

Rigid supersymmetry on 5-dimensional Riemannian manifolds and contact geometry

Yiwen Pan

*C.N. Yang Institute for Theoretical Physics,
Stony Brook, NY, 11790, U.S.A.*

E-mail: yiwen.pan@stonybrook.edu

ABSTRACT: In this note we generalize the methods of [1–3] to 5-dimensional Riemannian manifolds M . We study the relations between the geometry of M and the number of solutions to a generalized Killing spinor equation obtained from a 5-dimensional supergravity. The existence of 1 pair of solutions is related to almost contact metric structures. We also discuss special cases related to $M = S^1 \times M_4$, which leads to M being foliated by submanifolds with special properties, such as Quaternion-Kähler. When there are 2 pairs of solutions, the closure of the isometry sub-algebra generated by the solutions requires M to be S^3 or T^3 -fibration over a Riemann surface. 4 pairs of solutions pin down the geometry of M to very few possibilities. Finally, we propose a new supersymmetric theory for $\mathcal{N} = 1$ vector multiplet on K-contact manifold admitting solutions to the Killing spinor equation.

KEYWORDS: Differential and Algebraic Geometry, Extended Supersymmetry, Supersymmetric gauge theory

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1 Introduction

As discussed in [4], to obtain a supersymmetric theory on a Riemannian manifold M , one can first couple the desired multiplet to supergravity, then take the rigid limit, sending the Planck mass to infinite.

In the process of taking the limit, one keeps auxiliary fields instead of imposing their equations of motion. If a background of auxiliary fields and metric is invariant under the supergravity transformation, they actually give rise to rigid supersymmetry.

This line of reasoning has been utilized to study $4d \mathcal{N} = 1$ supersymmetry with/without $U(1)_R$ symmetry [1, 2, 5], and $3d \mathcal{N} = 2$ supersymmetry with $U(1)_R$ [3]. In these papers, the existence of a number of supercharges is proven to be related to the geometric structure of M . For instance, on any $4d$ Hermitian manifold there exists at least one supercharge [1, 5], and $3d$ manifolds with an almost contact structure admit at least one supercharge. Similar discussions for manifolds with Lorentz signatures in dimension 3 and 4 can be found in [6, 7].

In 5-dimension, there are rapidly growing literatures on constructing 5d supersymmetric theories, as well as their relations with 6d (2,0) theories and lower dimensional Chern-Simons theories.

For example, in [8], a supersymmetric gauge theory on S^5 is obtained from 5d supergravity, with the well-known Killing spinor equation

$$\nabla_m \xi_I = t_I^J \Gamma_m \xi_J. \quad (1.1)$$

Using the supersymmetry algebra, the author proposed adding a term $\delta \text{tr}((\delta \lambda)^\dagger \lambda)$ to the Lagrangian, and derived the localization condition. The localization condition is further used in [9, 10] to analyze physical and twisted supersymmetric gauge theories coupled with matter defined on a principal $U(1)$ bundle M_5 over a 4-dimensional manifold. In particular, the perturbative partition function on S^5 is computed. Their localization result leads to the derivation of N^3 -behavior of the free energy of 5d SYM, in the large 't Hooft coupling limit [11]. Complete localization of the partition function on S^5 is carried out in [12, 13]. The authors first computed the perturbative contribution and conjectured the non-perturbative contribution by requiring the full partition function to be identical to a 6d index [12]. In their subsequent work [13] the full partition function is computed which proves the conjecture.

There are also supersymmetric theories constructed by hand or by dimensional reduction from 6d. In [14] supersymmetric gauge theory on $S^1 \times S^4$ is obtained, and the localization is carried out. [15] derived a class of 5d SYM theories from 6d (2,0) supergravity. Further in [16–18], supersymmetric theories on $S^3 \times M_3$ obtained from M5-brane are shown to be equivalent to 3d complexified Chern-Simons theory. Supersymmetric theories on $\mathbb{CP}^2 \times \mathbb{R}$ were also obtained from 6d and studied in details in [19, 20].

A complete picture of the relation between supersymmetry and geometry of M , however, is not clear. It is reasonable to believe that methods similar to those in [3] can be straight-forwardly applied to 5-dimensional manifolds.

In this paper, we take a step towards such understanding, and as expected, the results turn out to be closely related to contact and almost contact structures on 5-manifolds.

We use the minimal off-shell 5d Supergravity discussed in [21] and focus on the Killing spinor equation (2.11)

$$\nabla_m \xi_I = t_I{}^J \Gamma_m \xi_J + \frac{1}{4} V^{pq} \Gamma_{mpq} \xi_I + \frac{1}{2} F_{mn} \Gamma^n \xi_I + (A_m)_I{}^J \xi_J, \quad (1.2)$$

coming from requiring supergravity variation $\delta\psi_m^I$ of gravitino ψ to vanish. We study many interesting necessary conditions for M to admit different number of solutions to the Killing spinor equation, by partially solving the auxiliary fields in terms of the bilinears, and discussing special cases which are related to well-known geometries and results in lower dimension supersymmetry. However, the results we obtain are not a complete classification of manifolds admitting solutions.

In the end, we propose a 5-dimensional supersymmetric theory for the $\mathcal{N} = 1$ vector multiplet, which can be defined on a contact manifold with an associated metric admitting solution to equation (5.1). However, it should be pointed out that this theory is not obtained directly from supergravity, since we started from 5-dimensional Weyl multiplet without coupling to matter. Therefore the rigid limit of the supergravity action on a fixed background gives a number rather than a supersymmetric theory. To obtain the final supersymmetric background, one also needs to require the background auxiliary fields to satisfy a more complicated differential equation from the vanishing of the supergravity variation of another spinor field in the Weyl multiplet. In this sense, the present work covers an important sector of the problem, but a complete analysis requires further exploration.

This paper is organized as follows:

- In section 2, we briefly review Zucker's 5d $\mathcal{N} = 1$,¹ minimal supergravity and the Killing spinor equation from the vanishing gravitino variation.
- In section 3, we study the bilinears constructed from one or two symplectic Majorana spinors. We see that when a global nowhere-vanishing section of $ad(P_{\text{SU}(2)})$ over M exists, M has an almost contact structure corresponding to each spinor.
- In section 4, we start with a general discussion of the Killing spinor equation (2.11)

$$\nabla_m \xi_I = t_I{}^J \xi_J + \frac{1}{2} V^{pq} \Gamma_{mpq} \xi_I + \frac{1}{2} F_{mn} \Gamma^n \xi_I + (A_m)_I{}^J \xi_J, \quad (1.3)$$

including its shifting symmetry and Weyl symmetry. We then analyze the necessary conditions on the geometry of M such that it admits certain number of solutions. For each (pair of) solutions, we see that the auxiliary fields can be partially solved in terms of the bilinears, and the Killing spinor equation is then simplified using a compatible connection $\hat{\nabla}$

$$\hat{\nabla}_m \hat{\xi}_I - (\hat{A}_m)_I{}^J \hat{\xi}_J = 0. \quad (1.4)$$

¹It is called " $\mathcal{N} = 2$ " in his paper, but it really means a theory with 8 supercharges, which supersymmetry parameter an $\text{SU}(2)$ -symplectic Majorana spinor ξ_I^α .

Some special cases related to product form $M = S^1 \times M_4$ are discussed: depending on the field configuration, one obtains geometrical restriction of M_4 being Kähler, Quaternion Kähler or HyperKähler, or a reduction of our 5d Killing spinor equation to 4d cases discussed in literatures [1, 2]. For 2 (pairs of) supercharges to exist, we will see that the geometry of M is heavily constrained by the isometry algebra to be T^3 or S^3 -fibration over Riemann surface Σ .

For 4 pairs of supercharges to exist, we will see that there are only 3 possible cases, which basically fixes the geometry of M .

- In section 5, we propose a new supersymmetric theory for the $\mathcal{N} = 1$ vector multiplet, which can be defined on K-contact manifolds (M, g, κ) admitting solutions to equation

$$D_m \xi_I = t_I{}^J \xi_J + \frac{1}{4} \mathcal{F}^{pq} \Gamma_{mpq} \xi_I + \frac{1}{2} \mathcal{F}_{mn} \Gamma^n \xi_I \quad (1.5)$$

with \mathcal{F} any “anti-self-dual” (defined later) closed 2-form.

- In section 7, we provide a few examples of solving Killing spinor equations on selected manifolds to illustrate some results obtained in previous sections.
- In the appendix, we review conventions on gamma matrices and differential geometry as well as necessary mathematical backgrounds on contact geometry. Useful formula are also listed.

2 $\mathcal{N} = 1$ minimal off-shell supergravity

In this section we will review the construction of minimal $\mathcal{N} = 1$ off-shell supergravity² proposed by Zuker [21].

The general idea of [21] is as follows. First one writes down the current multiplet \mathcal{J} from an on-shell abelian super-Yang-Mills theory. This current multiplet \mathcal{J} is then coupled to linearized supergravity multiplet \mathcal{G} , and using the already-known supersymmetry transformation (which is made local at this stage), one can postulate the linearized supergravity transformation of \mathcal{G} .

The abelian super-Yang-Mills multiplet consists of the field content (φ, a, λ') , where φ is a real scalar, a is a photon with field strength $f_{mn} = \partial_m a_n - \partial_n a_m$, and λ' is the gaugino.³

The Lagrangian reads

$$\mathcal{L} = -\frac{1}{4} f_{mn} f^{mn} + \frac{1}{2} \partial_m \varphi \partial^m \varphi + \frac{i}{2} \bar{\lambda}' \Gamma^m \partial_m \lambda'. \quad (2.1)$$

The Lagrangian is invariant under the on-shell supersymmetry transformation

$$\delta \varphi = i \bar{\epsilon} \lambda', \quad \delta a_m = i \bar{\epsilon} \Gamma_m \lambda', \quad \delta \lambda' = \frac{1}{2} f_{mn} \Gamma^{mn} \epsilon - \partial_m \varphi \Gamma^m \epsilon, \quad (2.2)$$

²It is called $\mathcal{N} = 2$ in [21], however, it actually has 8 supercharges following from the symplectic Majorana reality condition, and therefore is more sensible to call it $\mathcal{N} = 1$.

³We use here the similar Greek letter to denote fields as we will use in section 5.

which form a closed algebra modulo the equation of motion:

$$\Gamma^m \partial_m \lambda' = 0. \quad (2.3)$$

There are several symmetries of the theory:

- Spacetime symmetry, whose conserved current is the energy-momentum tensor

$$T_{mn} = -f_{mk} f_n^k + \frac{1}{2} \eta_{mn} f_{kl} f^{kl} + \partial_m \varphi \partial_n \varphi - \frac{1}{2} \eta_{mn} (\partial \varphi)^2 + \frac{i}{4} \bar{\lambda}' (\Gamma_m \partial_n + \Gamma_n \partial_m) \lambda'. \quad (2.4)$$

- Supersymmetry, whose the conserved current is

$$J^m = \Gamma^n \Gamma^m \lambda' \partial_n \varphi + \frac{1}{2} f_{nl} \Gamma^{nl} \Gamma^m \lambda'. \quad (2.5)$$

- SU(2) R -symmetry, whose the conserved R -current is

$$J_m^a = \bar{\lambda}' \tau^a \Gamma_m \lambda'. \quad (2.6)$$

All of these currents are conserved when equations of motion are imposed. These currents can form a supermultiplet if proper additional objects are added to close the algebra. The complete current multiplet consists of

$$\mathcal{J} = \{T^{mn}, w_{mn}, J_m, C', \zeta, X^a, J_m^a, j^a, \}. \quad (2.7)$$

Now one can couple this multiplet to the supergravity multiplet

$$\mathcal{G}_{\text{linear}} = \{h_{mn}, V_{mn}, \psi_m, C, \lambda, t_a, A_m^a, A_m\}, \quad (2.8)$$

where h_{mn} is the linearized metric, V_{mn} is a 2-form, A_m is the U(1)-gauge field with field strength $F_{mn} = \partial_m A_n - \partial_n A_m$, while $\{A_m^a, a = 1, 2, 3\}$ is a SU(2) $_{\mathcal{R}}$ gauge field. The linearized Lagrangian of \mathcal{G} coupled to the current multiplet reads

$$\mathcal{L} = \frac{1}{8} h_{mn} T^{mn} + \frac{i}{4} \bar{J}^m \psi_m - 4C' C - 2i \bar{\zeta} \lambda - \frac{1}{2} w_{mn} V^{mn} + X^a t^a + \frac{1}{2\sqrt{3}} A_m J_{(1)}^m + \frac{1}{4} J_m^a A_a^m. \quad (2.9)$$

In the following, we will replace the a index by *symmetric* IJ index, with $I, J = 1, 2$, and the conversion can be easily made using Pauli matrices $(\sigma^a)_{IJ}$

By requiring the linearly-coupled Lagrangian to be supersymmetric, one obtains supergravity transformation (with symplectic-Majorana spinor ξ_I) of \mathcal{G} . Further covariantizing the transformation gives the full supergravity transformation (here we only list schematically first few lines and omit coefficients in front of each term)

$$\left\{ \begin{array}{l} \delta e_m^i \sim \xi_I \Gamma^i \psi_m^I \\ \delta A_m \sim \xi_I \psi_m \\ \delta \psi_m \sim \mathcal{D}_m^{\hat{\omega}} \xi_I + F_{mn} \Gamma^n \xi_I + V^{pq} \Gamma_{mpq} \xi_I + (A_m)_I^J \xi_J + \dots \\ \delta \lambda_I = \nabla_m V^{mn} \Gamma_n \xi_I + *(V \wedge V)^m \Gamma_m \xi_I + V_{mnk} \Gamma^{mnk} \xi_I + \dots \end{array} \right., \quad (2.10)$$

where ... in the third line denotes terms that will vanish when taking rigid limit. In the last line we schematically show a few terms involving V , and use ... to denote remaining complicated terms.

The rigid limit procedure sets fermions to zero, keeping only the bosonic fields to some background which needs to be determined. If such background is invariant under certain supergravity transformation, in particular, $\delta\psi = 0$, one obtains a rigid supersymmetric background with the resulting metric.

The condition $\delta\psi = 0$ reads, with some coefficients reinstated without loss of generality,

$$\delta\psi_{mI} = \nabla_m \xi_I - t_I^J \Gamma_m \xi_J - \frac{1}{2} F_{mn} \Gamma^n \xi_I - \frac{1}{2} V^{pq} \Gamma_{mpq} \xi_I - (A_m)_I^J \xi_J = 0. \quad (2.11)$$

This is the Killing spinor equation we are going to analyze in the following sections.

To complete the whole rigid limit procedure, one also needs to also solve the equation from $\delta\lambda = 0$ in taking the rigid limit. Such equation will place various restrictions on the auxiliary fields. However, in this paper we focus on the simpler yet important Killing spinor equation (2.11), and leave equation $\delta\lambda = 0$ for future project.

3 Symplectic Majorana spinor and bilinears

In this section, we review the properties of symplectic Majorana spinor and their bilinears. Note that we consider bosonic spinors in the following discussions.

On a 5-dimensional Riemannian manifold M , one can define Hermitian Gamma matrices, the charge conjugation matrix and SU(2) symplectic Majorana spinors.⁴

Hermitian Gamma matrices are denoted as Γ

$$\{\Gamma_m, \Gamma_n\} = 2g_{mn}, \quad (3.1)$$

and hermiticity implies

$$\overline{\Gamma_m} = (\Gamma_m)^T. \quad (3.2)$$

The Charge conjugation matrix is denoted as C ,

$$C\Gamma^m C^{-1} = (\Gamma^m)^T = \overline{\Gamma_m}. \quad (3.3)$$

We also define the SU(2)-invariant tensor ϵ^{IJ} and ϵ_{IJ}

$$\epsilon^{12} = -\epsilon^{21} = -\epsilon_{12} = 1, \quad (3.4)$$

and raising and lowering convention

$$\epsilon_{IJ} X^J = X_I, \quad \epsilon^{IJ} X_J = X^I. \quad (3.5)$$

With these quantities we define the symplectic Majorana spinor condition as

$$\overline{\xi_I^\alpha} = \epsilon^{IJ} C_{\alpha\beta} \xi_J^\beta, \quad (3.6)$$

⁴Note that ordinary Majorana condition cannot be defined in 5d.

and a \mathbb{C} -valued inner product of any two spinors denoted by parenthesis $()$

$$(\xi\eta) \equiv \xi^\alpha C_{\alpha\beta} \eta^\beta, \quad (3.7)$$

and further a positive-definite inner product $(\ , \)$ between symplectic Majorana spinors ξ, η

$$(\xi, \eta) \equiv \epsilon^{IJ} (\xi_I \eta_J). \quad (3.8)$$

3.1 Bilinears from 1 symplectic Majorana spinor

Now we're ready to define bilinears constructed from one symplectic Majorana spinor ξ_I .

(1) Function $s \in C^\infty(M)$:

$$s \equiv \epsilon^{IJ} (\xi_I \xi_J) = 2 (\xi_1 \xi_2). \quad (3.9)$$

Note that this function is strictly positive if ξ is nowhere-vanishing:

$$s = \epsilon^{IJ} \xi_I^\alpha C_{\alpha\beta} \xi_J^\beta = \sum_\alpha \xi_I^\alpha \overline{\xi_I^\alpha} > 0. \quad (3.10)$$

(2) Vector field $R \in \Gamma(TM)$:

$$R_m \equiv \epsilon^{IJ} \xi_I \Gamma_m \xi_J, \quad (3.11)$$

and the corresponding 1-form

$$\kappa_m \equiv g_{mn} R^n, \quad (3.12)$$

which implies, when acting on $\Omega^p(M)$

$$\iota_R \circ * = (-1)^p * \circ (\kappa \wedge). \quad (3.13)$$

(3) 2-form⁵

$$\Theta_{mn}^{IJ} \equiv (\xi^I \Gamma_{mn} \xi^J), \quad (3.16)$$

with symmetry

$$\Theta^{IJ} = \Theta^{JI}. \quad (3.17)$$

Let t_{IJ} be an arbitrary triplet of functions, namely

$$t_{IJ} = t_{JI}, I = 1, 2; \quad (3.18)$$

then its contraction with Θ gives a real 2-form

$$(t\Theta) \equiv t^I{}_J (\Theta^J{}_I). \quad (3.19)$$

⁵One could of course go on defining higher forms $\Theta_{lmn}^{IJ} \equiv \xi^I \Gamma_{lmn} \xi^J$ and $\Theta_{mnpq}^{IJ} \equiv \xi^I \Gamma_{mnpq} \xi^J$, but duality of Gamma matrices gives

$$\Theta_{lmn}^{IJ} = -\frac{\sqrt{g}}{2!} \epsilon^{pq}{}_{lmn} \Theta_{pq}^{IJ}, \quad (3.14)$$

and

$$\Theta_{mnpq}^{IJ} = \sqrt{g} \epsilon^r{}_{mnpq} \Theta_r^{IJ}. \quad (3.15)$$

Using the Fierz identities one can derive useful relations between these quantities, which we list in appendix E.

Given the nowhere-vanishing 1-form κ and the vector field R , one can decompose the tangent bundle $TM = TM_H \oplus TM_V$, where at any point $p \in M$, $TM_H|_p$ is annihilated by κ , while TM_V is a trivial line bundle generated by R . Let's call TM_H , and similarly all tensors annihilated by κ (or R) “horizontal”, while those in the orthogonal complement “vertical”. In particular, one has decompositions

$$\Omega^2(M) = \Omega_V^2(M) \oplus \Omega_H^2(M) = \kappa \wedge \Omega_H^1(M) \oplus \Omega_H^2(M) \quad (3.20)$$

For an arbitrary nowhere-vanishing triplet of functions t_{IJ} with the property (readers may find conventions in appendix B)

$$t_{IJ} = t_{JI}, \quad \overline{t_{IJ}} = \epsilon^{I'J'} \epsilon^{JJ'} t_{I'J'}. \quad (3.21)$$

one can define a map $\varphi_t : \Gamma(TM) \rightarrow \Gamma(TM)$ as

$$(\varphi_t)_m{}^n \equiv \frac{1}{s} \sqrt{\frac{-2}{\text{tr}(t^2)}} (t\Theta)_m{}^n. \quad (3.22)$$

Obviously, one has

$$\varphi_t \circ \varphi_t = -1 + s^{-2} R \otimes \kappa, \quad (3.23)$$

and when restricted on TM_H , φ_t is some sort of a “complex” structure:

$$\varphi_t \circ \varphi_t|_{TM_H} = -1. \quad (3.24)$$

Together with the vector field $s^{-1}R$ and 1-form $s^{-1}\kappa$, φ_t defines an almost contact structure on M [22] (see also appendix D).

Finally, let us comment on the “(anti)self-dual” horizontal forms. Define operator $*_H \equiv s^{-1} \iota_R *$, which is the hodge dual “within” horizontal hyperplanes. It is easy to verify that acting on any horizontal p -forms

$$*_H^2 = (-1)^p. \quad (3.25)$$

In particular, we decompose the horizontal 2-forms into 2 subspaces according to their eigenvalues of $*_H$

$$\Omega_H^2 = \Omega_H^{2+} \oplus \Omega_H^{2-}, \quad *_H \omega_H^\pm = \pm \omega_H^\pm, \quad \forall \omega_H^\pm \in \Omega_H^\pm. \quad (3.26)$$

We call the horizontal forms in Ω_H^{2+} “self-dual”, while the others “anti-self-dual”. Clearly, these 2 notions are interchanged as one flips the sign of the vector field R , hence this notion of “self-duality” is not as intrinsic as the well-established notion of self-duality on 4-dimensional oriented manifolds.

Suppose Ω^+ is a self-dual 2-form. Then it satisfies, by definition,

$$\frac{\sqrt{g}}{2s} \epsilon^{pq}{}_{lmn} R^l \Omega_{pq}^+ = \Omega_{mn}^+. \quad (3.27)$$

It follows immediately that

$$\Omega_{mn}^+ \Gamma^{mn} \xi_I = 0, \quad (3.28)$$

using the fact that the inner product $(\psi, \psi) \equiv \epsilon^{IJ} (\psi_I \psi_J)$ is positive definite, and the action of Γ_{mn} preserve symplectic Majorana property.

3.2 Bilinears from 2 symplectic Majorana spinors

In this section, we consider the case when there are 2 symplectic Majorana spinors, and analyze their bilinears.

Denote the two spinors ξ_I and $\tilde{\xi}_I$. Obviously they each generates a set of quantities as we discussed in the previous sections: (s, R, κ, Θ) and $(\tilde{s}, \tilde{R}, \tilde{\kappa}, \tilde{\Theta})$.

In addition to these quantities, they form some new mixed bilinears. Conventions for IJ indices can be found in appendix B.

- Functions

$$u_{IJ} \equiv (\xi_I \tilde{\xi}_J), \quad (3.29)$$

with triplet-singlet decomposition

$$u_{IJ} = u_{(IJ)} + u_{[IJ]} = \hat{u}_{IJ} - \frac{1}{2}\epsilon_{IJ}u, \quad (3.30)$$

where

$$u \equiv \epsilon^{IJ}u_{IJ}. \quad (3.31)$$

Notice that

$$\overline{u_{IJ}} = \epsilon^{II'}\epsilon^{JJ'}u_{I'J'} \equiv u^{IJ}, \quad (3.32)$$

and in particular function u is real-valued

$$\bar{u} = u = \sum_I \xi_I^\alpha \tilde{\xi}_I^\alpha, \quad (3.33)$$

which results in positivity

$$u_{IJ}u^{IJ} = \sum u_{IJ}\overline{u_{IJ}} = \frac{1}{2}u^2 + \hat{u}_{IJ}\hat{u}^{IJ} \geq 0. \quad (3.34)$$

- Vector fields Q_{IJ}

$$Q_{IJ}^m \equiv (\xi_I \Gamma^m \tilde{\xi}_J), \quad (3.35)$$

with a decomposition

$$Q_{IJ} = \hat{Q}_{IJ} - \frac{1}{2}\epsilon_{IJ}Q, \quad (3.36)$$

where

$$Q^m \equiv \epsilon^{IJ}(\xi_I \Gamma^m \tilde{\xi}_J). \quad (3.37)$$

Note that similar to the function case, we have

$$\overline{Q_{IJ}} = Q^{IJ}, \quad (3.38)$$

and in particular a real vector field

$$\overline{Q} = Q. \quad (3.39)$$

We denote corresponding 1-forms

$$\tau_{IJ} \equiv (Q_{IJ})_m dx^m = \hat{\tau}_{IJ} - \frac{1}{2}\epsilon_{IJ}\tau. \quad (3.40)$$

- Two forms

$$\chi_{mn}^{IJ} \equiv (\xi^I \Gamma_{mn} \tilde{\xi}^J). \quad (3.41)$$

Also we define

$$\chi \equiv \epsilon^{IJ} \chi_{IJ}, \quad \hat{\chi}_{IJ} = \chi_{(IJ)}. \quad (3.42)$$

These bilinears satisfy various algebraic relations. Here we list some relevant formulas.

Norms and inner products of vector fields

(1)

$$R \cdot \tilde{R} = 4u_{IJ}u^{IJ} - s\tilde{s} \Rightarrow \begin{cases} |\tilde{s}R + s\tilde{R}|^2 = 8s\tilde{s}u_{IJ}u^{IJ} \\ |\tilde{s}R - s\tilde{R}|^2 = 4s\tilde{s}(s\tilde{s} - 2u_{IJ}u^{IJ}) \end{cases} \quad (3.43)$$

(2)

$$Q_{IJ} \cdot Q_{KL} = 2u_{IL}u_{KJ} - u_{IJ}u_{KL} - \frac{1}{2}\epsilon_{IK}\epsilon_{LJ}s\tilde{s} \quad (3.44)$$

In particular

$$\begin{cases} |u^{IJ}Q_{IJ}|^2 = \frac{1}{2}(u^{IJ}u_{IJ})s\tilde{s} \\ |Q|^2 = -2\hat{u}_{IJ}\hat{u}^{IJ} + s\tilde{s} \end{cases} \quad (3.45)$$

(3)

$$R \cdot Q_{IJ} = su_{IJ}, \quad \tilde{R} \cdot Q_{IJ} = \tilde{s}u_{IJ}. \quad (3.46)$$

Positivity of the norms implies

$$s\tilde{s} \geq 2u_{IJ}u^{IJ} = 2\hat{u}_{IJ}\hat{u}^{IJ} + u^2. \quad (3.47)$$

When $s\tilde{s} = 2u_{IJ}u^{IJ}$, we have R and \tilde{R} are parallel at such point, which in general we like to avoid.

(4) Using Fierz identity, one can shows

$$\tilde{s}R + s\tilde{R} = 4u_{IJ}Q^{IJ} = 2uQ + 4\hat{u}_{IJ}\hat{Q}^{IJ}, \quad (3.48)$$

$$\begin{cases} R_m\tilde{R}_n - R_n\tilde{R}_m = -4u_{IJ}\chi_{mn}^{IJ} \Rightarrow \kappa \wedge \tilde{\kappa} = -4u_{IJ}\chi^{IJ} \\ g_{mn} = -\frac{2s\tilde{s}}{|s\tilde{R} - \tilde{s}R|^2} [R_m\tilde{R}_n + R_n\tilde{R}_m - 4(Q_{IJ})_m(Q^{IJ})_n] \end{cases}, \quad (3.49)$$

where the last equation tells us that the metric is completely determined by the bilinears constructed from 2 solution.

Contraction between the vectors and 2-forms

$$\begin{cases} \iota_R(t\chi) = s(t\hat{\tau}) - (t\hat{u})\kappa \\ \iota_Q(t\Theta) = (t\hat{u})\kappa - s(t\hat{\tau}) \\ \iota_{\hat{u}\hat{Q}}(t\Theta) = (t\hat{u})(u\kappa + s\tau) \\ \iota_R(t^{IJ}\tilde{\Theta}_{IJ}) - \iota_{\tilde{R}}(t^{IJ}\Theta_{IJ}) = 4t^{IJ}(u\hat{\tau}_{IJ} - \hat{u}_{IJ}\tau) \end{cases} \quad (3.50)$$

where again t_{IJ} is arbitrary triplet of functions.

4 Killing spinor equation

In this section we will discuss what constraints will be imposed on geometry of M when there exists different number of solutions to the Killing spinor equation (2.11). We focus on situations where there are 1, 2, and 4 pairs of solutions to the equation.

Recall that the Killing spinor equation required by rigid limit of supergravity is

$$\delta\psi_{mI} = \nabla_m \xi_I - \Gamma_m t_I^J \xi_J - \frac{1}{2} V^{pq} \Gamma_{mpq} \xi_I - \frac{1}{2} F_{mn} \Gamma^n \xi_I - (A_m)_I^J \xi_J = 0, \quad (4.1)$$

where t_{IJ} is a triplet of scalars (or more precisely, a global section of the $ad(P_{\text{SU}(2)})$ where $P_{\text{SU}(2)}$ is an underlying principal $\text{SU}(2)_{\mathcal{R}}$ -bundle, with gauge field $(A_m)_I^J$), F is a closed 2-form, V is a 2-form.

The symplectic Majorana spinor ξ_I is a section of the $\text{SU}(2)_{\mathcal{R}}$ twisted spin bundle of M . In general the $\text{SU}(2)_{\mathcal{R}}$ -bundle P is non-trivial. We define the gauge-covariant derivative on t_{IJ}

$$\nabla_m^A t_I^J \equiv \nabla_m t_I^J - (A_m)_I^K t_K^J + t_I^K (A_m)_K^J, \quad (4.2)$$

and curvature of A as

$$(W_{mn})_I^J \equiv \nabla_m (A_n)_I^J - \nabla_n (A_m)_I^J - \left[(A_m)_I^K (A_n)_K^J - (A_n)_I^K (A_m)_K^J \right]. \quad (4.3)$$

Note that the Killing spinor equation is $\text{SU}(2)$ gauge covariant. It is also invariant under complex conjugation, provided that the auxiliary fields satisfies reality conditions: F and V are real,

$$\overline{t_{IJ}} = \epsilon^{II'} \epsilon^{JJ'} t_{I'J'}, \quad (4.4)$$

and similar for A . The reality condition on t_{IJ} and A is just saying that they are linear combinations of Pauli matrices with imaginary coefficients.

Apart from the above obvious symmetries, the equation further enjoys a shifting symmetry and a Weyl symmetry.

- Shifting symmetry: the equation is invariant under the shifting transformation of auxiliary fields V and F

$$\begin{cases} V \rightarrow V + \Omega^+ \\ F \rightarrow F + 2\Omega^+ \end{cases}, \quad (4.5)$$

where Ω^+ is any self-dual 2-form discussed in (3.26), following from the fact that

$$\Omega_{mn}^+ \Gamma^{mn} \xi_I = 0. \quad (4.6)$$

- Weyl symmetry: after rescaling the metric $g \rightarrow e^{2\sigma}g$, one can properly transform the auxiliary fields as well as the Killing spinor solution such that the Killing spinor equation is invariant. This can be seen by first rearranging the Killing spinor equation (2.11) into the form

$$\nabla_m \xi_I = \Gamma_m \tilde{\xi}_I + \frac{1}{2} P_{mn} \Gamma^n \xi_I, \quad (4.7)$$

where

$$\tilde{\xi}_I \equiv \left(t_I^J + \frac{1}{2} V_{pq} \Gamma^{pq} \delta_I^J \right) \xi_J, \quad P_{mn} \equiv F_{mn} - 2V_{mn}. \quad (4.8)$$

and we ignore the gauge field A_{IJ} for simplicity.

Focusing on (4.7) alone as an equation for pair $(\xi, \tilde{\xi})$ on any d -dimensional manifold, it is obvious that

$$\tilde{\xi}_I = \frac{1}{d} \Gamma^m \nabla_m \xi_I - \frac{1}{2d} P_{mn} \Gamma^{mn} \xi_I. \quad (4.9)$$

Substituting it back to (4.7), one obtains the equation

$$\mathcal{D}_m(g) \xi_I = -\frac{1}{2d} P_{pq} \Gamma_m \Gamma^{pq} \xi_I + \frac{1}{2} P_{mn} \Gamma^n \xi_I \quad (4.10)$$

where the well-known differential operator \mathcal{D}_g is defined as

$$\mathcal{D}_m(g) \equiv \nabla_m - \frac{1}{d} \Gamma_m \Gamma^n \nabla_n. \quad (4.11)$$

and depends on the metric g . It's easy to show that⁶

$$\mathcal{D}_m(e^{2\sigma}g) e^{\sigma/2} = e^{\sigma/2} \mathcal{D}_m(g). \quad (4.14)$$

Hence, equation (4.7) is invariant under rescaling

$$g \rightarrow e^{2\sigma}g, \quad P \rightarrow e^\sigma P, \quad \xi \rightarrow e^{\sigma/2} \xi. \quad (4.15)$$

Now we return to the equation (2.11), and compute the transformation of auxiliary fields under Weyl rescaling. Suppose the scaling function σ is constant along vector field R :

$$R^m \nabla_m \sigma = 0, \quad (4.16)$$

then one can see that the Killing spinor equation (2.11) is invariant under rescaling

$$g \rightarrow e^{2\sigma}g, \quad t_{IJ} \rightarrow e^{-\sigma} t_{IJ}, \quad V \rightarrow e^\sigma V - \frac{1}{2} e^\sigma (\kappa \wedge d\sigma), \quad F \rightarrow e^\sigma F - e^\sigma (\kappa \wedge d\sigma), \quad (4.17)$$

⁶Under Weyl rescaling $g \rightarrow e^{2\sigma}g$, the spin connection is shifted according to

$$\nabla_m^g \psi \rightarrow \nabla_m^{e^{2\sigma}g} \psi = \nabla_m^g \psi + \frac{1}{2} (\nabla_n^g \sigma) \Gamma_m{}^n \psi. \quad (4.12)$$

To prove the Weyl transformation rule for $\mathcal{D}(g)$, one just need to plug the above formula into

$$D(e^{2\sigma}g)(e^{\sigma/2}\psi) = \nabla_m^{e^{2\sigma}g}(e^{\sigma/2}\psi) - \frac{1}{d} \Gamma_m \Gamma^n \nabla_n^{e^{2\sigma}g}(e^{\sigma/2}\psi). \quad (4.13)$$

provided we also rescale $\xi \rightarrow e^{\sigma/2}\xi$. Note that the Weyl rescaling only affects the vertical part of F and V . One can therefore use this rescaling symmetry with appropriate σ to make F horizontal, namely

$$\iota_R F = 0. \quad (4.18)$$

However, unless explicitly stated, in most of the following discussions, we will keep the general F without exploiting the Weyl symmetry.

Let us comment on the reality condition defined earlier.

- (1) In 5 dimension Euclidean signature, the spinors belong to 2^2 dimensional pseudoreal representation of $\text{Spin}(5) \sim \text{Sp}(2)$, spinor $(\psi^*)_\alpha$ and $(C\psi)_\alpha \equiv C_{\alpha\beta}\psi^\beta$ transform in the same way. It is impossible to impose the usual Majorana condition, but one can impose the symplectic Majorana condition on spinors. In this sense, 4 complex (8 real) supercharges correspond to unbroken supersymmetry, namely $\mathcal{N} = 1$.

The reality conditions introduced above are required by the supergravity that we started from, where one is interested in a real-valued action. However, it is fine to relax the reality condition on the Killing spinors and auxiliary fields, as long as one is only interested in a formally supersymmetric invariant theory. It makes perfect sense to consider complexified Killing spinor equation. In particular, the reality condition is not used in many of the following discussion, for instance, section 4.1 actually can be carried out without assuming the reality condition (except for the shifting symmetry of Ω^+ which requires positivity following from reality condition). One only needs to work with \mathbb{C} -valued differential forms. Also, when we compare our 5d Killing spinor equation to the 4d equations appearing in [1, 2], we drop the reality requirement. However, in this paper we mainly restrict ourselves to the real case, and reality condition does help simplify certain discussions.

- (2) Solutions to equation (2.11) come in pairs. Suppose ξ is a solution, corresponding to one supercharge Q , then its complex conjugate ξ'

$$\xi'_1 = \xi_2 = \overline{\xi_1}, \quad \xi'_2 = -\xi_1, \quad (4.19)$$

automatically satisfies (2.11) corresponding to the supercharge \overline{Q} . The pair of solutions ξ_I and ξ'_I define the same scalar function s and vector field R , but 2-forms Θ with different sign.

In view of such “pair-production” of solutions, we focus on finding different number of pairs of solutions to (2.11), and discuss them separately in the following subsections. When analyzing the case when M admits 1 and 2 pairs of solutions, we will select one representative solution from each pair, say, ξ and $\tilde{\xi}$, and study the relation between the bilinears that can be formed by these representing Killing spinors. Generically, the vector fields R and \tilde{R} from separate pairs should not be parallel everywhere on M .

- (3) One may worry about possible zeroes of Killing spinors. Similar to that in [1], the Killing spinor equations are a first order homogeneous differential equation system,

whose set of solutions span a complex vector space $\mathbb{C}^{k \leq 4}$, with each solution completely specified by its value at a point $p \in M$. By the symplectic Majorana condition, $\xi_1(p) = 0$ implies $\xi_2(p) = 0$, and hence $\xi_I(\forall x \in M) = 0$. Therefore, any non-trivial solution of the Killing spinor equation must be nowhere-vanishing, which ensures that the many bilinears defined (especially the almost contact structure) will be global.

In some sense, our Killing spinor equation is a generalization of the well-known Killing spinor equation

$$\nabla_m \psi = \lambda \Gamma_m \psi, \quad (4.20)$$

The constant λ can be real, pure-imaginary or zero, and the equation is accordingly called real, imaginary Killing spinor equation and covariantly constant spinor equation. If a manifold admits a Killing spinor, its Ricci curvature must take the form

$$Ric = 4(n-1)\lambda^2 g, \quad (4.21)$$

hence Einstein. For λ pure imaginary, Baum gave a classification in [23, 24]. Prior to [25], manifolds with real Killing spinor are better known in low dimensions. For instance, 4-dimensional complete manifolds with real Killing spinor were shown to be isometric to the 4-sphere [26]. In 5-dimension, simply-connected manifolds with real Killing spinors were shown to be round S^5 or Sasaki-Einstein manifolds, with solutions coming down from covariantly constant spinors on their Calabi-Yau cone. In [25], these results were generalized to higher dimensions: in dimension $n = 4k + 1$, only S^{4n+1} and Sasaki-Einstein manifolds admits real Killing spinors, while in $n = 4n+3 \geq 11$ dimension, only the round sphere, Sasaki-Einstein and 3-Sasakian manifolds admit real Killing spinors.

Our generalized Killing spinor equation has milder constraints on the geometry of manifold. We will see that the existence of one Killing spinor requires some soft geometry structure, one being an almost contact structure, similar to [3]. Of course, as the number of solutions increase, the geometry will be more constrained.

4.1 Manifolds admitting 1 pair of supercharges

4.1.1 General result and ACMS structure

In this subsection we will analyze the case when there is one pair of solutions to the Killing spinor equation (2.11). We partially solve the auxiliary fields in terms of bilinears constructed, and rewrite the (2.11) into a simpler form. We will also briefly discuss 3 interesting cases with special auxiliary field configurations, which lead to geometrical restrictions of M being locally foliated by special manifolds, or dimensional reduction to known 4d equations.

By differentiating the bilinears and using (2.11), one arrives at the following differential constraints on the quantities:

- Derivative on real positive function s

$$ds = -\iota_R F. \quad (4.22)$$

- Derivative on real vector field R

$$\nabla_m R_n = 2(t\Theta)_{mn} - \sqrt{g}\epsilon^{rpq}{}_{nm} R_r V_{pq} + sF_{mn}. \quad (4.23)$$

- Derivative on the 2-form with any triplet r_{IJ}

$$\begin{aligned} \nabla_k (r^{IJ} \Theta_{IJ})_{mn} &= (\nabla_k^A r^{IJ}) (\Theta_{IJ})_{mn} \\ &\quad + \text{tr}(rt) (g_{nk} R_m - g_{mk} R_n) - 2r^{JI} t_I^K (*\Theta_{JK})_{kmn} \\ &\quad + 2 \left[(*V)_{nk}{}^l r^{IJ} (\Theta_{IJ})_{ml} - (*V)_{mk}{}^l (r^{IJ} \Theta_{IJ})_{nl} \right] \\ &\quad - F_k{}^p r^{IJ} (*\Theta_{IJ})_{mnp} \end{aligned} \quad (4.24)$$

Let us comment on the above relations. The first equation implies $s = \text{const}$ and can be normalized to $s = 1$ when F is horizontal. Recall that one can always use the Weyl symmetry of the equation to achieve this, although we keep the general situation. The second implies that R is a Killing vector field:

$$\nabla_m R_n + \nabla_n R_m = 0. \quad (4.25)$$

The 3rd relation can be simplified as one puts in the solutions to F and V_H .

Using the 2nd and 3rd equation, one can solve (partially) the auxiliary fields in terms of the bilinears (field V is decomposed as $V = V_H + \kappa \wedge \eta$):

$$\begin{aligned} F &= (2s)^{-1} d\kappa + 2s^{-1} \Omega^- + 2s^{-1} \Omega^+ \\ V_H &= -s^{-1} (t\Theta) - s^{-1} \Omega^- + s^{-1} \Omega^+ \\ \eta^m &= \frac{1}{4s^3} (\Theta^{IJ})^{mn} \nabla^k (\Theta_{IJ})_{nk} - \frac{3}{4} (\nabla^m s^{-1}) - \frac{1}{s^2} (A_n)_{IJ} (\Theta^{IJ})^{nm} \end{aligned} \quad (4.26)$$

where Ω^\pm are self-dual (+) and anti-self-dual (−) 2-forms respectively, satisfying extra condition

$$\mathcal{L}_R \Omega^\pm = 0. \quad (4.27)$$

From previous discussions, we know that Ω^+ corresponds to the arbitrary shifting symmetry of Killing spinor equation, so we may simply consider $\Omega^+ = 0$.

Ω^- is in general non-zero. For instance, the well-known Killing spinor equation (1.1) corresponds to

$$\Omega^- = -\frac{1}{4} d\kappa, \quad (4.28)$$

which is non-zero. Also, at the end of the paper we construct a supersymmetric theory for the $\mathcal{N} = 1$ vector multiplet using the Killing spinor equation corresponding to

$$\Omega^- = \frac{1}{4} d\kappa. \quad (4.29)$$

However, to highlight some interesting underlying geometry related to (2.11), we will consider

$$\Omega^- = 0, \quad (4.30)$$

in this section unless explicitly stated. It is straight forward to generalize to non-zero Ω^- , with slight modification to the following discussions.

Now that the auxiliary fields are partially solved, we can start simplifying the Killing spinor equation. As mentioned before, t_{IJ} is a global section of associate rank-3 vector bundle of $P_{\text{SU}(2)}$, it may have zeroes. Below we will focus on 2 cases corresponding to $t \neq 0$ and $t = 0$ everywhere on M .

First let us consider the case when $t_{IJ} \neq 0$.

(1) $t_{IJ} \neq 0$. Notice that the quantities $(g, s^{-1}R, s^{-1}\kappa, \varphi_t)$ actually form an almost contact metric structure (abbreviated as ACMS). Using the ACMS, one can further rewrite the Killing spinor equation:

$$\hat{\nabla}_m \hat{\xi}_I - (\hat{A}_m)_I^J \hat{\xi}_J = 0, \quad (4.31)$$

where we rescaled ξ

$$\hat{\xi}_I \equiv (\sqrt{s})^{-1} \xi_I, \quad (4.32)$$

$$\begin{aligned} (\hat{A}_m)_I^J &\equiv (A_m)_I^J + \frac{1}{s} R_m t_I^J + \frac{1}{\text{tr}(t^2)} (\nabla_m^A t^{JK}) t_{KI} + \eta \text{ terms} \\ &= \frac{1}{s} R_m t_I^J + \frac{1}{\text{tr}(t^2)} (\nabla_m t^{JK}) t_{KI} + \eta \text{ terms}, \end{aligned} \quad (4.33)$$

and $\hat{\nabla}$ being the compatible spin connection introduced in the appendix D.

$$\begin{aligned} \hat{\nabla}_m \xi_I &= \nabla_m \xi_I + \frac{1}{\text{tr}(t^2)} (T_m)^J_I \xi_J - \frac{1}{2s} \nabla_m R_n \Gamma^n \xi_I + \frac{1}{2} (\nabla_m \log s) \xi_I \\ &\quad - \frac{1}{\text{tr}(t^2)} \eta_q (t\Theta)^q_m t_I^J \xi_J + \frac{1}{2} (*V^V)_{mpq} \Gamma^{pq} \xi_I \end{aligned} \quad (4.34)$$

Notice that the new gauge connection is no longer $\text{SU}(2)$ connection, since the term

$$(T_m)_{IJ} \equiv (\nabla_m^A t_I^K) t_{KJ}, \quad (4.35)$$

might not be symmetric in I, J , but rather

$$T_m^{IJ} - T_m^{JI} = \frac{1}{2} \epsilon^{IJ} \nabla_m \text{tr}(t^2), \quad (4.36)$$

which corresponds to an new extra $\text{U}(1)$ gauge field. Fortunately this extra $\text{U}(1)$ part is in pure gauge,

$$\hat{A}_{U(1)}^{IJ} \sim \epsilon^{IJ} \nabla \ln \text{tr}(t^2), \quad (4.37)$$

and can be easily gauged away. Hence, let us choose a gauge

$$\nabla \text{tr}(t^2) = 0. \quad (4.38)$$

Before moving to the $t \equiv 0$ case, let us make a few remarks.

- (1) The appearing of ACMS has already been hinted in literatures. In [3], supersymmetric theory is obtained on any 3d almost contact metric manifold. [27] constructed twisted version of the super-Chern-Simons theory considered in [28] on any Seifert manifold M_3 . Their twisted theory is defined with a choice of contact structure on M_3 ,

with fermions replaced by differential forms. Note that the non-degenerate condition of a contact structure is crucial in defining the theory and the supersymmetry used for localization. Similar situations appear in [9, 10], where the authors constructed twisted YM-CS theory on any 5d K-contact manifold M .

- (2) There is an interesting configuration (among many similar ones). It corresponds to the case when

$$2V = F. \quad (4.39)$$

To make things even simpler one can use the Weyl rescaling symmetry to make field F as well as V horizontal, and ξ_I has length 1. The Killing spinor equation can be rewritten as

$$\nabla_m \xi_I = \Gamma_m \left(t_I^J + \frac{1}{4} F^{pq} \Gamma_{pq} \delta_I^J \right) \xi_J, \quad (4.40)$$

which takes the familiar form

$$\nabla_m \xi_I = \Gamma_m \tilde{\xi}_I, \quad (4.41)$$

with $\tilde{\xi}_I = (t_I^J + (1/4) F^{pq} \Gamma_{pq} \delta_I^J) \xi_J$. We will use this Killing spinor equation to construct a supersymmetric theory for the $\mathcal{N} = 1$ vector multiplet in section 5.

Note that this configuration allows a simultaneous removal of the self-dual part of both V and F , and therefore $d\kappa \in \Omega_H^-(M)$. Away from points where $d\kappa = 0$, κ, R defines a contact structure:

$$\kappa d\kappa \wedge d\kappa = -d\kappa \wedge *d\kappa \neq 0. \quad (4.42)$$

There are many examples of contact manifolds. For instance, any non-trivial $U(1)$ -bundle over a 4d Hodge manifold, with unit Reeb vector field R pointing along the $U(1)$ fiber is a contact manifold. One should note that trivially fibered S^1 -bundle, namely $M = S^1 \times N$ with Reeb vector field pointing along S^1 is not contact, because the non-degenerate condition cannot be satisfied. However, this type of manifold still serve as important examples admitting supersymmetry. Hence, we will have a brief discussion related to this type of manifold at the end of this section.

- (2) $t_{IJ} \equiv 0$. There is no natural ACMS arises in this case (although, if possible, one could choose by hand a nowhere-vanishing section of $ad(P_{SU(2)})$ to play the role of t_{IJ} , and similar calculations goes through. In this paper we do not consider this approach). The auxiliary fields F and V read

$$\begin{cases} F_{mn} = (2s)^{-1} (\nabla_m R_n - \nabla_n R_m) \\ V_{mn} = R_m \eta_n - R_n \eta_m \end{cases}, \quad (4.43)$$

and the Killing spinor equation reads

$$\nabla_m \hat{\xi}_I + \left[-\frac{1}{4s^2} (R_l \nabla_m R_n - R_n \nabla_m R_l) + \frac{1}{2} (\iota_R * \eta)_{mnl} \right] \Gamma^{nl} \hat{\xi}_I = (A_m \hat{\xi})_I. \quad (4.44)$$

Similar to the previous discussion, we again have a new connection $\hat{\nabla}$ defined as

$$\hat{\Gamma}_{mn}^l = \Gamma_{mn}^l + \frac{1}{s^2} \left(R^l \nabla_m R_n - R_n \nabla_m R^l \right) - 2(\iota_R * \eta)_{mn}^l, \quad (4.45)$$

satisfying

$$\hat{\nabla}_m (s^{-1} R^n) = 0, \quad (4.46)$$

although there is no obvious geometrical interpretation for this connection.

Again the Killing spinor equation can be rewritten as

$$\hat{\nabla}_m \hat{\xi}_I = (A_m)_I^J \hat{\xi}_J, \quad (4.47)$$

where $\hat{\xi} = \sqrt{s^{-1}} \xi$ has unit norm

To end this section, we discuss, in the following subsections, 3 special cases related to 5-manifolds of the form $M = S^1 \times M_4$, with the Reeb vector field R pointing along S^1 . As we will see there are 2 cases corresponding to two different types of auxiliary field configurations: V horizontal, F vertical and V, F both vertical. The first configuration leads to geometric restrictions on the sub-manifold M_4 , while the second corresponds to the dimensional-reduction of our 5d equation to 4d already discussed in the literatures.

For such product form (or foliation) to appear, one first needs the horizontal distribution TM_H to be integrable: the Frobenius integrability condition for κ reads

$$d\kappa \wedge \kappa = 0, \text{ or equivalently } d\kappa = \kappa \wedge \lambda, \lambda \in \Omega_H^1(M). \quad (4.48)$$

Recall that $F \propto d\kappa$ (Ω^- is assumed to be 0), one sees that the Frobenius integrability condition requires vertical F

$$F = \kappa \wedge (\dots). \quad (4.49)$$

4.1.2 Special manifold foliation

To proceed to the first class of special cases, let us define a local $SU(2)$ section of “almost complex structure”:

$$J^a \equiv \frac{i}{s} (\sigma^a)^I{}_J \Theta^J{}_I, \quad (4.50)$$

satisfying

$$J^a J^b = \epsilon^{abc} J^c - \delta^{ab} I + \delta^{ab} s^{-1} R \otimes s^{-1} \kappa. \quad (4.51)$$

It is immediate that when restricted on TM_H ,

$$J^a J^b = \epsilon^{abc} J^c - \delta^{ab} I. \quad (4.52)$$

Moreover, we have

$$\hat{\nabla}_k (J^a)_{mn} = (\hat{A}_k)^a{}_b (J^b)_{mn}, \quad (4.53)$$

where

$$(\hat{A}_m)^a{}_b \equiv (-i)^2 (\hat{A}_m)^I{}_K (\sigma^a)^J{}_I (\sigma_b)^K{}_J. \quad (4.54)$$

Note that we can solve the new connection in terms of “almost complex structures”:

$$(\hat{A}_k)^a{}_b = \frac{1}{4} (J_b)^{mn} \hat{\nabla}_k (J^a)_{mn}, \quad (4.55)$$

which, depending on whether $t_{IJ} = 0$, provides constraints on t_{IJ} or A .

These equations closely resemble that of Quaternion-Kähler geometry, where one has on manifold M a $SU(2)$ bundle of local almost complex structure J^a satisfying

$$J^a J^b = \epsilon^{abc} J^c - \delta^{ab} I, \quad (4.56)$$

and is parallel with respect to the gauged connection

$$\nabla J^a = A^a_b J^b, \quad (4.57)$$

with the Levi-civita connection ∇ and a $SU(2)$ gauge connection A .

However the situation here is slightly different. We do not have actually a manifold but rather a horizontal part of tangent bundle TM_H of 5-fold M .

Let us assume V is horizontal:

$$\eta = 0. \quad (4.58)$$

The induced connection (for $t \neq 0$ case; $t = 0$ case goes through similarly and yields the same conclusion) on TM_H is

$$\hat{\nabla}_X Y = \nabla_X Y - g(s^{-1}R, \nabla_X Y) s^{-1}R - \frac{1}{\text{tr}(t^2)} (\nabla_X^A t_I^K) t_{KJ} \Theta^{IJ}(Y), \quad \forall X, Y \in TM_H. \quad (4.59)$$

Consider the special case where the sub-bundle TM_H is integrable as the tangent bundle TM_4 of a co-dimension 1 sub-manifold M_4 , then $\hat{\nabla}$ reduces to a connection on M_4 . The first 2 terms of the connection combine to be the induced Levi-Civita connection ∇^{M_4} on M_4 ($s^{-1}R$ being the unit normal vector), while the third term add to it a torsion part:

$$\hat{\Gamma}_{mk}^n = \Gamma_{mk}^n + \gamma_{mk}^n, \quad (4.60)$$

where

$$\gamma_{mk}^n = -\frac{1}{\text{tr}(t^2)} (\nabla_m^A t_I^K) t_{KJ} (\Theta^{IJ})^n_k. \quad (4.61)$$

Rewrite the Quaternion-Kähler-like equation as

$$\hat{\nabla}_k^{M_4} J_{mn}^a = \nabla_k^{M_4} J_{mn}^a - \gamma_{km}^l J_{ln}^a - \gamma_{kn}^l J_{ml}^a = (\hat{A}_k)^a_b J_{mn}^b. \quad (4.62)$$

Now one can put back expression for both γ and J^a , and sees that the torsion terms gives

$$\gamma_{kn}^l J_{ml}^a - \gamma_{km}^l J_{nl}^a = \frac{1}{\text{tr}(t^2)} (\nabla_k^A t_I^K) t_{K^L} (\sigma^a)_K^J (\sigma_b)^I_J (J^b)_{mn} \equiv (B_k)^a_b (J^b)_{mn}. \quad (4.63)$$

This implies that the Quaternion-Kähler-like equation, restricted on a horizontal sub-manifold M_4 , actually reduces to Quaternion-Kähler equation (with newer version of gauge field $\hat{A} + B$)

$$\nabla^{M_4} J^a = (\hat{A} + B)^a_b J^b = \left((A_k)_I^J + R_k t_I^J \right) (\sigma^a)^I_K (\sigma_b)^K_J (J^b)_{mn}. \quad (4.64)$$

Thus, we see that for generic auxiliary fields t_{IJ} and A_m , provided that the horizontal distribution can be globally integrated to a sub-manifold M_4 , M_4 is actually a Quaternion-Kähler manifold. Of course, there are special combinations of t_{IJ} and A such that $\hat{A} + B$ vanish. In such case, M_4 is a HyperKähler manifold.

With the integrability condition satisfied, we see that M is now locally foliated by Quaternion-Kähler (or HyperKähler in special case) manifold. In particular, compact manifold M could be a direct product

$$M = S^1 \times M_4, \quad M_4 \text{ is Quaternion Kahler.} \quad (4.65)$$

In view of the fact that there are only 2 compact smooth Quaternion-Kähler manifolds in $4d$, possible examples are $M = S^1 \times \mathbb{C}P^2$, $S^1 \times S^4$, where the vector field R is chosen to be the unit vector field along S^1 , with gauge field A turned on on $\mathbb{C}P^2$ and S^4 . There are more examples when M_4 is allowed to be non-compact or orbifolds.

4.1.3 Normal ACMS, cosymplectic manifold and Kähler foliation

As mentioned above, there are 2 ways to define ACMS structure on M using the data coming from Killing spinors: with the nowhere-vanishing auxiliary field t_{IJ} or some other nowhere-vanishing section of $ad(P)$. In general the ACMS structure so defined does not have nice differential property. However, when some (rather strong) conditions are satisfied, the ACMS will behave nicer.

Let us focus on the case $t \neq 0$ and $(s^{-1}R, s^{-1}\kappa, \varphi_t)$ define ACMS on M .

One obtains

$$\mathcal{L}_{Rt}\Theta = \frac{1}{2} (\nabla_R^A t^{IJ}) (\Theta_{IJ}) + s \nabla^p \left(\frac{1}{s} R_m \right) (t\Theta)_{np} dx^m \wedge dx^n. \quad (4.66)$$

Setting

$$\nabla_R^A t = 0, \quad \nabla_m (s^{-1} R_n) = 0 \Leftrightarrow \nabla_m R_n \propto F_{mn} = 0, \quad (4.67)$$

one has $\mathcal{L}_{Rt}\Theta = 0$ and hence $\mathcal{L}_{s^{-1}R}\varphi_t = 0$.

If, in addition to the above, one further imposes V to be horizontal and $\nabla^A t = 0$, then it is easy to see that the ACMS satisfies

$$\nabla \varphi_t = 0, \quad (4.68)$$

and hence it is cosymplectic. In this case, the Levi-civita connection ∇ on M respects the ACMS, the restriction of ∇ on the horizontal distribution is automatically a connection on TM_H .

Note that $\nabla R = 0$ implies that the horizontal distribution is locally integrable. Therefore, restricted on the integral sub-manifold, ∇ is the induced Levi-civita connection, φ_t is an almost complex structure which can be shown to have vanishing Nijenhuis tensor and hence actually a complex structure. It is parallel with respect to induced Levi-civita connection, hence is Kähler.

In summary, we see that

$$\nabla^A t^{IJ} = 0, \quad F = 0, \quad V = V_H = -t\Theta, \quad (4.69)$$

implies a cosymplectic ACMS (namely $\nabla \varphi_t = 0$), and M is locally foliated by 4d Kähler manifold, with the Kähler structure provided by φ_t .

Recall that we had conclusion that M is locally foliated by Quaternion-Kähler manifold in the previous subsection, for configuration $F_H = 0$, $V = V_H$. Suppose $M = M_4 \times S^1$ with a Reeb vector field R from a Killing spinor pointing along S^1 , then we see that M_4 must be Quaternion-Kähler as well as Kähler. If M_4 is a smooth compact manifold, then this leaves only one possibility:

$$M = \mathbb{CP}^2 \times S^1. \quad (4.70)$$

Of course, for more general Reeb vector field pointing along other directions, one could have other possibilities of M_4 .

4.1.4 Reducing to 4d

Finally let us point out the reduction of (2.11) to 4d already discussed in literatures [1, 2]. Consider $M = M_4 \times S^1$ with spinor ξ_I and auxiliary fields independent on the S^1 coordinate. The 4d part of the Killing spinor equation reads

$$\nabla_\mu \xi_I = t_I^J \gamma_\mu \xi_J + \frac{1}{2} F_{\mu 5} \gamma^5 \xi_I + \frac{1}{2} V^{\nu 5} \gamma_{\mu\nu} \xi_I + \frac{1}{2} V^{\lambda\rho} \gamma_{\mu\lambda\rho} \xi_I + \frac{1}{2} F_{\mu\nu} \gamma^\nu \xi_I + (A_\mu)_I^J \xi_J, \quad (4.71)$$

and the S^1 part serves as direct constraints on auxiliary fields

$$\partial_5 \xi_I = t_I^J \xi_J + \frac{1}{2} F_{5\mu} \gamma^\mu \xi_I + \frac{1}{2} V^{\mu\nu} \gamma_{\mu\nu} \gamma_5 \xi_I + (A_5)_I^J \xi_J = 0. \quad (4.72)$$

There are now 2 different ways to reduce the equation, each gives rise to the Killing equation discussed in [1, 2]. The involved vertical condition $V_H = F_H = 0$ and requirement $t = 0$ or $t_{IJ} \propto \epsilon_{IJ}$ indeed imply the Frobenius Integrability condition

$$d\kappa \wedge \kappa = 0, \quad (4.73)$$

which is necessary for M to be a product.

I. Reduction to [1]. Setting $t = A = F_{\mu\nu} = V_{\mu\nu} = 0$, namely F and V are both vertical 2-forms, the equation simplifies to

$$\begin{cases} \nabla_\mu \xi_I = \frac{1}{2} F_{\mu 5} \gamma^5 \xi_I + \frac{1}{2} V^{\nu 5} \gamma_{\mu\nu} \xi_I \\ \partial_5 \xi_I = F_{5\mu} \gamma^\mu \xi_I = 0 \end{cases}, \quad (4.74)$$

or written in terms of Weyl components $\xi_I = (\zeta_I, \tilde{\zeta}_I)$,

$$\begin{cases} \nabla_\mu \zeta_I = \frac{1}{2} F_{\mu 5} \zeta_I + \frac{1}{2} V^{\nu 5} \sigma_{\mu\nu} \zeta_I \\ \nabla_\mu \tilde{\zeta}_I = -\frac{1}{2} F_{\mu 5} \tilde{\zeta}_I - \frac{1}{2} V^{\nu 5} \tilde{\sigma}_{\mu\nu} \tilde{\zeta}_I \end{cases}, \quad (4.75)$$

with constraint on $F_{\mu 5}$

$$F_{5\mu} \tilde{\sigma}^\mu \zeta_I = 0, \quad F_{5\mu} \sigma^\mu \tilde{\zeta}_I = 0. \quad (4.76)$$

Suppose we relax the reality condition on ξ and also F and V , and define new complex auxiliary vector fields A and V

$$\begin{cases} 2iA_\mu \equiv F_{\mu 5} - V_{\mu 5} = \partial_\mu a_5 - V_{\mu 5} \\ -2iV_\mu \equiv V_{\mu 5} \end{cases}, \quad (4.77)$$

then the above equation takes a familiar form

$$\begin{cases} \nabla_\mu \zeta_I = -i(V_\mu - A_\mu) \zeta_I - iV^\nu \sigma_{\mu\nu} \zeta_I \\ \nabla_\mu \tilde{\zeta}_I = i(V_\mu - A_\mu) \tilde{\zeta}_I + iV^\nu \tilde{\sigma}_{\mu\nu} \tilde{\zeta}_I \end{cases}, \quad (4.78)$$

which is just the Killing equations discussed in [1] for 2 separate pairs of chiral spinors $(\zeta_1, \tilde{\zeta}_1)$ and $(\zeta_2, \tilde{\zeta}_2)$. $V_{\mu 5}$ has to satisfy conservation condition $\nabla_\mu V^{\mu 5} = 0$, and $F_{\mu 5}$ is holomorphic w.r.t. $J_{\mu\nu}^I$ and $\tilde{J}_{\mu\nu}^I$ if any of them is non-zero. The conservation condition on $V_{\mu 5}$ is equivalent to d^* -closed condition on vertical 2-form V

$$\nabla_\mu V^{\mu 5} = 0 \Leftrightarrow \nabla^m V_{mn} = 0 \Leftrightarrow d * V = 0. \quad (4.79)$$

Now that we choose not to impose reality condition on auxiliary fields, it is also fine for ξ_I to be non-symplectic-Majorana, hence ξ_1 and ξ_2 are now unrelated complex spinors, and one of the two can vanish. This then leads to different numbers of Killing spinor solutions in 4d, ranging from 1 to 4. In [1], the cases when M_4 admits 1, 2 and 4 supercharges are discussed in detail. Here we list a few points and discuss their 5d interpretation. More results can be obtained similarly.

- (1) *2 supercharges of the form $(\zeta, 0)$ and $(\eta, 0)$* : then assuming M_4 is compact, M_4 has to be a Hyperhermitian manifold up to global conformal transformation. Moreover, the auxiliary fields satisfy

- a) $V_\mu - A_\mu$ is closed 1-form.
- b) $\partial_\mu V_\nu - \partial_\nu V_\mu$ is anti-self-dual 2-form.

Condition a) is obviously satisfied by definition: $V_\mu - A_\mu \sim \partial_\mu a_5$ is obviously closed. The condition b) reads in 5d point of view

$$\iota_R dV = - * dV, \quad (4.80)$$

- (2) *2 supercharges of the form $(\zeta, 0)$ and $(0, \tilde{\zeta})$* : there are 2 commuting Killing vector on M_4 , and hence M_4 is locally T^2 -fibration over Riemann surface Σ . The auxiliary fields $V_{\mu 5}$ and $F_{\mu 5}$ are given in terms of $J_{\mu\nu}$ and $\tilde{J}_{\mu\nu}$.

II. Reduction to [2]. Setting $A = F_{\mu\nu} = V_{\mu\nu} = 0$, $\frac{1}{2}F_{\mu 5} = \frac{1}{2}V_{\mu 5} = \frac{i}{3}b_\mu$, $t = (i/6)MI_{2 \times 2}$, one similarly obtains

$$\begin{cases} \nabla_\mu \zeta_I = \frac{i}{6}M \tilde{\sigma}_\mu \tilde{\zeta}_I + \frac{i}{3}b_\mu \zeta_I + \frac{i}{3}b^\nu \sigma_{\mu\nu} \zeta_I \\ \nabla_\mu \tilde{\zeta}_I = \frac{i}{6}M \sigma_\mu \zeta_I - \frac{i}{3}b_\mu \tilde{\zeta}_I - \frac{i}{3}b^\nu \tilde{\sigma}_{\mu\nu} \tilde{\zeta}_I \end{cases}, \quad (4.81)$$

which is the Killing spinor equation for 2 pairs of spinor $(\zeta_1, \tilde{\zeta}_1)$ and $(\zeta_2, \tilde{\zeta}_2)$ discussed in [2] for but with condition $M = \tilde{M}$.

Again, ξ_I are no longer symplectic Majorana, and solution of the 5d Killing spinor equation leads to different number of solutions to 4d Killing spinor equation. Let us list a few examples from the detail discussion in [2]. Interested reader can refer to their paper for more results.

- (1) *1 supercharge of the form $(\zeta, \tilde{\zeta})$* : any manifold (M_4, g) with a nowhere-vanishing complex Killing vector field K which squares to zero and commutes with its complex conjugate

$$K_\mu K^\mu = 0, [K, \bar{K}] = 0, \quad (4.82)$$

admits solution of the form $(\zeta, \tilde{\zeta})$ to the 4d Killing spinor equation. K and the metric can be used to build up a Hermitian structure on M_4 .

- (2) *2 supercharges of the form $(\zeta_1, 0)$ and $(\zeta_2, 0)$* : M_4 is anti-self-dual with $V_{5\mu}$ and $F_{5\mu}$ closed 1-forms, and hence in 5d point of view, they are closed vertical 2-forms. Moreover, the form of solution requires $\tilde{M} = 0$, and according to our reduction, $M = \tilde{M} = 0$. If $F = V$ are exact, then M_4 is locally conformal to a Calabi-Yau 2-fold. Otherwise, M_4 is locally conformal to $\mathbb{H}^3 \times \mathbb{R}$.
- (3) *2 supercharges of the form $(\zeta_1, 0)$ and $(0, \tilde{\zeta}_2)$* : one must have $M = \tilde{M} = 0$. This is equivalent to M_4 having solution $(\zeta_1, \tilde{\zeta}_2)$ with $M = \tilde{M} = 0$.

4.2 Manifolds admitting 2 pairs of supercharges

In this section we consider the case when 2 pairs of solutions to the (2.11) exist. We see that when certain assumptions on vectors Q_{IJ} are made, and if the Killing vector fields form closed algebra, the geometry of M will be heavily constrained. And in particular, all the resulting geometries admit contact metric structures.

The spinors ξ and $\tilde{\xi}$ satisfy equations:

$$\begin{aligned} \nabla_m \xi_I &= t_I^J \Gamma_m \xi_J + \frac{1}{2} V^{pq} \Gamma_{mpq} \xi_I + \frac{1}{2} F_{mp} \Gamma^n \xi_I + (A_m)_I^J \xi_J \\ \nabla_m \tilde{\xi}_I &= t_I^J \Gamma_m \tilde{\xi}_J + \frac{1}{2} V^{pq} \Gamma_{mpq} \tilde{\xi}_I + \frac{1}{2} F_{mn} \Gamma^n \tilde{\xi}_I + (A_m)_I^J \tilde{\xi}_J \end{aligned} \quad (4.83)$$

Similar to the previous section, we have

- Derivative on u_{IJ}

$$(1) \quad u^{IJ} du_{IJ} = \hat{u}^{IJ} d\hat{u}_{IJ} + \frac{1}{2} u du = -2t^{IJ} (\hat{u} \hat{\tau})_{IJ} - \iota_{(uQ)} F. \quad (4.84)$$

$$(2) \quad du = -\iota_Q F. \quad (4.85)$$

- Derivative on Q_{IJ}

$$\nabla_m Q_n + \nabla_n Q_m = 0, \quad (4.86)$$

namely, Q is a Killing vector.

The derivative on u_{IJ} implies relation

$$2u_{IJ} u^{IJ} = s\tilde{s} + C, \quad (4.87)$$

where the function C is invariant along R and \tilde{R} . When $t_{IJ} = 0$, C reduces to constant. Notice that when $C = 0$,

$$s\tilde{R} = \tilde{s}R, \quad (4.88)$$

and when $C = -s\tilde{s}$

$$\tilde{s}R = -s\tilde{R}, \quad (4.89)$$

which are degenerate cases that we do not consider in the following.

- Commutator between R and \tilde{R}

$$K \equiv [R, \tilde{R}]^m = 8(t\hat{u})Q^m - 8u(t\hat{Q})^m - 4(\iota_R \iota_{\tilde{R}} * V)^m + (\tilde{s}\iota_R F - s\iota_{\tilde{R}} F)^m. \quad (4.90)$$

Recall that we now have several Killing vector fields, R , \tilde{R} , K and Q . If some of them form closed Lie algebra, the geometry of M will be heavily constrained. In the rest of this section, we discuss several simplest possibilities where they form 2 or 3 dimensional Lie algebras.

1. R and \tilde{R} form 2-dimensional algebra. There exist only two 2-dimensional Lie algebras up to isomorphisms. One is the abelian algebra, the other is a unique non-abelian algebra.

When R and \tilde{R} commute, namely $K = 0$, one obtains the abelian algebra. If the orbits of R and \tilde{R} are closed, then M is acted freely by T^2 , and therefore M is a T^2 -fibration.

The non-abelian algebra corresponds to $K \neq 0$. Assume K is a linear combination of R and \tilde{R} , then

$$[R, \tilde{R}] = aR + b\tilde{R}. \quad (4.91)$$

Contracting with R and \tilde{R} it gives

$$\begin{cases} as^2 + b(s\tilde{s} + 2C) = s\iota_{\tilde{R}}\iota_R F \\ a(s\tilde{s} + 2C) + b\tilde{s}^2 = \tilde{s}\iota_{\tilde{R}}\iota_R F \end{cases}. \quad (4.92)$$

The determinant of the system is

$$\det = s^2\tilde{s}^2 - (s\tilde{s} + 2C)^2 = -4C(C + s\tilde{s}). \quad (4.93)$$

Notice that away from the degenerate cases when $C = 0$ and $C = -s\tilde{s}$, the determinate is non-zero. Therefore, when $\iota_R \iota_{\tilde{R}} F \neq 0$, the system allows solution (a, b)

$$\begin{cases} b = \frac{s\iota_{\tilde{R}}\iota_R F}{2(s\tilde{s} + C)} \\ a = \frac{\tilde{s}\iota_{\tilde{R}}\iota_R F}{2(s\tilde{s} + C)} \end{cases}. \quad (4.94)$$

Notice however that R , \tilde{R} and their commutator are all Killing vectors, therefore the coefficients a and b must be constant. This implies

$$\frac{s}{\tilde{s}} = \text{const}, \quad (4.95)$$

and further

$$\mathcal{L}_R \tilde{s} = \mathcal{L}_{\tilde{R}} s = 0 \Rightarrow \iota_R \iota_{\tilde{R}} F = 0, \quad (4.96)$$

hence

$$a = b = 0. \quad (4.97)$$

To summarize, if R and \tilde{R} form 2-dimensional algebra, it can only be trivial abelian algebra.

What remains is the Killing vector Q . Assume Q and the commuting R and \tilde{R} form 3 dimensional algebra:

$$\begin{cases} [R, \tilde{R}] = 0 \\ [Q, R] = aR + b\tilde{R} + mQ \\ [Q, \tilde{R}] = cR + d\tilde{R} + nQ \end{cases} \quad (4.98)$$

Let us make a Weyl rescaling to set $\iota_R F = 0$. Then it automatically implies $\iota_R \iota_Q F = \iota_{\tilde{R}} \iota_Q F = 0$ by previous arguments. Therefore,

$$\begin{cases} \mathcal{L}_R (u\tilde{s}) = \mathcal{L}_R (\tilde{R} \cdot Q) = \tilde{R} \cdot [R, Q] = 0 \\ \mathcal{L}_{\tilde{R}} (us) = \mathcal{L}_{\tilde{R}} (R \cdot Q) = R \cdot [\tilde{R}, Q] = 0 \end{cases} \quad (4.99)$$

It is immediate to see that the determinant of the above linear system is

$$\det \propto |s\tilde{R} - \tilde{s}R|^2 |Q|^2, \quad (4.100)$$

and hence non-trivial solution requires $Q = 0$ or $\tilde{s}R = s\tilde{R}$, which we do not consider. Therefore, one has Q, R, \tilde{R} forming abelian algebra, and M is a T^3 -fibration over Riemann surface Σ . Up to an overall rescaling factor, the metric can be written as

$$ds^2 = h_{\alpha\beta} dx^\alpha dx^\beta + \sum_{i=1}^3 (d\theta_i + \alpha_i(x))^2, \quad (4.101)$$

where θ_i are the periodic coordinates along R, \tilde{R} and Q provided their orbits are closed, and α_i are 1-forms that determine the fibration.

2. R, \tilde{R} and $[R, \tilde{R}]$ form 3-dimensional algebra. Assume that the algebra takes the form

$$\begin{cases} [R, \tilde{R}] = K \\ [R, K] = aR + b\tilde{R} + mK \\ [\tilde{R}, K] = cR + d\tilde{R} + nK \end{cases} \quad (4.102)$$

In general, $\iota_R \iota_{\tilde{R}} F$ does not vanish. However, we can make a Weyl rescaling to make, for instance, $\iota_R F = 0$, and in particular, s is constant and $\iota_R \iota_{\tilde{R}} F = 0$. This implies

$$R \cdot K = \tilde{R} \cdot K = 0. \quad (4.103)$$

It is then easy to solve the coefficients in the above linear relation:

$$\begin{cases} a = -\frac{1}{4C}|K|^2 \frac{s\tilde{s} + 2C}{s\tilde{s} + C} \\ b = \frac{1}{4C}|K|^2 \frac{s^2}{s\tilde{s} + C} \end{cases}, \quad \begin{cases} c = -\frac{1}{4C}|K|^2 \frac{s^2}{s\tilde{s} + C} \\ d = \frac{1}{4C}|K|^2 \frac{s\tilde{s} + 2C}{s\tilde{s} + C} \end{cases}. \quad (4.104)$$

The fact that all coefficients must be constants implies

$$\frac{s}{\tilde{s}} = \text{const}, \quad \frac{s^2}{s\tilde{s} + 2C} = \text{const}, \quad (4.105)$$

and therefore both \tilde{s} and C are constant as well.

It is then straight forward to renormalize and linearly recombine the vectors to form a standard $\mathfrak{su}(2)$ algebra. Therefore topologically M is a $\text{SU}(2)$ -fibration over a Riemann surface Σ ; however, there is no non-trivial $\text{SU}(2)$ bundle over a Riemann surface from the fact that the 3-skeleton of the classifying space $\text{BSU}(2)$ is a point), hence topologically $M = S^3 \times \Sigma$. Up to an overall scaling factor which was used to bring s to 1, the metric takes the form

$$ds_M^2 = ds_\Sigma^2 + ds_{S^3}^2 = h_{\alpha\beta}(x)dx^\alpha dx^\beta + \sum_{a=1}^3 e^a e^a, \quad (4.106)$$

where $e^a = \kappa, \tilde{\kappa}, \gamma$ are $\text{SU}(2)$ invariant 1-forms on $\text{SU}(2)$. Note that $\iota_R F = \iota_{\tilde{R}} F = 0$ implies F is a form on Σ :

$$F = \frac{1}{2} F_{\alpha\beta}(x) dx^\alpha \wedge dx^\beta. \quad (4.107)$$

Recall that there is one more Killing vector field Q . The metric has isometry subgroup $\text{SU}(2)_L \times \text{SU}(2)_R$, which comes from the isometry of S^3 . If $Q \notin \mathfrak{su}(2)_L \times \mathfrak{su}(2)_R$, then Q must generate continuous isometry in Σ , which implies $\Sigma = T^2$ or S^2 if M is compact. In this case, by requiring Q commutes and being orthogonal to R, \tilde{R} and K , one can derive new constraints on the auxiliary fields. For instance,

$$R \cdot Q = 0 \Leftrightarrow u = 0 \Leftrightarrow \iota_Q F = 0 \quad (4.108)$$

which, combining with the fact that F is a 2-form on Σ , implies actually $F = 0$.

4.3 Manifolds admitting 8 supercharges

In this section, we discuss the optimal case where the Killing spinor equation has full 4 complex dimensional space of solutions. This is done by taking the commutator of the ∇ , applying Killing spinor equation and matching the Gamma matrix structure on both sides. We will see that there are 3 cases corresponding to the survival of only one of the 3 auxiliary fields (t, V, F) , with the other two vanishes identically. Here we list main results that we will discuss in detail:

- $V \neq 0$: M is positively curved, with product structure $T^k \times G$ where G is a compact Lie group. The non-trivial example is then $T^2 \times \text{SU}(2)$ with standard bi-invariant metric.

- $F \neq 0$: M is locally of the form $M_3 \times \mathbb{H}^2$, where M_3 is a 3 dimensional flat manifold.
- $t \neq 0$: M is locally a space of constant curvature with positive scalar curvature, hence M is locally isometric to S^5 .
- $t = V = F = 0$: M has zero curvature, hence is locally isometric to \mathbb{R}^5 .

By explicitly writing down the commutator $[\nabla_m, \nabla_n]\xi_I$ using Killing spinor equation, one would obtain 2 immediate results:

- Terms independent of Γ .

$$(W_I^J)_{mn} \equiv \nabla_m(A_n)_I^J - \nabla_n(A_m)_I^J + (A_n)_I^K (A_m)_K^J - (A_m)_I^K (A_n)_K^J = 0. \quad (4.109)$$

For simply-connected 5-manifolds, flat connections must be gauge equivalent to trivial connections.

- Terms linear in Γ .

$$\begin{aligned} 0 = & (\nabla_m t_I^J) \Gamma_n - \left[(A_m t)_I^J - (t A_m)_I^J \right] \Gamma_n \\ & + \frac{1}{2} (\nabla_m F_{np}) \Gamma^p \delta_I^J - F_n^p (*V)_{mp}{}^q \Gamma_q \delta_I^J - 2t_I^J (*V)_{mn}{}^l \Gamma_l \\ & - (m \leftrightarrow n) \end{aligned} \quad (4.110)$$

The solutions to the equation are:

Case 1

$$\begin{cases} t_{IJ} = 0 \\ F = 0 \end{cases} \quad (4.111)$$

Case 2

$$V = 0 \quad (4.112)$$

Now we study 2 cases separately.

Case 1. Only $V \neq 0$.

The solution $t^{IJ} = 0$ and $F = 0$ implies (4.110) vanishes identically, no further condition on V is required.

Combining with previous section, we know that

$$ds = 0, \quad (4.113)$$

and we conveniently set $s = 1$.

By identifying the terms quadratic in Γ matrices, one sees that the

- The curvature tensor satisfies a flat condition:

$$\hat{R}_{mnkl}(\hat{\nabla}) = 0, \quad (4.114)$$

where \hat{R} is the curvature tensor of a metric connection with anti-symmetric torsion

$$\hat{\nabla}_m X^n = \nabla_m X^n + 2(*V)^n{}_{mk} X^k. \quad (4.115)$$

with ∇ the Levi-civita connection of g . This result is most easily understood by looking at the Killing spinor equation, where V can be absorbed into the metric connection as a totally anti-symmetric torsion.

- The Ricci curvature

$$Ric_{mn} = 4(*V)^{pq}{}_m(*V)_{pqn}. \quad (4.116)$$

- Scalar curvature

$$\mathcal{R} = +4(*V)^{kmn}(*V)_{kmn} \geq 0, \quad (4.117)$$

which indicates the manifold must have positive curvature. Moreover, compact manifolds admitting metric connection with anti-symmetric torsion are known to be products of $T^k \times G$ where G is a compact group. This leaves us only a few possibilities, the non-trivial one being

$$M = SU(2) \times T^2, \quad (4.118)$$

which has standard positive curvature.

Case 2. $V = 0$.

Putting back $V = 0$ into (4.110), one has

$$\begin{cases} g_{nk}(\nabla_m^A t_I^J) - g_{mk}(\nabla_n^A t_I^J) = 0 \\ (\nabla_m F_{nk}) - (\nabla_n F_{mk}) = 0 \end{cases}. \quad (4.119)$$

These 2 condition implies covariant-constantness of t_{IJ} and F :

$$\nabla_m^A t_I^J = 0, \quad \nabla_k F_{mn} = 0. \quad (4.120)$$

In particular,

$$d * F = 0, \quad dF = 0 \Leftrightarrow \Delta F = 0, \quad (4.121)$$

and 2nd/3rd Betti number is forced to be non-zero, if $F \neq 0$:

$$b^2 = b^3 \geq 1 \quad (4.122)$$

Compare the the terms quadratic in Γ , one obtains

$$\frac{1}{4}R_{mnpq}\Gamma^{pq}\delta_I^J = -2(t^2)_I^J \Gamma_{mn} - \frac{1}{2}F_{mp}F_{ns}\Gamma^{ps}\delta_I^J + [2t_I^J F_{pm}\Gamma^p_n - (m \leftrightarrow n)]. \quad (4.123)$$

The solutions are

$$t_{IJ} = 0 \quad \text{or} \quad F = 0. \quad (4.124)$$

i) $t = 0$ while $F \neq 0, t = 0$.

- Riemann tensor

$$R_{mnkl} = F_{ml}F_{nk} - F_{mk}F_{nl}. \quad (4.125)$$

Note that the expression satisfies interchange symmetry automatically, while the 1st Bianchi identity implies

$$F_{m[l}F_{nk]} - F_{m[k}F_{nl]} = 0 \Rightarrow F \wedge F = 0. \quad (4.126)$$

- Ricci tensor

$$Ric_{mn} = F_{mk}F^k_n. \quad (4.127)$$

.

- Scalar curvature

$$\mathcal{R} = -F_{mn}F^{mn}, \quad (4.128)$$

which is negative definite if $F \neq 0$. Also note that F is covariantly constant, hence R_{mnnk} is also covariantly constant.

Let's further constraint the form of curvature using the condition $F \wedge F = 0$. Noting that F_{mn} is a 5×5 antisymmetric matrix, we choose a coordinate where it takes block diagonal form:

$$F = F_{12}dx^1 \wedge dx^2 + F_{34}dx^3 \wedge dx^4. \quad (4.129)$$

Requiring that $F \wedge F = 0$ forces

$$F_{12}F_{34} = 0. \quad (4.130)$$

Assuming

$$F_{12} \neq 0, \quad (4.131)$$

with all other component zero, one arrives at a Riemann tensor with only one non-vanishing component:

$$R_{1212} = -(F_{12})^2 < 0. \quad (4.132)$$

Combining with the fact that F is parallel, this implies the 5-manifold M should locally be product manifold

$$M = T^3 \times \mathbb{H}^2, \quad (4.133)$$

where $F = F_{12}dx^1 \wedge dx^2$ serves as the volume form of \mathbb{H}^2 .

The metric of M can be written as

$$ds^2 = ds_{T^3}^2 + \frac{F_{12}}{y^2} (dx^2 + dy^2). \quad (4.134)$$

ii) The case where $t \neq 0, F = 0$.

- Riemann tensor

$$R_{mnkl} = 2\text{tr}(t^2)(g_{ml}g_{nk} - g_{mk}g_{nl}), \quad (4.135)$$

where interchange symmetry and first Bianchi identity are automatically satisfied.

The second Bianchi identity forces $\text{tr}(t^2)$ to be constant. The form of curvature implies that M is a space of constant curvature, and therefore it must be locally isometric to S^5 . This corresponds to the well-known fact that maximal number of solutions to the well-known Killing spinor equation can only be achieved on round S^5 .

5 Supersymmetric theory for vector multiplet

In section 4, we analyzed many properties of the proposed Killing spinor equation (2.11) from supergravity, and discussed some necessary geometric conditions on the underlying manifold for solutions to exist.

In this section, we propose a slightly generalized version of the supersymmetric theory for $\mathcal{N} = 1$ vector multiplet. It is not the most general one, as there are other known examples (constructed by dimensional reduction from $6d$, for instance) in recent literatures that does not completely fit in the following discussion.

Let us consider a simplified Killing spinor equation, where we set $F = 2V \equiv \mathcal{F}$ in (2.11)

$$D_m \xi_I = t_I^J \Gamma_m \xi_J + \frac{1}{4} \mathcal{F}^{pq} \Gamma_{mpq} \xi_I + \frac{1}{2} \mathcal{F}_{mn} \Gamma^n \xi_I. \quad (5.1)$$

D_m contains Levi-civita connection, spin connection, gauge field A_m from the vector multiplet and background $SU(2)$ -gauge field A_I^J , depending on the objects it acts on. The change of notation to \mathcal{F}_{mn} is to avoid confusion with the field strength of $\mathcal{N} = 1$ gauge field

$$F_{mn} \equiv \nabla_m A_n - \nabla_n A_m - i[A_m, A_n]. \quad (5.2)$$

We propose a supersymmetry transformation of $\mathcal{N} = 1$ vector multiplet with parameter ξ being solution to the (5.1) is

$$\begin{cases} \delta_\xi A_m = i\epsilon^{IJ}(\xi_I \Gamma_m \lambda_J) \\ \delta_\xi \sigma = i\epsilon^{IJ}(\xi_I \lambda_J) \\ \delta_\xi \lambda_I = -\frac{1}{2} F_{mn} \Gamma^{mn} \xi_I + (D_m \sigma) \Gamma^m \xi_I + \epsilon^{JK} \xi_J D_{KI} + 2\sigma t_I^J \xi_J + \frac{1}{2} \sigma \mathcal{F}^{pq} \Gamma_{pq} \xi_I \\ \delta_\xi D_{IJ} = -i(\xi_I \Gamma^m D_m \lambda_J) + [\sigma, (\xi_I \lambda_J)] + it_I^K (\xi_K \lambda_J) - \frac{i}{4} \mathcal{F}^{pq} (\xi_I \Gamma_{pq} \lambda_J) + (I \leftrightarrow J) \end{cases} \quad (5.3)$$

Using previous results we obtain

$$d\kappa = -4(t\Theta) + 2sV_V, \quad \mathcal{F} = -\frac{2}{s}(t\Theta) + \frac{2}{s}\Omega^- + V_V, \quad (5.4)$$

with V_V denoting the vertical part of field V .

As discussed in an earlier remark, we applying Weyl rescaling symmetry, one can eliminate V_V and set $s = 1$. The Reeb vector field is then compatible with κ :

$$\iota_R \kappa = 1, \iota_R d\kappa = 0. \quad (5.5)$$

For simplicity let us consider a special case where

$$\mathcal{F} = d\kappa, \quad (5.6)$$

namely $\Omega^- = 1/4 d\kappa$.

Then it is straight forward to prove that the following Lagrangian $S(\kappa, g)$ is invariant under (5.3):

$$S = \int_M \text{tr} \left[F \wedge *F - \kappa \wedge F \wedge F - d_A \sigma \wedge *d_A \sigma - \frac{1}{2} D_{IJ} D^{IJ} + i\lambda_I \not{D}_A \lambda^I - \lambda_I [\sigma, \lambda^I] \right. \\ \left. - it^{IJ} (\lambda_I \lambda_J) + 2\sigma t^{IJ} D_{IJ} + \frac{i}{2} \nabla_m \kappa_n (\lambda_I \Gamma^{mn} \lambda^I) + 2\sigma F \wedge *d\kappa + \frac{1}{4} \mathcal{R} \sigma^2 \right], \quad (5.7)$$

where \mathcal{R} is the scalar curvature of the manifold.

The detail proof will be presented in appendix F, but let us first make a few remarks here.

As already mentioned, in the explicit form (5.7) we took the choice to assume $\Omega^- = (1/4)d\kappa$, which is in fact a special case of a large family of supersymmetric theories in the following sense.

Under supersymmetry (5.3) with ξ satisfying (5.1) without imposing $\Omega^- = (1/4)d\kappa$, the Lagrangian without $\kappa \wedge F \wedge F$ has variation

$$\frac{i}{2} \mathcal{F}_{mn} F_{pq} (\xi_I \Gamma^{mnpq} \lambda^I) \quad (5.8)$$

Such term can be identified in two ways. If we assume \mathcal{F} is not only closed, but also exact

$$\mathcal{F} = d\mathcal{A} = \frac{1}{2} d\kappa + 2\Omega^- \quad (5.9)$$

for some global 1-form \mathcal{A} , then the term can be identified as variation of

$$\mathcal{A} \wedge F \wedge F \quad (5.10)$$

In such case, the theory is specified by κ and \mathcal{A} .

However, if we do not assume anything of \mathcal{F} , then the term can also be identified as variation of

$$\mathcal{F} \wedge \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \quad (5.11)$$

In such case, the theory is specified by a nowhere-vanishing 1-form κ and a closed anti-self-dual 2-form Ω^- , although the gauge invariance is not nicely manifested.

Following an analysis similar to that in [8], one can add to the Lagrangian (5.7) a δ -exact term $\delta_\xi V$ with

$$V = \text{tr} \left((\delta_\xi \lambda)^\dagger \lambda \right). \quad (5.12)$$

Then the localization locus is

$$F_H^- = \sigma d\kappa, \quad \iota_R F = 0, \quad d_A \sigma = 0, \quad D_{IJ} + 2\sigma t_{IJ} = 0 \quad (5.13)$$

For general Ω^- , the first equation will take a more general form

$$F_H^- = \sigma d\kappa + \sigma \Omega_H^-. \quad (5.14)$$

This localization locus is a generalization of the contact instanton in [9].

It would be interesting to perform a complete localization for the theory (5.7) with the above localization locus, which we leave for future study.

6 Discussion

So far we have obtained many constraints on geometry of M imposed by the existence of supercharges. For 1 pair of supercharges, generically M must be almost contact manifold, and using the compatible connection, the Killing spinor equation can be simplified to a compact form. We also discussed a few interesting cases related to product manifold, which leads to special foliation and reduction to known 4-dimensional Killing spinor equations. The presence of 2 pairs supercharges with 2 additional assumptions restricts the isometry algebra of M , forcing M to be S^3 or T^3 -fibration over Riemann surfaces. The presence of 4 pairs of supercharges allows for only 3 major possibilities, where the corresponding topologies and geometries are basically fixed.

There are several problems that are interesting to explore further.

- (1) We obtained necessary conditions for supercharges to exist, but not sufficient conditions. In 3 dimension [3], the general solution to Killing spinor equation is obtained from the special coordinate, which requires some integrability of the almost contact structure. However, we do not have such integrability for the almost contact structure we defined, partly because the definition involves auxiliary field t_{IJ} , and the differential property of t_{IJ} is not known a priori. Moreover, in the extreme case where $t_{IJ} = 0$, it is not obvious that M is still a almost contact manifold. Perhaps it is possible to define almost contact structure of M without referring to t_{IJ} .
- (2) We partially solved the auxiliary fields, but not all: gauge field A and t_{IJ} are entangled together. If t_{IJ} and A could be solved in terms of pure bilinears separately, the first problem above can also be solved.
- (3) In the discussions, we made a few assumptions and simplifications. For examples, we did not study all possible bilinears formed by all solutions, but focused on those formed by the representatives from each pair. One should be able to obtain more information of M by taking into account all of them. Also, to simplify computation we assumed $\Omega^- = 0$ in some discussions. It is straight-forward and interesting to reinstate general Ω^- , and understand its role in the almost contact metric structure.

- (4) We start from Zucker's off-shell supergravity [21]. However, it is not coupled to matter fields, and hence one do not automatically obtain any supersymmetric theory for matter multiplets. Our analysis, in this sense, is far from enough to obtain a complete picture. A next step one could try is to start from known 5-dimensional off-shell supergravity coupled with matter and take the rigid limit. For instance, one can start with $\mathcal{N} = 1$ supergravity coupled to Yang-Mills matters in [29], which is a natural extension of [21]. The Killing spinor equation will be the same as we have discussed. Also one can study [30], which was considered in [8]. After turning on auxiliary fields t_{IJ} and V_{mn} , the Killing spinor equation involved is then

$$\nabla_m \xi_I = t_I{}^J \Gamma_m \xi_J + \frac{1}{2} V_{pq} \Gamma^{mpq} \xi_I, \quad (6.1)$$

which is a special case of our more general equation.

7 Examples

In this section, we present simple explicit examples that illustrate some of the discussion before, by solving Killing spinor equations on selected manifolds and determining the auxiliary fields.

7.1 $M = S^1 \times S^4$

In earlier discussion, we discussed the possibility of having $M = S^1 \times N$ with N a 4d Quaternion-Kähler manifold. In this section, we consider the case where $N = S^4$.

Denote the coordinate along S^1 to be θ , x^μ are stereo-projection coordinates on S^4 . The metric of $S^1 \times S^4$ is simply

$$ds^2 = d\theta^2 + \frac{\delta_{\mu\nu} dx^\mu dx^\nu}{(1+r^2)^2} \quad (7.1)$$

with function $r^2 = \sum_{\mu=1}^4 (x^\mu)^2$

As discussed before, we partially fix the auxiliary fields

$$F = 0, \quad V = t\Theta \quad (7.2)$$

However, non-zero $t\Theta$ will generate globally defined almost complex structure on S^4 , which we already know does not exist, hence we can set $t = 0$ and $V = 0$. The only auxiliary fields allowed is thus $SU(2)$ gauge field A .

The Killing spinor equation (4.31) now reads

$$\begin{cases} \partial_\theta \xi_I = (\hat{A}_\theta)_I{}^J \xi_J \\ \nabla_\mu \xi_I = (\hat{A}_\mu)_I{}^J \xi_J \end{cases} \quad (7.3)$$

The gauge field A_μ is determined by the a choice of Quaternion-Kähler structure on S^4 . Denoting

$$z_1 \equiv x^1 + ix^2, \quad z_2 \equiv x^3 + ix^4 \quad (7.4)$$

one can define locally 3 almost complex structures as the basis,

$$\begin{cases} J^1 = \left(\frac{\partial}{\partial \bar{z}_1} \otimes dz_2 - \frac{\partial}{\partial \bar{z}_2} \otimes dz_1 \right) + h.c. \\ J^2 = \frac{1}{i} \left(\frac{\partial}{\partial \bar{z}_1} \otimes dz_2 - \frac{\partial}{\partial \bar{z}_2} \otimes dz_1 \right) + h.c. \\ J^3 = i \frac{\partial}{\partial z_i} \otimes dz_i - i \frac{\partial}{\partial \bar{z}_i} \otimes d\bar{z}_i \end{cases} \quad (7.5)$$

and determine the gauge field using (4.55).

We choose the Gamma matrices to be

$$\Gamma^i = \sigma^i \otimes \sigma^2, \Gamma^4 = I \otimes \sigma^1, \Gamma^5 = I \otimes \sigma_3, C = \Gamma^{13} \quad (7.6)$$

and the obvious vielbein

$$e^5 = -d\theta, \quad e^a = \frac{1}{1+r^2} \delta_\mu^a dx^\mu \quad (7.7)$$

solution is given as

$$\xi_1 = e^{i \int A_\theta d\theta} \chi_+ \otimes \chi_-, \quad \xi_2 = -e^{-i \int A_\theta d\theta} \chi_- \otimes \chi_- \quad (7.8)$$

7.2 $M = S^2 \times S^3$

Consider S^3 as a $U(1)$ bundle over S^2 . Let S^3 be embedded into \mathbb{C}^2 ,

$$S^3 = \left\{ |z_1|^2 + |z_2|^2 = 1 \mid (z_i) \in \mathbb{C}^2 \right\} \quad (7.9)$$

Similarly define

$$z_2 = \rho e^{i\theta}, \quad z \equiv \frac{z_1}{z_2} \Rightarrow \rho^2|_{S^3} = \frac{1}{1+|z|^2}, \quad z_1 = z z_2 = \rho e^{i\theta} z \quad (7.10)$$

and hence the induced round metric on S^3 can be written as

$$\begin{aligned} ds^2 &= dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 = \left[d\theta + i \frac{z d\bar{z} - \bar{z} dz}{2(1+|z|^2)} \right]^2 + \frac{dz d\bar{z}}{(1+|z|^2)^2} \\ &= (d\theta + a)^2 + g^1 \end{aligned} \quad (7.11)$$

where

$$g^1 = \frac{dz d\bar{z}}{(1+|z|^2)^2} \quad (7.12)$$

is the metric on $\mathbb{C}P^1 = S^2$ with radius 1/2. In coordinate,

$$g_{z\bar{z}}^1 = g_{\bar{z}z}^1 = \frac{1}{2(1+|z|^2)^2} = \frac{1}{2} \partial_z \partial_{\bar{z}} \ln(1+|z|^2) \equiv \partial_z \partial_{\bar{z}} K \quad (7.13)$$

and

$$\cdot \quad (7.14)$$

The vector field $R \equiv \partial_\theta$ is a Killing vector field, and its dual is $\kappa = d\theta + A$, such that $\iota_R \kappa = 1$.

Define the frame on S^3 to be

$$e^3 \equiv e^\theta = \kappa, \quad e^1 = \frac{\text{Re} dz}{1 + |z|^2}, \quad e^2 = \frac{\text{Im} dz}{1 + |z|^2} \quad (7.15)$$

s.t.

$$g = e^\theta e^\theta + e^1 e^1 + e^2 e^2 \quad (7.16)$$

then it is obvious that

$$\omega_{\theta ab} = 0, \quad a, b = \theta, 1, 2 \quad (7.17)$$

from the fact

$$de^\theta \sim \frac{idz \wedge d\bar{z}}{(1 + |z|^2)^2}. \quad (7.18)$$

The base manifold $S^2 \times S^2$ is complex, with natural complex structure and Kähler form. Setting the radius of the stand-alone S^2 to be l , with local complex coordinate w , the metric of $S^3 \times S^2$ reads

$$g = (d\theta + a)^2 + \frac{dz d\bar{z}}{(1 + |z|^2)^2} + \frac{4l^2 dw d\bar{w}}{(1 + |w|^2)^2} \quad (7.19)$$

with Kähler form on base manifold

$$\omega = \frac{idz \wedge d\bar{z}}{2(1 + |z|^2)^2} + \frac{i4l^2 dw \wedge d\bar{w}}{2(1 + |w|^2)^2} \quad (7.20)$$

or in components

$$\omega_{z\bar{z}} = -\omega_{\bar{z}z} = \frac{i}{2(1 + |z|^2)^2}, \quad \omega_{w\bar{w}} = -\omega_{\bar{w}w} = ig_{w\bar{w}} = \frac{il}{2(1 + |z|^2)^2}. \quad (7.21)$$

The 2 complex structures on both $\mathbb{C}P^1$ can form linear combination

$$\varphi_\pm \equiv J_1 \pm J_2 \quad (7.22)$$

which satisfies

$$\varphi_\pm^2 = -1 + R \otimes \kappa. \quad (7.23)$$

Let us now construct the auxiliary fields. We choose t_{IJ} such that $\text{tr}(t^2) = -\frac{1}{2}$, and therefore

$$4(t\Theta)^2 \sim -1 + \dots \quad (7.24)$$

We identify a combination of the 2 complex structures on $2 \mathbb{C}P^1$ as $t\Theta$. Recall that $t\Theta$ also satisfies $\iota_R * (t\Theta) = -(t\Theta)$, hence we identify

$$\varphi_- \sim 2(t\Theta) \quad (7.25)$$

or a 2-form equation

$$2(t\Theta) = \frac{dz \wedge d\bar{z}}{2(1+|z|^2)^2} - \frac{il^2 dw \wedge d\bar{w}}{2(1+|w|^2)^2} \quad (7.26)$$

Then we obtain F and V :

$$F = \frac{1}{2}d\kappa = \frac{dz \wedge d\bar{z}}{2(1+|z|^2)^2} \quad (7.27)$$

and

$$V = t\Theta = \frac{dz \wedge d\bar{z}}{4(1+|z|^2)^2} - \frac{il^2 dw \wedge d\bar{w}}{(1+|w|^2)^2}. \quad (7.28)$$

With these auxiliary fields, one can solve the Killing spinor equation

$$\hat{\nabla}_m \hat{\xi}_I = (\hat{A}_m)_I^J \hat{\xi}_J. \quad (7.29)$$

Denote $\alpha = w, \bar{w}$, and $\mu, \nu = z, \bar{z}$, we have

$$\begin{cases} \nabla_\alpha \xi_I = (A_\alpha)_I^J \xi_J \\ \nabla_\mu \xi_I - \frac{1}{2}(\nabla_\mu R_\nu) \Gamma^\nu \xi_I = (\hat{A}_\mu)_I^J \xi_J \\ \nabla_\theta \xi_I = (\hat{A}_\theta)_I^J \xi_J \end{cases} \quad (7.30)$$

where

$$R_\theta = 1, \quad R_z = \frac{1}{2} \frac{-i\bar{z}}{1+|z|^2} = -i\partial_z K, \quad R_{\bar{z}} = \frac{1}{2} \frac{iz}{1+|z|^2} = i\bar{\partial}_{\bar{z}} K \quad (7.31)$$

and we used

$$\nabla_\mu R_\theta = \nabla_\theta R_\theta = 0. \quad (7.32)$$

Choosing gauge field to be $(A_m)_I^J = (A_m)(\sigma_3)_I^J$,

$$iA_z = \frac{\bar{z}}{4(1+|z|^2)}, \quad iA_{\bar{z}} = -\frac{z}{4(1+|z|^2)}, \quad A_\theta = -\frac{1}{4} \quad (7.33)$$

and representation of Gamma matrices

$$\Gamma_{w,\bar{w}} \sim \sigma_{1,2} \otimes 1, \quad \Gamma_{z,\bar{z},\theta} \sim \sigma_3 \otimes \sigma_{1,2,3} \quad (7.34)$$

one obtains the chiral solution (ξ_2 is obtained from symplectic Majorana condition)

$$\xi_1 = e^{-\frac{i}{4}\theta} \chi_+ \otimes \chi_+. \quad (7.35)$$

The calculation can be easily generalized to $M = S^3 \times \Sigma$ for Riemann surface Σ .

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A Gamma matrices and Fierz identities

We denote the 5d Gamma matrices as Γ^m with defining anti-commutation relation

$$\{\Gamma^m, \Gamma^n\} = 2g^{mn}. \quad (\text{A.1})$$

We require them to be Hermitian

$$(\Gamma^m)^\dagger = \Gamma^m. \quad (\text{A.2})$$

Also we have charge conjugation matrix $C = C_+$

$$C\Gamma^m C^{-1} = (\Gamma^m)^T = \overline{\Gamma^m}. \quad (\text{A.3})$$

These matrices have the following symmetry properties:

$$C_{\alpha\beta} = -C_{\beta\alpha}, \quad (C\Gamma_m)_{\alpha\beta} = -(C\Gamma_m)_{\beta\alpha} \quad (\text{A.4})$$

$$(C\Gamma_{mn})_{\alpha\beta} = (C\Gamma_{mn})_{\beta\alpha}, \quad (C\Gamma_{lmn})_{\alpha\beta} = (C\Gamma_{lmn})_{\beta\alpha} \quad (\text{A.5})$$

and complex conjugation properties

$$\sum_{\beta} \overline{C_{\alpha\beta}} C_{\beta\gamma} = -\delta^\alpha_\gamma, \quad \overline{(\Gamma^m)^\alpha_\beta} = (\Gamma^m)^\beta_\alpha. \quad (\text{A.6})$$

The symmetry properties of $C\Gamma$ results in symmetry properties of bilinears of spinors:

$$\begin{aligned} (\xi_1 \xi_2) &= -(\xi_2 \xi_1), & (\xi_1 \Gamma_m \xi_2) &= -(\xi_2 \Gamma_m \xi_1) \\ (\xi_1 \Gamma_{mn} \xi_2) &= (\xi_2 \Gamma_{mn} \xi_1), & (\xi_1 \Gamma_{lmn} \xi_2) &= (\xi_2 \Gamma_{lmn} \xi_1). \end{aligned} \quad (\text{A.7})$$

In 5d, we have

$$\Gamma^1 \dots \Gamma^5 = 1 \Leftrightarrow \Gamma^{abcde} = \epsilon^{abcde} \quad (\text{A.8})$$

following from the fact that $[\Gamma^1 \dots \Gamma^5, \Gamma^a] = 0$ and Schur lemma.

This fact has the following duality consequence:

Proposition.

$$\Gamma^{abcd} = \epsilon^{abcde} \Gamma^e \Leftrightarrow \frac{1}{4!} \epsilon^{abcde} \Gamma_{abcd} = \Gamma^e \quad (\text{A.9})$$

and

$$\frac{1}{3!} \epsilon^{abcde} \Gamma_{abc} = -\Gamma^{de} \Leftrightarrow \frac{1}{2!} \epsilon^{abcde} \Gamma_{ab} = -\Gamma^{cde}. \quad (\text{A.10})$$

The Hodge star operator associated with metric g_{mn} is defined as

$$* dx^{i_1} \wedge \dots \wedge dx^{i_p} = \frac{\sqrt{g}}{(n-p)!} \epsilon^{i_1 \dots i_p}_{j_1 \dots j_{n-p}} dx^{j_1} \wedge \dots \wedge dx^{j_{n-p}} \quad (\text{A.11})$$

or equivalently for $\omega_{(p)} \equiv \frac{1}{p!} \omega_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$

$$(*\omega)_{j_1 \dots j_{n-p}} = \frac{\sqrt{g}}{p!} \epsilon^{i_1 \dots i_p}_{j_1 \dots j_{n-p}} \omega_{i_1 \dots i_p} \quad (\text{A.12})$$

$$*\omega = \frac{\sqrt{g}}{p! (n-p)!} \epsilon^{i_1 \dots i_p}_{j_1 \dots j_{n-p}} \omega_{i_1 \dots i_p} dx^{j_1} \wedge \dots \wedge dx^{j_{n-p}}. \quad (\text{A.13})$$

Let us define p -forms $\Theta_{(p)}$ as

$$\Theta_{(p)}^{IJ} = \frac{1}{p!} (\xi^I \Gamma_{i_1 \dots i_p} \xi^J) dx^{i_1} \wedge \dots \wedge dx^{i_p}. \quad (\text{A.14})$$

They satisfy

$$*\Theta_{(2)}^{IJ} = -\Theta_{(3)}^{IJ} \quad (\text{A.15})$$

$$*\Theta_{(1)}^{IJ} = \Theta_{(4)}^{IJ}. \quad (\text{A.16})$$

In components, they are

$$(\xi^I \Gamma_{lmn} \xi^J) = -(*\Theta_{(2)}^{IJ})_{lmn}, \quad (\xi^I \Gamma_{mnpq} \xi^J) = *(\Theta_{(1)}^{IJ})_{mnpq} = \frac{1}{2} \epsilon^{IJ} (*\kappa)_{mnpq}. \quad (\text{A.17})$$

For any 2 spinors ξ_1 and ξ_2 , we define their inner product as a complex number:

$$(\xi_1 \xi_2) \equiv \xi_1^\alpha C_{\alpha\beta} \xi_2^\beta \quad (\text{A.18})$$

with symmetry properties

$$(\xi_1 \xi_2) = -(\xi_2 \xi_1), \quad (\xi_1 \Gamma_m \xi_2) = -(\xi_2 \Gamma_m \xi_1) \quad (\text{A.19})$$

$$(\xi_1 \Gamma_{mn} \xi_2) = (\xi_2 \Gamma_{mn} \xi_1), \quad (\xi_1 \Gamma_{lmn} \xi_2) = (\xi_2 \Gamma_{lmn} \xi_1) \quad (\text{A.20})$$

and

$$((\Gamma_m \xi_1) \xi_2) = (\xi_1 \Gamma_m \xi_2). \quad (\text{A.21})$$

Fierz identity [8]: for any 3 Grassmann even spinors (ξ_1, ξ_2, ξ_3) , one has Fierz identity

$$\xi_1 (\xi_2 \xi_3) = \frac{1}{4} \xi_3 (\xi_2 \xi_1) + \frac{1}{4} \Gamma^m \xi_3 (\xi_2 \Gamma_m \xi_1) - \frac{1}{8} \Gamma^{mn} \xi_3 (\xi_2 \Gamma_{mn} \xi_1). \quad (\text{A.22})$$

It immediately follows from the above Fierz identity that

$$\Gamma^m \xi_1 (\xi_2 \Gamma_m \xi_3) + \xi_1 (\xi_2 \xi_3) = 2\xi_3 (\xi_2 \xi_1) - 2\xi_2 (\xi_3 \xi_1) \quad (\text{A.23})$$

$$\xi_1 (\xi_2 \xi_3) + \xi_2 (\xi_1 \xi_3) = -\frac{1}{4} \Gamma^{pq} \xi_3 (\xi_2 \Gamma_{pq} \xi_1). \quad (\text{A.24})$$

B SU(2) indices and some notations

In the main text we frequently have to deal with the SU(2) indices.

The SU(2)-invariant tensor ϵ is defined as $\epsilon^{12} = \epsilon_{21} = 1$, with contraction

$$\epsilon^I{}_K = \epsilon^{IJ} \epsilon_{JK} = -\epsilon^{IJ} \epsilon_{KJ} = -\epsilon_K{}^I = \delta^I{}_K \Rightarrow \epsilon^{IJ} \epsilon_{IJ} = -2 \quad (\text{B.1})$$

and raising/lowering rules

$$X^I = \epsilon^{IJ} X_J \Leftrightarrow X_I = \epsilon_{IJ} X^J. \quad (\text{B.2})$$

With this “metric”, we define for any 2 triplets of functions X^{IJ} and Y^{IJ} a product in a natural way:

$$(XY)^{IJ} \equiv \epsilon_{LK} X^{IK} Y^{LJ} = X^I{}_K Y^{KJ}. \quad (\text{B.3})$$

Note that this product has the following symmetry:

$$(XY)^{IJ} = -(YX)^{JI} \quad (\text{B.4})$$

and in particular

$$(X^2)^{IJ} = -(X^2)^{JI} = \frac{1}{2} \text{tr}(X^2) \epsilon^{IJ} \quad (\text{B.5})$$

where we define the trace for triplet products:

$$\text{tr}(XY) \equiv X_I^J Y_J^I = -X_{IJ} Y^{IJ} \quad (\text{B.6})$$

with cyclic symmetry

$$\text{tr}(XY) = \text{tr}(YX) . \quad (\text{B.7})$$

As an example, when $X_I^J = \frac{i}{2}(\sigma_3)_I^J$

$$\text{tr} X^2 = -\frac{1}{2} \quad (\text{B.8})$$

Note that if non-zero quantity X_{IJ} satisfies reality condition

$$\overline{X_{IJ}} = \epsilon^{II'} \epsilon^{JJ'} X_{I'J'} \quad (\text{B.9})$$

then the trace is negative definite

$$\text{tr}(X^2) < 0 . \quad (\text{B.10})$$

For objects of direct sum of triplet and singlet,

$$X_{IJ} \equiv \hat{X}_{IJ} - \frac{1}{2} \epsilon_{IJ} X, \quad X^{IJ} \equiv \hat{X}^{IJ} + \frac{1}{2} \epsilon^{IJ} X \quad (\text{B.11})$$

with

$$X = \epsilon^{IJ} X_{IJ} = -\epsilon_{IJ} X^{IJ} . \quad (\text{B.12})$$

C Differential geometry

In the main text, we denote Levi-civita connection on M as ∇ :

$$\nabla g = 0 \quad (\text{C.1})$$

with connection coefficients

$$\Gamma^k_{mn} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{ml}}{\partial x^n} + \frac{\partial g_{nl}}{\partial x^m} - \frac{\partial g_{mn}}{\partial x^l} \right) \quad (\text{C.2})$$

and curvature tensor

$$[\nabla_m, \nabla_n] X^k = R_{mnl}{}^k X^l . \quad (\text{C.3})$$

Ricci tensor is defined as

$$Ric_{mn} = R_{mkn}{}^k \quad (\text{C.4})$$

and covariant derivative on spinor is

$$\nabla_m \psi = \partial_m \psi - \frac{1}{4} \omega_{mab} \Gamma^{ab} \psi \quad (\text{C.5})$$

where the spin connection is defined as

$$\omega_{ma}{}^b \equiv e^b (\nabla_m e_a) = e^b{}_n \nabla_m e_a{}^n = e^b{}_n \partial_m e_a{}^n + \Gamma^b{}_{ma} . \quad (\text{C.6})$$

Lie derivative for (1,1), (0,2) tensor are defined as

$$\mathcal{L}_X T^m{}_n = X^k \nabla_k T^m{}_n - (\nabla_k X^m) T^k{}_n + (\nabla_n X^k) T^m{}_k \quad (\text{C.7})$$

$$\mathcal{L}_X T_{mn} = X^k \nabla_k T_{mn} + (\nabla_m X^k) T_{kn} + (\nabla_n X^k) T_{mk} \quad (\text{C.8})$$

with the obvious relation

$$\mathcal{L}_X T_{mn} = (\nabla_m X_l + \nabla_l X_m) T^l{}_n + g_{ml} \mathcal{L}_X T^l{}_n . \quad (\text{C.9})$$

D Contact, almost contact structure and compatible connection

In this appendix, we will introduce necessary background on contact geometry. It is the odd dimensional analog of symplectic geometry in even dimension. Compared to its even dimensional sister, contact geometry is much less studied. However, there are interesting developments in the past few years, on the existence and classification of contact structures, as well as Sasaki-Einstein structures.

The Euler number of any odd dimensional manifold M^{2n+1} is zero, therefore one can have nowhere vanishing vector fields. Contact geometry and almost contact geometry studies the behavior of these vector fields, or their corresponding hyperplane fields.

Suppose one has a nowhere vanishing 1-form κ on $d = 2n + 1$ -dimensional M . κ singles out a rank- $2n$ vector bundle TM_H as a sub-bundle of TM , such that at $p \in M$, $T_p M_H = \ker \kappa_p$. The sub-bundle TM_H is usually called oriented hyperplane field. As κ is nowhere-vanishing, the quotient line bundle $TM_V \equiv TM/TM_H$ is trivial. Let us call TM_H horizontal vector bundle, and TM_V as vertical bundle. Note that specifying a oriented hyperplane field is equivalent to specifying κ up to any nowhere vanishing real function factor.

Recall that TM_H is integrable if and only if

$$d\kappa \wedge \kappa = 0 \Leftrightarrow d\kappa = \kappa \wedge (\dots) \quad (\text{D.1})$$

and M is locally foliated by the integral manifold.

Contact structure, however, sits in the opposite extreme. It is completely non-integrable in the sense that the hyperplane fields cannot be smoothly patched together to be submanifolds. κ satisfies the non-degenerate condition

$$\kappa \wedge (d\kappa)^n \neq 0 \quad (\text{D.2})$$

which remains true when κ is rescaled by nowhere-vanishing function. This condition implies $d\kappa$ is of maximal rank $2n$, but κ and $d\kappa$ do not have common zero eigenvector. Therefore, at $\forall p \in M$ there exist a line in $T_p M$ such that on the line $d\kappa = 0$ but not κ . Then one can choose a vector along this line at each point such that the resulting vector field R satisfies

$$\kappa(R) = 1, \quad \iota_R d\kappa = 0. \quad (\text{D.3})$$

Such vector field R is called Reeb vector field.

In low dimensions, contact structure is ubiquitous. In 3 dimension, every orientable manifold admits a contact structure, thanks to Thurston's geometrization. In 5 dimension, contact structure exists on manifolds with vanishing third integral Stiefel-Whitney class [31]. However, it is not clear if similar holds true in higher dimension. We will comment on this after we discuss the almost contact structure.

For any contact structure, one can associate a metric g such that

$$g(R, \cdot) = \kappa(\cdot) \quad (\text{D.4})$$

and consequently κ and R have unit norm. Actually, there are infinitely many such associate metrics compatible with the contact structure. In this case, (κ, R, g) is called contact metric structure.

Another kind of similar structure exists on contact manifolds is called almost contact structure. It is defined by a triplet $(R^m, \kappa_m, \varphi_m^n)$ satisfying

$$\begin{cases} R^m \kappa_m = 1 \\ \varphi_m^n R^m = 0 \\ \varphi_m^n \kappa_n = 0 \\ \varphi_m^k \varphi_k^n = -\delta_m^n + R^n \kappa_m \end{cases} \quad (\text{D.5})$$

Given any contact structure, one can construct (many) geometric structures called almost contact structure by the procedure of polarization, although not all almost contact structure arise in this way. Given any almost contact structure, one can construct again many associated metric g in the sense that

$$\kappa(R) = g(R, \cdot), \quad g_{mk} \varphi_n^k = -g_{nk} \varphi_m^k. \quad (\text{D.6})$$

The structure (κ, R, φ, g) is called almost contact metric structure (ACMS).

If an ACMS arises from some contact metric structure, such that

$$(d\kappa)_{mn} = \varphi_{mn} \quad (\text{D.7})$$

then it is easy to see that

$$\iota_R * d\kappa = -d\kappa. \quad (\text{D.8})$$

The existence of contact structure in 5 dimension was not clear until very recently. It is proved that every almost contact manifold admits contact structures, moreover, there is

at least one contact structure within each homotopy class of almost contact structure [32]. There are also new results on distinguishing inequivalent contact structure as Boothby-Wang 5-manifolds [31].

Suppose ∇ is any affine connection on TM , then one can define new connection $\hat{\nabla}$ that preserves φ :

$$\hat{\nabla}_m X^n \equiv \nabla_m X^n + K^n_{mk} X^k \quad (\text{D.9})$$

where

$$K^n_{ml} \equiv -\frac{1}{2} \left(\nabla_m \varphi_l^k \right) \varphi_k^n - \frac{1}{2} \kappa_l \nabla_m R^n + R^n \nabla_m \kappa_l. \quad (\text{D.10})$$

If the affine connection ∇ is chosen to be the Levi-civita connection associated to the ACMS structure, then one has

$$K_{nml} = -K_{lmn}. \quad (\text{D.11})$$

As mentioned, we have

$$\hat{\nabla}_m \varphi_n^k = 0. \quad (\text{D.12})$$

Moreover, for any $X, Y \in \Gamma(TM_H)$, one has

$$g(\hat{\nabla}_X Y, R) = 0 \quad (\text{D.13})$$

which means $\hat{\nabla}_X Y \in \Gamma(TM_H)$, the restriction of $\hat{\nabla}$ on TM_H gives directly a connection $\hat{\nabla}|_{TM_H} \equiv \nabla^H$ on TM_H .

The connection coefficients are now

$$\hat{\Gamma}^n_{ml} = \Gamma^n_{ml} + K^n_{ml} \quad (\text{D.14})$$

and the corresponding change of spin connection

$$\Delta \omega_{ma}^b = \omega_{ma}^b + K^b_{ma} \quad (\text{D.15})$$

where we define the spin connection⁷

$$\omega_{ma}^b \equiv e^b(\nabla_m e_a) = e^b_n \nabla_m e_a^n = e^b_n \partial_m e_a^n + \Gamma^b_{ma}. \quad (\text{D.18})$$

In three dimension, where one has relation

$$\varphi_{mn} = \epsilon_{mnk} R^k, \quad R_m = \kappa_m \quad (\text{D.19})$$

K can be simplified as

$$K^n_{ml} = R^n \nabla_m R_l - R_l \nabla_m R^n. \quad (\text{D.20})$$

⁷Note that the position of the flat indices a and b indicates that

$$\nabla_m \psi = \partial_m \psi - \frac{1}{4} \omega_{mab} \Gamma^{ab} \psi \quad (\text{D.16})$$

as opposed to the frequently used notation $\omega_m^b{}_a$ which indicates

$$\nabla_m \psi = \partial_m \psi + \frac{1}{4} \omega_{mab} \Gamma^{ab} \psi. \quad (\text{D.17})$$

The covariant derivative on spinor with new connection is now

$$\hat{\nabla}_m \xi_I = \nabla_m \xi_I - \frac{1}{4} K_{lmn} \Gamma^{nl} \xi_I. \quad (\text{D.21})$$

Now, let's consider the ACMS data coming from $(s^{-1}R, s^{-1}\kappa, r(t)t\Theta, g)$, where

$$r(t) = \frac{1}{s} \sqrt{\frac{-2}{\text{tr}(t^2)}} \quad (\text{D.22})$$

such that

$$r(t)^2 (t\Theta)^2 = -1 + (s^{-2}R) \otimes \kappa. \quad (\text{D.23})$$

Substituting all these into definition of K , one has (with the assumption that $\Omega^- = 0$)

$$\begin{aligned} K_{nml} = & \frac{1}{s^2} (R_n \nabla_m R_l - R_l \nabla_m R_n) + \frac{1}{s} \frac{1}{\text{tr}(t^2)} (T_m)_{IJ} (\Theta^{IJ})_{ln} \\ & - r(t)^2 [(*V_V)_{kmr} (t\Theta)_l^r - (*V_V)_{lmr} (t\Theta)_k^r] (t\Theta)_n^k \end{aligned} \quad (\text{D.24})$$

where

$$(T_m)_{IJ} \equiv (\nabla_m^A t_I^K) t_{KJ}. \quad (\text{D.25})$$

Note that when $s = 1$, $K_{nml} = -K_{lmn}$.

To calculate the spin connection, one needs several convenient formula

$$(t\Theta)_{nm} \Gamma^n \xi_I = (s\Gamma_m - R_m) t_I^J \xi_J \quad (\text{D.26})$$

$$(t\Theta)_{nm} \Gamma^{kn} \xi_I = \Gamma^k (\Gamma_m s - R_m) t_I^J \xi_J - (t\Theta)_m^k \xi_I \Rightarrow (t\Theta)_{mn} \Gamma^{mn} \xi_I = -4st^J_I \xi_J \quad (\text{D.27})$$

$$R^m \Gamma_{nm} \xi_I = (s\Gamma_n - R_n) \xi_I. \quad (\text{D.28})$$

Finally, one has

$$\begin{aligned} \hat{\nabla}_m \xi_I = & \nabla_m \xi_I + \frac{1}{\text{tr}(t^2)} (T_m)^J_I \xi_J - \frac{1}{2s} \nabla_m R_n \Gamma^n \xi_I + \frac{1}{2} (\nabla_m \log s) \xi_I \\ & - \frac{1}{\text{tr}(t^2)} \eta_q (t\Theta)^q_m t_I^J \xi_J + \frac{1}{2} (*V^V)_{mpq} \Gamma^{pq} \xi_I. \end{aligned} \quad (\text{D.29})$$

Some remark. We used almost contact data φ defined as $\sim t\Theta$, but in fact one could use any $\text{SU}(2)$ triplet function λ to define $\varphi_\lambda \sim \lambda\Theta$, and in particular, one could choose $\lambda = \lambda_a \sigma^a$. It also has corresponding compatible connection $\hat{\nabla}_\lambda$, such that

$$\hat{\nabla}_\lambda \varphi_\lambda = 0 \quad (\text{D.30})$$

However, the tensor K_{lmn} would not have the above simple form.

E Useful identities

$$|R|^2 = R^m R_m = \iota_R \kappa = s^2 \quad (\text{E.1})$$

$$\iota_R \Theta^{IJ} = 0 \quad (\text{E.2})$$

$$\iota_R * \Theta^{IJ} = -s \Theta^{IJ} \Leftrightarrow R^k (\xi^I \Gamma_{mnk} \xi^J) = +s (\xi^I \Gamma_{mn} \xi^J) \quad (\text{E.3})$$

$$\kappa \wedge \Theta \wedge \Theta \neq 0 \quad (\text{E.4})$$

$$(\lambda_1 \Theta)_m{}^p (\lambda_2 \Theta)_p{}^n = s (\lambda_1)_I{}^K (\lambda_2)_{KJ} (\Theta^{IJ})_m{}^n + \frac{s^2}{2} \text{tr}(\lambda_1 \lambda_2) \delta_m{}^n - \frac{1}{2} \text{tr}(\lambda_1 \lambda_2) \kappa_m R^n \quad (\text{E.5})$$

$$(\lambda \Theta)^{mn} (\lambda \Theta)_{mn} = -2s^2 \text{tr}(\lambda^2) \quad (\text{E.6})$$

$$*(\lambda \Theta)_{nkl} (\lambda \Theta)_m{}^l = \frac{s}{2} \text{tr}(\lambda^2) [g_{mk} R_n - g_{mn} R_k] \quad (\text{E.7})$$

$$(*\lambda \Theta)^{mnk} (\lambda \Theta)_{mn} = 2 \text{tr}(\lambda^2) s R^k. \quad (\text{E.8})$$

Also there are several useful spinor identities

$$R^m \Gamma_m \xi_I = s \xi_I \quad (\text{E.9})$$

$$R^m \Gamma_{nm} \xi_I = (s \Gamma_n - R_n) \xi_I \quad (\text{E.10})$$

$$(\lambda \Theta)_{nm} \Gamma^n \xi_I = (R_m - s \Gamma_m) \lambda_I{}^J \xi_J \quad (\text{E.11})$$

$$(\lambda \Theta)_{nm} \Gamma^{kn} \xi_I = \Gamma^k (R_m - s \Gamma_m) \lambda^J{}_I \xi_J - (\lambda \Theta)_m{}^k \xi_I \Rightarrow (\lambda \Theta)_{mn} \Gamma^{mn} \xi_I = 4s \lambda^J{}_I \xi_J. \quad (\text{E.12})$$

F Proof of supersymmetry invariance

Let us focus on the part of Lagrangian (with trace left implicit) without the “topological term” $\kappa \wedge F \wedge F$ and the scalar curvature term:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} F_{mn} F^{mn} - D_m \sigma D^m \sigma - \frac{1}{2} D_{IJ} D^{IJ} + i \lambda_I \Gamma^m D_m \lambda^I - \lambda_I [\sigma, \lambda^I] \\ & - i t^{IJ} (\lambda_I \lambda_J) + 2 \sigma t^{IJ} D_{IJ} + \frac{i}{4} \mathcal{F}_{mn} (\lambda_I \Gamma^{mn} \lambda^I) + \mathcal{F}^{mn} F_{mn} \sigma. \end{aligned} \quad (\text{F.1})$$

Its supersymmetry variation (partial integration has been used for λ kinetic term)

$$\begin{aligned} \delta \mathcal{L} = & F^{mn} \delta F_{mn} - 2 D_m \sigma \delta (D^m \sigma) - D^{IJ} \delta D_{IJ} + 2 i \delta \lambda_I \Gamma^m D_m \lambda^I + \lambda_I \Gamma^m [\delta A_m, \lambda^I] \\ & - \{2 \delta \lambda_I [\sigma, \lambda^I] + \lambda_I [\delta \sigma, \lambda^I]\} - 2 i t^{IJ} (\delta \lambda_I \lambda_J) + 2 \{\delta \sigma t^{IJ} D_{IJ} + \sigma t^{IJ} \delta D_{IJ}\} \\ & + \frac{i}{2} \mathcal{F}_{mn} (\delta \lambda_I \Gamma^{mn} \lambda^I) + \{\mathcal{F}^{mn} \delta F_{mn} \sigma + \mathcal{F}^{mn} F_{mn} \delta \sigma\}. \end{aligned} \quad (\text{F.2})$$

I) Let us simplify the first row (contributed by flat SUSY Lagrangian) denoted by **I**.

$$\begin{aligned}
 \mathbf{I} = & 2F^{mn}D_m(i\xi_I\Gamma_n\lambda^I) - 2D^m\sigma\{D_m(i\xi_I\lambda^I) + [(\xi_I\Gamma_m\lambda^I), \sigma]\} \\
 & - 2D^{IJ}\left(-i\xi_I\Gamma^m D_m\lambda_J + [\sigma, (\xi_I\lambda_J)] + i(\tilde{\xi}_I\lambda_J)\right) \\
 & + 2i\left(-\frac{1}{2}F_{pq}\Gamma^{pq}\xi_I + (D_n\sigma)\Gamma^n\xi_I + D_I^J\xi_J + 2\sigma\tilde{\xi}_I\right)\Gamma^m D_m\lambda^I + i\lambda_I\Gamma^m[(\xi_J\Gamma_m\lambda^J), \lambda^I] \\
 & - \left\{2\left(-\frac{1}{2}F_{pq}\Gamma^{pq}\xi_I + (D_n\sigma)\Gamma^n\xi_I + D_I^J\xi_J + 2\sigma\tilde{\xi}_I\right)[\sigma, \lambda^I] + i\lambda_I[(\xi_J\lambda^J), \lambda^I]\right\}
 \end{aligned} \tag{F.3}$$

(1) Immediate cancelation between the **red** terms.

(2) The **blue** terms add up to

$$\begin{aligned}
 & 2F^{mn}D_m(i\xi_I\Gamma_n\lambda^I) + (2i)\left(\frac{1}{2}\right)F_{pq}(\xi_I\Gamma^{pq}\Gamma^m D_m\lambda^I) \\
 & = 2iF^{mn}(\tilde{\xi}_I\Gamma_{mn}\lambda^I) + iF_{pq}(\xi_I\Gamma^{pqm}D_m\lambda^I) \\
 & = -iF_{mn}(\tilde{\xi}_I\Gamma_{mn}\lambda^I)
 \end{aligned} \tag{F.4}$$

(3) The **green** terms add up to

$$\begin{aligned}
 & -2D^m\sigma D_m(i\xi_I\lambda^I) + 2i(D_n\sigma)(\xi_I\Gamma^n\Gamma^m D_m\lambda^I) - F_{pq}\xi_I\Gamma^{pq}[\sigma, \lambda^I] \\
 & = -2D^m\sigma(i\tilde{\xi}_I\Gamma_m\lambda^I) - 8i\sigma(\tilde{\xi}_I\Gamma^m D_m\lambda^I) - \delta_\xi\left(\frac{1}{4}\mathcal{R}\sigma^2\right)
 \end{aligned} \tag{F.5}$$

where the last term cancels the scalar curvature term in the Lagrangian.⁸

(4) Now, we can gather all the terms and obtain the leftovers

$$\mathbf{I} = -2iD^{IJ}(\tilde{\xi}_I\lambda_J) - 4i\sigma(\tilde{\xi}_I\Gamma^m D_m\lambda^I) - iF_{mn}(\tilde{\xi}_I\Gamma^{mn}\lambda^I) - 2i(D^m\sigma)(\tilde{\xi}_I\Gamma_m\lambda^I) \tag{F.6}$$

II) Now we try to simplify the 2nd and 3rd row, and denote it by **II**.

$$\begin{aligned}
 & -\{2\delta\lambda_I[\sigma, \lambda^I] + \lambda_I[\delta\sigma, \lambda^I]\} - 2it^{IJ}(\delta\lambda_I\lambda_J) + 2\{\delta\sigma t^{IJ}D_{IJ} + \sigma t^{IJ}\delta D_{IJ}\} \\
 & + \frac{i}{2}\mathcal{F}_{mn}(\delta\lambda_I\Gamma^{mn}\lambda^I) + \{\mathcal{F}^{mn}\delta F_{mn}\sigma + \mathcal{F}^{mn}F_{mn}\delta\sigma\}
 \end{aligned} \tag{F.7}$$

⁸Note that there are two ways of doing partial integration for the second term: one gives $[D_m, D_n]\sigma(\xi_I\Gamma^{mn}\lambda^I)$ while the other $\sigma(\xi_I\Gamma^{mn}[D_m, D_n]\lambda^I)$. The former way directly produces $\text{tr}(t_{IJ}t^{IJ}\sigma^2)$ which appears in [8], but the cancellation of other terms are relatively tricky. Hence we take the second way of doing partial integration, which gives instead $\text{tr}(\mathcal{R}\sigma^2)$.

Explicitly writing down all terms,

$$\begin{aligned}
 \mathbf{II} = & -2it^{IJ} \left(\frac{1}{2} F_{pq} \xi_I \Gamma^{pq} \lambda_J + (D_p \sigma) \xi_I \Gamma^p \lambda_J + D_I^K \xi_K \lambda_J + 2\sigma \tilde{\xi}_I \lambda_J \right) \\
 & + 2 \left(i \xi_K \lambda^K \right) t^{IJ} D_{IJ} \\
 & - 4i\sigma t^{IJ} (\xi_I \Gamma^m D_m \lambda_J) + 4i\sigma t^{IJ} (\tilde{\xi}_I \lambda_J) \\
 & + \frac{i}{2} \mathcal{F}_{mn} \left(\frac{1}{2} F_{pq} (\xi_I \Gamma^{pq} \Gamma^{mn} \lambda^I) + (D_p \sigma) \xi_I \Gamma^p \Gamma^{mn} \lambda^I + \boxed{D_I^K \xi_K \Gamma^{mn} \lambda^I} + 2\sigma \tilde{\xi}_I \Gamma^{mn} \lambda^I \right) \\
 & + \mathcal{F}^{mn} F_{mn} (i \xi_I \lambda^I) \\
 & + 2i \mathcal{F}^{mn} \sigma D_m (\xi_I \Gamma_n \lambda^I)
 \end{aligned} \tag{F.8}$$

Combining with leftover of \mathbf{I} , one sees

- (1) Some immediately cancels in red

$$2it^{IJ} D_{IJ} (\xi_K \lambda^K) - 2it^{IJ} D_I^K (\xi_K \lambda_J) + 2i D^{IJ} (\tilde{\xi}_I \lambda_J) = \frac{i}{2} \mathcal{F}_{mn} D^{IJ} (\xi_I \Gamma^{mn} \lambda_J) \tag{F.9}$$

where one needs identity

$$t^J{}_K (\xi^I \lambda^K) + t^I{}_K (\xi^J \lambda^K) + 2t^{IJ} (\xi^K \lambda_K) = t_K{}^I (\xi^K \lambda^J) + t_K{}^J (\xi^K \lambda^I) \tag{F.10}$$

The leftover $\mathcal{F} D \lambda$ terms cancels the boxed term

$$\frac{i}{2} \mathcal{F}_{mn} D_I^K (\xi_K \Gamma^{mn} \lambda^I) + \frac{i}{2} F_{mn} D^{IJ} (\xi_K \Gamma^{mn} \lambda_I) = 0 \tag{F.11}$$

- (2) The $tF\lambda$ terms and $F\mathcal{F}\lambda$ terms in blue add up to

$$\begin{aligned}
 & -2it^{IJ} \frac{1}{2} F_{pq} (\xi_I \Gamma^{pq} \lambda_J) - i F^{mn} (\tilde{\xi}_I \Gamma_{mn} \lambda^I) + \frac{i}{4} \mathcal{F}_{mn} F_{pq} (\xi_I \Gamma^{pq} \Gamma^{mn} \lambda^I) + \mathcal{F}^{mn} F_{mn} (i \xi_I \lambda^I) \\
 & = \frac{i}{4} \mathcal{F}_{mn} F_{pq} (\xi_I \Gamma^{mn} \Gamma^{pq} \lambda^I) + \frac{i}{4} \mathcal{F}_{mn} F_{pq} (\xi_I \Gamma^{pq} \Gamma^{mn} \lambda^I) + \mathcal{F}^{mn} F_{mn} (i \xi_I \lambda^I) \\
 & = \frac{i}{4} \mathcal{F}_{mn} F_{pq} (\xi_I \{ \Gamma^{mn}, \Gamma^{pq} \} \lambda^I) + \mathcal{F}^{mn} F_{mn} (i \xi_I \lambda^I) \\
 & = \frac{i}{2} \mathcal{F}_{mn} F_{pq} (\xi_I \Gamma^{mnpq} \lambda^I)
 \end{aligned} \tag{F.12}$$

where one needs identity

$$\{ \Gamma^{mn}, \Gamma^{pq} \} = 2 (g^{np} g^{mq} - g^{mp} g^{nq}) + 2 \Gamma^{mnpq} \tag{F.13}$$

The term remaining can actually be written as

$$\frac{i}{2} \mathcal{F}_{mn} F_{pq} (\xi_I \Gamma^{mnpq} \lambda^I) \sqrt{g} d^5 x = \delta \left(\frac{\sqrt{g}}{4} \epsilon^{mnpqr} \kappa_r F_{pq} F_{mn} \sqrt{g} d^5 x \right) = \delta (\kappa \wedge F \wedge F) . \tag{F.14}$$

This term cancels the “topological” terms in the proposed action.

(3) The remaining black terms reads

$$\begin{aligned}
& -2it^{IJ} (D_p \sigma) (\xi_I \Gamma^p \lambda_J) - 4i\sigma t^{IJ} (\xi_I \Gamma^m D_m \lambda_J) + \frac{i}{2} \mathcal{F}_{mn} (D_p \sigma) (\xi_I \Gamma^p \Gamma^{mn} \lambda^I) \\
& + \frac{i}{2} \mathcal{F}_{mn} 2\sigma (\tilde{\xi}_I \Gamma^{mn} \lambda^I) + 2i \mathcal{F}^{mn} \sigma D_m (\xi_I \Gamma_n \lambda^I) - 2i (D^m \sigma) (\tilde{\xi}_I \Gamma_m \lambda^I) - 4i\sigma (\tilde{\xi}_I \Gamma^m D_m \lambda^I).
\end{aligned}
\tag{F.15}$$

Explicit computation shows these terms cancel each other.

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