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RIGID LEFT NOETHERIAN RINGS

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We prove that any rigid left Noetherian ring is either a domain or isomorphic to some ring $\mathbb{Z}_{p^n}$ of integers modulo a prime power p^n.

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Let R be an associative ring. A map $\sigma : R \to R$ is called a ring endomorphism if $\sigma(x+y) = \sigma(x)+\sigma(y)$ and $\sigma(xy) = \sigma(x)\sigma(y)$ for all elements $a,b \in R$. A ring R is said to be rigid if it has only the trivial ring endomorphisms, that is, identity id_R and zero 0_R. Rigid left Artinian rings were described by Maxson [9] and McLean [11]. Friger [4, 6] has constructed an example of a noncommutative rigid ring R with the additive group R^+ of finite Prüfer rank. A characterization for rigid rings of finite rank was obtained by the author in [1]. Some aspects of a ring rigidity has been studied by Suppa [12, 13], Friger [5], and the author [2].

In this paper, we study rigid left Noetherian rings and prove the following theorem.

Theorem 1. *Let R be a left Noetherian ring. Then R is a rigid ring if and only if $R \cong \mathbb{Z}_{p^t}$ (p is a prime, $t \in \mathbb{N}$) or it is a rigid domain.*

All rings are assumed to be associative and, as a rule, with an identity element. For a ring R, $N(R)$ will always denote the set of all nil elements of R, $\mathrm{char}(R)$ the characteristic, and $\mathrm{Ann}(I) = \{a \in R \mid aI = Ia = \{0\}\}$ the annihilator of I in R. If R is a left order in Q (or equivalently, Q is the left quotient ring of R), then we will write $Q = Q(R)$. Any unexplained terminology is standard as in [10].

We recall that a ring R is reduced if $r^2 = 0$ implies $r = 0$ for any $r \in R$. Clearly, if R is a rigid reduced ring with an identity element, then either $\mathrm{char}(R) = 0$ or $\mathrm{char}(R) = p$ for some prime p.

Lemma 2. *Let R be a reduced left Goldie ring. If R is rigid, then it is a domain.*

Proof. Let R be a reduced rigid left Goldie ring. Assume that R is not a domain. From $bx = 0$ (resp., $xb = 0$), where $b,x \in R$, it holds that $(xb)^2 = 0$ (resp., $(bx)^2 = 0$) and thus a right (resp., left) annihilator of every element b in R coincides with $\mathrm{Ann}(b)$. Moreover, in view of [10, Lemma 2.3.2(i)], $\mathrm{Ann}(a)$ is a maximal left annihilator for some $a \in R$.

Assume that the quotient ring $R/\mathrm{Ann}(a)$ contains elements $\overline{x} = x+\mathrm{Ann}(a) \neq \overline{0}$, $\overline{y} = y+\mathrm{Ann}(a)$ such that

$$\overline{x}\,\overline{y} = \overline{0} \tag{1}$$

for some $x, y \in R$. Since $y \in \operatorname{Ann}(ax)$ and $\operatorname{Ann}(a) = \operatorname{Ann}(ax)$, we obtain that $\overline{y} = \overline{0}$. This means that $R/\operatorname{Ann}(a)$ is a domain.

By [10, Lemma 2.3.3], $I_a = Ra \oplus \operatorname{Ann}(a)$ is an essential left ideal of R and so by [10, Corollary 3.1.8], $Q(I_a) = Q(R)$. Then the map $\sigma : I_a \to I_a$ given by $\sigma(ra) = ra$ $(r \in R)$ and $\sigma(\operatorname{Ann}(a)) = \{0\}$ is a nontrivial ring endomorphism of I_a. If $\overline{\sigma} : Q(R) \to Q(R)$ is an extension of σ to $Q(R)$, then

$$\overline{\sigma}(r)a = \overline{\sigma}(ra) = ra \tag{2}$$

for any $r \in R$, in which case,

$$a(\overline{\sigma}(r) - r) = 0 = (\overline{\sigma}(r) - r)a. \tag{3}$$

Since $\overline{\sigma}(r) - r = q^{-1}t$ for some regular element $q \in R$ and some $t \in R$, we see that

$$q(\overline{\sigma}(r) - r) \in \operatorname{Ann}(a). \tag{4}$$

But $q \notin \operatorname{Ann}(a)$ and so $\overline{\sigma}(r) - r \in \operatorname{Ann}(a)$. This means that $\overline{\sigma}(R) \subseteq R$ and R has a nontrivial ring endomorphism, a contradiction. The lemma is proved. □

In the commutative case, we obtain that a commutative reduced rigid Noetherian ring R of finite exponent is isomorphic to some $\mathbb{Z}_p$.

Indeed, as it is noted above, $\operatorname{char}(R) = p$ for some prime p. A map $\omega : R \to R$ given by the rule $\omega(x) = x^p$ $(x \in R)$ is a ring endomorphism of R and so $x^p = x$ for all elements x of R. Assume that R is not a domain and then it follows that every prime ideal is maximal in R. Hence R is an Artinian ring by Krull-Akizuki theorem [14, Chapter IV, Section 2, Theorem 2] and by the theorem of [11], $R \cong \mathbb{Z}_p$, contrary to our assumption. This means that R is a domain and [9, Theorem 2.5] allows us to state that $R \cong \mathbb{Z}_p$.

Remark 3. Maxson [9] has proved that a rigid commutative domain of prime characteristic p is isomorphic to $\mathbb{Z}_p$. Rigid rings of finite rank were studied in [1]. A characterization of rigid commutative domains (in particular, rigid fields) R of characteristic 0 with the additive group R^+ of infinite (Prüfer) rank is not known. As it is noted in [8], from the result of Gaifman [7], it holds that there exist rigid Peano fields of arbitrary infinite cardinality. Moreover, it was proved by Dugas and Göbel [3] that each field can be embedded into a rigid field of arbitrary large cardinality.

Remark 4. There exist noncommutative rigid Noetherian domains of characteristic 0 (see [4, 6]).

Recall that a map $d : R \to R$ is called a derivation of R if

$$d(x+y) = d(x) + d(y), \qquad d(xy) = d(x)y + xd(y) \tag{5}$$

for all elements $x,y \in R$. A ring having no nonzero derivations is called differentially trivial (see [1]). Obviously, any differentially trivial ring is commutative.

Lemma 5. *Let R be a left Noetherian ring such that $N(R) \neq \{0\}$. If R is a rigid ring, then it is isomorphic to some $\mathbb{Z}_{p^t}$.*

Proof. Suppose that R is a rigid ring such that $N = N(R) \neq \{0\}$. Then $N \subseteq Z(R)$ (see [9, page 96]). Let d be any nonzero derivation of R. If $zd(R) = \{0\}$ for all elements $z \in N$ of the nilpotency indices $i < n-1$ and $ad(R) \neq \{0\}$ for some element $a \in N$ of the nilpotency index n, then the rule

$$\sigma(r) = r + ad(r), \quad r \in R, \tag{6}$$

determines a nontrivial ring endomorphism σ of R, a contradiction. Hence

$$N(R)d(R) = \{0\} \tag{7}$$

for every derivation d of R.

Let $K_0 = \{a \in N \mid (N \cap \operatorname{Ann}(N^2))a = \{0\}\}$. Then $N \cap \operatorname{Ann}(K_0) = N \cap \operatorname{Ann}(N^2)$. Assume that $\delta : R/K_0 \to R/K_0$ is a nonzero derivation of R/K_0 and therefore for every $r \in R$, there is an element $r_1 \in R$ such that

$$\delta(r + K_0) = r_1 + K_0. \tag{8}$$

Moreover, $a_1 \notin K_0$ for some $a \in R$. Writing I for the two-sided ideal of R generated by a_1, we see that $(N \cap \operatorname{Ann}(N^2))(K_0 + I) \neq \{0\}$. Thus there exists an element $m_0 \in N \cap \operatorname{Ann}(N^2)$ such that $m_0 a_1 \neq 0$ and so the rule $g(r) = m_0 r_1$, with $r \in R$ and r_1 as in (8), determines a nonzero derivation g of R. In view of (7) $g(r)g(t) = 0$, for any elements $r,t \in R$ and a map $\alpha : R \to R$ given by the rule $\alpha(r) = r + g(r)$, $(r \in R)$ is a nontrivial ring endomorphism of R, a contradiction with hypothesis. This gives that R/K_0 is differentially trivial and consequently commutative. Since $K_0 \subseteq N$ and $N \subseteq Z(R)$, R is a Noetherian ring and, as a consequence of [10, Theorem 4.1.9] and [9, Theorem 2.2], R is an Artinian ring. Finally, by the theorem from [11], $R \cong \mathbb{Z}_{p^t}$ for some prime p and integer t. This completes the proof. □

Proof of Theorem 1. It follows immediately from Lemmas 2 and 5. □

Corollary 6. *Any rigid simple left Goldie ring R is a field (or equivalently, any noncommutative simple left Goldie ring has a nontrivial automorphism).*

Proof. Since $N(R) \subseteq Z(R)$, R is a semiprime ring and so according to [10, Proposition 5.1.5] and Lemma 2, it is a domain. If q is any element of $Q(R) \setminus R$ and $A = q^{-1}Rq$, then A is a left order in $Q(R)$. Moreover, $qAq^{-1} = R$ and so A and R are equivalent left orders in $Q(R)$. By [10, Proposition 5.1.2], R is a maximal left order in $Q(R)$ and thus $A \subseteq R$, which implies $R \subseteq Z(Q(R))$, as required. □

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