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RIGIDITY OF GRADIENT ALMOST RICCI SOLITONS

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ABSTRACT. In this paper, we show that either, a Euclidean space $\mathbb{R}^n$, or a standard sphere $\mathbb{S}^n$, is the unique manifold with nonnegative scalar curvature which carries a structure of a gradient almost Ricci soliton, provided this gradient is a non trivial conformal vector field. Moreover, in the spherical case the field is given by the first eigenfunction of the Laplacian. Finally, we shall show that a compact locally conformally flat almost Ricci soliton is isometric to Euclidean sphere $\mathbb{S}^n$ provided an integral condition holds.

1. Introduction and statement of the results

The study of almost Ricci soliton was introduced by Pigola et al. [8], where essentially they modified the definition of Ricci solitons by adding the condition on the parameter λ to be a variable function, more precisely, we say that a Riemannian manifold (M^n, g) is an almost Ricci soliton, if there exist a complete vector field X and a smooth soliton function $\lambda : M^n \to \mathbb{R}$ satisfying

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X \mathrm{g} = \lambda \mathrm{g}, \tag{1.1}$$

where Ric and $\mathcal{L}$ stand, respectively, for the Ricci tensor and the Lie derivative. We shall refer to this equation as the fundamental equation of an almost Ricci soliton $(M^n, \mathrm{g}, X, \lambda)$. It will be called *expanding*, *steady* or *shrinking*, respectively, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. Otherwise, it will be called *indefinite*. When the vector field X is a gradient of a smooth function $f : M^n \to \mathbb{R}$ the manifold will be called a gradient almost Ricci soliton. In this case, the preceding equation becomes

$$\mathrm{Ric} + \nabla^2 f = \lambda \mathrm{g}, \tag{1.2}$$

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where $\nabla^2 f$ stands for the Hessian of f. Sometimes classical theory of tensorial calculus is more convenient to make computations. Then, we can write the fundamental equation in this language as follows

$$R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}. \tag{1.3}$$

Moreover, when either the vector field X is trivial, or the potential f is constant, the almost Ricci soliton will be called *trivial*, otherwise it will be a *nontrivial* almost Ricci soliton. We notice that when $n \geq 3$ and X is a Killing vector field an almost Ricci soliton will be a Ricci soliton, since in this case we have an Einstein manifold, from which we can apply Schur's lemma to deduce that λ is constant. Taking into account that the soliton function λ is not necessarily constant, certainly comparison with soliton theory will be modified. In particular, the rigidity result contained in Theorem 1.3 of [8] indicates that almost Ricci solitons should reveal a reasonably broad generalization of the fruitful concept of classical soliton. In fact, we refer the reader to [8] to see some of these changes.

In the direction to understand the geometry of almost Ricci soliton, Barros and Ribeiro Jr. proved in [1] that a compact gradient almost Ricci soliton with nontrivial conformal vector field is isometric to a Euclidean sphere. In the same paper, they proved an integral formula for compact case, which was used to prove several rigidity results, for more details see [1].

Next, we shall give examples of almost Ricci soliton whose soliton function λ is not constant.

Example 1 (Compact case). In this case, a simple example appeared in [1]. It was built over the standard sphere $(\mathbb{S}^n, g_0)$ endowed with the conformal vector field $X = a^{\top}$, where a is a fixed vector in $\mathbb{R}^{n+1}$ and $a^{\top}$ stands for its orthogonal projection over $T\mathbb{S}^n$. We notice that $a^{\top}$ is the gradient of the right function h_a; for more details see the quoted paper.

It is well known that all compact 2-dimensional Ricci solitons are trivial. However, the previous example gives that there exists a nontrivial compact 2-dimensional almost Ricci soliton. The next example concerns to a noncompact almost Ricci soliton.

Example 2 (Noncompact case). Let us consider the warped product manifold $M^{n+1} = \mathbb{R} \times_{\cosh t} \mathbb{S}^n$ with metric $g = dt^2 + \cosh^2 t g_0$, where g_0 is the standard metric of $\mathbb{S}^n$. Taking $(M^{n+1}, g, \nabla f, \lambda)$, where $f(x,t) = \sinh t$ and $\lambda(x,t) = \sinh t + n$, we can prove, by using Lemma 1.1 of [8], that $(M^{n+1}, g, \nabla f, \lambda)$ is an almost Ricci soliton.

In particular, in [8] it was proved that there are complete manifolds that do not support an almost soliton structure; see Example 1.4 in the quoted article.

Now we present a strong characterization to a gradient almost Ricci soliton. Moreover, on the compact case, essentially we have the manifold presented at Example 1.

THEOREM 1. *Let* $(M^n, g, \nabla f, \lambda), n \geq 3$, *be a gradient almost Ricci solitons with nonnegative scalar curvature. If* ∇f *is a nontrivial conformal vector field, then we have*:

(1) *Either*, M^n *is isometric to a Euclidean space* $\mathbb{R}^n$.
(2) *Or*, M^n *is isometric to a Euclidean sphere* $\mathbb{S}^n$. *In this case, up to constant,* f *is a first eigenfunction of the Laplacian and* $\lambda = -\frac{R}{n(n-1)} f + \kappa$, *where* κ *is a constant.*

As a consequence of this theorem, we obtain the following corollary.

COROLLARY 1. *Let* $(M^n, g, \nabla f, \lambda), n \geq 3$, *be a nontrivial compact gradient almost Ricci soliton. Then,* M^n *is isometric to a Euclidean sphere* $\mathbb{S}^n$ *and, up to constant,* f *is a first eigenfunction of the Laplacian and* $\lambda = -\frac{R}{n(n-1)} f + \kappa$, *where* κ *is a constant, provided*:

(1) M^n *has constant scalar curvature.*
(2) M^n *is homogeneous.*

Moreover, for a compact gradient almost Ricci soliton surface with nonpositive Gaussian curvature we have the following rigidity result.

THEOREM 2. *Every compact gradient almost Ricci soliton surface with nonpositive Gaussian curvature is trivial.*

In [3], Catino proved that a locally conformally flat gradient almost Ricci soliton, around any regular point of f, is locally a warped product with $(n-1)$-dimensional fibers of constant sectional curvature. Considering such a compact gradient almost Ricci soliton we have the following theorem.

THEOREM 3. *Let* $(M^n, g, \nabla f, \lambda)$ *be a locally conformally flat compact almost Ricci soliton. If* dV_g *denotes the Riemannian volume form of* M^n *and*

$$-\int_M R\Delta\lambda e^{-f}\, dV_g \geq n(n-1)\int_M |\nabla\lambda|^2 e^{-f}\, dV_g, \tag{1.4}$$

then M^n *isometric to a Euclidean sphere* $\mathbb{S}^n$.

For instance, it is an interesting problem to prove that assumption (1.4) in Theorem 3 can be removed. As a consequence of Theorem 3 we obtain the following corollary.

COROLLARY 2. *Let* $(M^n, g, \nabla f, \lambda)$ *be a compact almost Ricci soliton satisfying condition* (1.4). *If* Y *is a Killing vector field on* M, *then, either* $D_Y f$ *is constant or* M^n *is isometric to a Euclidean sphere* $\mathbb{S}^n$.

2. Preliminaries and some basic results

In this section, we shall present some preliminaries that will be useful for the establishment of the desired results. First, taking into account that $\operatorname{div}(hI)(Y) = \langle \nabla h, Y\rangle$, where $h : M^n \to \mathbb{R}$ is a smooth function and $Y \in \mathfrak{X}(M)$, we recall the next identity for an almost Ricci soliton $(M^n, \mathrm{g}, X, \lambda)$, that was proved by Barros and Ribeiro Jr. in [1]:

$$\frac{1}{2}\Delta_X |X|^2 = |\nabla X|^2 - \lambda |X|^2 - (n-2)\mathrm{g}(\nabla\lambda, X), \tag{2.1}$$

where $\Delta_X = \Delta - D_X$ is the diffusion operator.

As a consequence of this identity, we obtain the following corollary.

Corollary 3. *Let us suppose that* $(M^n, \mathrm{g}, X, \lambda), n \geq 3,$ *is an expanding almost Ricci soliton, for which* $|X|$ *achieves its maximum. If* $\mathrm{g}(\nabla\lambda, X) \leq 0$, *then* (M^n, g) *is an Einstein manifold. In particular, an expanding or steady Ricci soliton, for which* $|X|$ *attains its maximum is an Einstein manifold.*

Proof. We notice that we can apply the maximum principle to guarantee that $\nabla X = 0$. Thus $\mathcal{L}_X \mathrm{g} = 0$, which gives $\mathrm{Ric} = \lambda \mathrm{g}$, that is, (M^n, g) is an Einstein manifold. □

Now we claim that

$$\begin{aligned}\Delta R_{ik} = {} & \langle \nabla R_{ik}, \nabla f\rangle + \lambda R_{ik} - 2R_{ijks}R^{js} + R_{is}R^s_k \\ & + \nabla_k\nabla_i\left(\frac{R}{2} - \lambda\right) - \nabla_k R_{si}\nabla^s f + \Delta\lambda \mathrm{g}_{ik}.\end{aligned} \tag{2.2}$$

In fact, since $\Delta R_{ik} = \mathrm{g}^{jk}\nabla_k\nabla_j R_{ik} = \nabla^j\nabla_j R_{ik}$ we have

$$\begin{aligned}\Delta R_{ik} &= \nabla^j\left(\nabla_i R_{jk} + R_{ijks}\nabla^s f + \nabla_j\lambda \mathrm{g}_{ik} - \nabla_i\lambda \mathrm{g}_{jk}\right)\\
&= \nabla^j\nabla_i R_{jk} + \nabla^j R_{ijks}\nabla^s f + R_{ijks}\nabla^j\nabla^s f + \Delta\lambda \mathrm{g}_{ik} - \mathrm{g}^{js}\nabla_s\nabla_i\lambda \mathrm{g}_{jk}\\
&= \nabla^j\nabla_i R_{jk} + \nabla R_{ijks}\nabla^j\nabla^s f + \Delta\lambda \mathrm{g}_{ik} - \nabla_k\nabla_i\lambda\\
&= \nabla_i\nabla^j R_{jk} + R^j_{ijs}R^s_k + R^j_{iks}R^s_j - \nabla_k R_{si}\nabla^s f + \nabla_s R_{ki}\nabla^s f\\
&\quad + R_{ijks}\nabla^j\nabla^s f + \Delta \mathrm{g}_{ik} - \nabla_k\nabla_i\lambda\\
&= \nabla_i\nabla^j R_{jk} + R_{is}R^s_k + R^j_{iks}R^s_j - \nabla_k R_{si}\nabla^s f + \nabla_s R_{ki}\nabla^s f\\
&\quad + R_{ijks}\nabla^j\nabla^s f + \Delta\lambda \mathrm{g}_{ik} - \nabla_k\nabla_i\lambda\\
&= \frac{1}{2}\nabla_i\nabla k R + R_{is}R^s_k + R^j_{iks}R^s_j - \nabla_k R_{si}\nabla^s f + \langle\nabla R_{ik}, \nabla f\rangle\\
&\quad - R_{ijks}R^{js} + \lambda R_{ik} + \Delta\lambda \mathrm{g}_{ik} - \nabla_k\nabla_i\lambda\\
&= \langle\nabla R_{ik}, \nabla f\rangle + \lambda R_{ik} - 2R_{ijks}R^{js} + R_{is}R^s_k + \frac{1}{2}\nabla_k\nabla_i R\\
&\quad - \nabla_k R_{si}\nabla^s f + \Delta\lambda \mathrm{g}_{ik} - \nabla_k\nabla_i\lambda,\end{aligned}$$

which completes our claim.

The next proposition can be found in [1].

PROPOSITION 1. *For a gradient almost Ricci soliton* $(M^n, \mathrm{g}, \nabla f, \lambda)$ *the following formulae hold*:

(1) $R + \Delta f = n\lambda$
(2) $\nabla_i R = 2R_{ij}\nabla^j f + 2(n-1)\nabla_i \lambda$
(3) $\nabla_j R_{ik} - \nabla_i R_{jk} - R_{ijks}\nabla^s f = (\nabla_j \lambda)\mathrm{g}_{ik} - (\nabla_i \lambda)\mathrm{g}_{jk}$
(4) $\nabla(R + |\nabla f|^2 - 2(n-1)\lambda) = 2\lambda\nabla f$.

It is important to point out that assertion (4) is a generalization of a main equation derived by Hamilton in [4], that was used by Perelman in [7] to prove that a compact Ricci soliton is always gradient. We notice that assertion (2) of Proposition 1 yields for any $Z \in \mathfrak{X}(M)$

$$\mathrm{g}(\nabla R, Z) = 2\,\mathrm{Ric}(\nabla f, Z) + 2(n-1)\mathrm{g}(\nabla \lambda, Z). \tag{2.3}$$

As a consequence of this proposition, we shall prove the following lemma.

LEMMA 1. *For a gradient almost Ricci soliton* $(M^n, \mathrm{g}, \nabla f, \lambda)$ *the following formula holds*:

$$\Delta R_{ij} = \langle \nabla R_{ij}, \nabla f\rangle + 2\lambda R_{ij} - 2R_{ikjs}R^{ks} + (n-2)\nabla_j\nabla_i\lambda + \Delta\lambda \mathrm{g}_{ik}.$$

Proof. Using once more assertion (2) of Proposition 1, we infer

$$0 = \frac{1}{2}\nabla_k\big(\nabla_i R - 2R_{is}\nabla^s f - 2(n-1)\nabla_i\lambda\big),$$

which gives

$$\frac{1}{2}\nabla_k\nabla_i R - \nabla_k R_{is}\nabla^s f = (n-1)\nabla_k\nabla_i\lambda + R_{is}\nabla^s\nabla_k f.$$

Thus, using Equation (2.2), we have

$$\begin{aligned}
\Delta R_{ik} &= \langle \nabla R_{ik}, \nabla f\rangle + \lambda R_{ik} - 2R_{ijks}R^{js} + R_{is}R^s_k \\
&\quad + R_{is}\nabla^s\nabla_k f + (n-1)\nabla_k\nabla_i\lambda + \Delta\lambda \mathrm{g}_{ik} - \nabla_k\nabla_i\lambda \\
&= \langle \nabla R_{ik}, \nabla f\rangle + \lambda R_{ik} - 2R_{ijks}R^{js} + R_{is}R^s_k \\
&\quad + R_{is}\mathrm{g}^{sj}\nabla_j\nabla_k f + (n-1)\nabla_k\nabla_i\lambda + \Delta\lambda \mathrm{g}_{ik} - \nabla_k\nabla_i\lambda \\
&= \langle \nabla R_{ik}, \nabla f\rangle + \lambda R_{ik} - 2R_{ijks}R^{js} + R_{is}R^s_k + \lambda R_{is} \\
&\quad - R_{is}R^s_k + (n-1)\nabla_k\nabla_i\lambda + \Delta\lambda \mathrm{g}_{ik} - \nabla_k\nabla_i\lambda \\
&= \langle \nabla R_{ik}, \nabla f\rangle + 2\lambda R_{ik} - 2R_{ijks}R^{js} \\
&\quad + (n-2)\nabla_k\nabla_i\lambda + \Delta\lambda \mathrm{g}_{ik}.
\end{aligned}$$

From where we deduce

$$\Delta R_{ij} = \langle \nabla R_{ij}, \nabla f\rangle + 2\lambda R_{ij} - 2R_{ikjs}R^{ks} + (n-2)\nabla_j\nabla_i\lambda + \Delta\lambda \mathrm{g}_{ik}, \tag{2.4}$$

which finishes the proof of the lemma. □

In particular, taking trace of both members of identity (2.4), we have

$$\Delta R = \langle \nabla R, \nabla f\rangle + 2\lambda R - 2|\mathrm{Ric}|^2 + 2(n-1)\Delta\lambda. \tag{2.5}$$

This equation already appeared in [8], but by a different argument. By using a maximum principle and this last identity, we obtain the following corollary.

Corollary 4. *Let* $(M^n, \mathrm{g}, \nabla f, \lambda)$ *be a gradient almost Ricci soliton for which the following inequality holds*: $\lambda R + (n-1)\Delta\lambda \geq |\mathrm{Ric}|^2$. *Then* R *is constant in a neighborhood of any local maximum.*

Proof. In fact, using the assumption in Equation (2.5), we deduce

$$\frac{1}{2}\Delta_f R \geq 0.$$

Therefore, by the maximum principle for elliptic PDE's, we conclude that R is constant in a neighborhood of any local maximum. □

Taking into account assertion (1) of Proposition 1 and the diffusion operator $\Delta_f = \Delta - \nabla f$, we can rewrite (3.4) as follows:

$$\frac{1}{2}\Delta_f R = (n-1)\Delta\lambda + \left(\lambda - \frac{R}{n}\right)R - \left|\mathrm{Ric} - \frac{R}{n}\mathrm{g}\right|^2. \tag{2.6}$$

Using Equation (2.6), we obtain the following proposition.

Proposition 2. *Every steady almost Ricci soliton whose scalar curvature achieves its minimum is Ricci flat.*

Proof. First, we notice that at a minimum point of R, we can use Equation (2.6) to conclude

$$0 \leq \Delta_f R = -\frac{R^2}{n} - \left|\mathrm{Ric} - \frac{R}{n}\mathrm{g}\right|^2 \leq 0.$$

Thus $R = 0$ and $\mathrm{Ric} = 0$, therefore (M^n, g) is Ricci flat. □

Proceeding we obtain the following lemma.

Lemma 2. *Let* $(M^n, \mathrm{g}, \nabla f, \lambda)$ *be a gradient almost Ricci soliton. Then the following formulae hold*:

(1) $(\mathrm{div}\, Rm)_{jkl} = R_{lkjs}\nabla^s f + (\nabla_l\lambda)\mathrm{g}_{kj} - (\nabla_k\lambda)\mathrm{g}_{jl}$
(2) $\nabla_i(R_{ijkl}e^{-f}) = ((\nabla_l\lambda)\mathrm{g}_{kj} - (\nabla_k\lambda)\mathrm{g}_{lj})e^{-f}$
(3) $\nabla_i(R_{ik}e^{-f}) = ((n-1)\nabla_k\lambda)e^{-f}$.

Proof. In order to obtain identity (1) it is enough to use the Ricci identity and assertion (3) of Proposition 1. Indeed, we have

$$\begin{aligned}(\mathrm{div}\, Rm)_{jkl} &= \nabla_i(R_{ijkl}) = \nabla_i R_{klij} \\ &= -\nabla_k R_{liij} - \nabla_l R_{ikij}\end{aligned}$$

$$= -\nabla_k R_{lj} + \nabla_l R_{kj}$$
$$= R_{lkjs}\nabla^s f + (\nabla_l \lambda)g_{kj} - (\nabla_k \lambda)g_{lj},$$

which gives the first assertion. Next, using this identity, we obtain

$$\nabla_i\big(R_{ijkl}e^{-f}\big) = \nabla_i(R_{ijkl})e^{-f} - (\nabla_i f)R_{ijkl}e^{-f}$$
$$= \big((\nabla_l\lambda)g_{kj} - (\nabla_k\lambda)g_{jl}\big)e^{-f}.$$

Finally, taking trace of both members of the first identity, we derive

$$\nabla_i\big(R_{ik}e^{-f}\big) = (\nabla_i R_{ik})e^{-f} - (\nabla_i f)R_{ik}e^{-f}$$
$$= \big(R_{ki}\nabla^i f + (n-1)\nabla_k\lambda - \nabla_i f R_{ik}\big)e^{-f}$$
$$= (n-1)(\nabla_k f)e^{-f},$$

which completes the proof of the lemma. □

As a consequence of Lemma 2, we obtain the following integral formula.

Corollary 5. *Let $(M^n, g, \nabla f, \lambda)$ be a gradient almost Ricci soliton. Then we have*

$$\frac{1}{2}\int_M |\operatorname{div} Rm|^2 e^{-f}\, d_g$$
$$= -\int_M R\langle\nabla\lambda, \nabla f\rangle e^{-f}\, dV_g - \int_M R_{lkjs}\nabla_l\nabla^s f R_{kj} e^{-f}\, dV_g$$
$$- (n-1)\int_M |\nabla\lambda|^2 e^{-f}\, dV_g + \int_M \langle\nabla\lambda, \nabla R\rangle e^{-f}\, dV_g.$$

Proof. Using Lemma 2 and item (2) of Proposition 1, we have

$$\int_M |\operatorname{div} Rm|^2 e^{-f}\, dV_g$$
$$= \int_M R_{lkjs}\nabla^s f(-\nabla_k R_{lj} + \nabla_l R_{kj})e^{-f}\, dV_g$$
$$+ \int_M (\nabla_l\lambda g_{kj} - \nabla_k\lambda g_{lj})(-\nabla_k R_{lj} + \nabla_l R_{kj})e^{-f}\, dV_g$$
$$= -\int_M R_{lkjs}\nabla^s f\nabla_k R_{lj} e^{-f}\, dV_g + \int_M R_{lkjs}\nabla^s f \nabla_l R_{kj} e^{-f}\, dV_g$$
$$+ \int_M (\nabla_l\lambda g_{kj} - \nabla_k\lambda g_{lj})(-\nabla_k R_{lj} + \nabla_l R_{kj})e^{-f}\, dV_g$$
$$= -\int_M \nabla_l\big(R_{lkjs}e^{-f}\big)\nabla^s f R_{kj} e^{-f}\, dV_g + \int_M \nabla_k\big(R_{lkjs}e^{-f}\big)\nabla^s f R_{lj} e^{-f}\, dV_g$$
$$- \int_M R_{lkjs}\nabla_l\nabla^s f R_{kj} e^{-f}\, dV_g - \int_M R_{lkjs}\nabla_k\nabla^s f R_{lj} e^{-f}\, dV_g$$
$$+ \int_M \langle\nabla\lambda, \nabla R\rangle e^{-f}\, dV_g$$

$$
\begin{aligned}
&= -2\int_M R_{lkjs}\nabla_l\nabla^s f R_{lj}e^{-f}\,dV_g - 2\int_M \nabla_l\big(R_{lkjs}e^{-f}\big)\nabla^s f R_{kj}e^{-f}\,dV_g\\
&\quad + \int_M \langle\nabla\lambda,\nabla R\rangle e^{-f}\,dV_g\\
&= -2\int_M R\langle\nabla\lambda,\nabla f\rangle e^{-f}\,dV_g - 2\int_M R_{lkjs}\nabla_l\nabla^s f R_{kj}e^{-f}\,dV_g\\
&\quad + 2\int_M \mathrm{Ric}(\nabla f,\nabla\lambda)e^{-f}\,dV_g + \int_M \langle\nabla\lambda,\nabla R\rangle e^{-f}\,dV_g\\
&= -2\int_M R\langle\nabla\lambda,\nabla f\rangle e^{-f}\,dV_g - 2\int_M R_{lkjs}\nabla_l\nabla^s f R_{kj}e^{-f}\,dV_g\\
&\quad - 2(n-1)\int_M |\nabla\lambda|^2 e^{-f}\,dV_g + 2\int_M \langle\nabla\lambda,\nabla R\rangle e^{-f}\,dV_g,
\end{aligned}
$$

which concludes the proof of the corollary. □

Now, recall that for any Riemannian manifold, we have

$$\nabla_i\nabla_j R_{ik} - \nabla_j\nabla_i R_{ik} = R_{jm}R_{mk} - R_{ijkm}R_{im}, \tag{2.7}$$

for more details see [2]. Using Equation (2.7) and Corollary 5, we obtain the following lemma.

LEMMA 3. *Let $(M^n, g, \nabla f, \lambda)$ be a compact gradient almost Ricci soliton. Then*

$$
\begin{aligned}
\int_M |\operatorname{div} Rm|^2 e^{-f}\,dV_g &= \int_M |\nabla \mathrm{Ric}|^2 e^{-f}\,dV_g\\
&\quad - \int_M R\Delta\lambda e^{-f}\,dV_g - n(n-1)\int_M |\nabla\lambda|^2 e^{-f}\,dV_g.
\end{aligned}
$$

Proof. First, using (1.3), we deduce

$$
\begin{aligned}
&-2\int_M \nabla_k R_{jl}\nabla_l R_{jk}e^{-f}\,dV_g\\
&\quad = 2\int_M R_{jk}\nabla_l\nabla_k R_{jl}e^{-f}\,dV_g - 2\int_M R_{jk}\nabla_k R_{jl}\nabla_l f e^{-f}\,dV_g\\
&\quad = 2\int_M R_{jk}\nabla_i\nabla_j R_{ik}e^{-f}\,dV_g - 2\int_M R_{jk}\nabla_j R_{ik}\nabla_i f e^{-f}\,dV_g.
\end{aligned}
$$

Next, using item (2.7) and Lemma 2, we have

$$
\begin{aligned}
&-2\int_M \nabla_k R_{jl}\nabla_l R_{jk}e^{-f}\,dV_g\\
&\quad = 2\int_M R_{jk}(\nabla_j\nabla_i R_{ik} + R_{jm}R_{mk} - R_{ijkm}R_{im})e^{-f}\,dV_g\\
&\qquad + 2\int_M \nabla_j\big(R_{jk}e^{-f}\big)R_{ik}\nabla_i f + 2\int_M R_{jk}R_{ik}\nabla_j\nabla_i f e^{-f}\,dV_g
\end{aligned}
$$

$$
\begin{aligned}
&= -2\int_M \nabla_j\big(R_{jk}e^{-f}\big)\nabla_i R_{ik}\, dV_g + 2\int_M R_{jk}R_{jm}R_{mk}e^{-f}\, dV_g \\
&\quad - 2\int_M R_{ijkm}R_{im}R_{jk}e^{-f}\, dV_g + 2\int_M \nabla_j\big(R_{jk}e^{-f}\big)R_{ik}\nabla_i f\, dV_g \\
&\quad + 2\int_M R_{jk}R_{ik}\nabla_j\nabla_i f e^{-f}\, dV_g.
\end{aligned}
$$

Taking into account item (2) of Proposition 1 and the twice contracted second Bianchi identity, we obtain

$$
\begin{aligned}
&-2\int_M \nabla_k R_{jl}\nabla_l R_{jk}e^{-f}\, dV_g \\
&\quad = 2\int_M R_{jk}R_{ik}(R_{ij} + \nabla_i\nabla_j f)e^{-f}\, dV_g \\
&\qquad - \int_M \nabla_j\big(R_{jk}e^{-f}\big)\nabla_k R\, dV_g - 2\int_M R_{ijkm}R_{im}R_{jk}e^{-f}\, dV_g \\
&\qquad + 2\int_M \nabla_j\big(R_{jk}e^{-f}\big)\left(\frac{1}{2}\nabla_k R - (n-1)\nabla_k\lambda\right) dV_g \\
&\quad = 2\int_M \lambda|\mathrm{Ric}|^2 e^{-f}\, dV_g - 2\int_M R_{ijkm}R_{im}R_{jk}e^{-f}\, dV_g \\
&\qquad - 2(n-1)^2\int_M |\nabla\lambda|^2 e^{-f}\, dV_g.
\end{aligned}
$$

On the other hand, comparing the previous equation and Corollary 5 we have

$$
\begin{aligned}
&\int_M |\operatorname{div} Rm|^2 e^{-f}\, dV_g \\
&\quad = \int_M |-\nabla_k R_{lj} + \nabla_l R_{kj}|^2 e^{-f}\, dV_g \\
&\quad = 2\int_M |\nabla \mathrm{Ric}|^2 e^{-f}\, dV_g - 2\int_M \nabla_k R_{jl}\nabla_l R_{jk}e^{-f}\, dV_g \\
&\quad = 2\int_M |\nabla \mathrm{Ric}|^2 e^{-f}\, dV_g + 2\int_M \lambda|\mathrm{Ric}|^2 e^{-f}\, dV_g \\
&\qquad - 2\int_M R_{ijkm}R_{im}R_{jk}e^{-f}\, dV_g - 2(n-1)^2\int_M |\nabla\lambda|^2 e^{-f}\, dV_g.
\end{aligned}
$$

Using again item (2) of Proposition 1, we have

$$
\begin{aligned}
&\int_M |\operatorname{div} Rm|^2 e^{-f}\, dV_g \\
&\quad = \int_M |\nabla \mathrm{Ric}|^2 e^{-f} - \int_M R\langle\nabla\lambda, \nabla f\rangle e^{-f}\, dV_g \\
&\qquad + \int_M \langle\nabla R, \nabla\lambda\rangle e^{-f}\, dV_g - n(n-1)\int_M |\nabla\lambda|^2 e^{-f}\, dV_g.
\end{aligned} \tag{2.8}
$$

By using the divergence theorem, we have

$$\begin{aligned}\int_M \langle \nabla R, \nabla\lambda\rangle e^{-f}\, dV_{\mathrm{g}} &= \int_M \left\langle \nabla R, e^{-f}\nabla\lambda\right\rangle dV_{\mathrm{g}} \\ &= \int_M R\langle\nabla f, \nabla\lambda\rangle e^{-f}\, dV_{\mathrm{g}} - \int_M R\Delta\lambda e^{-f}\, dV_{\mathrm{g}}.\end{aligned}$$

Now we compare the last equation with (2.8) to finish the proof of the lemma. □

For any Riemannian manifold (M^n, g), let us consider the Weyl tensor as well as the Cotton tensor, which are given respectively, by

$$\begin{aligned}W_{ijkl} = R_{ijkl} &+ \frac{R}{(n-1)(n-2)}(\mathrm{g}_{il}\mathrm{g}_{jk} - \mathrm{g}_{ik}\mathrm{g}_{jl}) \\ &- \frac{1}{n-2}(R_{il}\mathrm{g}_{jk} + \mathrm{g}_{il}R_{jk} - R_{ik}\mathrm{g}_{jl} - \mathrm{g}_{ik}R_{jl})\end{aligned}$$

and

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(\nabla_i R\mathrm{g}_{jk} - \nabla_j R\mathrm{g}_{ik}). \tag{2.9}$$

It is easy to check that for $n \geq 4$, if the Weyl tensor of (M^n, g) vanishes, then the Cotton tensor also vanishes. Moreover, a classical result gives that W_{ijkl} is conformally invariant and $(M^n, \mathrm{g}), n \geq 4$, is locally conformally flat if and only if $W_{ijkl} = 0$.

3. Proofs of the results

3.1. Proof of Theorem 1.

Proof. First, we notice that for an almost Ricci soliton $(M^n, \mathrm{g}, \nabla f, \lambda)$ it holds

$$R + \Delta f = n\lambda. \tag{3.1}$$

Since ∇f is nontrivial it follows that $\mathcal{L}_{\nabla f}\mathrm{g} = 2\rho\mathrm{g}$, where $\rho \neq 0$. Moreover, from $\frac{1}{2}\mathcal{L}_{\nabla f}\mathrm{g} = \frac{\Delta f}{n}\mathrm{g}$ we have that $\Delta f \neq 0$. Now, using that ∇f is a conformal vector field we deduce $\mathrm{Ric} = (\lambda - \rho)\mathrm{g}$. In particular, Schur's lemma gives that $(\lambda - \rho)$ is constant, which yields $R = n(\lambda - \rho)$ is also constant. Supposing $R = 0$ we have that (M^n, g) is Ricci flat and by using Theorem 2 due to Tashiro [9] and fundamental Equation (1.2) we deduce that (M^n, g) is isometric to a Euclidean space $\mathbb{R}^n$. On the other hand, if $R \neq 0$, we can invoke Theorem 1 due to Nagano and Yano [5] to conclude that (M^n, g) is isometric to a Euclidean sphere $\mathbb{S}^n$. Now we invoke a well-known formula (see, e.g., [6, p. 56]), which gives

$$\Delta\rho + \frac{R}{n-1}\rho = 0. \tag{3.2}$$

On the other hand, we also have $\rho = \frac{1}{n}\Delta f$. Hence, we can use identity (1) of Proposition 1 to deduce $\lambda = \rho + \frac{R}{n}$. Taking into account that $\mathrm{Ric} = \frac{R}{n}\mathrm{g}$, we can use Lichnerowicz's theorem jointly with Equation (3.2) to deduce that the first eigenvalue of the Laplacian of M^n is $\lambda_1 = \frac{R}{n-1}$. Then, ρ is a first eigenfunction of the Laplacian of M^n. In particular, we also have $\Delta(\Delta f + \lambda_1 f) = 0$. Hence, $\Delta f + \lambda_1 f = c$, where c is constant. Now a straightforward computation gives $\lambda = -\frac{\lambda_1}{n} f + \kappa$, which completes the proof of the theorem. □

3.1.1. *Proof of Corollary 1.*

Proof. First we integrate formula (3.4) to obtain

$$\int_M \left|\mathrm{Ric} - \frac{R}{n}\mathrm{g}\right|^2 d\mu = -\frac{n-2}{2n}\int R\Delta f\, d\mu. \tag{3.3}$$

Now, we notice that under the assumptions of Corollary 1, R is constant. Therefore, we conclude from (3.3) that $\mathrm{Ric} = \frac{R}{n}\mathrm{g}$. By using (1.2), we deduce $\nabla^2 f = (\lambda - \frac{R}{n})\mathrm{g}$, which gives that ∇f is a conformal vector field. So, we can invoke Theorem 1 to complete the proof of the corollary. □

3.2. Proof of Theorem 2.

Proof. In [1] it was proved that for a gradient almost Ricci soliton the following equation is satisfied

$$\frac{1}{2}\Delta R + \left|\mathrm{Ric} - \frac{R}{n}\mathrm{g}\right|^2 = (n-1)\Delta\lambda + \frac{R}{n}\Delta f + \frac{1}{2}\langle\nabla R, \nabla f\rangle, \tag{3.4}$$

for more details, see Corollary 3 there.

Next, we notice that $\mathrm{Ric} = \frac{R}{2}\mathrm{g}$. So, the previous identity gives

$$\Delta\left(\frac{1}{2}R - \lambda\right) = \frac{1}{2}\big(R\Delta f + \langle\nabla R, \nabla f\rangle\big). \tag{3.5}$$

From where we have

$$\Delta(\Delta f) + R\Delta f + \langle\nabla R, \nabla f\rangle = \mathrm{div}(\nabla\Delta f + R\nabla f) = 0.$$

In particular,

$$\mathrm{div}\big(f(\nabla\Delta f + R\nabla f)\big) = \langle\nabla f, \nabla\Delta f\rangle + R\langle\nabla f, \nabla f\rangle.$$

On integrating this last identity, we obtain

$$\int_M R|\nabla f|^2\, d\mu = \int_M (\Delta f)^2\, d\mu. \tag{3.6}$$

Since $R \le 0$, we use (3.6) to conclude that f is constant, which finishes the proof of the theorem. □

3.3. Proof of Theorem 3.

Proof. Since $(M^n, \mathrm{g}, \nabla f, \lambda)$ is a locally conformally flat gradient almost Ricci soliton, it follows from (2.9) that

$$|\operatorname{div} Rm|^2 = \frac{|\nabla R|^2}{2(n-1)}. \tag{3.7}$$

On the other hand, comparing the assumption of the theorem with Lemma 3, we obtain the following inequality

$$\int_M |\operatorname{div} Rm|^2 e^{-f}\, dV_{\mathrm{g}} \geq \int_M |\nabla \mathrm{Ric}|^2 e^{-f}\, dV_{\mathrm{g}}. \tag{3.8}$$

Moreover, from Cauchy–Schwarz inequality we have $|\nabla \mathrm{Ric}|^2 \geq \frac{|\nabla R|^2}{n}$, which allows us to deduce jointly with (3.7) and (3.8) the inequality

$$\frac{1}{2(n-1)} \int_M |\nabla R|^2 e^{-f}\, dV_{\mathrm{g}} \geq \frac{1}{n} \int_M |\nabla R|^2 e^{-f}\, dV_{\mathrm{g}},$$

giving that R is constant. Therefore, we may apply Corollary 1 to conclude that M^n is isometric to a Euclidean sphere $\mathbb{S}^n$, which finishes the proof of the theorem. ☐

3.3.1. *Proof of Corollary 2.*

Proof. Since Y is a Killing field, we have $\mathcal{L}_Y \mathrm{g} = 0$. Taking into account that the flow associated to Y generates isometries, we also have $\mathcal{L}_Y \mathrm{Ric} = 0$. Therefore, we deduce

$$\operatorname{Hess} \mathcal{L}_Y f = \mathcal{L}_Y \operatorname{Hess} f = \mathcal{L}_Y \lambda \mathrm{g},$$

which gives

$$\Delta \mathcal{L}_Y f = n \mathcal{L}_Y \lambda. \tag{3.9}$$

Consequently, we conclude

$$\operatorname{Hess}(\mathcal{L}_Y f) = \frac{\Delta \mathcal{L}_Y f}{n} \mathrm{g}.$$

Now we are in conditions to apply Theorem 6.3 (p. 28 of Yano [10]) to conclude that, either $D_Y f$ is trivial, or M^n is conformally equivalent to a Euclidean sphere $\mathbb{S}^n$. Therefore, we conclude that M^n is locally conformally flat. Since we are supposing that relation (1.4) holds, we may apply Theorem 3 to conclude the proof of the corollary. ☐

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