Rigid Motion Invariant Classification of 3D-Textures

Saurabh Jain
Department of Mathematics
University of Houston

October 31st 2009
2009 Fall Southeastern Meeting, Boca Raton, Florida

Acknowledgements

This work has been performed in collaboration with Prof. Robert Azencott and Prof. Manos Papadakis.

We are grateful to Simon Alexander for his suggestions and many fruitful discussions.

Outline

Textures

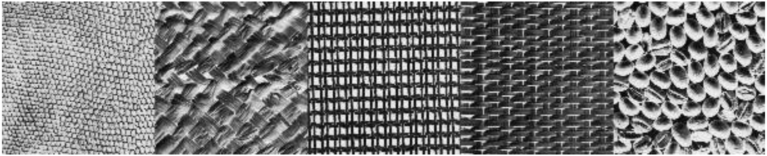


Figure: Examples of structural 2-D textures

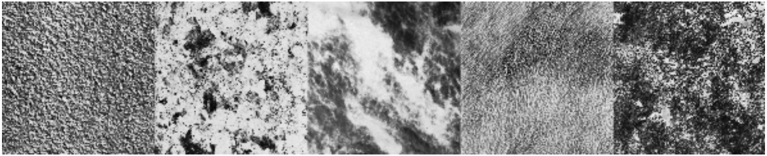
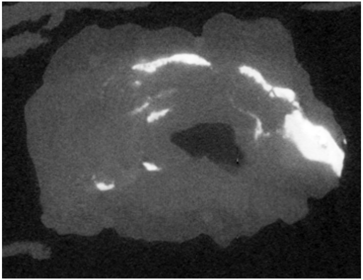
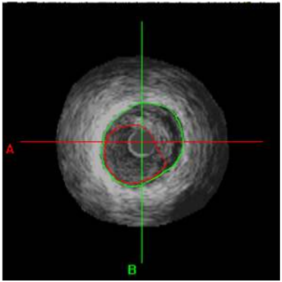


Figure: Examples of stochastic 2-D textures

Texture Examples from Biomedical Imaging



(a) 2D slice from 3D μCT x-ray data



(b) Slice from Intravascular Ultra Sound data

Figure: Examples of medical 3D data sets.

Outline

Definition

An IMRA is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ satisfying the following conditions:

- $\forall j \in \mathbb{Z},\ V_j \subset V_{j+1}$,
- $(D_{\mathfrak{M}})^j V_0 = V_j$,
- $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$,
- $\cap_{j\in\mathbb{Z}} V_j = \{0\}$,

Definition

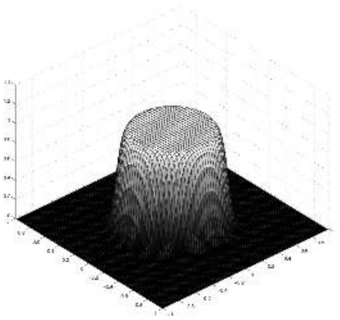
An IMRA is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ satisfying the following conditions:

- $\forall j \in \mathbb{Z},\ V_j \subset V_{j+1}$,
- $(D_{\mathfrak{M}})^j V_0 = V_j$,
- $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$,
- $\cap_{j\in\mathbb{Z}} V_j = \{0\}$,
- V_0 is invariant under translations by integers,
- V_0 is invariant under all rotations, i.e.,

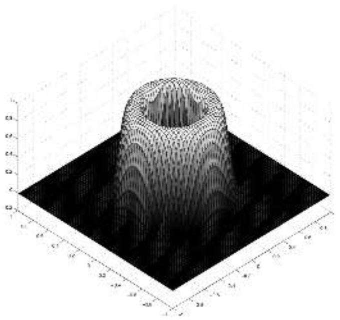
$$\mathcal{O}(R)V_0 = V_0 \qquad \text{for all } R \in SO(d),$$

where $\mathcal{O}(R)$ is the unitary operator given by $\mathcal{O}(R)f(x) := f(R^T x)$.

2D IMRA refinable function and wavelet



(a) Fourier transform of the refinable function $\hat{\phi}$



(b) Fourier transform of the wavelet $\hat{\psi}_1(2.)$

Outline

Texture Model

Let $\mathbf{X}_{cont}$ be a stationary Gaussian process on $\mathbb{R}^3$ and $\mathbf{X}$ be its samples on $\mathbb{Z}^3$.

Texture Model

Let $\mathbf{X}_{cont}$ be a stationary Gaussian process on $\mathbb{R}^3$ and $\mathbf{X}$ be its samples on $\mathbb{Z}^3$.
Let its autocovariance function ρ_{cont} be bandlimited to $\mathbb{B}_2$, the ball where $\widehat{\phi}$ is equal to 1.

Texture Model

Let $\mathbf{X}_{cont}$ be a stationary Gaussian process on $\mathbb{R}^3$ and $\mathbf{X}$ be its samples on $\mathbb{Z}^3$.
Let its autocovariance function ρ_{cont} be bandlimited to $\mathbb{B}_2$, the ball where $\widehat{\phi}$ is equal to 1.
Note that

$$\rho_{cont}(\mathbf{k}) = \mathbb{E}[\mathbf{X}_{cont}(\mathbf{k})\mathbf{X}_{cont}(0)] = \mathbb{E}[\mathbf{X}(\mathbf{k})\mathbf{X}(0)] = \rho(\mathbf{k})$$

Texture Model

Let $\mathbf{X}_{cont}$ be a stationary Gaussian process on $\mathbb{R}^3$ and $\mathbf{X}$ be its samples on $\mathbb{Z}^3$.
Let its autocovariance function ρ_{cont} be bandlimited to $\mathbb{B}_2$, the ball where $\widehat{\phi}$ is equal to 1.
Note that

$$\rho_{cont}(\mathbf{k}) = \mathbb{E}[\mathbf{X}_{cont}(\mathbf{k})\mathbf{X}_{cont}(0)] = \mathbb{E}[\mathbf{X}(\mathbf{k})\mathbf{X}(0)] = \rho(\mathbf{k})$$

Hence, $\rho_{cont} = \sum_{\mathbf{k}\in\mathbb{Z}^3} \rho(\mathbf{k})\, T_{\mathbf{k}}\phi$.

Rotation of Textures

Let $\mathcal{R}_{\boldsymbol{\alpha}}$ be the operator induced on $L^2(\mathbb{R}^3)$ by the rotation $\boldsymbol{\alpha} \in SO(3)$.

Rotation of Textures

Let $\mathcal{R}_{\boldsymbol{\alpha}}$ be the operator induced on $L^2(\mathbb{R}^3)$ by the rotation $\boldsymbol{\alpha} \in SO(3)$. The autocovariance function of $\mathcal{R}_{\boldsymbol{\alpha}}\mathbf{X}_{cont}$ is given by $\mathcal{R}_{\boldsymbol{\alpha}}\rho_{cont}$:

$$\begin{aligned} \mathbb{E}[\mathcal{R}_{\boldsymbol{\alpha}}\mathbf{X}_{cont}(\mathbf{s})\mathcal{R}_{\boldsymbol{\alpha}}\mathbf{X}_{cont}(0)] &= \mathbb{E}[\mathbf{X}_{cont}(\boldsymbol{\alpha}^T\mathbf{s})\mathbf{X}_{cont}((\boldsymbol{\alpha}^T 0)] \\ &= \rho_{cont}(\boldsymbol{\alpha}^T\mathbf{s}) = \mathcal{R}_{\boldsymbol{\alpha}}\rho_{cont}(\mathbf{s}). \end{aligned}$$

Rotation of Textures

Let $\mathcal{R}_{\boldsymbol{\alpha}}$ be the operator induced on $L^2(\mathbb{R}^3)$ by the rotation $\boldsymbol{\alpha} \in SO(3)$. The autocovariance function of $\mathcal{R}_{\boldsymbol{\alpha}} \mathbf{X}_{cont}$ is given by $\mathcal{R}_{\boldsymbol{\alpha}} \rho_{cont}$:

$$\begin{aligned} \mathbb{E}[\mathcal{R}_{\boldsymbol{\alpha}} \mathbf{X}_{cont}(\mathbf{s}) \mathcal{R}_{\boldsymbol{\alpha}} \mathbf{X}_{cont}(0)] &= \mathbb{E}[\mathbf{X}_{cont}(\boldsymbol{\alpha}^T \mathbf{s}) \mathbf{X}_{cont}((\boldsymbol{\alpha}^T 0)] \\ &= \rho_{cont}(\boldsymbol{\alpha}^T \mathbf{s}) = \mathcal{R}_{\boldsymbol{\alpha}} \rho_{cont}(\mathbf{s}). \end{aligned}$$

Now, the sequence of samples, $\langle \mathcal{R}_{\boldsymbol{\alpha}} \rho_{cont}, T_{\mathbf{k}} \phi \rangle \}_{\mathbf{k} \in \mathbb{Z}^3}$ is denoted by $\mathcal{R}_{\boldsymbol{\alpha}} \rho$.

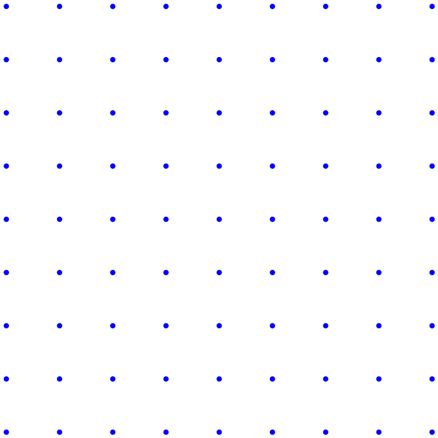
Rotation of Textures

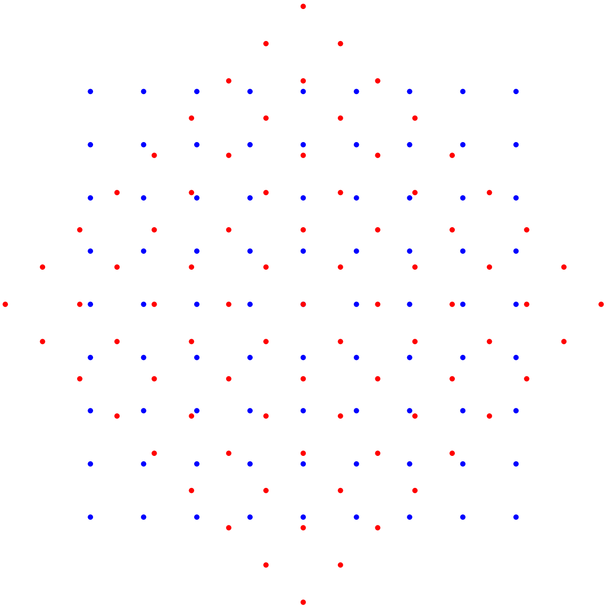
Let $\mathcal{R}_{\boldsymbol{\alpha}}$ be the operator induced on $L^2(\mathbb{R}^3)$ by the rotation $\boldsymbol{\alpha} \in SO(3)$. The autocovariance function of $\mathcal{R}_{\boldsymbol{\alpha}}\mathbf{X}_{cont}$ is given by $\mathcal{R}_{\boldsymbol{\alpha}}\rho_{cont}$:

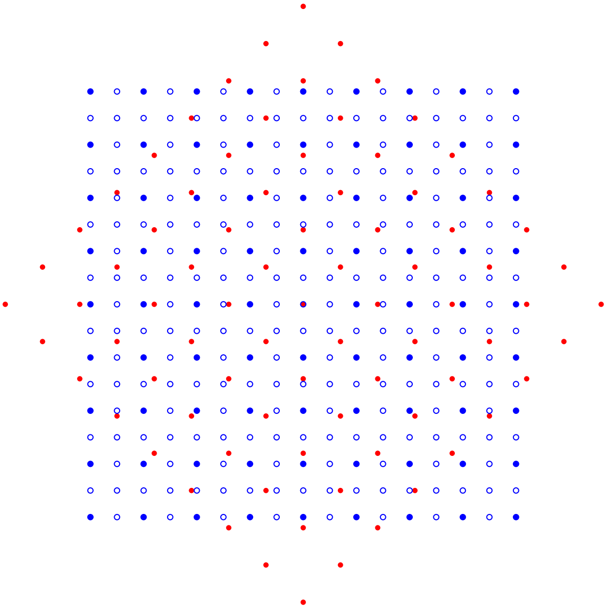
$$\begin{aligned} \mathbb{E}[\mathcal{R}_{\boldsymbol{\alpha}}\mathbf{X}_{cont}(\mathbf{s})\mathcal{R}_{\boldsymbol{\alpha}}\mathbf{X}_{cont}(0)] &= \mathbb{E}[\mathbf{X}_{cont}(\boldsymbol{\alpha}^T\mathbf{s})\mathbf{X}_{cont}((\boldsymbol{\alpha}^T 0)] \\ &= \rho_{cont}(\boldsymbol{\alpha}^T\mathbf{s}) = \mathcal{R}_{\boldsymbol{\alpha}}\rho_{cont}(\mathbf{s}). \end{aligned}$$

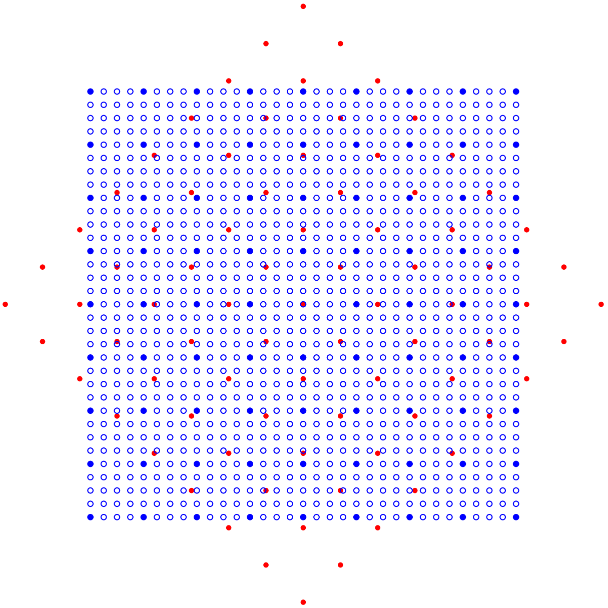
Now, the sequence of samples, $\langle \mathcal{R}_{\boldsymbol{\alpha}}\rho_{cont}, T_{\mathbf{k}}\phi\rangle\}_{\mathbf{k}\in\mathbb{Z}^3}$ is denoted by $\mathcal{R}_{\boldsymbol{\alpha}}\rho$.

$$\langle \mathcal{R}_{\boldsymbol{\alpha}}\rho_{cont}, T_{\mathbf{k}}\phi\rangle = \langle \rho_{cont}, \mathcal{R}^*_{\boldsymbol{\alpha}}T_{\mathbf{k}}\phi\rangle = \langle \rho_{cont}, T_{\boldsymbol{\alpha}\mathbf{k}}\phi\rangle$$









Gaussian Markov Random Field

A stochastic process $\mathbf{X}$ on $\mathbb{Z}^3$ is a stationary GMRF if a realization satisfies the following difference equation:

$$x_{\mathbf{k}} = \mu + \sum_{\mathbf{r} \in \eta} \theta_{\mathbf{r}} (x_{\mathbf{k}-\mathbf{r}} - \mu) + e_{\mathbf{k}}.$$

where the correlated Gaussian noise, $\mathbf{e} = (e_1, \ldots, e_{N_T})$, has the following structure:

$$\mathbb{E}[e_{\mathbf{k}} e_{\mathbf{l}}] = \begin{cases} \sigma^2, & \mathbf{k} = \mathbf{l}, \\ -\theta_{\mathbf{k}-\mathbf{l}} \sigma^2, & \mathbf{k} - \mathbf{l} \in \eta, \\ 0, & \text{else.} \end{cases}$$

Auto-covariance function

For a stationary random process $\mathbf{X}$ on $\mathbb{Z}^3$, the auto-covariance function is given by

$$\rho(\mathbf{l}) = \mathbb{E}[\mathbf{X}(\mathbf{l})\mathbf{X}(0)]$$

Auto-covariance function

For a stationary random process $\mathbf{X}$ on $\mathbb{Z}^3$, the auto-covariance function is given by

$$\rho(\mathbf{l}) = \mathbb{E}[\mathbf{X}(\mathbf{l})\mathbf{X}(0)]$$

Given a realization $\mathbf{x}$ on $\mathbf{\Lambda} \subset \mathbb{Z}^3$, ρ can be approximated by

$$\rho_0(\mathbf{l}) = \frac{1}{N_T} \sum_{\mathbf{r} \in \mathbf{\Lambda}} x_{\mathbf{r}} x_{\mathbf{r}+\mathbf{l}}, \quad \text{for all} \quad \mathbf{l} \in \mathbf{\Lambda}$$

for a sufficiently large $\mathbf{\Lambda}$; $N_T := |\mathbf{\Lambda}|$.

Auto-covariance function

For a stationary random process $\mathbf{X}$ on $\mathbb{Z}^3$, the auto-covariance function is given by

$$\rho(\mathbf{l}) = \mathbb{E}[\mathbf{X}(\mathbf{l})\mathbf{X}(0)]$$

Given a realization $\mathbf{x}$ on $\mathbf{\Lambda} \subset \mathbb{Z}^3$, ρ can be approximated by

$$\rho_0(\mathbf{l}) = \frac{1}{N_T} \sum_{\mathbf{r} \in \mathbf{\Lambda}} x_{\mathbf{r}} x_{\mathbf{r}+\mathbf{l}}, \quad \text{for all} \quad \mathbf{l} \in \mathbf{\Lambda}$$

for a sufficiently large $\mathbf{\Lambda}$; $N_T := |\mathbf{\Lambda}|$.
The parameters of the GMRF model fitted to the 'rotated texture', denoted by $\mathcal{R}_{\boldsymbol{\alpha}}\mathbf{x}$, can be calculated using $\mathcal{R}_{\boldsymbol{\alpha}}\rho$.

Rotationally Invariant Distance

We define the texture signature $\Gamma_{\mathbf{x}}$, via

$$\Gamma_{\mathbf{x}}(\boldsymbol{\alpha}) = \left[\widehat{\boldsymbol{\theta}}(\mathcal{R}_{\boldsymbol{\alpha}}\rho), \widehat{\sigma^2}(\mathcal{R}_{\boldsymbol{\alpha}}\rho)\right]$$

Rotationally Invariant Distance

We define the texture signature $\Gamma_{\mathbf{x}}$, via

$$\Gamma_{\mathbf{x}}(\boldsymbol{\alpha}) = \left[\widehat{\boldsymbol{\theta}}(\mathcal{R}_{\boldsymbol{\alpha}}\rho), \widehat{\sigma^2}(\mathcal{R}_{\boldsymbol{\alpha}}\rho)\right]$$

Now, we define a distance between two textures by the following expression:

$$\min_{\boldsymbol{\alpha}_0 \in SO(3)} \int_{SO(3)} \text{KLdist}\left(\Gamma_{\mathbf{x}_1}(\boldsymbol{\alpha}), \Gamma_{\mathbf{x}_2}(\boldsymbol{\alpha}\boldsymbol{\alpha}_0)\right) d\boldsymbol{\alpha}.$$

Experimental Results

	$\mathcal{T}_{1,0}$	$\mathcal{T}_{1,\frac{\pi}{2}}$	$\mathcal{T}_{2,0}$	$\mathcal{T}_{2,\frac{\pi}{2}}$
$\mathcal{T}_{1,0}$	0.0007	0.0005	0.0072	0.0137
$\mathcal{T}_{1,\frac{\pi}{2}}$	0.0010	0.0007	0.0101	0.0182
$\mathcal{T}_{2,0}$	0.0123	0.0128	0.0006	0.0004
$\mathcal{T}_{2,\frac{\pi}{2}}$	0.0093	0.0101	0.0012	0.0009

Table: Distances between two rotations of two distinct textures using the rotationally invariant distance and autocovariance resampled on $\frac{\mathbb{Z}^3}{4}$.

Experimental Results

	$\mathcal{T}_{1,0}$	$\mathcal{T}_{1,\frac{\pi}{2}}$	$\mathcal{T}_{2,0}$	$\mathcal{T}_{2,\frac{\pi}{2}}$
$\mathcal{T}_{1,0}$	0.0006	0.0006	0.0073	0.0136
$\mathcal{T}_{1,\frac{\pi}{2}}$	0.0013	0.0007	0.0100	0.0164
$\mathcal{T}_{2,0}$	0.0125	0.0203	0.0010	0.0004
$\mathcal{T}_{2,\frac{\pi}{2}}$	0.0119	0.0082	0.0007	0.0008

Table: Distances between two rotations of two distinct textures using the rotationally invariant distance and autocovariance resampled on $\frac{\mathbb{Z}^3}{2}$.

Experimental Results

	$\mathcal{T}_{1,0}$	$\mathcal{T}_{1,\frac{\pi}{2}}$	$\mathcal{T}_{2,0}$	$\mathcal{T}_{2,\frac{\pi}{2}}$
$\mathcal{T}_{1,0}$	0.0026	0.0812	0.0330	0.1750
$\mathcal{T}_{1,\frac{\pi}{2}}$	0.1118	0.0010	0.0852	0.0562
$\mathcal{T}_{2,0}$	0.0454	0.0694	0.0016	0.0108
$\mathcal{T}_{2,\frac{\pi}{2}}$	0.0607	0.0473	0.0246	0.0018

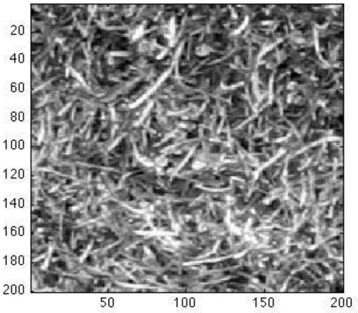
Table: Distances between two rotations of two distinct textures using the rotationally invariant distance and autocovariance sampled on the original grid $\mathbb{Z}^3$.

Experimental Results

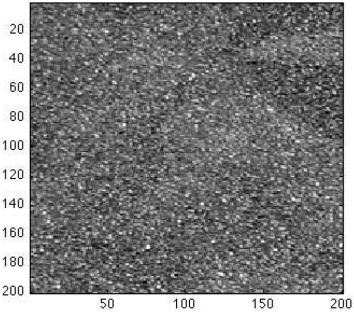
	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$
$\mathcal{T}_1$	0.0006	0.0073	0.4232	2.3180	1.7724
$\mathcal{T}_2$	0.0125	0.0010	0.4894	2.5227	1.8381
$\mathcal{T}_3$	0.4466	0.5134	0.0004	0.5208	0.4563
$\mathcal{T}_4$	2.4314	2.6315	0.5605	0.0021	0.3533
$\mathcal{T}_5$	1.8200	1.9227	0.4318	0.2540	0.0043

Table: Distances between five distinct textures using the rotationally invariant distance and autocovariance resampled on the grid $\frac{\mathbb{Z}^3}{2}$.

Experiments with 2-D Textures



(c) Grass



(d) Sand

Experiments with 2-D Textures

	grass	sand
grass	0.0200	0.0806
sand	0.0032	0.0443

	grass	sand
grass	0.0107	0.3418
sand	0.7174	0.0223

Table: Distances between the sand and grass textures for the original data (left) for the low pass component (right).