

The following examples show that $EOP(X)$ need not to be regular and there exists an element of $EOP(X)$ which is not left regular and right regular.

Example 1.2. *Let $A_1=\{3(k-1)+1:k\in\mathbb{Z}^+\}$, $A_2=\{3(k-1)+2:k\in\mathbb{Z}^+\}$ and $A_3=\{3k:k\in\mathbb{Z}^+\}$. Define $E=\bigcup_{i=1}^{3}A_i\times A_i$. It is clearly that E is an equivalence relation on $\mathbb{Z}^+$ and $\mathbb{Z}^+/E=\{A_1,A_2,A_3\}$. Define $\alpha:\mathbb{Z}^+\to\mathbb{Z}^+$ by*

$$x\alpha=\begin{cases}6k-5+r & \text{if } r=1,2;\\ 6k & \text{if } r=3,\end{cases}$$

where $x=3(k-1)+r$ for some $k,r\in\mathbb{Z}^+$ and $1\leq r\leq 3$. It is easy to verify that $\alpha\in EOP(\mathbb{Z}^+)$. Assume that $\alpha=\alpha\beta\alpha$ for some $\beta\in EOP(\mathbb{Z}^+)$. Since

$$2\alpha=2\alpha\beta\alpha=3\beta\alpha \text{ and } 3\alpha=3\alpha\beta\alpha=6\beta\alpha$$

and α is an injective, we deduce that $2=3\beta$ and $3=6\beta$. Because $3\leq 6$ and $(3,6)\in E$, we have $(2,3)=(3\beta,6\beta)\in E$ which is a contradiction. Therefore α is not regular of $EOP(\mathbb{Z}^+)$.

Suppose that $\alpha=\alpha^2\beta$ for some $\beta\in EOP(\mathbb{Z}^+)$. Since $\beta\in EOP(\mathbb{Z}^+)$ and $(3,6)\in E$ and $3\leq 6$, we have that

$$2=1\alpha=1\alpha^2\beta=2\alpha\beta=3\beta \text{ and } 3=2\alpha=2\alpha^2\beta=3\alpha\beta=6\beta.$$

It would follow that $(2,3)=(3\beta,6\beta)\notin E$ which is impossible. This proves that α is not right regular of $EOP(\mathbb{Z}^+)$.

Next, suppose that $\alpha=\beta\alpha^2$ for some $\beta\in EOP(\mathbb{Z}^+)$. Since α is injective and $1\alpha=1\beta\alpha^2=(1\beta\alpha)\alpha$, $1=1\beta\alpha$, we conclude that $1\in\mathbb{Z}^+\alpha$. This contradiction shows that α is not left regular of $EOP(\mathbb{Z}^+)$.

2 Main results

The aims of this section is to give a necessary and sufficient condition under which an element of $EOP(X)$ is regular, right regular and left regular, respectively.

Theorem 2.1. *Let $\alpha \in EOP(X)$. Then α is regular if and only if for all $A \in X/E$, there exists a partition π_A of A such that $(\pi_A, \preceq)$ is a totally ordered set and for each $P \in \pi_A$, there exists $x_P \in X$ corresponding to P such that $P \cap X\alpha \subseteq \{x_P\alpha\}$ and $P \preceq Q$ implies $x_P \leq x_Q$ and $(x_P, x_Q) \in E$ for $P, Q \in \pi_A$.*

Proof. Suppose that $\alpha = \alpha\beta\alpha$ for some $\beta \in EOP(X)$. Let $A \in X/E$. We note by Proposition 1.1 that

$$\pi(A, \beta) = \{P' \cap A : P' \in \pi_A(\beta)\}$$

is a totally ordered set. By the definition of $\pi(A, \beta)$, we have $\pi(A, \beta)$ is a partition of A. For each $P \in \pi(A, \beta)$, there exists $P' \in \pi_A(\beta)$ such that $P = P' \cap A$. We denote $x_P = P'\beta_*$. Let $P \in \pi(A, \beta)$ be such that $P \cap X\alpha \neq \emptyset$. We have that $P = P' \cap A$ for some $P' \in \pi_A(\beta)$. For arbitrary $x \in P \cap X\alpha$, $x = x'\alpha$ for some $x' \in X$. Hence

$$x = x'\alpha = x'\alpha\beta\alpha = x\beta\alpha = P'\beta_*\alpha = x_P\alpha.$$

This means that $P \cap X\alpha = \{x_P\alpha\}$. Let $P, Q \in \pi(A, \beta)$ be such that $P \preceq Q$. Then $P = P' \cap A$ and $Q = Q' \cap A$ for some $P', Q' \in \pi_A(\beta)$. Choose $x \in P$ and $y \in Q$. If $P = Q$, then $x_P = x_Q$. Assume that $P \neq Q$. By $P \preceq Q$, we have $x \leq y$. Since $(x, y) \in E$ and $x \leq y$, $(x_P, x_Q) = (x\beta, y\beta) \in E$ and $x_P = x\beta \leq y\beta = x_Q$.

For the converse, suppose that for all $A \in X/E$, there exists a partition π_A of A such that $(\pi_A, \preceq)$ is a totally ordered set and for each $P \in \pi_A$, there exists $x_P \in X$ corresponding to P such that $P \cap X\alpha \subseteq \{x_P\alpha\}$ and $P \preceq Q$ implies $x_P \leq x_Q$ and $(x_P, x_Q) \in E$ for $P, Q \in \pi_A$. We will construct $\beta \in EOP(X)$ in the following, let $x \in X$. Since X/E is a partition of X, $x \in A$ for some $A \in X/E$. We note by assumption that $x \in P_x$ for some $P_x \in \pi_A$. Define $\beta : X \to X$ by

$$x\beta = x_{P_x} \text{ for all } x \in X.$$

Clearly, β is well-defined. Let $x, y \in X$ be such that $(x, y) \in E$ and $x \leq y$. There exists $A \in X/E$ such that $x, y \in A$. Thus $x \in P_x$ and $y \in P_y$ for some $P_x, P_y \in \pi_A$. Since $x \leq y$ and by assumption, $P_x \preceq P_y$. It follows that $(x\beta, y\beta) = (x_{P_x}, x_{P_y}) \in E$ and $x\beta = x_{P_x} \leq x_{P_y} = y\beta$. Therefore $\beta \in EOP(X)$. Finally, let $x \in X$. Then $x\alpha \in A$ for some $A \in X/E$. It follows by the definition of β that $x\alpha\beta = x_{P_{x\alpha}}$ where $x\alpha \in P_{x\alpha}$ and $P_{x\alpha} \in \pi_A$. It is clear from assumption that $x\alpha \in P_{x\alpha} \cap X\alpha = \{x_{P_{x\alpha}}\alpha\}$. Thus $x\alpha\beta\alpha = x_{P_{x\alpha}}\alpha = x\alpha$.

Hence the theorem is terribly proved. □

This leads directly to the following corollary when X is finite.

Corollary 2.2. *[1] Let X be a finite set and $\alpha \in EOP(X)$. Then α is a regular element if and only if for every $A \in X/E$, there exists $B \in X/E$ such that $X\alpha \cap A \subseteq B\alpha$.*

Proof. Suppose that α is a regular element. Let $A \in X/E$. By Theorem 2.1, there exists a partition π_A of A such that $(\pi_A, \preceq)$ is a totally ordered set and for each $P \in \pi_A$, there exists $x_P \in X$ corresponding to P such that $P \cap X\alpha \subseteq \{x_P\alpha\}$ and $P \preceq Q$ implies $x_P \leq x_Q$ and $(x_P, x_Q) \in E$ for $P, Q \in \pi_A$. Let $P \in \pi_A$. Then $x_P \in B$ for some $B \in X/E$. Claim that $x_Q \in B$ for all $Q \in \pi_A$, let $Q \in \pi_A$. Since π_A is a totally ordered set, we assume that $P \preceq Q$. It follows by assumption and $x_P \in B$ that $x_Q \in B$, so we have the claim. Since π_A is a partition of A, $\cup_{P\in\pi_A} P = A$. We see that for each $P \in \pi_A$, $P \cap X\alpha \subseteq \{x_P\alpha\} \subseteq B\alpha$. Hence

$$A \cap X\alpha = (\cup_{P\in\pi_A} P) \cap X\alpha = \cup_{P\in\pi_A}(P \cap X\alpha) \subseteq B\alpha.$$

Conversely, assume that for each $A \in X/E$, there exists $B \in X/E$ such that $X\alpha \cap A \subseteq B\alpha$. We need to show that α is regular via Theorem 2.1. Let $A \in X/E$. Since X is a finite set, we order $A \cap X\alpha = \{a_1, a_2, \ldots, a_n\}$ where $a_1 < a_2 < \ldots < a_n$ for some $n \in \mathbb{N}$. Let $P_1 = \{x \in A : x \leq a_1\}$, $P_i = \{x \in A : a_{i-1} < x \leq a_i\}$ for all $i = 2, 3, \ldots, n-1$ and $P_n = \{x \in A : a_{n-1} < x\}$. It is easy to see that $\pi_A = \{P_i : i = 1, 2, \ldots, n\}$ is a partition of A. Moreover, $P_i \cap X\alpha = \{a_i\}$ for all $i = 1, 2, \ldots, n$. By assumption, we have $A \cap X\alpha \subseteq B\alpha$, then choose $x_i \in B$ such that $x_i\alpha = a_i$ for each $i = 1, 2, \ldots, n$. Hence $P_i \cap X\alpha = \{a_i\} = \{x_i\alpha\}$ for all $i = 1, 2, \ldots, n$. To verify $(\pi_A, \preceq)$ is a totally ordered set, let $P_i, P_j \in \pi_A$ be distinct. We assume that $a_i < a_j$ from X is a totally ordered set. This implies that $i < j$. Claim that $P_i \preceq P_j$, let $x \in P_i$ and $y \in P_j$. It follows by the definition of P_i and P_j that $x \leq a_i \leq a_{j-1} < y$. Hence $(\pi_A, \preceq)$ is a totally ordered set. Finally, let $P_i, P_j \in \pi_A$ be such that $P_i \preceq P_j$. Thus $a_i \leq a_j$. Clearly, $(x_i, x_j) \in E$. Since $x_i\alpha = a_i \leq a_j = x_j\alpha$, we conclude that $x_i \leq x_j$. By Theorem 2.1, we observe that α is regular. □

Theorem 2.3. *Let $\alpha \in EOP(X)$. Then α is a right regular element in $EOP(X)$ if and only if the following conditions are satisfied:*

(i) *$\alpha|_{X\alpha}$ is an injection*

(ii) *for all $A \in X/E$, there exists a partition π_A of A such that $(\pi_A, \preceq)$ is a totally ordered set and for each $P \in \pi_A$, there exists $x_P \in X$ corresponding to P such that $P \preceq Q$ implies $x_P \leq x_Q$ and $(x_P, x_Q) \in E$ for all $P, Q \in \pi_A$ and if $P \cap X\alpha^2 \neq \emptyset$, then $x_P \in X\alpha$ and $P \cap X\alpha^2 = \{x_P\alpha\}$.*

Proof. Suppose that $\alpha = \alpha^2\beta$ for some $\beta \in EOP(X)$. To verify that $\alpha|_{X\alpha}$ is an injection, let $x, y \in X\alpha$ be such that $x\alpha = y\alpha$. Since $x, y \in X\alpha$, $x = x'\alpha$ and $y = y'\alpha$ for some $x', y' \in X$. Consider,

$$x = x'\alpha = x'\alpha^2\beta = x\alpha\beta = y\alpha\beta = y'\alpha^2\beta = y'\alpha = y.$$

We then have $\alpha|_{X\alpha}$ is an injection. Let $A \in X/E$ and note by Proposition 1.1 that $(\pi(A, \beta) \preceq)$ is a totally ordered set. It is clear that $\pi(A, \beta)$ is a partition of A. For each $P \in \pi(A, \beta)$, we let $P' \in \pi_A(\beta)$ be such that $P = P' \cap A$ and denote $x_P = P'\beta_*$. Let $P, Q \in \pi(A, \beta)$ be such that $P \preceq Q$. If $P = Q$, then $x_P = x_Q$. Assume that $P \neq Q$. We note that $P = P' \cap A$ and $Q = Q' \cap A$ for some $P', Q' \in \pi_A(\beta)$. Let $x \in P$ and $y \in Q$. Since $P \preceq Q$ and $P \neq Q$, we have $x \leq y$ and $(x, y) \in E$. It follows by $\beta \in EOP(X)$ that $(x_P, x_Q) = (x\beta, y\beta) \in E$ and $x_P = x\beta \leq y\beta = x_Q$. Finally, let $P \in \pi(A, \beta)$ be such that $P \cap X\alpha^2 \neq \emptyset$. There exists $P' \in \pi_A(\beta)$ such that $P = P' \cap A$. Let $x \in P \cap X\alpha^2$, then $x = x'\alpha^2$ for some $x' \in X$. Consider,

$$x = x'\alpha^2 = x'\alpha^2\beta\alpha = x\beta\alpha = P'\beta_*\alpha = x_P\alpha \text{ and } x_P = x\beta = x'\alpha^2\beta = x'\alpha.$$

Thus $P \cap X\alpha^2 = \{x_P\alpha\}$ and $x_P \in X\alpha$.

Conversely, suppose that (i) and (ii) are hold. Let $x \in X$. Then by X/E is a partition of X, $x \in A$ for some $A \in X/E$. There exists $P_x \in \pi_A$ such that $x \in P_x$. Define $\beta : X \to X$ by

$$x\beta = x_{P_x} \text{ for all } x \in X.$$

Clearly, β is well-defined. Let $x, y \in X$ be such that $x \leq y$ and $(x, y) \in E$. There exists $A \in X/E$ such that $x, y \in A$. Then $x \in P_x, y \in P_y$ where $P_x, P_y \in \pi_A$. Since $x \leq y$ and by assumption, $P_x \preceq P_y$. Hence $x_{P_x} \leq x_{P_y}$ and $(x_{P_x}, x_{P_y}) \in E$. So we have $\beta \in EOP(X)$. Finally, let $x \in X$. Then $x\alpha^2 \in A$ for some $A \in X/E$. By the definition of β, $x\alpha^2\beta = x_{P_{x\alpha^2}}$ where $x\alpha^2 \in P_{x\alpha^2}$ and $P_{x\alpha^2} \in \pi_A$. We note that $x\alpha^2 \in P_{x\alpha^2} \cap X\alpha^2 = \{x_{P_{x\alpha^2}}\alpha\}$. That is $x\alpha^2 = x_{P_{x\alpha^2}}\alpha$. By (i) and $x_{P_{x\alpha^2}} \in X\alpha$, we conclude that $x\alpha = x_{P_{x\alpha^2}}$. Therefore $x\alpha^2\beta = x\alpha$. This shows that α is right regular in $EOP(X)$. □

Theorem 2.4. *Let $\alpha \in EOP(X)$. Then α is left regular if and only if for every $A \in X/E$ there exists $B \in X/E$ such that for each $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$.*

Proof. Suppose that $\alpha = \beta\alpha^2$ for some $\beta \in EOP(X)$. Let $A \in X/E$ and $a \in A$. Since X/E is a partition of X, $a\beta \in B$ for some $B \in X/E$. We claim that for each $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$. Let $P \in \pi_A(\alpha)$ and $x \in P \cap A$. Since X is a totally ordered set, we assume that $a \leq x$. From $(a, x) \in E$ and $a \leq x$, we then have $(a\beta, x\beta) \in E$. Since $a\beta \in B$, we conclude that $x\beta \in B$. Consider, $P\alpha_* = x\alpha = x\beta\alpha^2 = (x\beta\alpha)\alpha$. Therefore $x\beta\alpha \in P$ and $x\beta \in B$.

Conversely, suppose that for every $A \in X/E$ there exists $B \in X/E$ such that for each $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$. Hence each $A \in X/E$, we fix $A' \in X/E$ and $x_P \in A'$ corresponding to $P \in \pi_A(\alpha)$ such that $x_P\alpha \in P$. We will construct $\beta \in EOP(X)$ in the following, let $x \in X$. Since X/E is a partition of X, $x \in A$ for some $A \in X/E$. Then there exists $P_x \in \pi_A(\alpha)$ such that $x \in P_x$. Define $\beta : X \to X$ by

$$x\beta = x_{P_x} \text{ for all } x \in X.$$

To show that $\beta \in EOP(X)$, let $x, y \in X$ be such that $(x, y) \in E$ and $x \leq y$. We then have $x, y \in A$ for some $A \in X/E$ and $x \in P_x, y \in P_y$ where $P_x, P_y \in \pi_A(\alpha)$. Clearly, $(x\beta, y\beta) = (x_{P_x}, x_{P_y}) \in E$. If $P_x = P_y$, then $x_{P_x} = x_{P_y}$. Suppose that $P_x \neq P_y$. Claim that $x_{P_x} \leq x_{P_y}$, suppose not. Since X is a totally ordered set, we have $x_{P_y} < x_{P_x}$. Since $(x_{P_x}, x_{P_y}), (x, y) \in E$, $x_{P_y} < x_{P_x}$ and $x \leq y$, we conclude that $x_{P_y}\alpha \leq x_{P_x}\alpha, x\alpha \leq y\alpha$ and $(x_{P_y}\alpha, x_{P_x}\alpha) \in E$. Then $P_x\alpha_* = x\alpha \leq y\alpha = P_y\alpha_*$. We note by $P_x \neq P_y$ that $P_x\alpha_* < P_y\alpha_*$. Similarly, we see that $(x_{P_y}\alpha, x_{P_x}\alpha) \in E$ and $x_{P_y}\alpha \leq x_{P_x}\alpha$. Hence $x_{P_y}\alpha\alpha \leq x_{P_x}\alpha\alpha$. By assumption, we have $x_{P_y}\alpha \in P_y$ and $x_{P_x}\alpha \in P_x$. It follows that

$$x_{P_x}\alpha\alpha = P_x\alpha_* < P_y\alpha_* = x_{P_y}\alpha\alpha \leq x_{P_x}\alpha\alpha.$$

This is a contradiction. Thus $x_{P_x} \leq x_{P_y}$ and then $x\beta \leq y\beta$. Therefore $\beta \in EOP(X)$. We need to verify that $\alpha = \beta\alpha^2$, let $x \in X$. Hence $x\beta\alpha^2 = x_{P_x}\alpha\alpha = P_x\alpha_* = x\alpha$, so α is a left regular element of $EOP(X)$ as required. □

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