Rigid and finite type geometric structures

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Abstract. Rigid geometric structures on manifolds, introduced by Gromov, are characterized by the fact that their infinitesimal automorphisms are determined by their jets of a fixed order. Important examples of such structures are those given by an H-reduction of the first order frame bundle of a manifold, where the Lie algebra of H is of finite type; in fact, for structures given by reductions to closed subgroups of first order frame bundles, finite type implies rigidity. The goal of this paper is to generalize this to geometric structures defined by reductions of frame bundles of arbitrary order, and to give an algebraic characterization of the property of being rigid in terms of a suitable notion of finite type.

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1. Introduction

Geometric structures on a manifold M are given by Q-valued equivariant maps defined on some frame bundle of M, where Q is a manifold on which the structure group of the bundle acts; such structures are said to be of type Q. Gromov [3] introduced the notion of rigid geometric structure, which is characterized by the fact that every (infinitesimal) automorphism is completely determined by its jet of a fixed order. An important example of rigid geometric structures is given by H-reductions of the first order frame bundle of manifolds when H has Lie algebra of finite type as defined in Kobayashi [4]; such reductions considered as geometric structures are also called of finite type. We proved in our previous paper [2] that, for reductions to closed groups of first order frame bundles, being rigid is equivalent to being of finite

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type. This provided an algebraic characterization of rigid geometric structures, in terms of their type, for structures defined by reductions of first order frame bundles.

It is natural to consider the possibility of extending these results to geometric structures defined by reductions of frame bundles of arbitrary order. In other words, the proposed problem is to find, for such geometric structures, a characterization of the condition of being rigid and, furthermore, to prove that rigid is equivalent to a suitable notion of geometric structure of finite type.

The first goal of this work is to provide a characterization of rigid geometric structures, in terms of their type, for reductions of frame bundles of any order. In this regard, Theorem 3.8 shows that, for a geometric structure defined by an H-reduction, the condition of being rigid depends exclusively on the Lie algebra of H. Thus if some H-structure is rigid, then every other H-structure is rigid as well.

To prove Theorem 3.8, a notion of prolongation of a Lie subalgebra of $\mathfrak{gl}^{(k)}(n)$ (the Lie algebra of the structure group of the k-th order frame bundles) is introduced, generalizing the notion of prolongation for linear Lie algebras as found in Kobayashi [4]. Moreover, it is easily seen that Theorem 3.8 reduces to the above mentioned equivalence between rigid and finite type for reductions of first order frame bundles.

It is suggested by this that this notion of prolongation of a Lie subalgebra be used to introduce a concept of finite type structure for reductions of arbitrary frame bundles. By doing so it turns out that this extended concept of finite type reduces to the one found in [4] for order 1 and, with Theorem 3.8, there results the equivalence between rigid and finite type for reductions of frame bundles of arbitrary order (cf. Theorem 5.3).

The second goal of this paper is to show that this finite type structures have essentially the same basic features which those of order 1 are known to have (cf. Kobayashi [4]). This will be achieved by associating to an H-reduction of a frame bundle of any order a sequence of bundle prolongations with structure groups having Lie algebras isomorphic to the prolongations of the Lie algebra of H (a property known for reductions of first order frame bundles). Generalizing the case of structures of order 1, it will be shown that to any finite type structure there is a naturally associated complete parallelism on some bundle prolongation. This will be used to show that the study of the automorphisms and Killing fields of any finite type H-structure of any order, and so of any rigid structure, reduces to the study of the corresponding objects for complete parallelisms.

An application of this theory is that a proof of Gromov's centralizer theorem (cf. Gromov [3] and Zimmer [7]) for the pertinent geometric structures can now be given following the same outline of the proof presented in [2]. In this regard, the main contribution of this work is to extend the results of [2] and [7] to actions of simple Lie groups preserving a finite volume and a rigid geometric structure given by a reduction of some frame bundle. Note that [7] considers actions on compact manifolds and [2] considers actions preserving a finite volume and a rigid geometric structure given by a reduction of the first order frame bundle.

The organization of the article is as follows. Section 2 introduces some notation and basic properties of jet spaces. Section 3 considers rigid geometric structures, and Theorem 3.8 characterizes those rigid structures given by H-reductions. Section 4 introduces a notion of torsion of a connection and proves a structural equation for frame bundles of any order; this is needed to study the bundle prolongations considered latter. Section 5 contains the definition of finite type structures for any order and the proof that they have the same properties known in the first order case. Finally, Section 6 states Gromov's centralizer theorem and some applications to actions of simple Lie groups.

2. Preliminaries on jet bundles and Lie algebras

The results and constructions in this section are presented in the smooth category, but all of them can also be considered in the analytic category.

Two smooth maps $f, g : M \to Q$ between smooth manifolds have the same r-jet at $x \in M$ if $f(x) = g(x)$ and they have the same partial derivatives up to order r at x (in terms of local coordinate systems around x and $f(x)$). The r-jet determined by f at x is usually denoted by $j^r_x(f)$. If $l \geq k$, let π^l_k denote the natural jet projection sending $j^l_x(f)$ into $j^k_x(f)$; usually the context will determine the jet spaces on which π^l_k is being considered.

Let $J^r_n(Q)$ denote the smooth manifold of r-jets at the origin $0 \in \mathbb{R}^n$ of smooth maps $f : \mathbb{R}^n \to Q$. For simplicity in notation, the r-jet of a smooth map $f : \mathbb{R}^n \to Q$ at the origin $0 \in \mathbb{R}^n$ will be denoted by $j^r(f)$ instead of $j^r_0(f)$.

Note that if Q is a Lie group, then $J^r_n(Q)$ inherits a group structure defined by $j^r(g_1)j^r(g_2) = j^r(g_1 g_2)$.

For a vector space V, the jet space $J^r_n(V) = \Pi^r_{i=0} S_i(\mathbb{R}^n; V)$, where $S_i(\mathbb{R}^n; V)$ is the vector space of symmetric V-valued i-multilinear transformations on $\mathbb{R}^n$. This decomposition provides an alternative definition of the jet of a V-valued function f as an ordered set of multilinear transformations, which will be said to represent the jet of f.

Every smooth map $F: Q \to Q'$ naturally induces a smooth map $J^r_n(F): J^r_n(Q) \to J^r_n(Q')$. Moreover, if F realizes Q as a submanifold of Q', then $J^r_n(F)$ realizes $J^r_n(Q)$ as a submanifold of $J^r_n(Q')$. If Q and Q' are affine spaces and F is an affine inclusion, then $J^r_n(F)$ is the affine inclusion $\Pi^r_{i=0} S_i(\mathbb{R}^n; Q) \to \Pi^r_{i=0} S_i(\mathbb{R}^n; Q')$ obtained by simply viewing a Q-valued multilinear map as a Q'-valued multilinear map via the inclusion F; in other words, the ordered set of multilinear transformations that represents the r-th order jet of a Q-valued map is the same whether it is considered as an element of $J^r_n(Q)$ or of $J^r_n(Q')$.

LEMMA 2.1. *Let M be a submanifold of euclidean space $\mathbb{R}^N$ and let $j^r(h) \in J^r_n(M)$. If $j^{r-1}(h)$ is the jet of a constant map, then $j^r(h) = (h(0), 0, \ldots, 0, L)$ as an element of $J^r_n(\mathbb{R}^N)$, where the $\mathbb{R}^N$-valued r-multilinear symmetric form L has image contained in $T_{h(0)}M$.*

Proof. Locally at $h(0)$ there is a smooth vector-valued submersion π such that $M = \pi^{-1}(0)$, so it may be assumed that $\pi \circ h$ is constant. It follows, by applying the chain rule r times and using the fact that $j^{r-1}(h)$ is the jet of a constant map, that $d\pi_{h(0)} \circ D^{r-1} dh_0 = 0$, where D denotes the canonical connection on euclidean spaces. This proves the lemma since $T_{h(0)}M = \ker d\pi_{h(0)}$ and $j^r(h) = (h(0), 0, \ldots, 0, D^{r-1} dh_0)$ as an element of $J^r_n(\mathbb{R}^N)$.

In suitable coordinates, any given submersion can be written as a linear surjection. This fact implies the following result.

LEMMA 2.2. *Let $\pi: M \to N$ be a smooth submersion. If $g: \mathbb{R}^n \to M$ is a smooth function such that $j^r(\pi \circ g)$ is the jet of a constant map c, then there is a smooth function $h: \mathbb{R}^n \to \pi^{-1}(c)$ such that $j^r(g) = j^r(h)$.*

Let $\mathrm{Gl}^{(k)}(n)$ denote the group of k-jets at 0 of diffeomorphisms of $\mathbb{R}^n$ that fix 0. As a manifold $\mathrm{Gl}^{(k)}(n) = \mathrm{Gl}(n) \times \Pi^k_{i=2} S_i(\mathbb{R}^n; \mathbb{R}^n)$, and is in fact a Lie group. Note that $\mathrm{Gl}^{(1)}(n)$ is the general linear group $\mathrm{Gl}(n)$ and that for any pair of integers $l \geq k$ there is a canonical homomorphism $\pi^l_k: \mathrm{Gl}^{(l)}(n) \to \mathrm{Gl}^{(k)}(n)$. The kernel of π^k_{k-1} will be denoted by N^k.

Let $\mathfrak{gl}^{(k)}(n)$ denote the space of k-jets at 0 of vector fields on $\mathbb{R}^n$ that vanish at 0. The bracket of two elements $j^k(X), j^k(Y) \in \mathfrak{gl}^{(k)}(n)$ is defined by

$$[j^k(X), j^k(Y)]^k = -j^k([X, Y]).$$

A direct computation shows that, since both X and Y vanish at 0, the k-jet of $[X, Y]$ depends only on the k-jets of X and Y; thus $[\ ,\ ]^k$ defines a Lie algebra structure on $\mathfrak{gl}^{(k)}(n)$. Observe that $\mathfrak{gl}^{(1)}(n)$ can be naturally identified with the space of $n \times n$ real matrices and that, with the above bracket operation, it is canonically isomorphic to the general linear Lie algebra $\mathfrak{gl}(n)$.

The following lemma provides natural representations for $\mathrm{Gl}^{(k)}(n)$ and $\mathfrak{gl}^{(k)}(n)$, and shows that $\mathfrak{gl}^{(k)}(n)$ is the Lie algebra of $\mathrm{Gl}^{(k)}(n)$. Note that the space of $(k-1)$-jets at 0 of vector fields on $\mathbb{R}^n$ has been identified with $\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n)$.

LEMMA 2.3. *Let* $\Lambda\colon \mathrm{Gl}^{(k)}(n) \to \mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n))$ *and* $\lambda\colon \mathfrak{gl}^{(k)}(n) \to \mathfrak{gl}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n))$ *be defined by*

$$\begin{aligned}\Lambda(j^k(\varphi))(j^{k-1}(Z)) &= j^{k-1}(d\varphi(Z))\\ \lambda(j^k(X))(j^{k-1}(Z)) &= -j^{k-1}(L_X(Z)),\end{aligned}$$

for all $j^{k-1}(Z) \in \mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n)$. *Then both* Λ *and* λ *are faithful representations, and with respect to them* $\mathfrak{gl}^{(k)}(n)$ *is the Lie algebra of* $\mathrm{Gl}^{(k)}(n)$. *Moreover, the exponential map* $\exp\colon \mathfrak{gl}^{(k)}(n) \to \mathrm{Gl}^{(k)}(n)$ *is given by*

$$\exp(tj^k(X)) = j^k(d\varphi_t),$$

for $j^k(X) \in \mathfrak{gl}^{(k)}(n)$ *and* $t \in \mathbb{R}$, *where* φ_t *is the local flow of the vector field* X.

The above representations are isomorphisms when $k = 1$. Moreover, the homomorphism Λ realizes $\mathrm{Gl}^{(k)}(n)$ as a real algebraic subgroup of $\mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n))$.

The expression for λ given by this lemma and the definition of the Lie bracket on $\mathfrak{gl}^{(k-1)}(n)$ imply that $\lambda(A)|_{\mathfrak{gl}^{(k-1)}(n)} = \mathrm{ad}_{\mathfrak{gl}^{(k-1)}(n)}(\pi^k_{k-1}(A))$ for every $A \in \mathfrak{gl}^{(k)}(n)$. This in turn implies that $\Lambda(g)|_{\mathfrak{gl}^{(k-1)}(n)} = \mathrm{Ad}_{\mathrm{Gl}^{(k-1)}(n)}(\pi^k_{k-1}(g))$ for every $g \in \mathrm{Gl}^{(k)}(n)$.

Let $a : \mathrm{Gl}^{(k+r)}(n) \to J^r_n(\mathrm{Gl}^{(k)}(n))$ be the map defined as follows. If $g \in \mathrm{Gl}^{(k+r)}(n)$ is of the form $g = j^{(k+r)}(f)$, let $f_k : \mathbb{R}^n \to \mathrm{Gl}^{(k)}(n)$ be the map given by

$$f_k(x) = j^k(\tau_{-x} \circ f \circ \tau_{f^{-1}(x)}),$$

where $\tau_v(y) = y + v$ is the translation by v in $\mathbb{R}^n$, and set

$$a(g) = j^r(f_k).$$

This map a satisfies

$$a(g_1 g_2) = a(g_1)(a(g_2) \circ \pi^{k+r}_r(g_1^{-1})),$$

where '$\circ$' denotes the operation given by $j^r(f) \circ j^r(\varphi) = j^r(f \circ \varphi)$. It follows from this that if $\mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n))$ is the semi-direct product with group multiplication $(g,h)(g',h') = (gg', h(h' \circ g^{-1}))$, then the map $(\pi^{k+r}_r, a) : \mathrm{Gl}^{(k+r)}(n) \to \mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n))$ is a homomorphism of Lie groups.

For a transformation $L \in N^{k+r}$, let $L_{(r)}$ denote the N^k-valued r-multilinear symmetric transformation over $\mathbb{R}^n$ given by $(v_1, \ldots, v_r) \mapsto L(v_1, \ldots, v_r, \ \ldots\ )$.

LEMMA 2.4. *The homomorphism* $(\pi^{k+r}_r, a) : \mathrm{Gl}^{(k+r)}(n) \to \mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n))$ *maps* N^{k+r} *into* $J^r_n(N^k)$.

Proof. Each $g \in N^{k+r}$ can be represented as $g = j^{k+r}(\varphi)$, where $\varphi(y) = y + (1/(k+r)!)L(y, \ldots, y)$, L being a symmetric $\mathbb{R}^n$-valued $(k+r)$-multilinear map on $\mathbb{R}^n$. Correspondingly, $a(g) = j^r(f)$, where we can choose $f : \mathbb{R}^n \to N^k \subset \mathrm{Gl}^{(k)}(n)$ to be given by $f(x) = (I, 0, \ldots, 0, (1/r!)L_{(r)}(x, \ldots, x))$.

If $\mathrm{Gl}^{(k)}(n)$ acts smoothly (on the left) on a manifold Q, then there is a natural smooth action of $\mathrm{Gl}^{(k+r)}(n)$ on $J^r_n(Q)$ (on the left) which is called the r-prolongation of the action on Q, and which is constructed as follows.

There is a natural left action of the Lie group $\mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n))$ on $J^r_n(Q)$ given by $(g,h)q = h \cdot (q \circ g^{-1})$, where '$\circ$' is defined as before and '$\cdot$' is given by $j^r(f_1) \cdot j^r(f_2) = j^r(f_1 f_2)$. This induces, with the help of (π^{k+r}_r, a), a canonical action $\mathrm{Gl}^{(k+r)}(n) \times J^r_n(Q) \to J^r_n(Q)$ given by $gq = a(g) \cdot (q \circ \pi^{k+r}_r(g^{-1}))$.

3. Rigid geometric structures of type $\mathrm{Gl}^{(k)}(n)/H$

The purpose of this section is to show that, for geometric structures of type $\mathrm{Gl}^{(k)}(n)/H$, rigidity depends only on the algebraic properties of the Lie algebra of H. Some basic definitions and properties of geometric structures and their automorphisms will be presently reviewed.

For an n-dimensional manifold M we will denote with $L^{(k)}(M)$ its k-th order frame bundle, which is defined as the $\mathrm{Gl}^{(k)}(n)$-principal bundle over M of the k-jets at 0 of diffeomorphisms from a neighborhood of 0 in $\mathbb{R}^n$ onto some open set of M. If $f : M \to N$ is a (local) diffeomorphism, then $f_{(k)}$ will denote its lift as a (local) diffeomorphism $f_{(k)} : L^{(k)}(M) \to L^{(k)}(N)$. Note that $(f \circ g)_{(k)} = f_{(k)} \circ g_{(k)}$, for (local) diffeomorphisms $f, g : M \to M$. If X is a smooth vector field on a manifold M with local flow φ_t, then $(\varphi_t)_{(k)}$ is a local flow that induces a smooth vector field on $L^{(k)}(M)$ which will be denoted by $X_{(k)}$. Since the maps $(\varphi_t)_{(k)}$ are bundle diffeomorphisms, the vector field $X_{(k)}$ is

$\mathrm{Gl}^{(k)}(n)$-invariant. It is a simple matter to show that for $\alpha \in L^{(k)}(M)$ in the fiber of a point $x \in M$, the tangent vector $X_{(k)}(\alpha)$ depends only on $j^k_x(X)$.

DEFINITION 3.1. Let Q be a smooth manifold on which $\mathrm{Gl}^{(k)}(n)$ acts smoothly on the left. A smooth geometric structure of order k and type Q on an n-dimensional smooth manifold M is a smooth $\mathrm{Gl}^{(k)}(n)$-equivariant map $L^{(k)}(M) \to Q$.

A geometric structure of type $\mathrm{Gl}^{(k)}(n)/H$, where H is a closed subgroup of $\mathrm{Gl}^{(k)}(n)$, is a geometric structure associated to the natural action of $\mathrm{Gl}^{(k)}(n)$ on $\mathrm{Gl}^{(k)}(n)/H$. It is well known that there is a natural correspondence between geometric structures of type $\mathrm{Gl}^{(k)}(n)/H$ and H-reductions of k-th order frame bundles.

Let σ be a smooth geometric structure of order k and type Q on M. Then a (local) diffeomorphism φ of M is called a (local) automorphism if it (locally) satisfies $\sigma \circ \varphi_{(k)} = \sigma$. Also, for every $r \geq 0$, the jet $j^{k+r}_x(\varphi)$ of a local diffeomorphism φ is called an infinitesimal automorphism of order $k+r$ if $\sigma \circ \varphi_{(k)} = \sigma$ is satisfied up to order r at some (and hence at every) point in the fiber of $L^{(k)}(M)$ over x. Let $\mathrm{Aut}(M, \sigma)$ be the group of automorphisms of σ and $\mathrm{Aut}^{\mathrm{loc}}(M, \sigma)$ the pseudogroup of its local automorphisms. Let $\mathrm{Aut}^{k+r}(\sigma, x, y)$ denote the space of infinitesimal automorphisms (of order $k+r$) of σ which send x to y; and let $\mathrm{Aut}^{k+r}(\sigma, x, x) = \mathrm{Aut}^{k+r}(\sigma, x)$, which is in fact a Lie group.

Given a smooth geometric structure of order k and type Q on an n-dimensional smooth manifold M and given a non-negative integer r, there is a smooth geometric structure of order $k+r$ and type $J^r_n(Q)$ on M, called its r-th order prolongation, which is constructed as follows. If $\sigma : L^{(k)}(M) \to Q$ is the $\mathrm{Gl}^{(k)}(n)$-equivariant map that defines a smooth geometric structure on M, then the r-th order prolongation of σ is the geometric structure whose associated $\mathrm{Gl}^{(k+r)}(n)$-equivariant map is given by

$$\begin{aligned} \sigma^r : L^{(k+r)}(M) &\to J^r_n(Q) \\ j^{k+r}(\varphi) &\mapsto j^r(\sigma(j^k(\varphi \circ \tau_\bullet))) \end{aligned}$$

where $\sigma(j^k(\varphi \circ \tau_\bullet))$ denotes the map $v \in \mathbb{R}^n \mapsto \sigma(j^k(\varphi \circ \tau_v)) \in Q$. Note that $\sigma^0 = \sigma$ for any geometric structure σ. Also note that $j^{k+r}_x(\varphi) \in \mathrm{Aut}^{k+r}(\sigma, x, y)$ if and only if $\sigma^r \circ \varphi_{(k+r)} = \sigma^r$ on the fiber of $L^{(k+r)}(M)$ lying above x.

A Killing field for a smooth geometric structure σ on M is a smooth vector field on M whose local flow acts on M by local automorphisms of σ. The space of Killing fields and local Killing fields of a geometric structure σ are denoted by $\mathrm{Kill}(\sigma)$ and $\mathrm{Kill}^{\mathrm{loc}}(\sigma)$, respectively. It follows from the standard relation between a vector field and its local flow that a vector field X on M is a Killing field for σ if and only if $d\sigma_\alpha(X_{(k)}) = 0$ for every $\alpha \in L^{(k)}(M)$. An infinitesimal Killing field of order $k+r$ at x for σ is a $(k+r)$-jet $j_x^{k+r}(X)$ of a germ at x of a vector field X so that $d\sigma_\alpha^r(X_{(k+r)}) = 0$ for every $\alpha \in L^{(k+r)}(M)$ that lies in the fiber over x. Let $\mathrm{Kill}^{k+r}(\sigma, x)$ denote the space of infinitesimal Killing fields of σ of order $k+r$, and let $\mathrm{Kill}_0^{k+r}(\sigma, x)$ denote the subspace consisting of those vanishing at x.

Let $j_x^{k+r}(X)$ be the $(k+r)$-jet of a vector field on M. Then a straightforward computation shows that $j_x^{k+r}(X) \in \mathrm{Kill}^{k+r}(\sigma, x)$ if and only if the local flow φ_t of X satisfies $j_x^{k+r}(d\varphi_t) \in \mathrm{Aut}^{k+r}(\sigma, x, \varphi_t(x))$, for every t in a neighborhood of 0. If $j_x^{k+r}(X) \in \mathrm{Kill}_0^{k+r}(\sigma, x)$, then this last property holds for every $t \in \mathbb{R}$, from what it follows that $\mathrm{Kill}_0^{k+r}(\sigma, x)$ is the Lie algebra of $\mathrm{Aut}^{k+r}(\sigma, x)$.

The following definition of rigidity for geometric structures is due to Gromov [3].

DEFINITION 3.2. Let r be a non-negative integer. A smooth geometric structure σ of order k on a smooth manifold M is called r-rigid if, for every $x \in M$, the canonical jet projection $\pi_{k+r}^{k+r+1} \colon \mathrm{Aut}^{k+r+1}(\sigma, x) \to \mathrm{Aut}^{k+r}(\sigma, x)$ is injective. Also, σ is called Killing r-rigid if, for every $x \in M$, the projection $\pi_{k+r}^{k+r+1} \colon \mathrm{Kill}_0^{k+r+1}(\sigma, x) \to \mathrm{Kill}_0^{k+r}(\sigma, x)$ is injective.

The next result says that to determine whether a geometric structure is rigid or not it suffices to study the stabilizers of the actions of the groups N^{k+r}. Its proof can be found in [2].

PROPOSITION 3.3. *Let $\sigma : L^{(k)}(M) \to Q$ be the $\mathrm{Gl}^{(k)}(n)$-equivariant map defining a smooth geometric structure of order k and type Q on an n-dimensional smooth manifold M. Consider the action of N^{k+r+1} on $J_n^{r+1}(Q)$ induced by the $(r+1)$-prolongation of the action of $\mathrm{Gl}^{(k)}(n)$ on Q. Then σ is r-rigid ($r \geq 0$) if and only if the action of N^{k+r+1} on the image of σ^{r+1} is free. Also, σ is Killing r-rigid if and only if the action of N^{k+r+1} on the image of σ^{r+1} is locally free.*

If H is a closed subgroup of $\mathrm{Gl}^{(k)}(n)$, then every element of the space $J_n^r(\mathrm{Gl}^{(k)}(n)/H)$ can be written as $j^r(fH)$, for some smooth map $f \colon \mathbb{R}^n \to \mathrm{Gl}^{(k)}(n)$. Hence the following result computes the stabilizers of the action of N^{k+r} induced by the r-prolongation of a transitive action of $\mathrm{Gl}^{(k)}(n)$. This will be useful in understanding rigid structures.

PROPOSITION 3.4. *Let H be a closed subgroup of $\mathrm{Gl}^{(k)}(n)$, with Lie algebra $\mathfrak{h}$, and let $f\colon \mathbb{R}^n \to \mathrm{Gl}^{(k)}(n)$ be a smooth map. Then the stabilizer of $j^r(fH)$ under the action of N^{k+r} on $J^r_n(\mathrm{Gl}^{(k)}(n)/H)$ is the vector group of elements $L \in N^{k+r}$ that satisfy $L_{(r)}(v_1,\dots,v_r) \in \mathrm{Ad}_{\mathrm{Gl}^{(k)}(n)}(f(0))(\mathfrak{h}) \cap N^k$, for every $v_1,\dots,v_r \in \mathbb{R}^n$.*

Proof. It follows from Lemma 2.4 and its proof that the action of $L \in N^{k+r}$ on $j^r(fH)$ is given by $Lj^r(fH) = j^r(\bar{L}fH)$, where $\bar{L}\colon \mathbb{R}^n \to N^k$ is defined by $\bar{L}(x) = (1/r!)L_{(r)}(x,\dots,x)$. Hence, $L \in N^{k+r}$ belongs to the stabilizer of $j^r(fH)$ if and only if $j^r(f^{-1}\bar{L}fH) = j^r(eH)$.

Lemma 2.3 realizes N^k as a subgroup of $\mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n))$, hence as a submanifold of the vector space $\mathrm{End}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n))$. Under such inclusion N^k is either an open subset (if $k = 1$) or an affine subspace (if $k \geq 2$) of $\mathrm{End}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n))$. In particular, for any smooth function $g\colon \mathbb{R}^n \to N^k$, the ordered collection of multilinear transformations that represents the jet $j^r(g)$ is the same whether this is considered in $J^r_n(N^k)$ or in $J^r_n(\mathrm{End}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n)))$.

Choose $L \in N^{k+r}$. Since N^k is normal in $\mathrm{Gl}^{(k)}(n)$, the map $f^{-1}\bar{L}f$ is defined on a neighborhood of 0 in $\mathbb{R}^n$ and takes values in N^k, and thus in $\mathrm{End}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n))$. The ordered collection of multilinear transformations that represents $j^r(f^{-1}\bar{L}f)$ as an element of the space $J^r_n(\mathrm{End}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n)))$ will now be shown to be

$$j^r(f^{-1}\bar{L}f) = (I, 0, \dots, 0, \mathrm{Ad}_{\mathrm{Gl}^{(k)}(n)}(f(0)^{-1}) \circ L_{(r)}). \qquad (*)$$

Indeed, as sets of multilinear transformations representing a jet, it is easy to check that $j^{r-1}(\bar{L}) = 0$ in $J^{r-1}_n(N^k)$, from which it obtains that $j^{r-1}(f^{-1}\bar{L}f) = 0$ in $J^{r-1}_n(N^k)$, and thus also in $J^r_n(\mathrm{End}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n)))$. Given this, a simple calculation, using the properties of the product of matrices, can be performed to obtain the partial derivatives of order r of the map $f^{-1}\bar{L}f$, as an $\mathrm{End}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n))$-valued map. The result is that

$$j^r(f^{-1}\bar{L}f) = (I, 0, \dots, 0, \mathrm{Ad}_{\mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n))}(f(0)^{-1}) \circ L_{(r)}),$$

as an element of $J^r_n(\mathrm{End}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n)))$. But

$$\mathrm{Ad}_{\mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n))}(f(0)^{-1}) \circ L_{(r)} = \mathrm{Ad}_{\mathrm{Gl}^{(k)}(n)}(f(0)^{-1}) \circ L_{(r)}$$

(because $f(0) \in \mathrm{Gl}^{(k)}(n)$, $\mathrm{Gl}^{(k)}(n)$ is a subgroup of $\mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n))$, and $L_{(r)}$ is N^k-valued), confirming thus the expression for $j^r(f^{-1}\bar{L}f)$ stated above.

Suppose now that L stabilizes $j^r(fH)$. Then, by the previous discussion and by Lemma 2.2, there exists a smooth map $h\colon \mathbb{R}^n \to H$ such that $j^r(f^{-1}\bar{L}f) = j^r(h)$. It follows, by applying Lemma 2.1 with H considered as a submanifold of the vector space $\mathrm{End}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n))$ and using expression $(*)$ obtained above for $j^r(f^{-1}\bar{L}f)$, that the multilinear transformation $\mathrm{Ad}_{\mathrm{Gl}^{(k)}(n)}(f(0)^{-1}) \circ L_{(r)}$ is $\mathfrak{h}$-valued. In other words, $L_{(r)}(v_1, \ldots, v_r) \in \mathrm{Ad}_{\mathrm{Gl}^{(k)}(n)}(f(0))(\mathfrak{h}) \cap N^k$ for every $v_1, \ldots, v_r \in \mathbb{R}^n$.

Conversely, if $L \in N^{k+r}$ is such that $L_{(r)}$ is $\mathrm{Ad}_{\mathrm{Gl}^{(k)}(n)}(f(0))(\mathfrak{h}) \cap N^k$-valued, then, using local coordinates, it is easy to check the existence of a smooth function $h\colon \mathbb{R}^n \to H$ for which we have $j^r(h) = (I, 0, \ldots, 0, \mathrm{Ad}_{\mathrm{Gl}^{(k)}(n)}(f(0)^{-1}) \circ L_{(r)})$ as an element of $J^r_n(\mathrm{End}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n)))$. Therefore $j^r(f^{-1}\bar{L}f) = j^r(h)$, which implies that L stabilizes $j^r(fH)$.

The previous proposition suggests the following definition.

DEFINITION 3.5. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{gl}^{(k)}(n)$, and let r be a nonnegative integer. The r-th prolongation of $\mathfrak{h}$ is the vector space $\mathfrak{h}_r$ of multilinear transformations $L \in N^{k+r}$ such that $L_{(r)}(v_1, \ldots, v_r) \in \mathfrak{h} \cap N^k$ for every $v_1, \ldots, v_r \in \mathbb{R}^n$.

If $k = 1$, then this definition reduces to the notion of r-th prolongation of a linear Lie algebra as found in Kobayashi [4]. Note that the r-th prolongation of a Lie subalgebra as above is indeed a vector subspace of N^{k+r}, and so it can be considered as a commutative subalgebra of $\mathfrak{gl}^{(k+r)}(n)$. Also, for $\mathfrak{h}$ as above, the 0-th prolongation of $\mathfrak{h}$ is simply $\mathfrak{h} \cap N^k$.

Given this notion of prolongation, Proposition 3.4 states that the stabilizer of $j^r(fH)$ for the action of N^{k+r} on $J^r_n(\mathrm{Gl}^{(k)}(n)/H)$ is the vector group whose Lie algebra is $\mathrm{Ad}_{\mathrm{Gl}^{(k)}(n)}(f(0))(\mathfrak{h})_r$.

The following result is a linear algebra exercise using the definition of prolongation of a Lie subalgebra.

LEMMA 3.6. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{gl}^{(k)}(n)$. Then for every nonnegative integers r, s the prolongations $\mathfrak{h}_{r+s}$ and $(\mathfrak{h}_r)_s$ are isomorphic.*

The next result is a consequence of the computation of the stabilizer of prolongations of transitive actions.

LEMMA 3.7. *If $\mathfrak{h}$ is the Lie algebra of a closed subgroup H of $\mathrm{Gl}^{(k)}(n)$, then*

$$\mathrm{Ad}_{\mathrm{Gl}^{(k)}(n)}(\pi^{k+r}_k(g))(\mathfrak{h})_r = \mathrm{Ad}_{\mathrm{Gl}^{(k+r)}(n)}(g)(\mathfrak{h}_r),$$

for every $g \in \mathrm{Gl}^{(k+r)}(n)$.

Proof. Fix $g = j^{k+r}(\varphi) \in \mathrm{Gl}^{(k+r)}(n)$. Then the definition of the prolongation of an action shows that $gj^r(eH) = j^r(\varphi_k H)$, where the smooth map $\varphi_k\colon \mathbb{R}^n \to \mathrm{Gl}^{(k)}(n)$ is defined as in the remarks that follow Lemma 2.3, so that in particular $\varphi_k(0) = \pi^{k+r}_k(g)$. It follows from Proposition 3.4 that $\mathfrak{h}_r$ and $\mathrm{Ad}_{\mathrm{Gl}^{(k)}(n)}(\pi^{k+r}_k(g))(\mathfrak{h})_r$ are the Lie algebras of the stabilizers of $j^r(eH)$ and $j^r(\varphi_k H)$, respectively, for the action of N^{k+r} on $J^r_n(\mathrm{Gl}^{(k)}(n)/H)$. Because $gj^r(eH) = j^r(\varphi_k H)$, the stabilizer of $j^r(\varphi_k H)$ under the action of $\mathrm{Gl}^{(k+r)}(n)$ on $J^r_n(\mathrm{Gl}^{(k)}(n)/H)$ is the g-conjugate of the stabilizer of $j^r(eH)$. Hence the result follows from the normality of N^{k+r} in $\mathrm{Gl}^{(k+r)}(n)$.

The next theorem establishes that, for geometric structures of type $\mathrm{Gl}^{(k)}(n)/H$, the property of being rigid depends only on the Lie algebra of the group H. Therefore, some geometric structure of type $\mathrm{Gl}^{(k)}(n)/H$ on some n-dimensional smooth manifold is r-rigid if and only if every geometric structure of type $\mathrm{Gl}^{(k)}(n)/H$ on every n-dimensional smooth manifold is r-rigid.

THEOREM 3.8. *Let H be a closed subgroup of $\mathrm{Gl}^{(k)}(n)$ with Lie algebra $\mathfrak{h}$. Then a smooth geometric structure σ of type $\mathrm{Gl}^{(k)}(n)/H$ is r-rigid if and only if $\mathfrak{h}_{r+1} = 0$, i.e., the $(r+1)$-th prolongation of the Lie algebra of $\mathfrak{h}$ vanishes.*

Proof. By Propositions 3.3 and 3.4 the geometric structure σ is r-rigid if and only if $\mathrm{Ad}_{\mathrm{Gl}^{(k)}(n)}(g)(\mathfrak{h})_{r+1} = 0$ for any $g \in \mathrm{Gl}^{(k)}(n)$ such that there is a smooth map $f\colon \mathbb{R}^n \to \mathrm{Gl}^{(k)}(n)$ with $f(0) = g$ and $j^{r+1}(fH)$ in the image of σ^{r+1}. Because of Lemma 3.7, this is equivalent to $\mathfrak{h}_{r+1} = 0$.

4. Torsion of a connection

This section develops some properties of higher order frame bundles which will be used in the next one to define and study finite type geometric structures in order higher than one. The main result is a generalization to higher orders of the structural equation involving the torsion of a connection on the first order frame bundle.

Let M be an n-dimensional manifold. Then, it is known that the assignment $j^k_x(X) \mapsto X_{(k)}(\alpha)$ defines an isomorphism of the space of k-jets at x of vector fields on M onto $T_\alpha L^{(k)}(M)$. In particular, such assignment defines a natural isomorphism of $\mathbb{R}^n \oplus \mathfrak{gl}^{(k)}(n) = J^k_n(\mathbb{R}^n)$ onto $T_0 L^{(k)}(\mathbb{R}^n)$, where 0 in $L^{(k)}(\mathbb{R}^n)$ denotes the k-jet at 0 of the identity map on $\mathbb{R}^n$. Observe that if $\alpha = j^k(\varphi) \in L^{(k)}(M)$, then the differential at 0 of the local diffeomorphism $\varphi_{(k-1)}\colon L^{(k-1)}(\mathbb{R}^n) = \mathbb{R}^n \times \mathrm{Gl}^{(k-1)}(n) \to L^{(k-1)}(M)$ depends only on α. Such differential will be denoted by $\widehat{\alpha}\colon \mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n) = T_0 L^{(k-1)}(\mathbb{R}^n) \to T_{\pi^k_{k-1}(\alpha)} L^{(k-1)}(M)$, which is then an isomorphism.

For every n-dimensional smooth manifold M and for any of its frame bundles $L^{(k)}(M)$ there is a smooth $\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n)$-valued one-form θ_k on $L^{(k)}(M)$ called the canonical form of $L^{(k)}(M)$, which is defined as follows. If π^k_{k-1} denotes the natural jet projection $L^{(k)}(M) \to L^{(k-1)}(M)$, then θ_k at $\alpha \in L^{(k)}(M)$ is the linear map $T_\alpha L^{(k)}(M) \to \mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n)$ given by the expression $\theta_k|_{T_\alpha L^{(k)}(M)} = \widehat{\alpha}^{-1} \circ d\pi^k_{k-1}|_{T_\alpha L^{(k)}(M)}$. It is easy to see from this definition that if f is a diffeomorphism of M, then $f^*_{(k)}\theta_k = \theta_k$ on the k-th order frame bundle of M.

LEMMA 4.1. *For every k and $g \in \mathrm{Gl}^{(k)}(n)$ the diagram*

$$\begin{array}{ccc} T_0 L^{(k-1)}(\mathbb{R}^n) & \xrightarrow{dR_{\pi^k_{k-1}(g^{-1})} \circ \widehat{g}} & T_0 L^{(k-1)}(\mathbb{R}^n) \\ \uparrow & & \uparrow \\ \mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n) & \xrightarrow{\Lambda(g)} & \mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n) \end{array}$$

is commutative, where the vertical arrows are given by the natural isomorphism $j^{k-1}(X) \mapsto X_{(k-1)}(0)$, Λ is as in Lemma 2.3 and $\widehat{g}$ is given as above.

Proof. The identity $(\varphi \circ \psi)_{(k-1)} = \varphi_{(k-1)} \circ \psi_{(k-1)}$ can be used to show that if φ is a local diffeomorphism of $\mathbb{R}^n$ and X is a vector field on $\mathbb{R}^n$, then $d\varphi(X)_{(k-1)} = d\varphi_{(k-1)}(X_{(k-1)})$. The result now follows from this, the definition of Λ and the $\mathrm{Gl}^{(k-1)}(n)$-invariance of $X_{(k-1)}$.

The next result states basic properties of the canonical form; the case $k = 1$ is already given in Kobayashi and Nomizu [5]. If $A \in \mathfrak{gl}^{(k)}(n)$, let A^* denote the vertical vector field over $L^{(k)}(M)$ whose flow is given by the action on the right of the one-parameter subgroup $t \mapsto \exp(tA)$ of $\mathrm{Gl}^{(k)}(n)$.

PROPOSITION 4.2. *The canonical form θ_k of $L^{(k)}(M)$ satisfies the following properties:*

1. $\theta_k(A^*) = \pi^k_{k-1}(A)$ *for every* $A \in \mathfrak{gl}^{(k)}(n)$,
2. $R^*_g\theta_k = \Lambda(g^{-1}) \circ \theta_k$ *for every* $g \in \mathrm{Gl}^{(k)}(n)$,
3. $L_{A^*}\theta_k = -\lambda(A) \circ \theta_k$ *for every* $A \in \mathfrak{gl}^{(k)}(n)$,

where $\pi^k_{k-1}\colon \mathfrak{gl}^{(k)}(n) \to \mathfrak{gl}^{(k-1)}(n)$ *denotes the natural jet projection and* Λ*,* λ *are given as in Lemma 2.3.*

Proof. Property (1) is a direct consequence of the definition of θ_k. Also observe that (2) implies (3).

To prove (2), fix $g \in \mathrm{Gl}^{(k)}(n)$, $\alpha = j^k(\varphi) \in L^{(k)}(M)$ and $\xi \in T_\alpha L^{(k)}(M)$. Then a straightforward computation yields

$$\begin{aligned}
(R_g^*\theta_k)_\alpha(\xi) &= (\theta_k)_{\alpha g}(dR_g(\xi)) \\
&= \widehat{g}^{-1} \circ (d(\varphi_{(k-1)})_{(0,\pi^k_{k-1}(g))})^{-1} \circ d(\pi^k_{k-1})_{\alpha g}(dR_g(\xi)) \\
&= \widehat{g}^{-1} \circ dR_{\pi^k_{k-1}(g)} \circ \widehat{\alpha}^{-1} \circ d(\pi^k_{k-1})_\alpha(\xi) \\
&= \Lambda(g^{-1}) \circ \widehat{\alpha}^{-1} \circ d(\pi^k_{k-1})_\alpha(\xi) \\
&= (\Lambda(g^{-1}) \circ \theta_k)_\alpha(\xi),
\end{aligned}$$

where the third identity follows from the equivariance of both $\varphi_{(k-1)}$ and the projection $L^{(k)}(M) \to L^{(k-1)}(M)$ with respect to the projection $\mathrm{Gl}^{(k)}(n) \to \mathrm{Gl}^{(k-1)}(n)$, and the fourth identity follows from Lemma 4.1.

If H is a Lie subgroup of $\mathrm{Gl}^{(k)}(n)$ and P is a smooth H-reduction of a frame bundle $L^{(k)}(M)$ of an n-dimensional smooth manifold M, then the canonical form, θ_k, of $L^{(k)}(M)$ can be restricted to P. Such restriction, which will be denoted by the same symbol, inherits the corresponding properties stated in the previous result. By Lemma 2.3, the Lie algebra $\mathfrak{gl}^{(k)}(n)$ can be realized as a Lie subalgebra of $\mathfrak{gl}(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n))$, and thus the Lie algebra $\mathfrak{h}$ of H can be so realized. In particular, if ω is a connection on P (considered as an $\mathfrak{h}$-valued one-form), then the representation λ defined in Lemma 2.3 realizes $\omega(\xi)$ as a linear map on $\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n)$, for every $\xi \in TP$. This property will be used below.

With the above setup, given a connection on P and a vector v in $\mathbb{R}^n$ there exists a unique horizontal H-invariant vector field on P whose value at every $\alpha \in P$ is the horizontal lift of $\pi^k_1(\alpha)(v) \in TM$, where $\pi^k_1(\alpha) \in L(M)$ is the natural jet projection of α. This vector field is called the standard horizontal vector field associated to v, and is denoted by $B(v)$.

DEFINITION 4.3. Let ω be a smooth connection on a smooth reduction P of the frame bundle $L^{(k)}(M)$ of order k of an n-dimensional smooth manifold M. The torsion of ω is the $\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n)$-valued two-form on P given by $\Theta = d\theta_k \circ (h,h)$, where h denotes the horizontal projection $TP \to TP$ defined by ω.

The following result is a higher order generalization of the usual structural equation of a connection on the first order frame bundle.

PROPOSITION 4.4. *Let ω be a smooth connection on a smooth reduction P of the frame bundle $L^{(k)}(M)$ of order k of an n-dimensional smooth manifold M, and let Θ be the torsion of ω. Then*

$$d\theta_k = -\omega \wedge \theta_k + \frac{1}{2}\pi^k_{k-1}([\omega, \omega]) + \Theta.$$

More precisely,

$$\begin{aligned} d\theta_k(\xi_1, \xi_2) = & -\frac{1}{2}(\omega(\xi_1)\theta_k(\xi_2) - \omega(\xi_2)\theta_k(\xi_1)) \\ & + \frac{1}{2}\pi^k_{k-1}([\omega(\xi_1), \omega(\xi_2)]) + \Theta(\xi_1, \xi_2), \end{aligned}$$

for every $\xi_1, \xi_2 \in T_\alpha P$ and $\alpha \in P$, where π^k_{k-1} denotes the natural Lie algebra homomorphism $\mathfrak{gl}^{(k)}(n) \to \mathfrak{gl}^{(k-1)}(n)$.

Proof. It suffices to prove the statement for $P = L^{(k)}(M)$, for this case implies the result for any smooth reduction by simply restricting the forms involved. Furthermore, it suffices to verify the second identity for ξ_1, ξ_2 either vertical or horizontal, so there are three cases.

First, if both ξ_1 and ξ_2 are horizontal, then the identity is trivial by the definition of Θ and because ω vanishes on horizontal tangent vectors.

Second, given $A, B \in \mathfrak{gl}^k(n)$ with vertical fields A^* and B^*, respectively, then $\Theta(A^*, B^*) = 0$. It follows from the properties of θ_k that

$$\begin{aligned} 2d\theta_k(A^*, B^*) &= A^*(\theta_k(B^*)) - B^*(\theta_k(A^*)) - \theta_k([A^*, B^*]) \\ &= -\theta_k([A^*, B^*]) \\ &= -\theta_k([A, B]^*) \\ &= -\pi^k_{k-1}([A, B]). \end{aligned}$$

On the other hand, the properties of λ given in Lemma 2.3 imply that

$$\begin{aligned} \omega(A^*)\theta_k(B^*) &= \lambda(A)(\pi^k_{k-1}(B)) \\ &= [\pi^k_{k-1}(A), \pi^k_{k-1}(B)] \\ &= \pi^k_{k-1}([A, B]), \end{aligned}$$

from which the identity obtains in this case because $[\omega(A^*), \omega(B^*)] = [A, B]$.

And third, if A^* is a vertical vector field on $L^{(k)}(M)$, and $B(v)$ is a horizontal one, with $A \in \mathfrak{gl}^{(k)}(n)$ and $v \in \mathbb{R}^n$, then

$$\begin{aligned}2d\theta_k(A^*, B(v)) &= A^*(\theta_k(B(v))) - B(v)(\theta_k(A^*)) - \theta_k([A^*, B(v)]) \\ &= (L_{A^*}\theta_k)(B(v)) \\ &= -\lambda(A) \circ \theta_k(B(v)) \\ &= -\omega(A^*)\theta_k(B(v)),\end{aligned}$$

where the third identity follows from Proposition 4.2. This proves the required identity in this case because both $\Theta(A^*, B(v))$ and $\omega(B(v))$ vanish.

5. Finite type geometric structures

The characterization of rigid geometric structures given in Section 3 suggests to introduce a notion of finite type structure for reductions of any frame bundle that generalizes the one given in Kobayashi [4] for reductions of the first order frame bundle. This will be done in this section.

DEFINITION 5.1. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{gl}^{(k)}(n)$ is said to be of finite type if there exists an integer r such that $\mathfrak{h}_r = 0$.

As in Kobayashi [4], if H is a Lie subgroup of $\mathrm{Gl}^{(k)}(n)$, then define an H-structure (of order k) on an n-dimensional smooth manifold M to be a smooth H-reduction of $L^{(k)}(M)$. Note that if H is closed in $\mathrm{Gl}^{(k)}(n)$, then an H-structure is just a geometric structure of type $\mathrm{Gl}^{(k)}(n)/H$; however, if H is not closed, an H-structure does not define a geometric structure in the sense of Definition 3.1.

DEFINITION 5.2. Let H be a Lie subgroup of $\mathrm{Gl}^{(k)}(n)$. An H-structure on an n-dimensional manifold M is said to be of finite type if the Lie algebra of H is of finite type.

The following result is a corollary to Theorem 3.8 and these definitions.

THEOREM 5.3. *Let H be a closed subgroup of* $\mathrm{Gl}^{(k)}(n)$. *A smooth H-structure is rigid if and only if it is of finite type.*

As shown in Kobayashi [4, I.6], there is a construction which associates a tower of principal bundles to a given H-structure of order 1. This construction makes it possible to reduce the study of the properties of a finite type structure to the study of a complete parallelism. It will now be shown that such construction can be extended to the case of reductions of higher order frame bundles.

Let H a Lie subgroup of $\mathrm{Gl}^{(k)}(n)$, and let P be a smooth H-structure given as a reduction of the frame bundle $L^{(k)}(M)$ of an n-dimensional smooth manifold M. Write the canonical form of P as $\theta_k = (\theta_k^0, \ldots, \theta_k^{k-1})$ corresponding to the natural decomposition of $\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n)$ as the direct sum of spaces of multilinear maps. Observe that a subspace $\mathcal{H}$ of $T_\alpha P$, where $\alpha \in P$, is horizontal for some connection if and only if $\theta_k^0|_{\mathcal{H}} \colon \mathcal{H} \to \mathbb{R}^n$ is an isomorphism.

For $\alpha \in P$ and a horizontal space $\mathcal{H} \subset T_\alpha P$, let $c(\alpha, \mathcal{H})$ denote the N^{k-1}-valued antisymmetric bilinear map on $\mathbb{R}^n$ given by

$$c(\alpha, \mathcal{H})(v_1, v_2) = d\theta_k^{k-1}(\theta_k^0|_{\mathcal{H}}^{-1}(v_1), \theta_k^0|_{\mathcal{H}}^{-1}(v_2)),$$

for every $v_1, v_2 \in \mathbb{R}^n$. If Θ is the torsion of a connection on P for which $\mathcal{H}$ is horizontal, then

$$c(\alpha, \mathcal{H})(v_1, v_2) = \Theta_\alpha^{k-1}(\theta_k^0|_{\mathcal{H}}^{-1}(v_1), \theta_k^0|_{\mathcal{H}}^{-1}(v_2)),$$

where Θ^{k-1} denotes the N^{k-1}-component of Θ.

Let $\mathfrak{h}$ be the Lie algebra of H and let $\partial \colon \mathfrak{h} \cap N^k \otimes \mathbb{R}^{n*} \to N^{k-1} \otimes \wedge^2 \mathbb{R}^{n*}$ be the linear map defined by

$$\begin{aligned} \partial f(v_1, v_2) &= \frac{1}{2}(f(v_1)_{(1)} v_2 - f(v_2)_{(1)} v_1) \\ &= \frac{1}{2}(\lambda(f(v_1)) v_2 - \lambda(f(v_2)) v_1) \end{aligned}$$

where $f \in \mathfrak{h} \cap N^k \otimes \mathbb{R}^{n*}$, $v_1, v_2 \in \mathbb{R}^n$ and λ is given as in Lemma 2.3. Note that use was made of the fact that $\lambda(T)(v) = T_{(1)}(v) \in N^{k-1}$, for every $T \in N^k$ and $v \in \mathbb{R}^n$, an identity which is an easy consequence of the definition of λ.

The assignment $L \mapsto L_{(1)}$ realizes $\mathfrak{h}_1$ as subspace of $\mathfrak{h} \cap N^k \otimes \mathbb{R}^{n*}$, and $\ker(\partial) = \mathfrak{h}_1$ under this identification.

Choose a vector space $\mathcal{C}$ such that $N^{k-1} \otimes \wedge^2 \mathbb{R}^{n*} = \partial(\mathfrak{h} \cap N^k \otimes \mathbb{R}^{n*}) \oplus \mathcal{C}$, and choose a vector space $\mathfrak{H}$ such that $\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n) = \pi_{k-1}^k(\mathfrak{h}) \oplus \mathfrak{H}$, where $\pi_{k-1}^k(\mathfrak{h})$ is the natural projection of $\mathfrak{h}$ into $\mathfrak{gl}^{(k-1)}(n)$. Such choices will be fixed for the rest of this section.

DEFINITION 5.4. With the above setup, the first prolongation P_1 of P is the subset of $L(P)$ consisting of frames $\mathbb{R}^n \oplus \mathfrak{h} \to T_\alpha P$ of the form

$(v, A) \mapsto \theta_k^0|_{\mathcal{H}}^{-1}(v) + A_\alpha^*$, where $\mathcal{H}$ is a horizontal space tangent to P at α that satisfies

1. $c(\alpha, \mathcal{H}) \in \mathcal{C}$,
2. $\theta_k(\mathcal{H}) \subset \mathfrak{H}$.

Observe that the first prolongation of a reduction depends on the choices of vector spaces $\mathcal{C}$ and $\mathfrak{H}$ made above.

The group introduced in the following definition is fundamental in the description of the first prolongation of a reduction.

DEFINITION 5.5. Let H be a Lie subgroup of $\mathrm{Gl}^{(k)}(n)$ with Lie algebra $\mathfrak{h}$. The first prolongation H_1 of H is the closed subgroup of $\mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{h})$ consisting of the linear transformations of the form

$$\begin{aligned} \mathbb{R}^n \oplus \mathfrak{h} &\to \mathbb{R}^n \oplus \mathfrak{h} \\ (v, A) &\mapsto (v, L_{(1)}(v) + A), \end{aligned}$$

where $L \in \mathfrak{h}_1$.

It is a simple matter to show that H_1 is a vector group with Lie algebra isomorphic to $\mathfrak{h}_1$.

The following auxiliary results will be required for understanding the basic properties of the first prolongation.

LEMMA 5.6. *Let P be a smooth H-structure on an n-dimensional smooth manifold M, where H is a Lie subgroup of $\mathrm{Gl}^{(k)}(n)$. Fix $\alpha \in P$. If ω and ω' are connections on P with horizontal spaces $\mathcal{H}$ and $\mathcal{H}'$ at α, respectively, such that $\theta_k(\mathcal{H}), \theta_k(\mathcal{H}') \subset \mathfrak{H}$, then*

1. *$\omega(\mathcal{H}') \subset \mathfrak{h} \cap N^k$,*
2. *$c(\alpha, \mathcal{H}) - c(\alpha, \mathcal{H}') = \partial f$, where $f \colon \mathbb{R}^n \to \mathfrak{h} \cap N^k$ is given by $f = \omega \circ \theta_k^0|_{\mathcal{H}'}^{-1}$.*

In particular, if both $c(\alpha, \mathcal{H})$ and $c(\alpha, \mathcal{H}')$ are in $\mathcal{C}$, then $f \in \mathfrak{h}_1$.

Proof. Let $\xi' \in \mathcal{H}'$ be given and write $\xi' = \xi + A_\alpha^*$, where $\xi \in \mathcal{H}$ and $A \in \mathfrak{h}$. Then, by the properties of θ_k,

$$\theta_k(\xi') - \theta_k(\xi) = \theta_k(A_\alpha^*) = \pi_{k-1}^k(A).$$

This is a vector that lies both in $\mathfrak{H}$ and in $\pi_{k-1}^k(\mathfrak{h})$ and it thus vanishes. Hence $A \in \mathfrak{h} \cap N^k$ and (1) follows since $\omega(\xi') = \omega(A_\alpha^*) = A$.

Note that (1) implies that $\omega(\theta_k^0|_{\mathcal{H}'}^{-1}(u)) \in \mathfrak{h} \cap N^k$, for every $u \in \mathbb{R}^n$. From the definition of the representation λ considered in Lemma 2.3 it is easy to prove that $\lambda(N^k)(\mathfrak{gl}^{(k-1)}(n)) = 0$, and so

$$\begin{aligned} \omega(\theta_k^0|_{\mathcal{H}'}^{-1}(u))(\theta_k(\theta_k^0|_{\mathcal{H}'}^{-1}(v))) &= \omega(\theta_k^0|_{\mathcal{H}'}^{-1}(u))(\theta_k^0(\theta_k^0|_{\mathcal{H}'}^{-1}(v))) \\ &= \omega(\theta_k^0|_{\mathcal{H}'}^{-1}(u))(v) \end{aligned}$$

It also follows from (1) that $[\omega(\mathcal{H}'), \omega(\mathcal{H}')] = 0$. Thus property (2) is obtained by first replacing $\xi_1 = \theta_k^0|_{\mathcal{H}'}^{-1}(u)$ and $\xi_2 = \theta_k^0|_{\mathcal{H}'}^{-1}(v)$ in the structural equation from Proposition 4.4, then projecting the resulting identity from $\mathfrak{gl}^{(k-1)}(n)$ into N^{k-1}, and finally using the definitions of $c(\alpha, \mathcal{H})$ and $c(\alpha, \mathcal{H}')$.

In particular, if both $c(\alpha, \mathcal{H})$ and $c(\alpha, \mathcal{H}')$ are in $\mathcal{C}$, then $\partial f = 0$, which implies that $f \in \mathfrak{h}_1$.

LEMMA 5.7. *Let P be a smooth H-structure on an n-dimensional smooth manifold M, where H is a Lie subgroup of* $\mathrm{Gl}^{(k)}(n)$*. Then for every $\alpha \in P$ there exists a horizontal space $\mathcal{H} \subset T_\alpha P$ such that $c(\alpha, \mathcal{H}) \in \mathcal{C}$ and $\theta_k(\mathcal{H}) \subset \mathfrak{H}$.*

Proof. Let $\mathcal{H}_1$ be any horizontal subspace of $T_\alpha P$ and choose a basis $\xi_1, \ldots \xi_n$ for $\mathcal{H}_1$. Write $\theta_k(\xi_i) = v_i + \pi_{k-1}^k(A_i)$, with $v_i \in \mathfrak{H}$ and $A_i \in \mathfrak{h}$, corresponding to the decomposition $\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n) = \mathfrak{H} \oplus \pi_{k-1}^k(\mathfrak{h})$. It easily follows that the space $\mathcal{H}_2$ generated by $\xi_1 - (A_1)_\alpha^*, \ldots, \xi_n - (A_n)_\alpha^*$ is horizontal and satisfies $\theta_k(\mathcal{H}_2) \subset \mathfrak{H}$.

Write $c(\alpha, \mathcal{H}_2) = \partial g + C$, with $g \in \mathfrak{h} \cap N^k \otimes \mathbb{R}^{n*}$ and $C \in \mathcal{C}$, corresponding to the decomposition $N^{k-1} \otimes \wedge^2 \mathbb{R}^{n*} = \partial(\mathfrak{h} \cap N^k \otimes \mathbb{R}^{n*}) \oplus \mathcal{C}$. Choose a connection ω on P so that at α it is given by

$$\begin{aligned} T_\alpha P &\rightarrow \mathfrak{h} \\ \xi' + A_\alpha^* &\mapsto -g(\theta_k^0(\xi')) + A \end{aligned}$$

where $\xi' \in \mathcal{H}_2$ and $A \in \mathfrak{h}$. Such connection exists because the above expression maps $A_\alpha^* \mapsto A$ for every $A \in \mathfrak{h}$. Let $\mathcal{H}$ be the horizontal space determined by ω at α. It will now be shown that $\mathcal{H}$ satisfies the required conditions.

For $\xi \in \mathcal{H}$, write $\xi = \xi' + A_\alpha^*$ with $\xi' \in \mathcal{H}_2$ and $A \in \mathfrak{h}$. Then

$$0 = \omega(\xi) = -g(\theta_k^0(\xi')) + A,$$

which implies that $A \in \mathfrak{h} \cap N^k$. Hence

$$\theta_k(\xi) = \theta_k(\xi') + \pi_{k-1}^k(A) = \theta_k(\xi') \in \mathfrak{H},$$

and so $\theta_k(\mathcal{H}) \subset \mathfrak{H}$.

On the other hand, note that $\omega\circ\theta^0_k|^{-1}_{\mathcal{H}_2} = -g$, and so, by Lemma 5.6(2),

$$c(\alpha,\mathcal{H}) = c(\alpha,\mathcal{H}_2) - \partial g = C \in \mathcal{C},$$

thus concluding the proof.

The next result shows that the first prolongation of a smooth reduction is also a smooth reduction.

PROPOSITION 5.8. *Let P a smooth H-structure on an n-dimensional smooth manifold M, where H is a Lie subgroup of $\mathrm{Gl}^{(k)}(n)$. Then the first prolongation, P_1, is a smooth reduction of $L(P)$ to H_1.*

Proof. Let $\pi\colon L(P) \to P$ be the canonical projection. It follows from Lemma 5.7 that the set $\pi^{-1}(\alpha) \cap P_1$ is nonempty for every $\alpha \in P$.

Fix $\alpha \in P$, let $\mathcal{H}$ be a horizontal space tangent to P at α that defines an order 1 frame of P lying in P_1, and let $T_{\mathcal{H}}\colon \mathbb{R}^n \oplus \mathfrak{h} \to T_\alpha P$ denote the frame defined by $\mathcal{H}$.

If $\mathcal{H}'$ is any other horizontal space tangent to P at α that determines an element of P_1, then a straightforward computation using Lemma 5.6(1) shows that

$$T_{\mathcal{H}'} = T_{\mathcal{H}} \circ T,$$

where $T_{\mathcal{H}'}$ is the frame defined by $\mathcal{H}'$ and T is the linear isomorphism given by

$$\begin{array}{rcl} \mathbb{R}^n \oplus \mathfrak{h} & \to & \mathbb{R}^n \oplus \mathfrak{h} \\ (v, A) & \mapsto & (v, \omega(\theta^0_k|^{-1}_{\mathcal{H}'}(v)) + A). \end{array}$$

By the choice of $\mathcal{H}$ and $\mathcal{H}'$ it follows from Lemma 5.6 that $\omega\circ\theta^0_k|^{-1}_{\mathcal{H}'} \in \mathfrak{h}_1$ and so $T \in H_1$, from which it obtains that $\pi^{-1}(\alpha) \cap P_1$ is contained in an H_1-orbit.

Conversely, let $T \in H_1$ be given and consider the frame $T_{\mathcal{H}} \circ T$. Then there exists $L \in \mathfrak{h}_1$ such that $T(v, A) = (v, L_{(1)}(v) + A)$, for every $(v, A) \in \mathbb{R}^n \oplus \mathfrak{h}$. Hence

$$T_{\mathcal{H}} \circ T(v, A) = \theta^0_k|^{-1}_{\mathcal{H}}(v) + L_{(1)}(v)^*_\alpha + A^*_\alpha,$$

for every $(v, A) \in \mathbb{R}^n \oplus \mathfrak{h}$. Then it is easy to check that $\mathcal{H}'$, defined as the image of the map $v \mapsto \theta^0_k|^{-1}_{\mathcal{H}}(v) + L_{(1)}(v)^*_\alpha$, is a horizontal space tangent to P at α which defines a frame $T_{\mathcal{H}'}$ lying in P_1 and satisfying $T_{\mathcal{H}'} = T_{\mathcal{H}} \circ T$. It follows that $\pi^{-1}(\alpha) \cap P_1$ is an H_1-orbit.

Finally, note that the proof of Lemma 5.7 can be modified to obtain a smooth local section of $L(P)$ defined in a neighborhood of α and with image lying in P_1, thus proving P_1 is a smooth H_1-reduction.

The previous result permits to inductively define higher prolongations for any given reduction.

DEFINITION 5.9. Let H be a Lie subgroup of $\mathrm{Gl}^{(k)}(n)$ and P a smooth H-structure on an n-dimensional smooth manifold M. Let $P_0 = P$ and $P_{-k} = M$. For $r \geq 1$, define the r-th prolongation P_r of P to be the reduction of $L(P_{r-1})$ given by the first prolongation of P_{r-1}.

The following result is a consequence of Lemma 3.6 and of the above discussion.

PROPOSITION 5.10. *Let P be a smooth H-structure on an n-dimensional smooth manifold M, where H is a Lie subgroup of $\mathrm{Gl}^{(k)}(n)$. Then, for every $r \geq 1$, the r-th prolongation P_r of P is a smooth reduction of $L(P_{r-1})$ to a vector subgroup H_r whose Lie algebra is isomorphic to $\mathfrak{h}_r$, the r-th prolongation of the Lie algebra $\mathfrak{h}$ of H.*

The prolongations of a reduction depend on the choices of the vector spaces $\mathfrak{H}$ and $\mathcal{C}$. However, since the construction of such prolongations is given in terms of the canonical form, a diffeomorphism that preserves the reduction also preserves each of its prolongations. The following statement makes this precise.

PROPOSITION 5.11. *Let P be a smooth H-structure on an n-dimensional smooth manifold M, where H is a Lie subgroup of $\mathrm{Gl}^{(k)}(n)$. If φ a diffeomorphism of M that preserves P (that is, such that $\varphi_{(k)}(P) = P$), then there is sequence of diffeomorphisms $\varphi_i \colon P_i \to P_i$ such that $\varphi_{i+1} = (\varphi_i)_{(1)}|_{P_{i+1}}$. Moreover, if $\mathfrak{h}$ is of finite type, then there exists an integer r such that P_r carries a complete parallelism which is preserved by the map φ_r induced by φ.*

Proof. This is a simple exercise in the definition of prolongation. It uses the fact that if f is a diffeomorphism of a manifold, then the induced map $f_{(k)}$ on the k-th order frame bundle preserves the canonical form θ_k.

Because of Theorem 5.3, if H is closed in $\mathrm{Gl}^{(k)}(n)$, then this result permits to reduce the study of the automorphisms of any rigid geometric structure of type $\mathrm{Gl}^{(k)}(n)/H$ to the study of the automorphisms of a complete parallelism. The next goal is to extend this sort of property to infinitesimal Killing fields, which will be accomplished following the scheme that was used in [2].

Let H be a Lie subgroup of $\mathrm{Gl}^{(k)}(n)$, and let $\mathfrak{h}$ be its Lie algebra. If P a smooth H-structure on an n-dimensional smooth manifold M, then define the principal prolongation $W^1(P)$ of P as the space of

1-jets at $(0, I)$ of local bundle automorphisms $\mathbb{R}^n \times H \to P$. Then $W^1(P)$ is a smooth reduction of $L(P)$ (as a bundle over P) to the group $\mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*}$ (whose semidirect product structure is given by $(A, T)(A', T') = (AA', T \circ A' + T')$, where both T, T' are viewed as linear maps $\mathbb{R}^n \to \mathfrak{h}$). A proof of these properties can be found in Kolář, Michor and Slovák [6]. Observe that the injective homomorphism $\rho\colon \mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*} \to \mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{h})$ given by $\rho(A, T)(v, X) = (Av, Tv + X)$ realizes $\mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*}$ as a closed subgroup of $\mathrm{Gl}(\mathbb{R}^n \oplus \mathfrak{h})$ containing H_1.

Choose vector spaces $\mathcal{C}$ and $\mathfrak{H}$ as in Definition 5.4, and consider the map from the quotient $(\mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*})/H_1$ into the space

$$\mathrm{Gl}(n) \times \frac{(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n)) \otimes \mathbb{R}^{n*}}{\mathfrak{H} \otimes \mathbb{R}^{n*}} \times \frac{N^{k-1} \otimes \wedge^2 \mathbb{R}^{n*}}{\mathcal{C}}$$

given by the assignment

$$(A, T)H_1 \mapsto (A, [\pi^k_{k-1} \circ T], [\partial(\pi^k \circ T)]),$$

where $\pi^k_{k-1}\colon \mathfrak{gl}^{(k)}(n) \to \mathfrak{gl}^{(k-1)}(n)$ and $\pi^k\colon \mathfrak{gl}^{(k)}(n) \to N^k$ are the natural projections. This map is a $\mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*}$-equivariant diffeomorphism with respect to the smooth action of $\mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*}$ on the target given by

$$(A, T)(B, [L], [\Lambda]) = (AB, [\pi^k_{k-1}(T \circ B) + L], [\partial(\pi^k \circ T \circ B) + \Lambda]).$$

With such diffeomorphism, the next result describes the first prolongation as a reduction of the principal prolongation. Its proof uses the same kind of arguments as those found in the proof of Lemma 7.8 in [2] and requires only some minor extra calculations.

LEMMA 5.12. *Let P be a smooth H-structure on an n-dimensional smooth manifold M, where H is a Lie subgroup of $\mathrm{Gl}^{(k)}(n)$ with Lie algebra $\mathfrak{h}$. Let $\mathcal{C}$, $\mathfrak{H}$ be vector spaces as in Definition 5.4, and let σ' be the smooth map given by*

$$\begin{aligned} \sigma'\colon W^1(P) &\to \mathrm{Gl}(n) \times \frac{(\mathbb{R}^n \oplus \mathfrak{gl}^{(k-1)}(n)) \otimes \mathbb{R}^{n*}}{\mathfrak{H} \otimes \mathbb{R}^{n*}} \times \frac{N^{k-1} \otimes \wedge^2 \mathbb{R}^{n*}}{\mathcal{C}} \\ j^1_{(0,I)}(\lambda) &\mapsto (d\bar{\lambda}_0^{-1} \circ \lambda(0, I), [\theta_k \circ d\widehat{\lambda}_0 \circ d\bar{\lambda}_0^{-1} \circ \lambda(0, I)], [c(\lambda(0, I), \mathcal{H}_\lambda)]) \end{aligned}$$

where $\bar{\lambda}$ is the local diffeomorphism covered by λ and $\mathcal{H}_\lambda$ is the horizontal space tangent to P given by the image of the differential at 0 of the map $x \mapsto \widehat{\lambda}(x) = \lambda(x, I)$. Then σ' is $\mathrm{Gl}(n) \ltimes \mathfrak{h} \otimes \mathbb{R}^{n*}$-equivariant and realizes P_1 as the H_1-reduction of $W^1(P)$ given by $P_1 = \sigma'^{-1}(I, 0, 0)$.

The next result shows that for rigid geometric structures of type $\mathrm{Gl}^{(k)}(n)/H$ the study of infinitesimal Killing fields can be reduced to the study of infinitesimal Killing fields for complete parallelisms. Its proof uses the previous lemma and arguments like those in [2, Section 7].

PROPOSITION 5.13. *Let P be a smooth H-structure on an n-dimensional smooth manifold M, where H is a closed subgroup of $\mathrm{Gl}^{(k)}(n)$, and let σ denote the corresponding geometric structure of order k and type $\mathrm{Gl}^{(k)}(n)/H$. Let $x_{-k} = x \in M$, let $x_0 \in P$ be in the fiber over x, and, for each $i \geq 1$, let $x_i \in P_i$ be a point lying in the fiber of P_i over x_{i-1}. Then, for every infinitesimal Killing field $j^s_x(X) \in \mathrm{Kill}^s(\sigma, x)$ and for every $i = -k, 0, 1, \ldots, s-k-1$, there exists a vector field X_i defined in a neighborhood of x_i in P_i such that:*

1. *$X_{-k} = X$,*
2. *X_0 and $(X_{-k})_{(k)}|_P$ coincide up to order $s-k$ at x_0 as sections of $TL^{(k)}(M)|_P$,*
3. *X_i and $(X_{i-1})_{(1)}|_{P_i}$ coincide up to order $s-k-i$ at x_i as sections of $TL(P_{i-1})|_{P_i}$, for $i = 1, \ldots, s-k-1$,*
4. *$j^{s-k-i}_{x_i}(X_i)$ is an infinitesimal Killing field for the geometric structure σ_{i+1} on P_i defined by P_{i+1}, for $i = 0, 1 \ldots, s-k-1$.*

Moreover, for every $i = -k, 0, 1 \ldots, s-k-1$, the assignment $j^s_x(X) \mapsto j^{s-k-i}_{x_i}(X_i)$ defines a linear map $\mathrm{Kill}^s(\sigma, x) \to \mathrm{Kill}^{s-k-i}(\sigma_{i+1}, x_i)$. In particular, when σ is rigid (hence of finite type) and r is the first integer for which the prolongation P_{r+1} is a trivialization of $L(P_r)$, if $s > k+r$ and $j^s_x(X) \in \mathrm{Kill}^s(\sigma, x)$, then $j^{s-k-r}_{x_r}(X_r)$ is an infinitesimal Killing field for the corresponding complete parallelism of P_r.

This implies the following result regarding the extension of infinitesimal Killing fields to local ones for rigid geometric structures.

PROPOSITION 5.14. *Let M be an analytic manifold endowed with a rigid analytic geometric structure σ of order k defined by a reduction of $L^{(k)}(M)$. For every $x \in M$ there exists an integer $r(x)$ such that if $r \geq r(x)$ and $j^r_x(X) \in \mathrm{Kill}^r(\sigma, x)$, then there is a unique analytic local Killing vector field Y defined in a neighborhood of x so that $j^r_x(Y) = j^r_x(X)$.*

The proof of this proposition uses the fact that the problem of extending infinitesimal Killing fields to local ones, for the rigid geometric structures considered in the statement, can be reduced to the corresponding problem for complete parallelisms. It also depends on the fact that every construction and claim obtained up to this point in the smooth category remains valid in the analytic category, without need of further arguments. Details are omitted because they follow essentially the same steps used in [2, Section 7].

6. Gromov's centralizer theorem

The proof of the next result, Gromov's centralizer theorem, is obtained following the arguments used in [2, Section 9] together with Proposition 5.14. The proof also uses the fact that the arguments in Amores [1] can be easily modified to extend local Killing fields to global Killing fields for the setup considered below. The reader is referred to [2] for the definition of a Zariski measure; here it is noted that any smooth measure on an analytic manifold is a Zariski measure.

THEOREM 6.1. *Let M be a connected analytic manifold endowed with an analytic rigid geometric structure σ defined by a reduction of a frame bundle to an algebraic group.*

Let G be a connected, noncompact, simple Lie group acting analytically on M, preserving both σ and a finite Zariski measure.

Let $\mathcal{G}$ be the Lie algebra of Killing vector fields on the universal cover $\widetilde{M}$ induced by the action of the universal cover $\widetilde{G}$ of G. If $\mathcal{V}$ denotes the space of analytic Killing vector fields on $\widetilde{M}$ that centralize $\mathcal{G}$, then

1. *$\mathcal{V}$ is $\pi_1(M)$-invariant,*
2. *$\mathcal{V}$ is finite dimensional,*
3. *there exists an open, conull subset $\widetilde{U}$ of $\widetilde{M}$, invariant under both $\widetilde{G}$ and $\pi_1(M)$, on which $\widetilde{G}$ acts locally freely and such that $\mathrm{ev}_x(\mathcal{V}) \supset T_x\widetilde{G}x$ for every $x \in \widetilde{U}$.*

Here, for a point $x \in M$ and a vector field X defined in a neighborhood of x, $\mathrm{ev}_x(X) = X_x$ is the evaluation map. Also, if the measure on M is smooth, then $\widetilde{U}$ can be assumed to be dense.

This theorem has a variety of consequences for the structure of actions of simple Lie groups. Two fundamental corollaries are given below and further details can be found in [2, Section 10], all of whose statements extend with the same proof to the present setting.

In the following two results, the manifold M and the group G are as in Theorem 6.1 and are assumed to satisfy its hypothesis.

THEOREM 6.2. *There is a representation* $\rho\colon \pi_1(M) \to \mathrm{Gl}(q)$ *such that the Zariski closure of* $\rho(\pi_1(M))$ *contains a subgroup locally isomorphic to* G.

THEOREM 6.3. *Suppose that* G *has finite center and finite fundamental group. Then the action of* G *on* M *is topologically engaging on a conull, open subset of* M. *Moreover, there exists a conull, open set* $\widetilde{U} \subset \widetilde{M}$, *which is invariant under both* $\widetilde{G}$ *and* $\pi_1(M)$, *and such that the* $\widetilde{G}$-*orbit of each of its points is closed in* $\widetilde{U}$. *Furthermore, if the measure is smooth, then the open sets in both* M *and* $\widetilde{M}$ *where the topological engagement condition is satisfied can be assumed to be dense.*

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