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Research Article

Right k-Weakly Regular Γ-Semirings

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The concepts of a k-idempotent Γ-semiring, a right k-weakly regular Γ-semiring, and a right pure k-ideal of a Γ-semiring are introduced. Several characterizations of them are furnished.

1. Introduction

Γ-semiring was introduced by Rao in [1] as a generalization of a ring, a Γ-ring, and a semiring. Ideals in semirings were characterized by Ahsan in [2], Iséki in [3, 4], and Shabir and Iqbal in [5]. Properties of prime and semiprime ideals in Γ-semirings were discussed in detail by Dutta and Sardar [6]. Henriksen in [7] defined more restricted class of ideals in semirings known as k-ideals. Some more characterizations of k-ideals of semirings were studied by Sen and Adhikari in [8, 9]. k-ideal in a Γ-semiring was defined by Rao in [1] and in [6] Dutta and Sardar gave some of its properties. Author studied k-ideals and full k-ideals of Γ-semirings in [10]. The concept of a bi-ideal of a Γ-semiring was given by author in [11].

In this paper efforts are made to introduce the concepts of a k-idempotent Γ-semiring, a right k-weakly regular Γ-semiring, and a right pure k-ideal of a Γ-semiring. Discuss some characterizations of a k-idempotent Γ-semiring, a right k-weakly regular Γ-semiring, and a right pure k-ideal of a Γ-semiring.

2. Preliminaries

First we recall some definitions of the basic concepts of Γ-semirings that we need in sequel. For this we follow Dutta and Sardar [6].

Definition 1. Let S and Γ be two additive commutative semigroups. S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \rightarrow S$ denoted by $a\alpha b$, for all $a, b \in S$ and $\alpha \in \Gamma$ satisfying the following conditions:

(i) $a\alpha(b + c) = (a\alpha b) + (a\alpha c)$;

(ii) $(b + c)\alpha a = (b\alpha a) + (c\alpha a)$;

(iii) $a(\alpha + \beta)c = (a\alpha c) + (a\beta c)$;

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Definition 2. An element 0 in a Γ-semiring S is said to be an absorbing zero if $0\alpha a = 0 = a\alpha 0$, $a + 0 = 0 + a = a$, for all $a \in S$ and $\alpha \in \Gamma$.

Definition 3. A nonempty subset T of a Γ-semiring S is said to be a sub-Γ-semiring of S if $(T, +)$ is a subsemigroup of $(S, +)$ and $a\alpha b \in T$, for all $a, b \in T$ and $\alpha \in \Gamma$.

Definition 4. A nonempty subset T of a Γ-semiring S is called a left (resp., right) ideal of S if T is a subsemigroup of $(S, +)$ and $x\alpha a \in T$ (resp., $a\alpha x \in T$) for all $a \in T$, $x \in S$ and $\alpha \in \Gamma$.

Definition 5. If T is both left and right ideal of a Γ-semiring S, then T is known as an ideal of S.

Definition 6. A right ideal I of a Γ-semiring S is said to be a right k-ideal if $a \in I$ and $x \in S$ such that $a + x \in I$; then $x \in I$.

Similarly we define a left k-ideal of a Γ-semiring S. If an ideal I is both right and left k-ideal, then I is known as a k-ideal of S.

Example 7. Let N_0 denote the set of all positive integers with zero. $S = N_0$ is a semiring and with $\Gamma = S$, S forms a Γ-semiring. A subset $I = 3N_0 \setminus \{3\}$ of S is an ideal of S but not a k-ideal. Since 6 and $9 = 3 + 6 \in I$ but $3 \notin I$.

Example 8. If $S = N$ is the set of all positive integers, then (S, max., min.) is a semiring and with $\Gamma = S$, S forms a Γ-semiring. $I_n = \{1, 2, 3, \ldots, n\}$ is a k-ideal for any $n \in I$.

Definition 9. For a nonempty I of a Γ-semiring S,

$$\overline{I} = \{a \in S \mid a + x \in I, \text{ for some } x \in I\}. \quad (1)$$

$\overline{I}$ is called k-closure of I.

Now we give a definition of a bi-ideal.

Definition 10 (see [11]). A nonempty subset B of a Γ-semiring S is said to be a bi-ideal of S if B is a sub-Γ-semiring of S and $B\Gamma S\Gamma B \subseteq B$.

Example 11. Let N be the set of natural numbers and $\Gamma = 2N$. Then both N and Γ are additive commutative semigroups. An image of a mapping $N \times \Gamma \times N \to N$ is denoted by $a\alpha b$ and defined as $a\alpha b$ = product of a, α, b, for all $a, b \in N$ and $\alpha \in \Gamma$. Then N forms a Γ-semiring. $B = 4N$ is a bi-ideal of N.

Example 12. Consider a Γ-semiring $S = M_{2\times 2}(N_0)$, where N_0 denotes the set of natural numbers with zero and $\Gamma = S$. Define $A\alpha B$ = usual matrix product of A, α and B, for $A, \alpha, B \in S$.

$$P = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \mid x, y \in N_0 \right\} \quad (2)$$

is a bi-ideal of a Γ-semiring S.

Definition 13. An element 1 in a Γ-semiring S is said to be an unit element if $1\alpha a = a = a\alpha 1$, for all $a \in S$ and for all $\alpha \in \Gamma$.

Definition 14. A Γ-semiring S is said to be commutative if $a\alpha b = b\alpha a$, for all $a, b \in S$ and for all $\alpha \in \Gamma$.

Some basic properties of k-closure are given in the following lemma.

Lemma 15. *For nonempty subsets A and B of S, we have the following.*

(1) *If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.*

(2) *$\overline{A}$ is the smallest (left k-ideal, right k-ideal) k-ideal containing (left k-ideal, right k-ideal) k-ideal A of S.*

(3) *$\overline{A} = A$ if and only if A is a (left k-ideal, right k-ideal) k-ideal of S.*

(4) *$\overline{\overline{A}} = \overline{A}$, where A is a (left k-ideal, right k-ideal) k-ideal of S.*

(5) *$\overline{\overline{A}\Gamma\overline{B}} = \overline{A\Gamma B}$, where A and B are (left k-ideals, right k-ideals) k-ideals of S.*

Some results from [11] are stated which are useful for further discussion.

Result 1. For each nonempty subset X of S, the following statements hold.

(i) $S\Gamma X$ is a left ideal of S.

(ii) $X\Gamma S$ is a right ideal of S.

(iii) $S\Gamma X\Gamma S$ is an ideal of S.

Result 2. For $a \in S$, the following statements hold.

(i) $S\Gamma a$ is a left ideal of S.

(ii) $a\Gamma S$ is a right ideal of S.

(iii) $S\Gamma a\Gamma S$ is an ideal of S.

Now onwards S denotes a Γ-semiring with an absorbing zero and an unit element unless otherwise stated.

3. k-Idempotent Γ-Semiring

In this section we introduce and characterize the notion of a k-idempotent Γ-semiring.

Definition 16. A subset I of a Γ-semiring S is said to be k-idempotent if $I = \overline{I\Gamma I}$.

Definition 17. A Γ-semiring S is said to be k-idempotent if every k-ideal of S is k-idempotent.

Theorem 18. *In S the following statements are equivalent.*

(1) *S is k-idempotent.*

(2) *For any $a \in S$, $a \in \overline{S\Gamma a\Gamma S\Gamma a\Gamma S}$.*

(3) *For every $A \subseteq S$, $A \subseteq \overline{S\Gamma A\Gamma S\Gamma A\Gamma S}$.*

Proof. (1) $\Rightarrow$ (2). Suppose that S is a k-idempotent Γ-semiring. For any $a \in S$, $(a) = a\Gamma S + S\Gamma a + S\Gamma a\Gamma S + N_0 a$. Then $a \in \overline{(a)} = \overline{a\Gamma S + S\Gamma a + S\Gamma a\Gamma S + N_0 a}$. Hence by assumption $\overline{(a)} = \overline{\overline{(a)}\Gamma\overline{(a)}} = \overline{(\overline{a\Gamma S + S\Gamma a + S\Gamma a\Gamma S + N_0 a})\Gamma(\overline{a\Gamma S + S\Gamma a + S\Gamma a\Gamma S + N_0 a})}$.

Therefore $\overline{(a)} = (\overline{a\Gamma S + S\Gamma a + S\Gamma a\Gamma S + N_0 a})\Gamma(\overline{a\Gamma S + S\Gamma a + S\Gamma a\Gamma S + N_0 a})$.

Hence $\overline{(a)} \subseteq \overline{S\Gamma a\Gamma S\Gamma a\Gamma S}$. Therefore $a \in \overline{S\Gamma a\Gamma S\Gamma a\Gamma S}$.

(2) $\Rightarrow$ (3). Let $A \subseteq S$ and $a \in A$. Hence by assumption we have $a \in \overline{S\Gamma a\Gamma S\Gamma a\Gamma S}$. Therefore $a \in \overline{S\Gamma A\Gamma S\Gamma A\Gamma S}$. Thus we get $A \subseteq \overline{S\Gamma A\Gamma S\Gamma A\Gamma S}$.

(3) $\Rightarrow$ (1). Let A be any k-ideal of S. Then by assumption $A \subseteq \overline{S\Gamma A\Gamma S\Gamma A\Gamma S} \subseteq \overline{A\Gamma S\Gamma A} \subseteq \overline{A\Gamma A}$. As A is a k-ideal of S, $\overline{A\Gamma A} \subseteq A$. Therefore $\overline{A\Gamma A} = A$, which shows that S is a k-idempotent Γ-semiring. □

Definition 19. A sub-Γ-semiring I of S is a k-interior ideal of S if $\overline{S\Gamma I\Gamma S} \subseteq I$ and if $a \in I$ and $x \in S$ such that $a + x \in I$, then $x \in I$.

Theorem 20. *If S is a k-idempotent Γ-semiring, then a subset of S is a k-ideal if and only if it is a k-interior ideal.*

Proof. Let S be a k-idempotent Γ-semiring. As every k-ideal is a k-interior ideal, one part of theorem holds. Conversely, suppose a subset I of S is a k-interior ideal of S. To show I is a k-ideal of S, let $x \in I$ and $t \in S$. As S is a k-idempotent Γ-semiring, $x \in \overline{S\Gamma x\Gamma S\Gamma x\Gamma S}$ (see Theorem 18). Therefore, for any $\alpha \in \Gamma$, $x\alpha t \in \overline{S\Gamma x\Gamma S\Gamma x\Gamma S\Gamma S} \subseteq \overline{S\Gamma x\Gamma S\Gamma x\Gamma S\Gamma S} \subseteq \overline{S\Gamma I\Gamma S\Gamma I\Gamma S\Gamma S} \subseteq \overline{S\Gamma I\Gamma S} \subseteq I$. Similarly we can show that $t\alpha x \in I$. Therefore I is a k-ideal of S. □

Theorem 21. *S is k-idempotent if and only if $\overline{A\Gamma B} = A \cap B$, for any k-interior ideals A and B of S.*

Proof. Suppose a Γ-semiring S is k-idempotent. Let A and B be any two k-interior ideals of S. Then, by Theorem 20, A and B are two k-ideals of S. Hence $A\Gamma B \subseteq A$ and $\overline{A\Gamma B} \subseteq B$. Therefore $\overline{A\Gamma B} \subseteq A \cap B$. By assumption $A \cap B = \overline{(A \cap B)^2} = \overline{(A \cap B)\Gamma(A \cap B)} \subseteq \overline{A\Gamma B}$. Therefore $\overline{A\Gamma B} = A \cap B$. Conversely, let A be any k-ideal of S. As every k-ideal of S is a k-interior ideal of S, by assumption, $\overline{A\Gamma A} = A \cap A = A$. Therefore S is a k-idempotent Γ-semiring. □

4. Right k-Weakly Regular Γ-Semiring

Definition 22. A Γ-semiring S is said to be right k-weakly regular if, for any $a \in S$, $a \in \overline{(a\Gamma S)^2}$.

Theorem 23. *In S, the following statements are equivalent.*

(1) *S is right k-weakly regular.*

(2) *$\overline{R^2} = R$, for each right k-ideal R of S.*

(3) *$R \cap I = \overline{R\Gamma I}$, for a right k-ideal R and a k-ideal I of S.*

Proof. (1) ⇒ (2). For any right k-ideal R of S, $R^2 = R\Gamma R \subseteq R\Gamma S \subseteq R$. Hence $\overline{R^2} = \overline{R\Gamma R} \subseteq R$. For the reverse inclusion, let $a \in R$. As S is right k-weakly regular, $a \in \overline{(a\Gamma S)^2} = \overline{(a\Gamma S)\Gamma(a\Gamma S)} \subseteq \overline{(R\Gamma S)\Gamma(R\Gamma S)} \subseteq \overline{R\Gamma R} = \overline{R^2}$. Thus $\overline{R^2} = R$, for each right k-ideal R of S.

(2) ⇒ (1). For any $a \in S$, $a \in a\Gamma S \subseteq \overline{a\Gamma S}$ and $\overline{a\Gamma S}$ is a right k-ideal of S, then by assumption $\overline{(a\Gamma S)^2} = (a\Gamma S)$. Therefore $a \in \overline{(a\Gamma S)^2}$. Hence S is right k-weakly regular.

(2) ⇒ (3). Let R be a right k-ideal and I be a k-ideal of S. Then $R \cap I$ is a right k-ideal of S. By assumption $\overline{(R \cap I)^2} = R \cap I$. Consider $R \cap I = \overline{(R \cap I)^2} = \overline{(R \cap I)\Gamma(R \cap I)} \subseteq \overline{R\Gamma I}$. Clearly $R\Gamma I \subseteq R$ and $R\Gamma I \subseteq I$. Then $\overline{R\Gamma I} \subseteq \overline{R} = R$ and $\overline{R\Gamma I} \subseteq \overline{I} = I$, since R is a right k-ideal and I is a two sided k-ideal of S. Therefore $\overline{R\Gamma I} \subseteq R \cap I$. Hence $R \cap I = \overline{R\Gamma I}$.

(3) ⇒ (2). Let R be a right k-ideal of S and let (R) be two sided ideal generated by R. Then $(R) = S\Gamma R\Gamma S$. By assumption $R \cap \overline{(R)} = \overline{R\Gamma(R)}$. Hence $R = \overline{(R\Gamma S)\Gamma(R\Gamma S)} \subseteq \overline{R\Gamma R} = \overline{R^2}$. Therefore $\overline{R^2} = R$. □

Theorem 24. *S is right k-weakly regular if and only if every right k-ideal of S is semiprime.*

Proof. Suppose S is right k-weakly regular. Let R be a right k-ideal of S such that $\overline{A\Gamma A} \subseteq R$, for any right k-ideal A of S. $A = \overline{A\Gamma A}$. Then $\overline{A\Gamma A} \subseteq \overline{R} = R$. Therefore $A \subseteq R$. Hence R is a semiprime right k-ideal of S. Conversely, suppose every right k-ideal of S is semiprime. Let R be a right k-ideal of S. $\overline{R\Gamma R}$ is also a right k-ideal of S. By assumption $\overline{R\Gamma R}$ is a semiprime right k-ideal of S. $\overline{R\Gamma R} \subseteq \overline{R\Gamma R}$ implies $R \subseteq \overline{R\Gamma R}$. Therefore $R = \overline{R} \subseteq \overline{R\Gamma R} = \overline{R^2}$. Therefore $\overline{R^2} = R$. Hence S is right k-weakly regular by Theorem 23. □

Definition 25 (see [12]). A lattice $\mathscr{L}$ is said to be Brouwerian if, for any $a, b \in \mathscr{L}$, the set of all $x \in \mathscr{L}$ satisfying the condition $a \wedge x \leq b$ contains the greatest element.

If c is the greatest element in this set, then the element c is known as the pseudocomplement of a relative to b and is denoted by $a : b$.

Thus a lattice $\mathscr{L}$ is a Brouwerian if $a : b$ exists for all $a, b \in \mathscr{L}$.

Let $\mathscr{L}_S$ denote the family of all k-ideals of S. Then $\langle \mathscr{L}_S, \subseteq \rangle$ is a partially ordered set. As $\{0\}, S \in \mathscr{L}_S$ and $\bigcap_{\alpha\in\Delta} I_\alpha \in \mathscr{L}_S$, for all $I_\alpha \in \mathscr{L}_S$ and Δ is an indexing set, we have $\mathscr{L}_S$ is a complete lattice under $\wedge$ and $\vee$ defined by $I \vee J = \overline{I + J}$ and $I \wedge J = I \cap J$. Further we have the following.

Theorem 26. *If S is a right k-weakly regular Γ-semiring, then $\mathscr{L}_S$ is a Brouwerian lattice.*

Proof. Let B and C be any two k-ideals of S. Consider the family of k-ideals $K = \{I \in \mathscr{L}_S \mid I \cap B \subseteq C\}$. Then by Zorn's lemma there exists a maximal element M in K. Select $I \in \mathscr{L}_S$ such that $B \cap I \subseteq C$. By Theorem 23, we have $\overline{B\Gamma I} \subseteq C$. To show that $\overline{B\Gamma(\overline{I + M})} \subseteq C$. Let $x \in B\Gamma(\overline{I + M})$. Then $x = \sum_{i=1}^{n} b_i\alpha_i x_i$, where $b_i \in B$, $\alpha_i \in \Gamma$, and $x_i \in \overline{I + M}$. Therefore $a_i + x_i \in I + M$ for some $a_i \in I + M$ (see Definition 9).

$x = \sum_{i=1}^{n} b_i\alpha_i(a_i + x_i) \in B\Gamma(I + M) = (B\Gamma I) + (B\Gamma M) \subseteq C$, as $B\Gamma I \subseteq C$ and $B\Gamma M \subseteq C$. Hence $B\Gamma(\overline{I + M}) \subseteq C$ implies $\overline{B\Gamma(\overline{I + M})} \subseteq \overline{C} = C$, since C is a k-ideal. Therefore, by Theorem 23, $B \cap (\overline{I + M}) \subseteq C$. But, by the maximality, we have $\overline{I + M} \subseteq M$ which implies $I \subseteq M$. Hence $\mathscr{L}_S$ is Brouwerian. □

As $\mathscr{L}_S$ satisfies infinite meet distributive property property of lattice, we have the following.

Corollary 27. *If S is a right k-weakly regular Γ-semiring, then $\mathscr{L}_S$ is a distributive lattice (see Birkoff [12]).*

Theorem 28. *If S is a right k-weakly regular Γ-semiring, then a k-ideal P of S is prime if and only if P is irreducible.*

Proof. Let S be a right k-weakly regular Γ-semiring and let P be a k-ideal of S. If P is a prime k-ideal of S, then clearly P is an irreducible k-ideal. Suppose P is an irreducible k-ideal of S. To show that P is a prime k-ideal. Let A and B be any two k-ideals of S such that $\overline{A\Gamma B} \subseteq P$. Then, by Theorem 23, we have $A \cap B \subseteq$

P. Hence $\overline{(A \cap B) + P} = P$. As $\mathscr{L}_S$ is a distributive lattice, we have $\overline{(A + P)} \cap \overline{(B + P)} = P$. Therefore P is an irreducible k-ideal implies $\overline{A + P} = P$ or $\overline{B + P} = P$. Then $A \subseteq P$ or $B \subseteq P$. Therefore P is a prime k-ideal of S. □

As a generalization of a fully prime semiring defined by Shabir and Iqbal in [5], we define a fully k-prime Γ-semiring in [10] as follows.

A Γ-semiring S is said to be a fully k-prime Γ-semiring if each k-ideal of S is a prime k-ideal.

Theorem 29. *A Γ-semiring S is a fully k-prime Γ-semiring if and only if S is right k-weakly regular and the set of k-ideals of S is a totally ordered set by the set inclusion.*

Proof. Suppose that a Γ-semiring S is a fully k-prime Γ-semiring. Therefore every k-ideal of S is a prime k-ideal. As every prime k-ideal is a semiprime k-ideal, we have S which is a right k-weakly regular Γ-semiring by Theorem 24. For any two k-ideals A and B of S, $\overline{A\Gamma B} \subseteq A \cap B$. By assumption $A \cap B$ is a prime k-ideal and hence we have $A \subseteq A \cap B$ or $B \subseteq A \cap B$. But then $A \cap B = A$ or $A \cap B = B$. Hence either $A \subseteq B$ or $B \subseteq A$. This shows that the set of k-ideals of S is a totally ordered set by the set inclusion. Conversely, assume S is right k-weakly regular and the set of k-ideals of S is a totally ordered set by set inclusion. To show that S is a fully k-prime Γ-semiring. Let P be a k-ideal of S and $\overline{A\Gamma B} \subseteq P$, for any k-ideals A and B of S. Then $\overline{A\Gamma B} \subseteq P$. Hence by Theorem 23, we have $A \cap B = \overline{A\Gamma B} \subseteq P$. By assumption either $A \subseteq B$ or $B \subseteq A$. Therefore $A \cap B = A$ or $A \cap B = B$. Hence either $A \subseteq P$ or $B \subseteq P$. This shows that P is a prime k-ideal of S.

Therefore S is a fully k-prime Γ-semiring. □

Now we define a k-bi-ideal of a Γ-semiring.

Definition 30. A nonempty subset B of a Γ-semiring S is said to be a k-bi-ideal of S if B is a sub-Γ-semiring of S, $\overline{B\Gamma S\Gamma B} \subseteq B$, and for $a \in B$ and $x \in S$ such that $a + x \in B$; then $x \in B$.

Theorem 31. *S is right k-weakly regular if and only if $B \cap I \subseteq \overline{B\Gamma I}$, for any k-bi-ideal B and k-ideal I of S.*

Proof. Suppose S is a right k-weakly regular Γ-semiring. Let B be a k-bi-ideal and let I be a k-ideal of S. Let $a \in B \cap I$. Then $a \in \overline{(a\Gamma S)^2} = \overline{(a\Gamma S)\Gamma(a\Gamma S)} \subseteq \overline{B\Gamma(S\Gamma I\Gamma S)} \subseteq \overline{B\Gamma I}$. Therefore $B \cap I \subseteq \overline{B\Gamma I}$. Conversely, let R be a right k-ideal of S. Then R itself is a k-bi-ideal of S. Hence by assumption $R = R \cap \overline{(R)} \subseteq \overline{R\Gamma\overline{(R)}} = \overline{R\Gamma(S\Gamma R\Gamma S)} = \overline{(R\Gamma S)\Gamma(R\Gamma S)} \subseteq \overline{R\Gamma R}$. Therefore $R = \overline{R\Gamma R} = \overline{R^2}$. Hence, by Theorem 23, S is a right k-weakly regular Γ-semiring. □

Theorem 32. *S is right k-weakly regular if and only if $B \cap I \cap R \subseteq \overline{B\Gamma I\Gamma R}$, for any k-bi-ideal B, k-ideal I, and a right k-ideal R of S.*

Proof. Suppose S is a right k-weakly regular Γ-semiring. Let B be a k-bi-ideal, let I be a k-ideal, and let R be a right k-ideal of S. Let $a \in B \cap I \cap R$. Then $a \in \overline{(a\Gamma S)^2} = \overline{(a\Gamma S)\Gamma(a\Gamma S)} \subseteq \overline{(a\Gamma S)\Gamma\overline{(a\Gamma S)\Gamma(a\Gamma S)}\Gamma S} \subseteq \overline{B\Gamma(S\Gamma I\Gamma S)\Gamma(R\Gamma S)} \subseteq \overline{B\Gamma I\Gamma R}$. Therefore $B \cap I \cap R \subseteq \overline{B\Gamma I\Gamma R}$. Conversely, for a right k-ideal R of S, R itself is being a k-bi-ideal and S itself is being a k-ideal of S. Then by assumption $R \cap S \cap R \subseteq \overline{R\Gamma S\Gamma R} \subseteq \overline{R\Gamma R}$. Therefore $R \subseteq \overline{R\Gamma R}$. Therefore $R = \overline{R\Gamma R} = \overline{R^2}$. Then, by Theorem 23, S is right k-weakly regular. □

5. Right Pure k-Ideals

In this section we define a right pure k-ideal of a Γ-semiring S and furnish some of its characterizations.

Definition 33. A k-ideal I of a Γ-semiring S is said to be a right pure k-ideal if, for any $x \in I$, $x \in \overline{x\Gamma I}$.

Theorem 34. *A k-ideal I of S is right pure if and only if $R \cap I = \overline{R\Gamma I}$, for any right k-ideal R of S.*

Proof. Let I be a right pure k-ideal and let R be a right k-ideal of S. Then clearly $\overline{R\Gamma I} \subseteq R \cap I$. Now let $a \in R \cap I$. As I is a right pure k-ideal, $a \in \overline{a\Gamma I} \subseteq \overline{R\Gamma I}$. This gives $R \cap I \subseteq \overline{R\Gamma I}$. By combining both the inclusions we get $R \cap I = \overline{R\Gamma I}$. Conversely, assume the given statement holds. Let I be k-ideal of S and $a \in I$. $\overline{(a)_r}$ denotes a right k-ideal generated by a and $\overline{(a)_r} = \overline{N_0 a + a\Gamma I}$, where N_0 denotes the set of nonnegative integers. Then $a \in \overline{(a)_r} \cap I = \overline{\overline{(a)_r}\Gamma I} = \overline{\overline{(N_0 a + a\Gamma I)}\Gamma I} = \overline{(N_0 a + a\Gamma I)\Gamma I} \subseteq \overline{a\Gamma I}$ (see Result 2). Therefore I is a right pure k-ideal of S. □

Theorem 35. *The intersection of right pure k-ideals of S is a right pure k-ideal of S.*

Proof. Let A and B be right pure k-ideals of S. Then for any right k-ideal R of S we have $R \cap A = \overline{R\Gamma A}$ and $R \cap B = \overline{R\Gamma B}$. We consider $R \cap (A \cap B) = (R \cap A) \cap B = (\overline{R\Gamma A}) \cap B = \overline{(\overline{R\Gamma A})\Gamma B} = \overline{R\Gamma(A\Gamma B)} = \overline{R\Gamma(A \cap B)}$. Therefore $A \cap B$ is a right pure k-ideal of S. □

Characterization of a right k-weakly regular Γ-semiring in terms of right pure k-ideals is given in the following theorem.

Theorem 36. *S is right k-weakly regular if and only if any k-ideal of S is right pure.*

Proof. Suppose that S is a right k-weakly regular Γ-semiring. Let I be a k-ideal and let R be a right k-ideal of S. Then, by Theorem 23, $R \cap I = \overline{R\Gamma I}$. Hence, by Theorem 34, any k-ideal I of S is right pure. Conversely, suppose that any k-ideal of S is right pure. Then, from Theorems 34 and 23, we get that S is a right k-weakly regular Γ-semiring. □

6. Space of Prime k-Ideals

Let S be a Γ-semiring and let $\wp_S$ be the set of all prime k-ideals of S. For each k-ideal I of S define $\Theta_I = \{J \in \wp_S \mid I \nsubseteq J\}$ and

$$\zeta(\wp_S) = \{\Theta_I \mid I \text{ is a } k\text{-ideal of } S\}. \quad (3)$$

Theorem 37. *If S is a right k-weakly regular Γ-semiring, then $\zeta(\wp_S)$ forms a topology on the set $\wp_S$. There is an isomorphism between lattice of k-ideals $\mathscr{L}_S$ and $\zeta(\wp_S)$ (lattice of open subsets of $\wp_S$).*

Proof. As $\{0\}$ is a k-ideal of S and each k-ideal of S contains $\{0\}$, we have the following:

$\Theta_{\{0\}} = \{J \in \wp_s \mid \{0\} \nsubseteq J\} = \Phi$. Therefore $\Phi \in \zeta(\wp_S)$. As S itself is a k-ideal, $\Theta_S = \{J \in \wp_S \mid S \nsubseteq J\} = \wp_S$ imply $\wp_S \in \zeta(\wp_S)$. Now let $\Theta_{I_k} \in \zeta(\wp_S)$ for $k \in \Lambda$; Λ is an indexing set, and I_k is a k-ideal of S. Therefore $\Theta_{I_k} = \{J \in \wp_S \mid I_k \nsubseteq J\}$. As $\bigcup_k \Theta_{I_k} = \{J \in \wp_S \mid \overline{\sum_k I_k} \nsubseteq J\}$, $\overline{\sum_k I_k}$ is a k-ideal of S. Therefore $\bigcup_k \Theta_{I_k} = \Theta_{\overline{\sum_k I_k}} \in \zeta(\wp_S)$. Further let $\Theta_A, \Theta_B \in \zeta(\wp_S)$.

Let $J \in \Theta_A \bigcap \Theta_B$; J is a prime k-ideal of S. Hence $A \nsubseteq J$ and $B \nsubseteq J$. Suppose that $A \cap B \subseteq J$. In a right weakly regular Γ-semiring, prime k-ideals and strongly irreducible k-ideals coincide. Therefore J is a strongly irreducible k-ideal of S. As J is a strongly irreducible k-ideal of S, $A \subseteq J$ or $B \subseteq J$, which is a contradiction to $A \nsubseteq J$ and $B \nsubseteq J$. Hence $A \cap B \nsubseteq J$ implies $J \in \Theta_{A\cap B}$. Therefore $\Theta_A \bigcap \Theta_B \subseteq \Theta_{A\cap B}$. Now let $J \in \Theta_{A\cap B}$. Then $A\cap B \nsubseteq J$ implies $A \nsubseteq J$ and $B \nsubseteq J$. Therefore $J \in \Theta_A$ and $J \in \Theta_B$ imply $J \in \Theta_A \bigcap \Theta_B$. Thus $\Theta_{A\cap B} \subseteq \Theta_A \bigcap \Theta_B$. Hence $\Theta_A \bigcap \Theta_B = \Theta_{A\cap B} \in \zeta(\wp_S)$. Thus $\zeta(\wp_s)$ forms a topology on the set $\wp_S$.

Now we define a function $\phi : \mathscr{L}_S \to \zeta(\wp_S)$ by $\phi(I) = \Theta_I$, for all $I \in \mathscr{L}_S$. Let $I, K \in \mathscr{L}_S$. Consider $\phi(I \cap K) = \Theta_{I\cap K} = \Theta_I \bigcap \Theta_K = \phi(I) \cap \phi(K)$.

Consider $\phi(\overline{I + K}) = \Theta_{\overline{I+K}} = \Theta_I \cup \Theta_K = \phi(I) \cup \phi(K)$. Therefore ϕ is a lattice homomorphism. Now consider $\phi(I) = \phi(K)$. Then $\Theta_I = \Theta_K$.

Suppose that $I \neq K$. Then there exists $a \in I$ such that $a \notin K$. As K is a proper k-ideal of S, there exists an irreducible k-ideal J of S such that $K \subseteq J$ and $a \notin J$ (see Theorem 6 in [10]). Hence $I \nsubseteq J$. As S is a right k-weakly regular Γ-semiring, J is a prime k-ideal of S by Theorem 28. Therefore $J \in \Theta_K = \Theta_I$ implies $I \subseteq J$, which is a contradiction. Therefore $I = K$. Thus $\phi(I) = \phi(K)$ implies $I = K$ and hence ϕ is one-one. As ϕ is onto the result follows. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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