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RIGIDITY OF MEASURES — THE HIGH ENTROPY CASE AND NON-COMMUTING FOLIATIONS

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To Hillel Furstenberg with friendship and admiration

ABSTRACT

We consider invariant measures for partially hyperbolic, semisimple, higher rank actions on homogeneous spaces defined by products of real and p-adic Lie groups. In this paper we generalize our earlier work to establish measure rigidity in the high entropy case in that setting. We avoid any additional ergodicity-type assumptions but rely on, and extend the theory of conditional measures.

1. Introduction

This paper is a part of the program of studying invariant measures for hyperbolic actions of higher rank abelian groups: $\mathbb{Z}^k$ and $\mathbb{R}^k$ for $k \geq 2$, based on the considerations of entropy and conditional measures on invariant foliations. This program was initiated in [18] and continued in [16, 6, 17]. We precede the description of our results by a general discussion of problems that motivated our work.

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1.1. THE FURSTENBERG CONJECTURE. In his seminal paper [11] H. Furstenberg showed that the action of the multiplicative semigroup $\Sigma_{m,n} \subset \mathbb{N}$ generated by $\times m, \times n$ (with m, n not powers of the same integer) on $\mathbb{R}/\mathbb{Z}$ has only one infinite closed Σ-invariant set, namely $\mathbb{R}/\mathbb{Z}$ itself. Since there are many closed sets that are invariant under $\times m$ (or $\times n$) this is a remarkable rigidity property of the joint action, which was subsequently generalized by D. Berend [1, 2] to the higher dimensional torus and other groups.

Furstenberg also raised the question for the measure theoretic extensions of this.

CONJECTURE 1.1 (Furstenberg): *Let μ be an $\Sigma_{m,n}$-invariant and ergodic probability measure on $\mathbb{R}/\mathbb{Z}$. Then μ is either atomic supported by finitely many rational periodic points or μ is the Lebesgue measure.*

The first partial result for the measure problem on $\mathbb{T}$ was given by Lyons [23] under a strong additional assumption. D. Rudolph [33] and A. Johnson [15] weakened this assumption considerably, and proved that μ must be the Lebesgue measure provided that $\times m$ (or $\times n$) has positive entropy with respect to μ.

1.2. NUMBER THEORY AND DYNAMICS. There are numerous connections between number theory and dynamical systems. In fact Furstenberg's result mentioned earlier about $\Sigma_{m,n}$-invariant closed sets can be viewed in that light: Given any two multiplicatively independent integers $m, n \geq 2$ and an irrational $\alpha \in \mathbb{R}$, the set $\{m^i n^j \alpha \colon i, j \in \mathbb{N}\}$ is dense modulo 1.

A slightly different setting is the following. Dynamics on the space of unimodal lattices in $\mathbb{R}^k$, or equivalently on $\mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$, play a key role for many problems in the theory of diophantine approximations.

The long-standing Oppenheim Conjecture was solved by G. Margulis [24] through the study of the action of a certain subgroup H on the space of unimodal lattices in $\mathbb{R}^3$. This conjecture, posed by A. Oppenheim in 1929, deals with density properties of the values of indefinite quadratic forms in three or more variables. So far there is no proof known of this result in its entirety which avoids the use of dynamics of homogeneous actions.

An important property of the acting group H as above is that it is generated by unipotents: i.e. by elements of $\mathrm{SL}(k,\mathbb{R})$ all of whose eigenvalues are one. Another situation where the action of a unipotent subgroup appears is the horocycle flow on (the unit tangent bundle of) quotients of the hyperbolic plane. Here H. Furstenberg [12] showed earlier that the horocycle flow is uniquely ergodic if the quotient is compact. For non-compact quotients this is not true, but the only other ergodic measures are those supported by periodic horocycles as was shown by S. Dani and J. Smillie [5].

The dynamical results proved by Margulis, Furstenberg, Dani and Smillie were special cases of a conjecture of M. S. Raghunathan regarding the actions of general unipotents groups; if $H \subseteq G$ is a subgroup of an arbitrary connected Lie group G that is generated by unipotents and Γ is a lattice in G, then the left action of H on the homogeneous space G/Γ shows topological and measurable rigidity in the sense that the only possible H-orbit closures and H-ergodic probability measures are of an algebraic type. Raghunathan's conjecture was proved in full generality by M. Ratner in a landmark series of papers ([27, 28] and others; see also the expository papers [29, 30]).

A. Borel and G. Prasad [3] pointed out that a natural generalization of Raghunathan's conjecture to the case where G is an S-algebraic group (i.e. a product of real and p-adic algebraic groups) would imply an S-arithmetic analogue of the Oppenheim conjecture, and they proved a result in that direction. The S-algebraic Raghunathan's conjecture was proved in full independently by G. Margulis and G. Tomanov [25] and by M. Ratner [31].

Notice that rigidity of measures or orbit closures holds in the above setting regardless of the size of the acting group as long as it is generated by unipotent elements.

Another long-standing conjecture that is intimately linked to dynamics on $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ is the following.

CONJECTURE 1.2 (Littlewood (c. 1930)): *For every $u, v \in \mathbb{R}$,*

$$\liminf_{n\to\infty} n\langle nu\rangle\langle nv\rangle = 0, \tag{1.1}$$

where $\langle w\rangle = \min_{n\in\mathbb{Z}} |w-n|$ is the distance of $w \in \mathbb{R}$ to the nearest integer.

However, here it is the left action of the group A of positive diagonal matrices on $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ that gives the connection. This is a particular case of a **Weyl chamber flow** whose dynamics is not as well understood as that of unipotent actions. Notice that, similar to the case of the $\Sigma_{m,n}$ action on $\mathbb{R}/\mathbb{Z}$, one-parameter subgroups of the Weyl chamber flow are **partially hyperbolic** and do not show rigidity. The analogue to Furstenberg's result resp. conjecture are given by two conjectures of G. Margulis, both of which would imply Littlewood's conjecture. E. Lindenstrauss and B. Weiss [22] have obtained a partial result regarding the topological conjecture, and more recently we showed in joint work with E. Lindenstrauss the following

THEOREM 1.3 ([7, Thm. 1.3]): *Let μ be an A-invariant and ergodic measure on $X = \mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$ for $k \geq 3$. Assume that there is some one parameter subgroup of A which acts on X with positive entropy. Then μ is algebraic.*

Moreover, we applied this to Littlewood's conjecture and proved that the set of exceptions has Hausdorff dimension zero [7, Thm. 1.5].

1.3. MEASURE RIGIDITY, LOW ENTROPY, AND HIGH ENTROPY. Rudolph's result [33] has subsequently been proved using slightly different methods by J. Feldman [10] and W. Parry [26] but positive entropy remained a crucial assumption. A further extension was then given by B. Host [13].

When Rudolph's result appeared, the second author suggested another test model for the measure rigidity: two commuting hyperbolic automorphisms of the three-dimensional torus. In joint work with R. Spatzier the second author developed a more geometric technique [18, 19], which was subsequently extended by B. Kalinin and the second author [16] as well as by B. Kalinin and R. Spatzier [17].

This method is based on the study of conditional measures induced by a given invariant measure μ on certain invariant foliations. The foliations considered include stable and unstable foliations of various elements of the actions, as well as intersections of such foliations, and are related to the Lyapunov exponents of the action. For Weyl chamber flows these foliations are given by orbits of unipotent subgroups normalized by the action.

Unless there is an element of the action which acts with positive entropy with respect to μ, these conditional measures are well-known to be δ-measure supported on a single point, and do not reveal any additional meaningful information about μ. Hence this and later techniques are limited to study actions where at least one element has positive entropy. Under ideal situations, such as the original motivating case of two commuting hyperbolic automorphisms of the three torus, no further assumptions are needed, and a result entirely analogous to Rudolph's theorem can be proved using the method of [18] (see also [16]).

However, for Weyl chamber flows, an additional assumption is needed for the proof [18] to work. This assumption is satisfied, for example, if the flow along every singular direction in the Weyl chamber is ergodic (though a weaker hypothesis is sufficient, see also [17]). This additional assumption, which unlike the entropy assumption is not stable under weak* limits, precludes applying the results from [18] in many cases.

Recently, two new methods of proofs were developed, which overcome this difficulty.

The first method was developed by the authors [6], following an idea mentioned at the end of [18]. This idea uses the non-commutativity of the above-mentioned foliations (or more precisely, of the corresponding unipotent groups). This paper deals with general $\mathbb{R}$-split simple Lie groups; in particular it is shown there that if μ is an A-invariant measure on $X = \mathrm{SL}(k,\mathbb{R})/\Gamma$, and if the entropies of μ with respect to *all* one parameter groups are positive, then μ is the Haar measure. It should be noted that for this method the properties of the lattice do not play any role, and indeed this is true not only for $\Gamma = \mathrm{SL}(k,\mathbb{Z})$ but for every discrete subgroup Γ. Subsequently this was called the **high entropy case** and the corresponding method was one of the main tools for Theorem 1.3 and the above mentioned partial result on Littlewood's conjecture [7]. A second key argument which appeared in [6] the first time was the product structure of the conditional measures.

A different approach was developed by E. Lindenstrauss [21] and was used to prove a special case of the quantum unique ergodicity conjecture. A special case of the main theorem of [21] is the following: Let A be an $\mathbb{R}$-split Cartan subgroup of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Any A-ergodic measure on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})/\Gamma$ for which some one parameter subgroup of A acts with positive entropy is algebraic. Here Γ is, e.g., an irreducible lattice in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Since the foliations under consideration in this case do commute, the methods of [6] are not applicable. This was the other method used for Theorem 1.3 and Littlewood's conjecture in [7] and was applied in the case where very few conditional measures are not δ-measures; this is the **low entropy case**. Here the earlier mentioned product structure of the conditional measures was proved in a more formal setting and was crucial to the argument.

1.4. Generalizing the high entropy argument. In this paper we generalize the method [6] for the high entropy case to the case of a semisimple action on a locally homogeneous space defined by a product of real and p-adic Lie groups. This generalization is given by two separate theorems which give the product structure resp. translation invariance of the conditional measures. We expect that these two theorems together with a generalization of the method [21] for the low entropy case will likely again lead to a full understanding of the positive entropy case in this setting.

The importance of the understanding of conditional measures for measure rigidity lies in two central facts.

- Positive entropy for a partially hyperbolic map is equivalent to the conditional measures not being atomic a.e. For the high entropy case it is important that this statement can be made more quantitative; entropy can only be high if the support of the conditional measure is big; see also Section 9.1.

- Secondly, translation invariance of the measure in question can be characterized by the conditional measures; see Proposition 5.7.

These two facts allow one to show in certain cases that positive entropy implies translation invariance of the measure.

We now state the assumptions to the two main technical theorems of the paper which are stated below (not using some of the abbreviations defined in the course of the paper); see Section 3–4 for more details on the preliminary material needed.

Let S be a finite set of places, i.e. a set containing rational primes and the symbol ∞ (that stands for the Archimedean norm on $\mathbb{Q}$). We write σ for the elements of S, unless we want to specify that we talk about a rational prime p or about the Archimedean place ∞. So $\mathbb{Q}_\sigma$ denotes either the field of p-adic integers $\mathbb{Q}_p$ or the real numbers $\mathbb{R}$ accordingly.

Let G_σ be a Lie group over $\mathbb{Q}_\sigma$ for $\sigma \in S$ and define G_S to be the direct product of these. Let X be a locally compact second countable metric space. Assume that G_S acts locally free by homeomorphisms of X, and write $(h, x) \mapsto hx$ for the action. Moreover, assume that α is a $\mathbb{Z}^k$-action by homeomorphisms of X and θ is a $\mathbb{Z}^k$-action by automorphisms of G_S such that

$$\alpha^{\mathbf{n}}(hx) = \theta^{\mathbf{n}}(h)\alpha^{\mathbf{n}}(x) \quad \text{for } x \in X, h \in G_S, \text{ and } \mathbf{n} \in \mathbb{Z}^k.$$

Then the derivative of θ gives a (coordinate-wise linear) $\mathbb{Z}^k$-action A — **the adjoint action** — on the product $\mathfrak{g}_S$ of the Lie algebras $\mathfrak{g}_\sigma$ of G_σ for $\sigma \in S$. Local properties of α with respect to the G_S-leaf can be formulated in terms of the θ-action and ultimately in terms of the linear action A. Recall that A is semisimple if it is a direct product of diagonalizable linear actions (where we do not assume that the eigenvalues lie in $\mathbb{Q}_\sigma$).

For instance, let $\mathbf{m} \in \mathbb{Z}^k$ and let $\mathfrak{h}$ be a sum of A-invariant Lie subalgebras $\mathfrak{h}_\sigma \subset \mathfrak{g}_\sigma$ such that $A^{\mathbf{m}}$ contracts every element of $\mathfrak{h}$. The **m-stable subgroup** H **of** G_S (associated to $\mathfrak{h}$) is the minimal θ-invariant subgroup of G_S that is a product of (not necessarily closed) Lie subgroups H_σ with Lie algebra $\mathfrak{h}_\sigma$ for $\sigma \in S$; see Proposition 4.11. The associated leaves Hx for $x \in X$ are part of the stable 'manifold' of x with respect to $\alpha^{\mathbf{m}}$.

If a subgroup H' is in fact $\mathbf{n}$-stable for all $\mathbf{n}$ that lie in an open halfspace, then we call H' a **coarse Lyapunov subgroup**. We determine the halfspace by its outward normal ray Λ and write $H^\Lambda = H'$ for such a group. The basic Corollary 4.13 shows that every $\mathbf{m}$-stable subgroup H can be written as a product of coarse Lyapunov subgroups $H^{\Lambda_1}, \ldots, H^{\Lambda_\ell}$, where $\Lambda_i \neq \Lambda_j$ for $i \neq j$. (This homeomorphism is in general not a group isomorphism.)

Let μ be an α-invariant measure on X. Then there exists a family of conditional measures μ_x^H for the foliation into H-orbits that are almost surely locally finite measure on H. Roughly speaking, these measures describes the behavior (including its dimension) of the original measure (near x) along the direction of H; see Section 5 for a formal definition and [21] for the construction.

We are now ready to state the main technical theorems.

THEOREM 8.4: *Let $\alpha, X, \theta, G_S, A$ and $\mathfrak{g}_S$ be as above, and suppose the adjoint action A is a semisimple. Let H be an $\mathbf{m}$-stable subgroup of G_S, let $H^{\Lambda_1}, \ldots, H^{\Lambda_\ell}$ be the different coarse Lyapunov subgroups of H, and let $\phi\colon H^{\Lambda_1} \times \cdots \times H^{\Lambda_\ell} \to H$ be the homeomorphism defined by $\phi(g_1, \ldots, g_\ell) = g_1 \cdots g_\ell$. Then any α-invariant probability measure μ on X satisfies*

$$\mu_x^H \propto \phi_*(\mu_x^{\Lambda_1} \times \cdots \times \mu_x^{\Lambda_\ell}) \quad \text{a.e.},$$

where μ_x^H and $\mu_x^{\Lambda_i}$ are the conditional measures for the H-leaves and the H^{Λ_i}-leaves for $i = 1, \ldots, \ell$ respectively.

This decomposition helps to understand the structure of the conditional measures (and therefore of the original measure) since the conditional measures for coarse Lyapunov subgroups are easier to study. For these there exists sequences $\mathbf{n}_j$ for which the restriction of $\theta^{\mathbf{n}_j}$ are approximate isometries.

A particular case of this theorem for abelian G_S, namely Theorem 8.2, is one of the tools for a full analogue of Rudolph's theorem for the higher dimensional torus as announced [8] by E. Lindenstrauss and the first author.

In case the coarse Lyapunov subgroups do not commute with each other, we obtain the following generalization of the high entropy argument.

THEOREM 8.5: *With the same notation and assumptions as before, for any α-invariant probability measure μ there exist for a.e. x two subgroups*

$$H_x \subseteq P_x \subseteq H$$

with the following properties:

(1) *μ_x^H is supported by P_x.*

(2) *μ_x^H is left- and right-invariant under multiplication with elements of H_x.*

(3) *The subgroups H_x and P_x are both products of subgroups of G_σ for $\sigma \in S$. The latter are images under* exp *of Lie subalgebras of $\mathfrak{g}_\sigma$. Moreover, H_x and P_x allow weight decompositions.*

(4) *H_x is a normal subgroup of P_x and any elements $g \in P_x \cap H^{\Lambda_r}$ and $h \in P_x \cap H^{\Lambda_s}$ of different coarse Lyapunov subgroups ($r \neq s$) satisfy that gH_x and hH_x commute with each other in P_x/H_x.*

(5) *$\mu_x^{\Lambda_i}$ is left- and right-invariant under multiplication with elements of $H_x \cap H^{\Lambda_i}$ for $i = 1, \ldots, \ell$.*

The notion 'weight decomposition' is defined in Definition 6.1 — in the case of a real Lie group and real eigenvalues of A it is equivalent to H_x and P_x being normalized by θ.

In Section 2 we give a few corollaries of the above structure theorems for conditional measures. Moreover, E. Lindenstrauss and the first named author [9] apply Theorem 8.5 to show algebraicity of ergodic joinings for certain higher rank semisimple actions.

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The results presented in this paper were obtained jointly. The actual text of the paper (including exact formulations of principal theorems) was almost completely written by the first named author.

2. Measure rigidity in the high entropy case

We start by a few definitions. Let S denote as before a finite set of places.

Definition 2.1: For every $\sigma \in S$ let G_σ be a Lie group over $\mathbb{Q}_\sigma$ with Lie algebra $\mathfrak{g}_\sigma$. Then $G_S = \prod_{\sigma\in S} G_\sigma$ is an S-**Lie group** and $\mathfrak{g}_S = \sum_{\sigma\in S}\mathfrak{g}_\sigma$ its corresponding S-**Lie algebra**.

Notions like S-Lie subalgebra are defined similarly as products of the corresponding objects over $\mathbb{Q}_\sigma$ for $\sigma \in S$.

Let $\Gamma \subset G_S$ be a discrete subgroup, and let $\alpha\colon \mathbb{Z}^k \to G_S$ be a homomorphism. Then α induces a left action on $X = G_S/\Gamma$ by letting

$$\alpha^{\mathbf{n}}(g\Gamma) = (\alpha(\mathbf{n})g)\Gamma \quad \text{where } g\Gamma \in X.$$

Furthermore, α gives rise to an $\mathbb{Z}^k$-action θ by automorphisms of G_S and an adjoint $\mathbb{Z}^k$-action A on $\mathfrak{g}_S$ by letting $\theta^{\mathbf{n}}(g) = \alpha^{\mathbf{n}} g \alpha^{-\mathbf{n}}$ and $A^{\mathbf{n}} = \mathrm{Ad}_{\alpha(\mathbf{n})}$ for $g \in G_S$ and $\mathbf{n} \in \mathbb{Z}^k$, see Section 3.2.

Definition 2.2: Let $\mathfrak{h}^- \subset \mathfrak{g}_S$ be an A-invariant S-Lie subalgebra such that all eigenvalues of $A^{\mathbf{m}}$ restricted to $\mathfrak{h}^-$ have absolute value less than one, and let H^- be the corresponding $\mathbf{m}$-stable S-Lie subgroup. Then $\mathrm{mod}(\alpha^{\mathbf{m}}, H^-)$ denotes the negative logarithm of the module of $A^{\mathbf{m}}$ restricted to $\mathfrak{h}$ (with respect to the Haar measure $m_{\mathfrak{h}}$ of $\mathfrak{h}$), i.e.

$$m_{\mathfrak{h}}(A^{\mathbf{m}}B) = e^{-\mathrm{mod}(\alpha^{\mathbf{m}}, H^-)} m_{\mathfrak{h}}(B) \quad \text{for any measurable } B \subset \mathfrak{h}.$$

A Lyapunov weight is a linear functional $\lambda: \mathbb{R}^k \to \mathbb{R}$ (possibly zero) such that, for some joint eigenspace of the linear action A, the eigenvalues $t_{\mathbf{m}}$ of $A^{\mathbf{m}}$ satisfies $\log |t_{\mathbf{m}}|_\sigma = \lambda(\mathbf{m})$ for all $\mathbf{m} \in \mathbb{Z}^k$; see Section 4. The corresponding Lyapunov weight space is the sum of all eigenspaces that give rise to the same Lyapunov weight; it is denoted by $\mathfrak{g}_S^\lambda$, resp. by $\mathfrak{h}^\lambda = \mathfrak{h} \cap \mathfrak{g}_S^\lambda$ if $\mathfrak{h}$ is an A-invariant S-Lie subalgebra. In the case of a Cartan action and a real Lie group the Lyapunov weights coincides with the real part of the roots, and the weight spaces with the sum of the root spaces that give rise to the same Lyapunov weight.

We now state the main assumption on the action needed for our theorem. Note that the condition is local in its nature and independent of Γ.

Definition 2.3: The restriction of the adjoint action A to an A-invariant S-Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}_S$ has a **rank one factor** if there exists an $\mathfrak{h}' \subseteq \mathfrak{h}$ with the following properties:

(1) $\mathfrak{h}'$ is an S-Lie ideal.
(2) $\mathfrak{h}'$ is invariant under A.
(3) Any two nonzero Lyapunov weights λ_1 and λ_2 of $\mathfrak{h}$, whose weight spaces $\mathfrak{h}^{\lambda_1}, \mathfrak{h}^{\lambda_2}$ are both not contained in $\mathfrak{h}'$, are proportional.
(4) $\mathfrak{h}' \neq \mathfrak{h}$.

In particular, if A restricted to $\mathfrak{h}$ has no rank factors, then there are two linearly independent Lyapunov weights λ_1, λ_2 (set $\mathfrak{h}' = \{0\}$), and so $k \geq 2$.

For every $\mathbf{m} \in \mathbb{Z}^k$ there exists a unique maximal $\mathbf{m}$-stable S-Lie subgroup H of G_S; see Section 4. The notion of (maximal) $\mathbf{m}$-unstable S-Lie subgroups is defined similarly.

THEOREM 2.4: *Let G_S be an S-Lie group, let $\Gamma \subset G_S$ be a discrete subgroup, and let $X = G_S/\Gamma$. Let α be a $\mathbb{Z}^k$-action on X by left multiplication with elements of G_S such that the adjoint action on the Lie algebra $\mathfrak{g}_S$ is semisimple.*

Let $\mathbf{m} \in \mathbb{Z}^k$, and let $\mathfrak{h}$ be the S-Lie algebra that is generated by the maximal $\mathbf{m}$-stable S-Lie subalgebra $\mathfrak{h}^-$ and the maximal $\mathbf{m}$-unstable S-Lie subalgebra $\mathfrak{h}^+$ of $\mathfrak{g}_S$. Assume that $\mathfrak{h}$ has no rank one factors. Then there exists some $q < 1$ (which is independent of Γ) such that every α-invariant and ergodic probability measure μ on X with $\mathsf{h}_\mu(\alpha^{\mathbf{m}}) > q \operatorname{mod}(\alpha^{\mathbf{m}}, H^-)$ satisfies in fact $\mathsf{h}_\mu(\alpha^{\mathbf{m}}) = \operatorname{mod}(\alpha^{\mathbf{m}}, H^-)$ and that μ is invariant under left multiplication by elements of H^- and H^+.

Clearly, if we know additionally that G_S is generated by H^- and H^+, then Γ has to be a lattice in G_S and $\mu = m_X$ is the Haar measure of X.

2.1. Twisted Weyl chamber flows. Let G be a semisimple real connected Lie group, and let $\alpha\colon \mathbb{R}^k \to G$ be a homomorphism into a Cartan subgroup of G (so that $\mathrm{Ad}_{\alpha(\mathbf{t})}$ is semisimple for every $\mathbf{t} \in \mathbb{R}^k$). For every discrete subgroup $\Gamma \subset G$ and $\mathbf{t} \in \mathbb{R}^k$ we identify $\alpha^{\mathbf{t}} = \alpha(\mathbf{t})$ with its left action on $X = G/\Gamma$ and obtain the **Weyl chamber flow** α on X.

Every root λ of G can be restricted to the image of the derivative $D\alpha$ (with base point $0 \in \mathbb{R}^k$), and induces in this way a linear map $\lambda^{(\alpha)} = \lambda \circ D\alpha\colon \mathbb{R}^k \to \mathbb{C}$. For measure rigidity in the high entropy case we need the following condition.

Definition 2.5: We say that α has **no local rank one factors** if for every simple factor of G there exist two roots λ_1, λ_2 of that factor such that $\operatorname{Re} \lambda_1^{(\alpha)}$ and $\operatorname{Re} \lambda_2^{(\alpha)}$ are linearly independent.

Clearly, the above condition implies that none of the simple factors of G are compact. Notice, furthermore, that the above is a purely local condition that does not depend on the discrete subgroup Γ.

Before we state the theorem we extend the above setting as follows. Let $\rho\colon G \to \mathrm{SL}(n, \mathbb{R})$ be a linear representation, and define the group structure on $G_{\mathrm{tw}} = \mathbb{R}^n \rtimes G$ by

$$(u, g) \cdot (v, h) = (u + \rho(g)v, gh).$$

Let $\Gamma \subset G$ be a discrete subgroup (lattice) and suppose $\rho(\Gamma)(\mathbb{Z}^n) = \mathbb{Z}^n$; then $\Gamma_{\mathrm{tw}} = \mathbb{Z}^n \rtimes \Gamma$ is a discrete subgroup (lattice) of G_{tw}. We define $X_{\mathrm{tw}} = G_{\mathrm{tw}}/\Gamma_{\mathrm{tw}}$ and identify $\alpha^{\mathbf{t}}$ for $\mathbf{t} \in \mathbb{R}^k$ again with its left action on X_{tw} so that

$$\alpha^{\mathbf{t}}((u, g)\Gamma_{\mathrm{tw}}) = ((\rho(\alpha^{\mathbf{t}})u, \alpha^{\mathbf{t}} g))\Gamma_{\mathrm{tw}}.$$

This defines a **twisted Weyl chamber flow**. The projection map

$$\pi\colon X_{\mathrm{tw}} = G_{\mathrm{tw}}/\Gamma_{\mathrm{tw}} \to X = G/\Gamma$$

defined by $\pi((u,g)\Gamma_{\mathrm{tw}}) = g\Gamma$ is a factor map from the twisted Weyl chamber flow to the corresponding Weyl chamber flow with tori as fibers.

Definition 2.6: The twisted Weyl chamber flow α on X_{tw} acts **without center on the torus fibers** if there is no nonzero $u \in \mathbb{R}^n$ that is fixed under $\rho(G)$.

THEOREM 2.7: *Let α be a (twisted) Weyl chamber flow on X (on X_{tw}) that has no local rank one factors (and that acts without center on the torus fibers). Furthermore, let $\mathbf{t} \in \mathbb{R}^k$ be such that for every simple factor of G there exists a root λ with $\operatorname{Re}\lambda^{(\alpha)}(\mathbf{t}) \neq 0$. Then there exists some $q < 1$ such that for any α-invariant and ergodic probability measure μ with*

$$\mathsf{h}_\mu(\alpha^{\mathbf{t}}) > q \sum_{\operatorname{Re}(\lambda^{(\alpha)}(\mathbf{t}))>0} \operatorname{Re}(\lambda^{(\alpha)}(\mathbf{t}))d(\lambda)$$

in fact μ is the unique G-invariant Haar measure on X (G_{tw}-invariant Haar measure on X_{tw}). In this case Γ is a lattice in G. Here the above sum goes over all roots λ of G (and all weights λ of the representation ρ) and $d(\lambda)$ is the real dimension of the root (weight) space to λ.

The sum in the above theorem is the entropy of $\alpha^{\mathbf{t}}$ with respect to the Haar measure on X (X_{tw}) in case Γ is a lattice in G. So the theorem states that if the entropy is close to this maximal value, then in fact the entropy is equal and the invariant measure is the Haar measure.

Without further assumptions on the discrete subgroup Γ Rees's example [32], [6, Sect. 9] shows even in the case of the Weyl chamber flow on $\mathrm{SL}(3,\mathbb{R})/\Gamma$ that positive entropy alone is not sufficient to guarantee algebraicity of the invariant measure μ.

2.2. THE CARTAN ACTION FOR PRODUCTS OF $\mathrm{SL}(k+1,\mathbb{Q}_\sigma)$. Because of the close connection to number theory dynamics on the homogeneous space, $\mathrm{SL}(k+1,\mathbb{R})/\mathrm{SL}(k+1,\mathbb{Z})$ is especially interesting. We consider here the S-algebraic analogue of the Cartan action by diagonal matrices.

Let $k \geq 2$, let S be a finite set of places, let $G_S = \prod_{\sigma\in S}\mathrm{SL}(k+1,\mathbb{Q}_\sigma)$, and let $\Gamma \subset G_S$ be a discrete subgroup. For any $\mathbf{m} \in \mathbb{Z}^k$ let $\alpha_\infty^{\mathbf{m}}$ be the diagonal matrix with entries $e^{m_1},\ldots,e^{m_k},e^{-(m_1+\cdots+m_k)}$ along the diagonal, similarly let $\alpha_p^{\mathbf{m}}$ be the diagonal matrix with entries $p^{m_1},\ldots,p^{m_k},p^{-(m_1+\cdots+m_k)}$ along the diagonal. Each α_σ for $\sigma \in S$ defines a $\mathbb{Z}^k$-action on $X = G_S/\Gamma$ by left multiplication. Note that α_∞ naturally extends to an $\mathbb{R}^k$-action, but by only considering the $\mathbb{Z}^k$-action we get a slightly stronger result.

Let E_{ij} be the matrix with only one nonzero entry, namely a one in row i, column j, and let I_{k+1} denote the identity matrix. It is easy to see that each subgroup $H_\sigma^{(i,j)} = I_{k+1} + \mathbb{Q}_\sigma E_{ij}$ for $1 \le i,j \le k+1$ and $i \ne j$ is normalized by $\alpha_\sigma^{\mathbf{m}}$ for $\mathbf{m} \in \mathbb{Z}^k$. Moreover, the Lie algebra $\mathbb{Q}_\sigma E_{ij}$ is one of the common eigenspaces for the adjoint maps $\mathrm{Ad}_{\alpha_\sigma^{\mathbf{m}}}$ for $\mathbf{m} \in \mathbb{Z}^k$, i.e. one of the root spaces. We denote the conditional measures of μ with respect to the foliation into $H_\sigma^{(i,j)}$-orbits by $\mu_x^{\sigma,(i,j)}$; see Section 5 for a definition. High entropy can only occur if $\mu_x^{\sigma,(i,j)}$ are non-atomic a.e. (i.e. the dimension of μ along $H_\sigma^{(i,j)}$-orbits is positive) for many or all pairs (i,j); see also Section 9.1 for the exact relation between entropy and the conditional measures.

THEOREM 2.8: *Let $X = G_S/\Gamma$ and let α_σ for some fixed $\sigma \in S$ be as above. Let μ be an α_σ-invariant probability measure on $X = G_S/\Gamma$. If all conditional measures $\mu_x^{\sigma,(i,j)}$ are non-atomic a.e. for $i \ne j$, then μ is actually invariant under left multiplication by any element of $\mathrm{SL}(k+1,\mathbb{Q}_\sigma)$.*

Note that we do not assume ergodicity of α_σ here; this has the advantage that we can apply the above for a measure invariant and ergodic under the joint action of all α_σ with $\sigma \in S$.

It is clear that if μ satisfies the conclusion of Theorem 2.8 for all $\sigma \in S$, then Γ is actually a lattice in G_S and $\mu = m$ is the Haar measure of $X = G_S/\Gamma$. Instead of that we can also obtain a stronger conclusion by assuming ergodicity for α_σ and combining our result with measure rigidity for groups generated by unipotent subgroups; see [25] or [31].

For this recall that a measure μ on X is **algebraic** if there exists a closed subgroup $H \subset G_S$ and some $x \in X$ such that $\mu(Hx) = 1$ and μ is the unique H-invariant probability measure on the (necessarily closed) orbit Hx.

COROLLARY 2.9: *Assume that μ is an α_σ-invariant and ergodic probability measure on $X = G_S/\Gamma$. Suppose that at least one of the following conditions is satisfied.*

(1) *All conditional measures $\mu_x^{\sigma,(i,j)}$ are non-atomic a.e. for $i \ne j$.*
(2) *The measure theoretic entropy $\mathsf{h}_\mu(\alpha_\sigma^{\mathbf{m}}) > 0$ with respect to μ is positive for all nonzero $\mathbf{m} \in \mathbb{Z}^k$.*

Then μ is algebraic.

Note that [6, Thm. 4.2] gives various additional statements for the homogeneous space $X = \mathrm{SL}(3,\mathbb{R})/\Gamma$ which can (with the tools provided here) easily be shown to hold in the S-algebraic setting discussed above as well.

Another way to obtain a rigidity result for G_S is to replace the assumption on the conditional measures in Theorem 2.8 or the above assumption on entropy for all elements of the action, by the assumption that the entropy of some element

$\alpha^{\mathbf{m}}$ of the action is close to the maximal value (which is determined only by k and $\mathbf{m}$). In this case it is enough to assume invariance with respect to an arbitrary $\mathbb{Z}^2$-subaction of α_σ for some $\sigma \in S$. This is a consequence of Theorem 2.4. Moreover, we can apply this theorem also for a (sufficiently generic) higher rank subaction of the joint action of all α_σ with $\sigma \in S$.

2.3. OUTLINE OF THE PAPER. In Sections 3 and 4 we recall some basic material on p-adic numbers and Lie groups, and develop a basic theory of Lyapunov weights for real and p-adic Lie groups.

In Section 5 we recall the definition and basic properties of conditional measures from [21] for (T, H)-spaces, which generalizes the notion of the foliation into G_S-orbits considered above. The main difference is that the T-leaves of a (T, H)-space do not have a canonical 'coordinate map' from the space T to the leaf corresponding to a base point x.

In Section 6 we consider conditional measures for H-orbits, and show that for a.e. x the subgroup under which the conditional measure is translation invariant has a special structure: it allows a weight decomposition.

One tool which was used in an essential way, both in the high and low entropy case, is the product structure of the conditional measures. In Section 7 we use the framework of (T, H)-spaces to show Theorem 7.5 that generalizes [21, Prop. 6.4]. In the algebraic case this theorem states that the conditional measure of an invariant measure μ for the foliation into H-orbits is a product measure if $H = ST$ is itself a product of two subgroups S, T such that T is normal in H and (asymptotically) some part of the action acts isometrically on the induced S-orbits while the T-orbits are contracted.

In the case of a semisimple higher rank action on a homogeneous space we prove in Section 8 the Theorems 8.4 and 8.5, which we already discussed in Section 1.4.

Theorem 2.4, 2.7, and 2.8 all rely heavily on Theorem 8.5 and in part also on the relation between the conditional measures and entropy. The latter we recall from [25] and slightly extend in Section 9, where we will also prove the results presented in this section.

3. Preliminaries on local fields and Lie groups

3.1. p-ADIC NUMBERS. For any rational prime number p the p-adic field of rational numbers $\mathbb{Q}_p$ is defined as the completion of $\mathbb{Q}$ with respect to the norm

$|\cdot|_p$ defined by

$$|0|_p = 0,$$
$$\left|p^\ell \frac{n}{m}\right|_p = p^{-\ell} \quad \text{for } \ell, n, m \in \mathbb{Z} \text{ and } p \nmid nm.$$

Furthermore, $\mathbb{Q}_p$ is a locally compact field and every $t \in \mathbb{Q}_p$ can be written as a converging sum

$$t = \sum_{i=n}^{\infty} t_i p^i$$

for some $n \in \mathbb{Z}$ and $t_i \in \{0, 1, \ldots, p-1\}$. It is easy to check that $|\cdot|_p$ and its extension to $\mathbb{Q}_p$ satisfy

$$|s|_p \in \{p^k : k \in \mathbb{Z}\} \cup \{0\} \subset \mathbb{R}_{\geq 0},$$
$$|s+t|_p \leq \max(|s|_p, |t|_p),$$
$$|st|_p = |s|_p |t|_p \quad \text{for all } s, t \in \mathbb{Q}_p.$$

The second of these properties is the ultrametric triangle inequality.

The closure of $\mathbb{Z}$ in $\mathbb{Q}_p$ is the compact ring $\mathbb{Z}_p$ of p-adic integers and consists of all $t \in \mathbb{Q}_p$ allowing a representation as above but with $n \geq 0$. Moreover, every ball $B_r^{\mathbb{Q}_p}$ of arbitrary finite radius r and center 0 allows a similar description and is a compact open subgroup that is isomorphic to $\mathbb{Z}_p$.

Another main difference between real numbers and p-adic numbers is the multiplicative structure. It is easy to see that $\mathbb{Z}_p^\times$ is a compact open subgroup of $\mathbb{Q}_p^\times$. The p-adic logarithm log is defined on a neighborhood of 1 in $\mathbb{Z}_p^\times$ and has the same Taylor series expansion as for the reals. Its inverse map is the p-adic exponential map which is defined on a neighborhood of 0, and again has the same Taylor series expansion as for the reals. Therefore, the multiplicative group $\mathbb{Q}_p^\times$ is locally isomorphic to the additive group $\mathbb{Z}_p$. However, $\mathbb{Q}_p^\times$ also contains the cyclic subgroup $p^{\mathbb{Z}} = \{p^n : n \in \mathbb{Z}\}$. Together, these two subgroups generate a finite index subgroup of $\mathbb{Q}_p^\times$ that is isomorphic to $\mathbb{Z} \times \mathbb{Z}_p$.

For notational simplicity we write $\mathbb{Q}_\infty = \mathbb{R}$ and $|t|_\infty = |t|$ for the usual norm. We will refer to ∞ and to rational prime numbers as places and use the letter σ to denote a place. Note that for any of the fields $\mathbb{Q}_\sigma$ the Haar measure m_σ satisfies $m_\sigma(tB) = |t|_\sigma m_\sigma(B)$ for any measurable $B \subseteq \mathbb{Q}_\sigma$. Suppose $\mathbb{K}$ is a field extension of $\mathbb{Q}_\sigma$ of degree d; then we extend $|\cdot|_\sigma$ to $\mathbb{K}$ normalized such that $m_\sigma(tB) = |t|_\sigma^d m_\sigma(B)$ for any measurable $B \subseteq \mathbb{K}$.

3.2. Lie groups over local fields. In this section we recall some of the basic facts about real and p-adic Lie groups; see [4]. Let σ be a place, let $\mathbb{K}$ be a finite field extension of $\mathbb{Q}_\sigma$, and let $G_{\mathbb{K}}$ be a Lie group over $\mathbb{K}$ with Lie algebra $\mathfrak{g}_{\mathbb{K}}$, i.e. $\mathfrak{g}_{\mathbb{K}}$ is the tangent space of $G_{\mathbb{K}}$ at the identity element $e \in G_{\mathbb{K}}$ with an induced commutator map $[\cdot,\cdot]\colon \mathfrak{g}_{\mathbb{K}}^2 \to \mathfrak{g}_{\mathbb{K}}$. Then we can consider $G_\sigma = G_{\mathbb{K}}$ as a Lie group over $\mathbb{Q}_\sigma$ and $\mathfrak{g}_\sigma = \mathfrak{g}_{\mathbb{K}}$ as the corresponding Lie algebra over $\mathbb{Q}_\sigma$. Therefore, it is enough to consider Lie groups over $\mathbb{Q}_\sigma$.

Just as for the multiplicative group $\mathbb{Q}_\sigma^\times$ in Section 3.1 there exist two locally defined maps between G_σ and $\mathfrak{g}_\sigma$ (see [4, Ch. III, §7.2, Prop. 3]): the exponential map

$$\exp\colon B_\delta^{\mathfrak{g}_\sigma} \to G_\sigma$$

is defined on some ball around $0 \in \mathfrak{g}_\sigma$ and has as its local inverse the logarithm map

$$\log\colon B_R^{G_\sigma} \to \mathfrak{g}_\sigma$$

which is defined on some ball around $e \in G_\sigma$. (Note that when $\sigma = \infty$ the exponential map is of course defined on the whole of $\mathfrak{g}_\infty$.)

Recall that the commutator $[\cdot,\cdot]\colon \mathfrak{g}_\sigma^2 \to \mathfrak{g}_\sigma$ is skew-symmetric, bilinear, and satisfies the Jacobi-identity

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0 \quad \text{for all } u,v,w \in \mathfrak{g}_\sigma. \tag{3.1}$$

Another useful fact is the Campbell–Hausdorff formula which allows one to express $u*v = \log((\exp u)(\exp v))$ for sufficiently small $u,v \in \mathfrak{g}_\sigma$ as a converging sum

$$u * v = u + v + \frac{1}{2}[u,v] + \cdots = \sum_{n=1}^{\infty} F_n(u,v), \tag{3.2}$$

where each $F_n(u,v)$ is a finite sum of expressions of the form

$$[w_1,[w_2,[\cdots[w_{n-1},w_n]\cdots]]] \tag{3.3}$$

with universal (rational) coefficients. If $\sigma = p$ is a prime number it is possible to choose $\rho > 0$ such that $B_\rho^{\mathfrak{g}_p} * B_\rho^{\mathfrak{g}_p} \subseteq B_\rho^{\mathfrak{g}_p}$. In other words, the image of $B_\rho^{\mathfrak{g}_p}$ under exp is an open compact subgroup $G_p(\rho) \subset G_p$ (see [4, Ch. III, §7.1, Thm. 1]). (This ultimately goes back to the ultrametric triangle inequality.)

Of particular interest to us is the adjoint representation of G_σ on $\mathfrak{g}_\sigma$. For any $a \in G_\sigma$ the conjugation map $h \mapsto aha^{-1}$ for $h \in G_\sigma$ fixes e and its derivative at e,

$$A = \mathrm{Ad}_a\colon \mathfrak{g}_\sigma \to \mathfrak{g}_\sigma,$$

is the adjoint representation of $a \in G_\sigma$. More generally, if $\theta\colon G_\sigma \to G_\sigma$ is a group automorphism, we will consider its derivative

$$A = d_e\theta\colon \mathfrak{g}_\sigma \to \mathfrak{g}_\sigma.$$

In both cases this linear automorphism of $\mathfrak{g}_\sigma$ satisfies

$$A[u,v] = [Au, Av] \quad \text{for all } u, v \in \mathfrak{g}_\sigma \tag{3.4}$$

and

$$\exp(Au) = \theta(\exp(u)) \quad \text{for all sufficiently small } u \in \mathfrak{g}_\sigma. \tag{3.5}$$

3.3. S-LIE GROUPS AND ACTIONS PRESERVING G_S-LEAVES. Recall that G_S denotes a direct product of Lie groups G_σ over $\mathbb{Q}_\sigma$ for $\sigma \in S$, where S is a finite set of places.

We will use $\exp u$, $\log g$, $[u,v]$ for $u, v \in \mathfrak{g}_S$ and $g \in G_S$, and adjoint maps Ad_a for $a \in G_S$ freely; these are all defined as product maps. We identify G_σ and $\mathfrak{g}_\sigma$ with the corresponding subgroup of G_S resp. the corresponding subspace of $\mathfrak{g}_S$.

We define the "norm" $\|\cdot\|$ on $\mathfrak{g}_S$ by

$$\|v\| = \max_{\sigma \in S} \|v_\sigma\|_\sigma,$$

where $\|\cdot\|_\sigma$ denotes some fixed norm on $\mathfrak{g}_\sigma$ (which we will specify later).

It is easy to see that automorphisms of nilmanifolds and (twisted) Weyl chamber flows give actions of the following type.

Definition 3.1: Let X be a locally compact, second countable, metric space and suppose G_S acts continuously and locally free on X. Furthermore, let α be a $\mathbb{Z}^k$-action by homeomorphisms of X. Then α **preserves the G_S-leaves** if for every $\mathbf{n} \in \mathbb{Z}^k$ there exists an automorphism $\theta^{\mathbf{n}}$ of G_S such that

$$\alpha^{\mathbf{n}}(gx) = \theta^{\mathbf{n}}(g)\alpha^{\mathbf{n}}x \quad \text{for } x \in X, g \in G_S.$$

Let $A^{\mathbf{n}} = \mathrm{d}_e\theta^{\mathbf{n}}$ be the derivative of $\theta^{\mathbf{n}}$ at $e \in G_S$. Then A is the **adjoint action to α**.

We will use the adjoint action to study the behavior of the action along the G_S-leaves.

4. Lyapunov weights and weight subspaces

In this section we define Lyapunov weights and study their properties. This will lead to the notion of coarse Lyapunov subgroup G_S^Λ. Since we are only interested in this case, we assume from now on that the adjoint action to α is semisimple.

4.1. Semisimple linear maps.

Definition 4.1: Let V be a finite dimensional vector space over $\mathbb{Q}_\sigma$, and let $A\colon V \to V$ be a linear map. Then A is **semisimple** if the minimal polynomial of A is a product of distinct irreducible polynomials. An action by linear automorphisms is semisimple if this is the case for all of its elements.

Lemma 4.2: *Let V be a finite dimensional vector space over some field $\Bbbk$. Let $A\colon V \to V$ be linear. Then A is semisimple if and only if we can find A-invariant subspaces V_i (for $i = 1, \ldots, \ell$) such that V is the direct sum $\sum_{i=1}^{\ell} V_i$, each V_i can be given a vector space structure with respect to some finite field extension $\mathbb{K}_i$ of $\Bbbk$, and $A(v) = t_i v$ for some $t_i \in \mathbb{K}_i$ and all $v \in V_i$. Moreover, each V_i can be defined as the kernel of $q_i(A)$ for some irreducible factor $q_i(T) \in \Bbbk[T]$ of the minimal polynomial of A.*

Proof: This is an easy exercise in algebra. We only note that the linear map A gives V a module structure over the ring of polynomials $\Bbbk[T]$. Since $\Bbbk[T]$ is a principal ideal domain, V allows a decomposition into submodules V_i annihilated by a power $q_i^{n_i}(T)$ of some irreducible polynomial $q_i(T)$ (see [14, Thm. 6.12(ii)]). Since A is semisimple we must have $n_i = 1$ and we can give V_i a vector space structure over $\mathbb{K}_i = \Bbbk[T]/(q_i(T))$. ∎

We need to extend this to several commuting semisimple linear maps.

Proposition 4.3: *Let V be a vector space over $\mathbb{Q}_\sigma$, and let $A_1, \ldots, A_k$ be commuting semisimple linear maps on V. Then there exist subspaces V_i (for $i = 1, \ldots, \ell$) such that V is the direct sum $\sum_{i=1}^{\ell} V_i$, each V_i is invariant under $A_1, \ldots, A_k$, each V_i can be given a vector space structure with respect to some finite field extension $\mathbb{K}_i$ of $\mathbb{Q}_\sigma$, and $A_j(v) = t_i(j)v$ for some fixed $t_i(j) \in \mathbb{K}_i$, for all $v \in V_i$, and for $j = 1, \ldots, k$.*

Proof: The proposition follows by induction on k using Lemma 4.2. In every step the space $V_i \subseteq V$ is defined using only $A_1, \ldots, A_{k-1}$ and is therefore invariant under A_k. Moreover, the vector space structure on V_i over $\mathbb{K}_i$ is also defined using $A_1, \ldots, A_{k-1}$ and the restriction of A_k becomes a linear map on

V_i over $\mathbb{K}_i$. Applying the lemma to the restriction of A_k to V_i and the field $\mathbb{K}_i$ we refine the decomposition of V if necessary. ∎

4.2. (Coarse) Lyapunov weights. Let α be a $\mathbb{Z}^k$-action on X that preserves the G_S-leaves. Assume the adjoint action (i.e. the restriction of A to every $\mathfrak{g}_\sigma$ for $\sigma \in S$) is semisimple. Then we can decompose every $\mathfrak{g}_\sigma = \sum_i \mathfrak{g}_{\sigma,i}$ into common eigenspaces, so that $\mathfrak{g}_{\sigma,i}$ is a vector space over $\mathbb{K}_{\sigma,i}$ and $\mathrm{Ad}_{\alpha(\mathbf{n})}(v) = t^{\mathbf{n}}_{\sigma,i} v$ for every $v \in \mathfrak{g}_{\sigma,i}$ and $\mathbf{n} \in \mathbb{Z}^k$, where $t^{\mathbf{n}}_{\sigma,i} = t_{\sigma,i}(1)^{n_1} \cdots t_{\sigma,i}(k)^{n_k}$ and $t_{\sigma,i}(1), \ldots, t_{\sigma,i}(k) \in \mathbb{K}_{\sigma,i}$.

Definition 4.4: For every eigenspace $\mathfrak{g}_{\sigma,i}$ as above we define the **Lyapunov weight** $\lambda = \lambda_{\sigma,i}\colon \mathbb{Z}^k \to \mathbb{R}$ by

$$\lambda_{\sigma,i}(\mathbf{n}) = \log |t^{\mathbf{n}}_{\sigma,i}|_\sigma = \sum_{j=1}^{k} n_j \log |t_{\sigma,i}(j)|_\sigma.$$

Here $|\cdot|_\sigma$ is the extension to $\mathbb{K}_{\sigma,i}$ as in Section 3.1. Clearly λ can be extended to a linear map $\lambda\colon \mathbb{R}^k \to \mathbb{R}$. Next we group the eigenspaces together in two different ways according to their Lyapunov weights and obtain subspaces with dynamical significance.

Definition 4.5: For a Lyapunov weight λ the **Lyapunov weight subspace** $\mathfrak{g}^\lambda_S$ is the sum of all subspaces $\mathfrak{g}_{\sigma,i}$ for which $\lambda = \lambda_{\sigma,i}$. Moreover, let $\mathfrak{g}^\lambda_\sigma = \mathfrak{g}^\lambda_S \cap \mathfrak{g}_\sigma$ for any $\sigma \in S$.

Note that $\mathfrak{g}^\lambda_S = \sum_{\sigma\in S} \mathfrak{g}^\lambda_\sigma$ for any Lyapunov weight λ and that $\mathfrak{g}_S = \sum_\lambda \mathfrak{g}^\lambda_S$, where both sums are direct sums. The same holds similarly for the following notion.

Definition 4.6: For a nonzero Lyapunov weight λ the **coarse Lyapunov weight subspace** is defined by $\mathfrak{g}^\Lambda_S = \sum_{\zeta\in\Lambda} \mathfrak{g}^\zeta_S$ where $\Lambda = \{t\lambda : t > 0\} = \mathbb{R}^+\lambda$ is the ray from the origin through λ. Similarly let $\mathfrak{g}^\Lambda_\sigma = \mathfrak{g}^\Lambda_S \cap \mathfrak{g}_\sigma$ for any $\sigma \in S$.

4.3. Basic properties of (coarse) Lyapunov weight subspaces.

Lemma 4.7: *We can choose the norms $\|\cdot\|_\sigma$ on $\mathfrak{g}_\sigma$ for $\sigma \in S$ such that the induced norm*

$$\|v\| = \max_\sigma \|v_\sigma\|_\sigma \quad \textit{for } v \in \mathfrak{g}_S$$

satisfies

$$\|A^{\mathbf{n}} v\| = e^{\lambda(\mathbf{n})} \|v\| \tag{4.1}$$

for all $\mathbf{n} \in \mathbb{Z}^k$, $v \in \mathfrak{g}_S^\lambda$, *and all Lyapunov weights* λ.

Proof: For any eigenspaces $\mathfrak{g}_{\sigma,i}$ we fix some bases $v_1, \ldots, v_\ell$ over $\mathbb{K}_{\sigma,i}$ and define

$$\left\| \sum_{j=1}^{\ell} t_j v_j \right\|_{\sigma,i} = \max_{j=1}^{\ell} |t_j|_\sigma.$$

For some fixed $\sigma \in S$ and $u = \sum_i u_i \in \mathfrak{g}_\sigma$ with $u_i \in \mathfrak{g}_{\sigma,i}$ we define $\|u\|_\sigma = \max_i \|u_i\|_{\sigma,i}$. It is easy to check that $\|\cdot\|$ satisfies the lemma. ∎

The next lemma characterizes the (coarse) weight subspaces dynamically using the adjoint action.

LEMMA 4.8: *For a Lyapunov weight* λ *we have* $u \in \mathfrak{g}_S^\lambda$ *if and only if there exists some constant* $c > 0$ *such that*

$$\|A^{\mathbf{n}} u\| \le c e^{\lambda(\mathbf{n})} \|u\| \quad \textit{for all } \mathbf{n} \in \mathbb{Z}^k. \tag{4.2}$$

Furthermore, $u \in \mathfrak{g}_S^\Lambda$ *for* $\Lambda = (0,\infty)\lambda$ *if and only if there exist* $c, c_1, c_2 > 0$ *such that*

$$\|A^{\mathbf{n}} u\| \le c \max(e^{c_1 \lambda(\mathbf{n})}, e^{c_2 \lambda(\mathbf{n})}) \|u\| \quad \textit{for all } \mathbf{n} \in \mathbb{Z}^k. \tag{4.3}$$

Moreover, it is possible to set $c = 1$ *in (4.2) and (4.3).*

Proof: From Lemma 4.7 it is clear that elements of $\mathfrak{g}_S^\lambda$ satisfy (4.2). So suppose that $u \in \mathfrak{g}_S$ satisfies (4.2), and let $u = \sum_{\sigma,i} u_{\sigma,i}$ be the decomposition of u according to the eigenspaces $\mathfrak{g}_{\sigma,i}$. Since $\mathfrak{g}_S$ is a direct sum of these eigenspaces, it is clear that every $u_{\sigma,i}$ satisfies

$$\|A^{\mathbf{n}} u_{\sigma,i}\| \le c e^{\lambda(\mathbf{n})} \|u\|.$$

Note that $u_{\sigma,i}$ also satisfies (4.1) for $\lambda_{\sigma,i}$. This implies that

$$e^{\lambda_{\sigma,i}(\mathbf{n})} \|u_{\sigma,i}\| \le c e^{\lambda(\mathbf{n})} \|u\|$$

for all $\mathbf{n} \in \mathbb{Z}^k$ and so $\lambda_{\sigma,i} = \lambda$ unless $u_{\sigma,i} = 0$. The proof of the second statement is similar. ∎

By definition it is clear that every (coarse) weight subspace is invariant under the adjoint action. However, $\mathfrak{g}_S$ is also an S-Lie algebra and we study next how each of these decompositions respects the commutator. (There is no reason to expect some kind of linearity of $[\cdot,\cdot]$ with respect to the vector space structure of $\mathfrak{g}_{\sigma,i}$ over $\mathbb{K}_{\sigma,i}$.)

PROPOSITION 4.9: *Let $\alpha\colon \mathbb{Z}^k \to G_S$ be an* Ad*-semisimple homomorphism, and let λ_1, λ_2 be Lyapunov weights of α. Then for $u \in \mathfrak{g}_S^{\lambda_1}$ and $v \in \mathfrak{g}_S^{\lambda_2}$ we have $[u,v] \in \mathfrak{g}_S^{\lambda_1+\lambda_2}$. Therefore, any coarse weight subspace $\mathfrak{g}_S^{\Lambda}$ is an S-Lie subalgebra, i.e. $\mathfrak{g}_\sigma^{\Lambda}$ is a Lie subalgebra of $\mathfrak{g}_\sigma$ over $\mathbb{Q}_\sigma$ for every $\sigma \in S$.*

Proof: It is enough to consider $u, v \in \mathfrak{g}_\sigma$ for some $\sigma \in S$. It is clear that there exists some $r > 0$ such that $[B_1^{\mathfrak{g}_\sigma}, B_1^{\mathfrak{g}_\sigma}] \subseteq B_r^{\mathfrak{g}_\sigma}$. Let $\mathbf{n} \in \mathbb{Z}^k$ and choose $s \in \mathbb{Q}_\sigma$ such that $|s|_\sigma \geq \|A^{\mathbf{n}}u\|$ and $|s|_\sigma$ is minimal with this property. If $\sigma = \infty$ then

$$|s|_\infty = \|A^{\mathbf{n}}u\| = \|u\|e^{\lambda_1(\mathbf{n})} = c_1 e^{\lambda_1(\mathbf{n})},$$

otherwise $\sigma = p$ is some rational prime and

$$|s|_p \leq p\|A^{\mathbf{n}}u\| = p\|u\|e^{\lambda_1(\mathbf{n})} = c_1 e^{\lambda_1(\mathbf{n})}.$$

We choose $t \in \mathbb{Q}_\sigma$ similarly such that $|t|_\sigma \geq \|A^{\mathbf{n}}v\|$ and $|t|_\sigma \leq c_2 e^{\lambda_2(\mathbf{n})}$. From (3.4) we see that

$$\begin{aligned}\|A^{\mathbf{n}}[u,v]\| &= |st|_\sigma \|s^{-1}t^{-1}A^{\mathbf{n}}[u,v]\| \\ &\leq c_1c_2 e^{(\lambda_1+\lambda_2)(\mathbf{n})}\|[s^{-1}A^{\mathbf{n}}u, t^{-1}A^{\mathbf{n}}v]\| \leq rc_1c_2e^{(\lambda_1+\lambda_2)(\mathbf{n})}.\end{aligned}$$

Now Lemma 4.8 shows $[u,v] \in \mathfrak{g}_S^{\lambda_1+\lambda_2}$. The second statement follows since $\Lambda = (0,\infty)\lambda$ is closed under addition. ∎

4.4. COARSE LYAPUNOV SUBGROUPS, $\mathbf{m}$-STABLE, $\mathbf{m}$-UNSTABLE SUBGROUPS. In the next proposition we show that the exponential map has a canonical extension to coarse Lyapunov weight subspaces and find the subgroups corresponding to the coarse Lyapunov weight subspaces. A more general context gives the following definition.

Definition 4.10: An S-Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}_S$ is an **$\mathbf{m}$-stable S-Lie subalgebra** if $\mathfrak{h}$ is a closed under $[\cdot,\cdot]$, invariant under the adjoint action A, and $\mathfrak{h} = \sum_{j=1}^{\ell} \mathfrak{h}^{\lambda_j}$ is a direct sum of Lyapunov weight spaces $\mathfrak{h}^{\lambda_j} = \mathfrak{h} \cap \mathfrak{g}_S^{\lambda_j}$ of $\mathfrak{h}$ such that $\lambda_j(\mathbf{m}) < 0$ for $j = 1, \ldots, \ell$. Similarly, we define $\mathbf{m}$-unstable Lie subalgebras by requiring that $\lambda_j(\mathbf{m}) > 0$ for $j = 1, \ldots, \ell$ instead.

Recall that $\mathfrak{h}$ is nilpotent if there exists some $n \geq 1$ such that all Lie polynomials of degree n as in (3.3) with $w_1, \ldots, w_n \in \mathfrak{h}$ vanish.

It follows from Proposition 4.9 that the S-Lie subalgebra generated by two $\mathbf{m}$-stable S-Lie subalgebras is also $\mathbf{m}$-stable. Therefore, there exists a unique maximal $\mathbf{m}$-stable S-Lie subalgebra.

PROPOSITION 4.11: *Let α be a $\mathbb{Z}^k$-action on X that preserves the G_S-leaves such that its adjoint action A on $\mathfrak{g}_S$ is semisimple. Then every* $\mathbf{m}$*-(un)stable Lie subalgebra $\mathfrak{h}$ is nilpotent as an S-Lie algebra, i.e. a direct sum of nilpotent Lie algebras $\mathfrak{h}_\sigma = \mathfrak{h} \cap \mathfrak{g}_\sigma$ over $\mathbb{Q}_\sigma$ for $\sigma \in S$. The exponential map can be uniquely extended to the whole of $\mathfrak{h}$ such that*

$$\theta^{\mathbf{n}}(\exp(u)) = \exp(A^{\mathbf{n}}u) \quad \text{for all } \mathbf{n} \in \mathbb{Z}^k, u \in \mathfrak{g}_S^\Lambda, \tag{4.4}$$

where θ is as in Definition 3.1. The image $H = \exp \mathfrak{h}$ is a Lie subgroup of G_S. The inverse of the exponential map is the logarithm map

$$\log\colon H \to \mathfrak{h},$$

the map $u{*}v = \log((\exp u)(\exp v))$ is defined on $\mathfrak{h}^2$, and the Campbell–Hausdorff formula (3.2) expresses $u * v$ as a finite linear combination of expressions as in (3.3).

Furthermore, if $\mathfrak{h} = \sum_{i=1}^{\ell} \mathfrak{h}_i$ is a direct sum of $\mathbf{m}$*-stable Lie subalgebras and $H_i = \exp \mathfrak{h}_i$ are the corresponding subgroups of H, then $\phi\colon H_1 \times \cdots \times H_\ell \to H$ defined by $\phi(g_1, \ldots, g_\ell) = g_1 \cdots g_\ell$ is a homeomorphism.*

In the above proposition we do not require that the Lie subgroup H carries the induced topology.

Recall that an element $g \in G_S$ is unipotent if its adjoint Ad_g has only 1 as its eigenvalues, and that a subgroup is unipotent if all of its elements are unipotent. We only note that it is not too difficult to extend the above proposition: H is actually a unipotent subgroup of G_S.

By Proposition 4.9 the coarse Lyapunov weight subspace $\mathfrak{g}_S^\Lambda$ is an $\mathbf{m}$-stable Lie subalgebra for some $\mathbf{m} \in \mathbb{Z}^k$, whenever $\Lambda = \mathbb{R}^+\lambda$ and $\lambda \neq 0$ is a Lyapunov weight.

Definition 4.12: The image $G_S^\Lambda = \exp \mathfrak{g}_S^\Lambda$ of a coarse Lyapunov weight subspace is a **coarse Lyapunov subgroup** and the image $H = \exp \mathfrak{h}$ of an $\mathbf{m}$-(un)stable Lie algebra as in the proposition is an **$\mathbf{m}$-(un)stable Lie subgroup**.

Any $\mathbf{m}$-stable Lie subalgebra $\mathfrak{h}$ we can decompose into coarse Lyapunov weight subspaces $\mathfrak{h}^\Lambda = \mathfrak{h} \cap \mathfrak{g}_S^\Lambda$ which are $\mathbf{m}$-stable Lie subalgebras of $\mathfrak{h}$ by Proposition 4.9. Therefore we immediately obtain the following corollary.

COROLLARY 4.13: *Any* $\mathbf{m}$*-stable subgroup H is homeomorphic to the direct product of its coarse Lyapunov subgroups via the map that sends $(h_1, \ldots, h_\ell) \in H^{\Lambda_1} \times \cdots \times H^{\Lambda_\ell}$ to $h_1 \cdots h_\ell$.*

Proof of Proposition 4.11: Note that (4.4) already holds for sufficiently small u by (3.5). This shows that we can find for every $u \in \mathfrak{h}$ some $k \geq 0$ such that $\exp(A^{k\mathbf{m}}u)$ is already defined and that

$$\exp(u) = \theta^{-k\mathbf{m}}(\exp(A^{k\mathbf{m}}u))$$

does not depend on k. Similarly, we show that this extended map exp is invertible and get the extension of log.

We claim that $\mathfrak{h}$ is nilpotent. By definition $\mathfrak{h}$ is a direct sum over Lyapunov weight spaces. Therefore and since $[\cdot,\cdot]$ is bilinear, it is enough to consider $w_i \in \mathfrak{g}^{\lambda_i}$ for $i = 1, \ldots, n$. By Proposition 4.9 we know that the expression w in (3.3) belongs to $\mathfrak{g}_S^{\lambda_1+\cdots+\lambda_n}$. However, since all λ_ℓ satisfy $\lambda_\ell(\mathbf{m}) < 0$ and there are only finitely many weights, we conclude that for large enough n the sum $\lambda_1 + \cdots + \lambda_n$ cannot be a weight and $w = 0$ as claimed.

This shows that (3.2) is actually a finite sum and is well defined on the whole of $\mathfrak{h}$. It follows that $H = \exp \mathfrak{h}$ is a subgroup. We define the topology on H by requiring exp to be a homeomorphism.

For $\sigma \in S$ the restriction ϕ_σ of ϕ to $H_1 \times \cdots \times H_\ell$ has an invertible derivative at the identity and so is a local diffeomorphism. Therefore ϕ is a local homeomorphism. However, as for the exponential map we can use $\theta^{\mathbf{m}}$ to conclude that ϕ is a homeomorphism. ∎

4.5. A METRIC ON $\mathbf{m}$-STABLE SUBGROUPS. Let H be an $\mathbf{m}$-stable subgroup. As before we will use subscripts and superscripts for $\mathfrak{h}$ and H as for $\mathfrak{g}_S$ and G_S, e.g. $\mathfrak{h}_\sigma = \mathfrak{g} \cap \mathfrak{g}_\sigma$ and $H^\Lambda = H \cap G_S^\Lambda$. Furthermore, we say λ is a weight of H if $\mathfrak{h}^\lambda \neq 0$.

In this subsection we define a right invariant metric

$$d(g,h) = \max_{\sigma \in S} d_\sigma(g_\sigma, h_\sigma)$$

for $g, h \in H$ by specifying for any $\sigma \in S$ a right invariant metric $d_\sigma(\cdot,\cdot)$ on H_σ. We will show that there exists $\chi < 1$ such that

$$d(\theta^{\mathbf{m}}(g), \theta^{\mathbf{m}}(h)) < \chi d(g,h), \tag{4.5}$$

for any $g, h \in H$, and equivalently if H is an $\mathbf{m}'$-unstable subgroup then there exists $\chi > 1$ with

$$d(\theta^{\mathbf{m}'}(g), \theta^{\mathbf{m}'}(h)) > \chi' d(g,h). \tag{4.6}$$

LEMMA 4.14: *For every $\sigma \in S$ there exists a right invariant metric $d_\sigma(\cdot,\cdot)$ satisfying (4.5), (4.6), and the following additional property. If λ is a weight of H such that there exists a $\mathbf{t} \in \mathbb{R}^k$ with $\lambda(\mathbf{t}) = 0$ and $\xi(\mathbf{t}) < 0$ for any weight ξ of H that is linearly independent to λ, then*

$$d_\sigma(\theta^{\mathbf{n}}(g), e) \leq \max(e^{c_1\lambda(\mathbf{n})}, e^{c_2\lambda(\mathbf{n})}) d_\sigma(g, e)$$

for some $c_1, c_2 > 0$, all $\mathbf{n} \in \mathbb{Z}^k$, and all $g \in H_\sigma^\Lambda$ where $\Lambda = \mathbb{R}^+\lambda$.

Proof for $\sigma = \infty$: Let $\|\cdot\|$ be a norm on $\mathfrak{g}_\infty$ derived from an inner product that satisfies that all weight spaces are orthogonal to each other. Let d_∞ denote the right invariant Riemannian metric on H_∞ derived from $\|\cdot\|$.

Let λ be as in the lemma. We claim that d_∞ induces the Riemannian metric d_Λ on H_∞^Λ (which again is induced by the restriction of $\|\cdot\|$ to $\mathfrak{h}_\infty^\Lambda$). Clearly $d_\infty(g,e) \leq d_\Lambda(g,e)$ for any $g \in H_\infty^\Lambda$ (since any path in H_∞^Λ is also a path in H_∞). To show the opposite inequality define the Lie subalgebra $\mathfrak{h}' = \sum_{\xi(\mathbf{t})<0} \mathfrak{h}_\infty^\xi$ of $\mathfrak{h}_\infty$. Then $\mathfrak{h}_\infty = \mathfrak{h}_\infty^\Lambda + \mathfrak{h}'$ and $\mathfrak{h}_\infty/\mathfrak{h}'$ is metrically isomorphic to $\mathfrak{h}_\infty^\Lambda$. Moreover, $H' = \exp \mathfrak{h}'$ is a normal subgroup of H_∞ and H_∞/H' is isomorphic to H_∞^Λ. Any path γ in H_∞ connecting e to $g \in H_\infty^\Lambda$ induces a path in H/H', and so in H_∞^Λ which again connects e to g. In this process the length of the path does not increase, and so $d_\infty(e,g) = d_\Lambda(e,g)$ for all $g \in H_\infty^\Lambda$.

Let $c_1, c_2 > 0$ be such that all weights in $\Lambda = (0,\infty)\lambda$ are in fact elements of $[c_1, c_2]\lambda$. Let $g \in H_\infty^\Lambda$, let $\mathbf{n} \in \mathbb{Z}^k$, and let γ be a path connecting e to g within H_∞^Λ. Then we apply $\theta^{\mathbf{n}}$ to γ and obtain a new path $\theta^{\mathbf{n}} \circ \gamma$ connecting e to $\theta^{\mathbf{n}}(g)$, and the lengths of the paths satisfy $\text{length}(\theta^{\mathbf{n}} \circ \gamma) \leq \max(e^{c_1\lambda(\mathbf{n})}, e^{c_2\lambda(\mathbf{n})})\,\text{length}(\gamma)$. This shows the desired inequality.

The proof of (4.5) is similar to the above. ∎

Proof for $\sigma = p$: Suppose H is an $\mathbf{m}'$-unstable subgroup. We claim that it is possible to choose for any weight a constant $c_\lambda > 0$ such that

$$\kappa(u) = \max_\lambda c_\lambda \|u_\lambda\|^{1/\lambda(\mathbf{m}')}$$

satisfies $\kappa(u * v) \leq \max(\kappa(u), \kappa(v))$ for any $u, v \in \mathfrak{h}_p$.

Note first that $\kappa(u+v) \leq \max(\kappa(u), \kappa(v))$ holds independent of the choice of the constants. Because of that it is enough to show that $\kappa(tw) \leq \max(\kappa(u), \kappa(v))$ whenever w is as in (3.3) and t is one of the universal constants. Let n be the degree of w. For $n = 1$ there is nothing to prove, so assume $n > 1$. By using the bi-linearity of $[\cdot,\cdot]$ we can reduce to the situation where w is defined by various weight components $u_\zeta \in \mathfrak{h}^\zeta$ and $v_\xi \in \mathfrak{h}^\xi$ of $u = \sum_\zeta u_\zeta$ resp. $v = \sum_\xi v_\xi$. Suppose that $\zeta_1, \ldots, \zeta_\ell$ are the weight for which u_{ζ_i} appears in w, where we list a weight as often as the corresponding component appears. Let $\xi_1, \ldots, \xi_{n-\ell}$ be the corresponding weights for v. Then $w \in \mathfrak{h}_p^\eta$ for

$$\eta = \sum_{i=1}^{\ell} \zeta_i + \sum_{i=1}^{n-\ell} \xi_i \tag{4.7}$$

by Proposition 4.9. Since $[\cdot, \cdot]$ is bilinear there exists some constant $C > 0$ such that $\|tw\| \leq C \prod_i \|u_{\zeta_i}\| \prod_i \|v_{\xi_i}\|$. Therefore

$$\kappa(tw) = c_\eta \|tw\|^{1/\eta(\mathbf{m}')} \leq c_\eta c \prod_{i=1}^{\ell} \kappa(u)^{\zeta_i(\mathbf{m}')/\eta(\mathbf{m}')} \prod_{i=1}^{n-\ell} \kappa(v)^{\xi_i(\mathbf{m}')/\eta(\mathbf{m}')}$$

where $c = c(C, c_{\zeta_1}, \ldots, c_{\xi_{n-\ell}})$ is some combination of all these constants. Here the right hand side is $c_\eta c$ times a geometric mean of $\kappa(u)$ and $\kappa(v)$, so for the claim all we need is

$$c_\eta c(C, c_{\zeta_1}, \ldots, c_{\xi_{n-\ell}}) \leq 1. \tag{4.8}$$

We can now define c_λ inductively: For a weight λ with minimal value $\lambda(\mathbf{m}')$ we set $c_\lambda = 1$. Suppose the constants are already defined for weights λ with $\lambda(\mathbf{m}') < r$ and let η be a weight with $\eta(\mathbf{m}') = r$; then we can choose c_η small enough so that (4.8) holds for all ways to express η as a sum as in (4.7).

We define $d_p(g,h) = \kappa(\log(gh^{-1}))$ for $g, h \in H \cap G_p$. Right invariance is obvious, the triangle inequality follows from

$$\begin{aligned} d_p(g,e) &= \kappa(\log(g)) = \kappa(\log(gh^{-1}) * \log(h)) \\ &\leq \max(\kappa(\log(gh^{-1}), \kappa(\log(h))) = \max(d_p(g,h), d_p(h,e)) \end{aligned}$$

and symmetry is similar. The last statement of the lemma follows from (4.3) and the construction of the metric d_p. Properties (4.5) and (4.6) follow similarly. ∎

5. Conditional measures

In this section we provide the framework of conditional measures, which we will need for the main technical results in the following sections, namely the generalization of [6, Prop. 8.3] and [21, Prop. 6.4] which both state in different settings that certain conditional measures are product measures. In [21] the framework of (T, H)-spaces and their conditional measures was developed and applied to the proof of arithmetic cases of the Quantum Unique Ergodicity conjecture, where the above mentioned fact was one of the tools for the proof.

5.1. (T,H)-SPACES AND CONDITIONAL MEASURES. First we recall the notion of atoms $[x]_{\mathcal{A}}$ and conditional measures $\mu_x^{\mathcal{A}}$ for a countably generated σ-ring $\mathcal{A}$. Clearly there exists a maximal element $A \in \mathcal{A}$ and $\mathcal{A}|_A$ is a σ-algebra. If $\mathcal{A}$ is generated by $A_1, \ldots, A_i, \ldots$, the atom of x is defined by

$$[x] = \bigcap_{i:x \in A_i} A_i \cap \bigcap_{i:x \notin A_i} A \setminus A_i.$$

Then $\mu_x^{\mathcal{A}}$ is a probability measure on the atom $[x]_{\mathcal{A}}$ for a.e. $x \in A$, such that the conditional expectation can be expressed as an integral

$$E(f|_A|\mathcal{A})(x) = \int_A f(y) \mathrm{d}\mu_x^{\mathcal{A}}(y)$$

for all integrable functions f.

In the following T is a locally compact second countable metric space with a distinguished point $e \in T$, and H is a subgroup of the *group of isometries* $\mathrm{Isom}(T)$ that acts transitively on T. We will write $B_r^Y(a) = \{b \in Y : d(a, b) < r\}$ for the ball of radius r in the metric space Y and center a. If the center is $e \in T$ we also write $B_r^T = B_r^T(e)$.

Definition 5.1: A locally compact second countable metric space X is said to be a (T, H)-**space** if there is some countable open cover $\mathfrak{T}$ of X by relatively compact sets, and for every $U \in \mathfrak{T}$ a continuous map $t_U \colon T \times U \to X$ with the following properties:

(1) For every $x \in U \in \mathfrak{T}$, we have $t_U(e, x) = x$.

(2) For any $x \in U \in \mathfrak{T}$, any $t \in T$ and any $V \in \mathfrak{T}$ containing $y = t_U(t, x)$, there exists a $\phi \in H$ with $\phi(e) = t$ and

$$t_V(\cdot, y) = t_U(\cdot, x) \circ \phi. \tag{5.1}$$

(3) There is some $r_U > 0$ so that for any $x \in U$ the map $t_U(\cdot, x)$ is injective on $\overline{B_{r_U}^T}$.

We define $\mathfrak{T}(x) = \{U \in \mathfrak{T} : x \in U\}$. Property (2) above shows that the *leaf* $t_U(T, x)$ is independent of $U \in \mathfrak{T}(x)$ and furthermore how the two parameterizations $t_U(\cdot, x)$ and $t_V(\cdot, y)$ of $t_U(T, x)$ differ. It also implies that $B_r^T(x) = t_U(B_r^T, x)$ is independent of $U \in \mathfrak{T}(x)$. The following lemma will be useful later.

LEMMA 5.2: *Let $x_0 \in U \in \mathfrak{T}$. Suppose*

$$t_U(\cdot, x_0) \text{ is injective on } \overline{B^T_{20r}}. \tag{5.2}$$

Then for small enough $\epsilon > 0$ any $y \in A = t_U(B^T_r, B_\epsilon(x))$ satisfies the following two properties:

(1) *$B^T_{10r}(y) \cap A \subset B^T_{4r}(y)$ and*
(2) *$t_V(\cdot, y)$ is injective on B^T_{19r} where $V \in \mathfrak{T}(y)$.*

Proof: The first statement was shown in [21, Lemma 3.2 (1)].

We claim that (5.2) holds for small enough $\epsilon > 0$, in fact for all $x \in B_\epsilon(x_0)$. Assume by contradiction that for every $\epsilon > 0$ there exists an x_ϵ with $d(x_\epsilon, x_0) \le \epsilon$ such that (5.2) fails for x_ϵ. Then there exist two different $t_\epsilon, t'_\epsilon \in \overline{B^T_{20r}}$ with $y_\epsilon = t_U(t_\epsilon, x_\epsilon) = t_U(t'_\epsilon, x_\epsilon)$. Note that here y_ϵ belongs to a fixed compact set which we can cover by finitely many $V \in \mathfrak{T}$, and let $r_K > 0$ be the minimum over the corresponding r_V as in Definition 5.1 (3). Choose V with $y_\epsilon \in V$ and $\phi \in H$ such that $t_V(\cdot, y_\epsilon) = t_U(\phi(\cdot), x_\epsilon)$ and $\phi(t_\epsilon) = e$. Since $\phi(t'_\epsilon) \neq e$ but $t_V(\phi(t'_\epsilon), y_\epsilon) = y_\epsilon$ it follows that $d(t_\epsilon, t'_\epsilon) \ge r_V \ge r_K$.

By choosing a converging subsequence we now obtain $t_0, t'_0 \in \overline{B^T_{20r}}$ with $d(t_0, t'_0) \ge r_K$ and $t_U(t_0, x_0) = t_U(t'_0, x_0)$, which contradicts the assumption of the lemma. This shows the claim. To see that this implies the second part of lemma, let $y = t_U(t, x)$ for some $t \in B^T_r$ and note that the element $\phi \in H$ as in Definition 5.1 (2) maps B^T_{19r} into B^T_{20r}. ∎

The following algebraic case of this structure is of special interest to us.

Definition 5.3: Let X be a locally compact second countable metric space X, let H be a locally compact second countable metric group H with a right invariant metric. Then an **H-space** is given by a locally free action of H on X, which we write as $(h, x) \mapsto hx$.

An H-space gives an example of an (H, H)-space, where H acts on itself by right translation $R_g(s) = sg$ for $g, s \in H$, and $t_U(h, x) = hx$ is independent of $U \in \mathfrak{T}(x)$.

In [21, Thm. 3.3] the family of conditional measures for the (T, H)-space was constructed (see also [18, Sect. 4] and [16, Sect. 1.4]). Since this is fundamental for what follows, we state this result but first we recall two more necessary definitions.

Definition 5.4: A set $D \subset X$ is an **open T-plaque** if for any $x \in D$: $D \subseteq t_V(B^T_r, x)$ for some $r > 0$, and $t_V(\cdot, x)^{-1}D$ is open in T for some (all) $V \in \mathfrak{T}(x)$.

Definition 5.5: A pair $(\mathcal{A}, A)$ with $\mathcal{A} \subseteq \mathcal{B}$ a countably generated σ-ring and $A \subseteq X$ its maximal element is called an r, T-**flower with center** $C \subseteq X$ if

(1) A is open and relatively compact, and $C \subseteq A$.

(2) For every $y \in A$ the atom $[y]_{\mathcal{A}}$ is an open T-plaque; in fact for $V \in \mathfrak{T}(y)$ we require that
$$[y]_{\mathcal{A}} = A \cap t_V(B_{4r}^T, y).$$

(3) If $y \in C$ and $V \in \mathfrak{T}(y)$ then $[y]_{\mathcal{A}} \supset t_V(B_r^T, y)$.

The existence of r, T-flowers with a small disc as a base has been shown in [21, Cor. 3.5].

For the following it is convenient to write $\nu_1 \propto \nu_2$ if two measures ν_1, ν_2 are equal up to a multiplicative constant, i.e. if there exists a constant $C > 0$ with $\nu_1(B) = C\nu_2(B)$ for all measurable B. Furthermore, we will use for a measure ν on Y and a measurable function $f: Y \to Z$ the push forward $f_*\nu$ which is defined by $f_*\nu(A) = \nu(f^{-1}A)$ for any measurable $A \subseteq Z$.

We let $\mathcal{M}_\infty(T)$ be the set of all locally finite Borel measures on T, equipped with the weak* topology defined by $I_f(\mu) = \int_T f \mathrm{d}\mu$ for $f \in C_c(T)$. Note that $\mathcal{M}_\infty(T)$ (unlike the full dual of $C_c(T)$) is a metrizable, separable space with this topology.

PROPOSITION 5.6 ([21, Thm. 3.6]): *Let X be a (T, H)-space, let μ be a Borel probability measure on X, and suppose that*

$$t_U(\cdot, x) \text{ is injective for every } U \in \mathfrak{T}, \text{ and a.e. } x \in U. \tag{5.3}$$

Then the conditional measures $\mu_{T,x}^U$ for $U \in \mathfrak{T}(x)$ are Radon measures on T with the following properties:

(1) *The unit ball B_1^T has measure one.*

(2) *For any countably generated σ-ring $\mathcal{A}$ with maximal element A whose atoms are open T-plaques and for a.e. $x \in A$ and every $U \in \mathfrak{T}(x)$, we have*
$$t_U(\cdot, x)_*^{-1} \mu_x^{\mathcal{A}} \propto \mu_{x,T}^U|_{t_U(\cdot,x)^{-1}[x]_{\mathcal{A}}}.$$

(3) *There is a set $X_0 \subseteq X$ of full μ-measure so that for every $x, y \in X_0$, $t \in T$ with $y = t_U(t, x)$, $U \in \mathfrak{T}(x)$, and $V \in \mathfrak{T}(y)$ we have $\mu_{x,T}^U \propto \phi_* \mu_{y,T}^V$ where $\phi \in H$ is the isometry satisfying $\phi(e) = t$ and $t_V(\cdot, y) = t_U(\cdot, x) \circ \phi$.*

Furthermore, the map $x \mapsto \mu_{T,X}^U$ from U to the set $\mathcal{M}_T$ of Radon measures on T is measurable.

In the case of an H-space the conditional measure does not depend on the set $U \in \mathfrak{T}$, and we simply write $\mu_x^H = \mu_{x,H}^U$. Note that the existence of the conditional measures in the sense of Proposition 5.6 is linked to the assumption (5.3). When we speak below of a measure μ with conditional measures $\mu_{T,x}^U$ for $U \in \mathfrak{T}(x)$, we implicitly assume that (5.3) is satisfied.

Note that Proposition 5.6 (2) shows that the conditional measure does not depend on the choice of the metric $d(\cdot,\cdot)$ on T (assuming the topology is induced by the metric and all properties of T-spaces are satisfied).

It is well known that translation invariance of the conditional measures μ_x^H implies translation invariance of the global measure μ (see [18], [6, Prop. 3.3] or [21, Prop. 4.3]).

PROPOSITION 5.7: *Let X be an H-space, and let μ be a probability measure such that the H-action for a.e. base point is free. Then μ is H-invariant if, and only if, for μ-a.e. x the conditional measure μ_x^H is a left invariant Haar measure on H.*

The following lemma follows easily from the construction of the conditional measures $\mu_{x,T}^U$ (see [21, Lemma 3.7]) and is the reason why we can impose the normalization (1) in Proposition 5.6. Recall that the support $\operatorname{supp}\nu$ of a measure ν on Y is the complement of the biggest open set in Y that is also a null set with respect to ν.

LEMMA 5.8: *Let X be a (T,H)-space, let μ be a probability measure with conditional measures $\mu_{x,T}^U$ for $U \in \mathfrak{T}(x)$. Then for a.e. $x \in X$ and all $U \in \mathfrak{T}(x)$ we have $e \in \operatorname{supp}\mu_{x,T}^U$.*

Another corollary of the construction of the conditional measures (Proposition 5.6 (2) and the properties of μ_x^A) is the following (see [6, Lemma 3.1]).

LEMMA 5.9: *Let X be a (T_i,H_i)-space for $i \in I$, where I is a finite or countable index set. Let μ be a probability measure with conditional measures μ_{x,T_i}^U for $U \in \mathfrak{T}_i(x)$ and $i \in I$. Let N be a null set. Then there exists a null set $N' \supset N$ such that $\mu_{x,T_i}^{U_i}(t_{U_i}(\cdot,x)^{-1}N') = 0$ for all $x \in X \setminus N'$, $i \in I$, and $U_i \in \mathfrak{T}_i(x)$.*

5.2. FIRST DYNAMICAL PROPERTIES OF THE CONDITIONAL MEASURES. Let X be a (T,H)-space. A homeomorphism α: $X \to X$ **preserves the T-leaves** if

$$\alpha \circ t_U(x,\cdot) = t_V(\alpha x,\cdot) \circ \theta_x^{U,V} \tag{5.4}$$

for all $U \in \mathfrak{T}(x)$, $V \in \mathfrak{T}(\alpha x)$ and some homeomorphism $\theta_x^{U,V}$ of T fixing e. In the case where X is an H-space, α **preserves the H-leaves** if (5.4) holds for some fixed group automorphism θ of H (just as in Definition 3.1).

Clearly, this implies that there exists for every $n \in \mathbb{Z}$, $x \in X$, $U \in \mathfrak{T}(x)$, and $V \in \mathfrak{T}(\alpha^n x)$ a homeomorphism (group automorphism) $\theta_{n,x}^{U,V}$ of T fixing e with

$$\alpha^n \circ t_U(x,\cdot) = t_V(\alpha^n x,\cdot) \circ \theta_{n,x}^{U,V}. \tag{5.5}$$

We say α acts **isometrically** on the T-leaves if additionally

$$d(\theta_x^{U,V} s, \theta_x^{U,V} s') = d(s,s')$$

for all $s,s' \in T$, $x \in X$, and U,V as above. Furthermore, α **uniformly expands** the T-leaves if it preserves them and there exists a constant $\chi > 1$ so that

$$d(\theta_x^{U,V} s, \theta_x^{U,V} s') > \chi d(s,s')$$

for all $s,s' \in T$, and $x \in X$. Similarly α **uniformly contracts** the T-leaves if α^{-1} uniformly expands them. A more general group action preserves the leaves (or acts isometrically on the leaves) if this is true for every element of the action.

Note that, if α uniformly expands (or contracts) the T-leaves and preserves the probability measure μ, then (5.3), which is needed for the construction of the conditional measures, is automatically satisfied. To see this, note that by Poincaré recurrence for a.e. $x \in U \in \mathfrak{T}$ there exists arbitrary large $n > 0$ with $\alpha^{-n}x \in U$. By Definition 5.1 (3), $t_U(\cdot,\alpha^{-n}x)$ is injective on $B_{r_U}^T$. By (5.5) and expansion, this implies that $t_U(\cdot,x)$ is injective on $B_{r_U\chi^n}^T$.

If μ is an α-invariant measure and $\mu_{x,T}^U$ are the conditional measures as in the last section, it follows that

$$\mu_{\alpha x,T}^V \propto (\theta_x^{U,V})_* \mu_{x,T}^U \tag{5.6}$$

for a.e. $x \in X$. This can be seen as in the proof of [21, Prop. 5.2], which considers the case of an isometry: If α acts isometrically on the (T,H)-leaves then equality holds by the normalization in Proposition 5.6 (1).

LEMMA 5.10: *Let X be an H-space, and suppose α uniformly contracts the H-leaves. Let μ be an α-invariant probability measure. Let L_h: $H \to H$ denote the left translation with $h \in H$, i.e. $L_h(g) = hg$ for all $g \in H$. Then there exists a null set $N \subset X$ with the following property. If $x \in X \setminus N$ satisfies that μ_x^H is left translation invariant by some $h \in H$ in the affine sense, i.e. $(L_h)_*\mu_x^H \propto \mu_x^H$, then in fact μ_x^H is translation invariant by h, i.e. $(L_h)_*\mu_x^H = \mu_x^H$. (The same holds similarly for right translation $R_h(g) = gh$.)*

Proof: Since $\mu_x^H(B_2^H) < \infty$ for a.e. x, we can find for every $m > 0$ a set K_m with $\mu(K_m) > 1-1/m$ such that $\mu_x^H(B_2^H) < M_m$ for some $M_m > 0$ independent of $x \in K_m$. Without loss of generality we can assume that (5.6) holds for all $x \in K_m$ and all α^n for $n \geq 1$. Let $K'_m \subseteq K_m$ be the set of points x for which there exists infinitely many $n > 1$ with $\alpha^n x \in K_m$. By the Poincaré recurrence $\mu(K'_m) = \mu(K_m)$, so that $N = X \setminus \bigcup_m K'_m$ is a null set.

By continuity of the group multiplication in H, there exists $\epsilon > 0$ with $hB_1^H \subset B_2^H$ for all $h \in B_\epsilon^H$. Let $x \in X \setminus N$, and suppose μ_x^H is translation invariant in the affine sense by L_h. Then $x \in K'_m$ for some m, and so $\alpha^n x$ belongs to K_m infinitely often. Since θ is assumed to be a group automorphism of H, (5.6) implies that $\mu_{\alpha^n x}^H$ is translation invariant in the affine sense by $L_{\theta^n h}$. Here the multiplicative constant C remains unchanged. Fix some $\ell \geq 1$. Since α uniformly contracts the H-leaves, we can find $n > 0$ with $\alpha^n x \in K_n$ and $\theta^n h^\ell \in B_\epsilon^H$. It follows that

$$M_m \geq \mu_x^H(B_2^H) \geq \mu_x^H((\theta^n h)^\ell B_1^H) = C^\ell.$$

Since this holds for all $\ell \geq 1$ (and some fixed m), we conclude that $C \leq 1$. By applying the above to h^{-1} we see that $C = 1$. ∎

6. The structure of the subgroup leaving the conditional measure invariant

Let α be a $\mathbb{Z}^k$-action on X, and let G_S act continuously and locally free on X such that α preserves the G_S-leaves (as in Section 3.3). Let θ be the corresponding $\mathbb{Z}^k$-action by automorphisms of G_S describing the action of α on the G_S-leaves, i.e. such that $\alpha^{\mathbf{n}}(gx) = \theta^{\mathbf{n}}(g)\alpha^{\mathbf{n}}(x)$ for $x \in X$ and $\mathbf{n} \in \mathbb{Z}^k$. Finally, let A be the adjoint $\mathbb{Z}^k$-action on $\mathfrak{g}_S$ and assume that A is semisimple.

In this section we begin our study of α-invariant probability measures on X. We will show that the maximal subgroup leaving conditional measures invariant has a special structure.

For this let H be an $\mathbf{m}$-stable subgroup of G_S and let $d(\cdot,\cdot)$ be the metric defined in Section 4.5. The metric is right invariant as required in Definition 5.3. Since $\mathfrak{h}$ is invariant under the adjoint action A, it is easy to check that the induced H-space structure is also preserved by the $\mathbb{Z}^k$-action α. Moreover, (4.5) shows that $\alpha^{\mathbf{m}}$ uniformly contracts the H-leaves. As we saw in Section 5.2 this implies that the conditional measures μ_x^H exist for every α-invariant probability measure μ on X. We study in this section the maximal subgroup H_x of H that leaves μ_x^H invariant by multiplication from the left (or right).

Definition 6.1: A closed subgroup $H' \subseteq H = \exp \mathfrak{h}$ of an $\mathbf{m}$-stable subgroup H allows a **weight decomposition** if $H' = \exp \mathfrak{h}'$ for some $\mathfrak{h}' \subseteq \mathfrak{h}$ with the following properties:

(1) $[\mathfrak{h}', \mathfrak{h}'] \subseteq \mathfrak{h}'$,
(2) $\mathfrak{h}' = \sum_{\sigma \in S} \mathfrak{h}' \cap \mathfrak{h}_\sigma$,
(3) $\mathfrak{h}' \cap \mathfrak{h}_\sigma = \sum_\lambda \mathfrak{h}' \cap \mathfrak{h}_\sigma^\lambda$ for all $\sigma \in S$, and
(4) $\mathfrak{h}' \cap \mathfrak{h}_\sigma^\lambda$ is a vector space over $\mathbb{Q}_\sigma$ for all Lyapunov weights λ and all $\sigma \in S$.

In other words $H' = \exp \mathfrak{h}'$ is an S-Lie subgroup of H with S-Lie algebra $\mathfrak{h}'$ such that $\mathfrak{h}'$ allows a decomposition into subspaces of Lyapunov weight spaces of $\mathfrak{h}$. Note that we do not require invariance of $\mathfrak{h}'$ under the adjoint action A, and that in the case of a real Lie group the above requirements show in particular that H' is connected.

PROPOSITION 6.2: *Let X, α, G_S, and A be as above, and let $H = \exp \mathfrak{h}$ be an $\mathbf{m}$-stable subgroup of G_S. For any α-invariant probability measure μ on X the conditional measures μ_x^H exist a.e. and the subgroup*

$$H_x = \{h \in H : (L_h)_* \mu_x^H = \mu_x^H\}$$

allows a weight decomposition. (The same holds similarly for the subgroup defined using right multiplication R_h.)

The following easy facts will be useful for the proof of the proposition.

LEMMA 6.3: *Let $h_j, h \in H$ and ν_j, ν be locally finite measures on H such that $h_j \to h$ and $\nu_j \to \nu$ for $j \to \infty$, where we use the weak* topology induced by all continuous functions with compact support. Then*

$$(L_{h_j})_* \nu_j \to (L_h)_* \nu \quad \text{for } j \to \infty,$$

where $L_h(g) = hg$ is left multiplication. (The same holds similarly for right multiplication $R_h(g) = gh$.) In particular, H_x as in Proposition 6.2 is closed.

LEMMA 6.4: *Let μ and H be as in Proposition 6.2. Then for a.e. $x \in X$ and all $\mathbf{m} \in \mathbb{Z}^k$ we have $H_{\alpha^{\mathbf{m}} x} = \theta^{\mathbf{m}}(H_x)$.*

Proof: Let $\mathbf{m} \in \mathbb{Z}^k$. The lemma follows from the relationship between μ_x^H and $\mu_{\alpha^{\mathbf{m}} x}^H$ in (5.6). Recall that in our situation $\theta_{\mathbf{m},x}^{U,V} = \theta^{\mathbf{m}}$ is a fixed automorphism of H, so that $\mu_{\alpha^{\mathbf{m}} x} = \theta_*^{\mathbf{m}} \mu_x$ a.e. ∎

LEMMA 6.5: *Any subgroup $H' \subseteq H$ satisfying (2)–(4) of Definition 6.1, also satisfies (1).*

Proof: Note first that (2)–(4) imply that $\mathfrak{h}' = \log(H')$ is an additive subgroup of $\mathfrak{h}$. Since $[\cdot,\cdot]$ is bilinear and any element of $\mathfrak{h}'$ can be written as a sum of elements in $\mathfrak{h}' \cap \mathfrak{h}_\sigma^\lambda$ for various $\sigma \in S$ and weights λ, it is enough to consider $u, v \in \mathfrak{h}'$ with $u \in \mathfrak{h}_\sigma^\xi$ and $v \in \mathfrak{h}_\sigma^\zeta$ for some weights ξ, ζ. Since H' is a subgroup, we have $u*v \in \mathfrak{h}'$. By (3.2) we can express $u*v$ as a combination of $u, v, [u,v], \ldots$, where each of these expressions belongs to a particular weight subspace $\mathfrak{h}_\sigma^\lambda$ as in Proposition 4.9. In particular, $[u,v] \in \mathfrak{h}_\sigma^{\xi+\zeta}$ and this term is the only one in that particular weight subspace (using that H is $\mathbf{m}$-stable). However, Definition 6.1 (3) and (4) now show that $[u,v] \in \mathfrak{h}'$. ∎

Proof of Proposition 6.2: Suppose H is $\mathbf{m}$-unstable for an $\mathbf{m} \in \mathbb{Z}^k$ such that additionally $\lambda_1(\mathbf{m}) \neq \lambda_2(\mathbf{m})$ for any two different weights λ_1, λ_2 of H. We define $\beta = \alpha^{\mathbf{m}}$; then it is enough to show the proposition for the $\mathbb{Z}$-action defined by β. We denote the corresponding map on H by θ, and the adjoint action by A. Weights can be identified with real numbers, and so $\mathfrak{h}^r$ will denote the weight space corresponding to $r \in \mathbb{R}$. Moreover, we let

$$\mathfrak{h}^I = \sum_{r \in I} \mathfrak{h}^r \quad \text{for any interval } I \subseteq \mathbb{R}^+.$$

By our choice of $\mathbf{m}$ we are still considering the same weight subspaces, in fact $\mathfrak{h}^\lambda = \mathfrak{h}^{\lambda(\mathbf{m})}$.

Let N be a null set such that μ_x^H is well defined and satisfies Lemma 6.4 for $x \notin N$. Let $K \subseteq X \setminus N$ be a compact set with $\mu(K) > 1 - \epsilon$ such that μ_x^H depends continuously on x within K (Luzin's theorem). By Poincaré recurrence there exists a set $K' \subseteq K$ of equal measure such that for all $x \in K'$ there exists a diverging sequence n_j with $\beta^{n_j} x \in K$ and $\beta^{n_j} x \to x$ for $j \to \infty$. We can require that this holds for some sequence $n_j \to \infty$ as well as for some sequence $n_j \to -\infty$ for $j \to \infty$. Since ϵ is arbitrary, it is enough to show the proposition for any $x \in K'$.

Let $\mathfrak{h}_{S_f} = \sum_{p \in S \setminus \{\infty\}} \mathfrak{h} \cap \mathfrak{g}_p$. Our first step towards the linear structure of $\mathfrak{h}_x = \log H_x$ is contained in the next lemma.

LEMMA 6.6: *Let $x \in K'$ and $r > 0$ be fixed. Then there exists a group homomorphism*

$$\psi_r : \operatorname{dom} \psi_r = (\mathfrak{h}_\infty^{[r,\infty)} + \mathfrak{h}_{S_f}) \to \mathfrak{h}_\infty^r$$

which is linear on $\mathfrak{h}_\infty^{[r,\infty)} = \mathfrak{h}^{[r,\infty)} \cap \mathfrak{g}_\infty$ *and has*

$$\ker \psi_r = (\mathfrak{h}_\infty^{(r,\infty)} + \mathfrak{h}_{S_f})$$

as its kernel, such that $\mathbb{R}\psi_r(v) \subseteq \mathfrak{h}_x$ *for all* $v \in \mathfrak{h}_x \cap \operatorname{dom} \psi_r$.

Proof: Let $n_j \to -\infty$ be such that $\beta^{n_j} x \in K$ and $\beta^{n_j} x \to x$ for $j \to \infty$. Recall that every β^{-1} uniformly contracts the H-leaves. We define $\varphi_j = M_j A^{n_j}$, where $M_j = \lfloor e^{-n_j r} \rfloor$ and A is the adjoint action to β. Let $V = \mathfrak{h}_\infty^{[r,\infty)} + \mathfrak{h}_{S_f}$. Then $\varphi_j|_V$ has only eigenvalues of absolute value less than or equal to one and its eigenvalues on $\mathfrak{h}_\infty^r$ are the only ones bounded away from zero. To see this, note first that the natural number M_j has norm less than or equal to one with respect to all the non-Archimedean norms. Therefore the eigenvalues of φ_j restricted to $\mathfrak{h}_{S_f}$ are at least as small as the ones of A^{n_j} and approach zero for $j \to \infty$. For the real part $\mathfrak{h}_\infty$ we have chosen M_j such that the eigenvalues have precisely the stated behavior. We assume without loss of generality that $\varphi_j \to \psi_r$ on $V = \operatorname{dom} \psi_r$, where ψ_r is a group homomorphism as in the lemma.

Suppose now $v \in \mathfrak{h}_x \cap D$ and $t \in \mathbb{R}$. Let $g = \exp(v) \in H_x$. Since $M_j \to \infty$ we can choose some $q_j \in \mathbb{Z}$ with $t = \lim(q_j/N_j)$. Define $v_j = q_j A^{n_j}(v)$, then $\exp(v_j) = \theta^{n_j}(g^{q_j}) \in H_{a^{n_k} x}$ by Lemma 6.4. Since $v_j = \frac{q_j}{M_j}\varphi_j(v) \to t\psi_r(v)$ and $\mu^H_{a^{n_j} x} \to \mu^H_x$ by construction of K, we conclude from Lemma 6.3 that $\exp(t\psi_r(v)) \in H_x$. ∎

LEMMA 6.7: *Let* $x \in K'$. *Then* $\mathfrak{h}_x \cap \mathfrak{h}_\infty^r$ *is a real vector space for all* $r > 0$, *and* $\mathfrak{h}_x = \sum_r \mathfrak{h}_x \cap \mathfrak{h}_\infty^r + \mathfrak{h}_x \cap \mathfrak{h}_{S_f}$.

Proof: Fix some $r > 0$ and let ψ_r be as in Lemma 6.6. Choose a maximal list $v_1, \ldots, v_d \in \mathfrak{h}_x \cap \operatorname{dom} \psi_r$ of vectors such that $\psi_r(v_i)$ for $i = 1, \ldots, d$ are linearly independent over $\mathbb{R}$. Let

$$W = \langle \psi_r(v_i) : i = 1, \ldots, d \rangle$$

denote the linear span. Since ψ_r has image in $\mathfrak{h}_\infty^r$ on which it is also injective, it follows that any $v \in \mathfrak{h}_x \cap \operatorname{dom} \psi_r$ can be expressed as $v = v_r + v'$ where $v_r \in W$ and $v' \in \ker \psi_r$. For otherwise we would have a list of $d+1$ vectors with linearly independent image.

We claim that $W \subseteq \mathfrak{h}_x$ is ψ_r-invariant. For invariance note that $\psi_r^2(v_i) \in \mathfrak{h}_x$, and therefore $\psi_r^2(v_i) \in W$ by the first paragraph. Since ψ_r is injective on $\mathfrak{h}_\infty^r$, this shows that $\psi_r(W) = W$. It remains to show that $W \subseteq \mathfrak{h}_x$. So let $v \in W$ and find $t_1, \ldots, t_d \in \mathbb{R}$ with

$$v = t_1\psi_r^2(v_1) + \cdots t_d\psi_r^2(v_d).$$

By Lemma 6.6 we already know $t_i\psi_r(v_i) \in \mathfrak{h}_x$. Since H_x is a group, we have $u = (t_1\psi_r(v_1)) * \cdots * (t_d\psi_r(v_d)) \in \mathfrak{h}_x$. By (3.2) and Proposition 4.9, $u = u_r + u_{>r}$ where

$$u_r = t_1\psi_r(v_1) + \cdots t_d\psi_r(v_d)$$

and $u_{>r} \in \mathfrak{h}_\infty^{(r,\infty)}$. By Lemma 6.6, $v = \psi_r(u_r) = \psi_r(u) \in \mathfrak{h}_x$. This proves the claim and the first statement of the lemma.

For the second statement we show by induction that for all $r > 0$

$$\mathfrak{h}_x \cap \operatorname{dom}\psi_r = \sum_{s\in[r,\infty)} \mathfrak{h}_x \cap \mathfrak{h}_\infty^s + \mathfrak{h}_x \cap \mathfrak{h}_{S_f}. \tag{6.1}$$

For large enough r this is trivial because then $\operatorname{dom}\psi_r = \mathfrak{h}_{S_f}$. So suppose for the inductive step that $v \in \mathfrak{h}_x \cap \operatorname{dom}\psi_r$. Then we can decompose $v = v_r + \sum_{s>r} v_s + v_{S_f}$ according to the weights for the real part and the remaining non-Archimedean parts. The first two paragraphs show that $v_r \in W \subseteq \mathfrak{h}_x$. We need to show that $v_s, v_{S_f} \in \mathfrak{h}_x$ for all $s > r$. Since H_x is a group, we have $u = v*(-v_r) = v - v_r - \frac{1}{2}[v, v_r] + \cdots \in \mathfrak{h}_x$ by (3.2). Note that $u = \sum_{s>r} u_s + v_{S_f}$ already satisfies (6.1) by the inductive assumptions. This shows immediately that $v_{S_f} \in \mathfrak{h}_x$. Let $s > r$. Then

$$u_s = v_s - \frac{1}{2}[v_{s-r}, v_r] + \cdots \in \mathfrak{h}_x \cap \mathfrak{h}_\infty^s \tag{6.2}$$

by Proposition 4.9, where all other terms are $[\cdot,\cdot]$-polynomials in v_r and maybe several v_t for $t \in (r,s)$. Therefore $v_s = u_s \in \mathfrak{h}_x$ when s is the smallest weight bigger than r. Suppose we already know $v_t \in \mathfrak{h}_x$ for all $t \in (r,s)$. Then $v_r * v_t * (-v_r) \in \mathfrak{h}_x$ and the inductive assumptions shows $[v_r, v_t] \in \mathfrak{h}_x$ (just as in the proof of Lemma 6.5). Therefore, u_s and all the additional terms on the right of (6.2) belong to the vector space $\mathfrak{h}_x \cap \mathfrak{h}_\infty^s$. We conclude that $v_s \in \mathfrak{h}_x$ for all s. For small enough r the second statement of the lemma is exactly (6.1). ∎

Lemma 6.7 already shows that $\mathfrak{h}_x$ satisfies Definition 6.1 (3)–(4) hold for $\sigma = \infty$ and moreover that $\mathfrak{h}_x = \mathfrak{h}_x \cap \mathfrak{h}_\infty + \mathfrak{h}_x \cap \mathfrak{h}_{S_f}$. For Definition 6.1 (2) we still need to show that

$$\mathfrak{h}_x \cap \mathfrak{h}_{S_f} = \sum_{p\in S_f} \mathfrak{h}_x \cap \mathfrak{h}_p.$$

So suppose $v \in \mathfrak{h}_x \cap \mathfrak{h}_{S_f}$; then $Mv = \log((\exp v)^M) \in \mathfrak{h}_x \cap \mathfrak{h}_{S_f}$ for every integer M. Let $p \in S_f$; then we can find a sequence $M_{p,j}$ for $j = 1, \dots$ such that $M_{p,j} \to 1$ with respect to $|\cdot|_p$ but $M_{p,j} \to 0$ with respect to $|\cdot|_q$ for any $q \in S_f \setminus \{p\}$. Since $\mathfrak{h}_x$ is closed, $M_{p,j} v \to v_p \in \mathfrak{h}_x \cap \mathfrak{h}_p$. Taking the sum we find that $v = \sum_{p \in S_f} v_p \in \mathfrak{h}_x \cap \mathfrak{h}_{S_f}$ decomposes as claimed.

Similar to the real case we will establish Definition 6.1 (3)–(4) for $p \in S_f$ in two lemmata.

LEMMA 6.8: *Let $p \in S_f$, $x \in K'$, and $r > 0$ be fixed. Then there exists a $\mathbb{Q}_p$-linear map*

$$\psi_r\colon \operatorname{dom} \psi_r = \mathfrak{h}_p^{(0,r]} \to \mathfrak{h}_p^r$$

with kernel

$$\ker \psi_r = \mathfrak{h}_p^{(0,r)}$$

such that $\mathbb{Q}_p \psi_r(v) \subseteq \mathfrak{h}_x$ for all $v \in \mathfrak{h}_x \cap \operatorname{dom} \psi_r$.

Proof: By definition of K' there exists a sequence $n_j \to \infty$ such that $\beta^{n_j} x \in K$ and $\beta^{n_j} x \to x$ for $j \to \infty$. We choose some integer sequence M_j such that the eigenvalues of $p^{M_j} A^{n_j}$ restricted to $\mathfrak{h}_p^r$ stay bounded and bounded away from zero. By choosing a subsequence if necessary we can assume that these eigenvalues have a nonzero limit. Note that the eigenvalues of A^{n_j} restricted to $\mathfrak{h}_p^{(0,r)}$ grow at a smaller rate. We denote the limit of the restriction of $p^{M_j} A^{n_j}$ to $\mathfrak{h}_p^{(0,r]}$ by ψ_r.

Suppose $v \in \mathfrak{h}_x \cap \mathfrak{h}_p^{(0,r]}$ and $M \in \mathbb{Z}$. Then $g = \exp(v) \in H_x$ and $\theta^{n_j}(g) \in H_{\beta^{n_j} x}$ by Lemma 6.4. Since

$$(\theta^{n_j}(g))^{p^{M_j - M}} \to \exp(p^{-M} \psi_r(v)) \quad \text{for } j \to \infty,$$

we get $p^{-M} \psi_r(v) \in \mathfrak{h}_x$ from Lemma 6.3. Therefore $m p^{-M} \psi_r(v) \in \mathfrak{h}_x$ for all $m \in \mathbb{Z}$, which implies that $\mathbb{Q}_p \psi_r(v) \subseteq \mathfrak{h}_x$. ∎

LEMMA 6.9: *Let $x \in K'$ and $p \in S_f$. Then $\mathfrak{h}_x \cap \mathfrak{h}_p^r$ is a vector space over $\mathbb{Q}_p$ for all $r > 0$, and $\mathfrak{h}_x \cap \mathfrak{h}_p = \sum_r \mathfrak{h}_x \cap \mathfrak{h}_p^r$.*

Proof: We prove by induction that $\mathfrak{h}_x \cap \mathfrak{h}_p^s$ is a vector space for $s \geq r$ and that for any $v = \sum_{s>0} v_s \in \mathfrak{h}_x$ with $v_s \in \mathfrak{h} \cap \mathfrak{h}_p^s$ we have in fact $v_s \in \mathfrak{h}_x$ for $s \geq r$. For large enough r this statement is vacuous, and for $r = 0$ it is a reformulation of the lemma. So it is enough to prove the inductive step.

Let $v_1, \dots, v_d \in \mathfrak{h}_x \cap \mathfrak{h}_p^r$ be a maximal set of linearly independent vectors, and let $t_1, \dots, t_d \in \mathbb{Q}_p$. By Lemma 6.8, $t_i \psi_r(v_i) \in \mathfrak{h}_x$ for $i = 1, \dots, d$, and so

$w = (t_1\psi_r(v_1)) * \cdots * (t_d\psi_r(v_d)) \in \mathfrak{h}_x$. Clearly $w = \sum_{s\geq r} w_s$ with $w_s \in \mathfrak{h}_p^s$ for $s \geq r$. By the inductive assumption we know $w_s \in \mathfrak{h}_x$ for $s > r$. Suppose $t > r$ is the smallest weight with $w_t \neq 0$; then $w' = w * (-w_t) \in \mathfrak{h}_x$ satisfies that $w'_r = w_r$ and $w'_t = 0$. Continuing like that we finally show that $w_r = t_1\psi_r(v_1) + \cdots + t_d\psi_r(v_d) \in \mathfrak{h}_x$. Since $v_1, \ldots, v_d$ is a maximal list of linearly independent vectors, we conclude that $\psi_r(v_1), \ldots, \psi_r(v_d)$ must have the same linear span over $\mathbb{Q}_p$ and that $\mathfrak{h}_x \cap \mathfrak{h}_p^r$ is a d-dimensional ψ_r-invariant vector space over $\mathbb{Q}_p$.

To conclude the proof we need to show that $v_r \in \mathfrak{h}_x$ whenever $v = \sum_{s>0} v_s \in \mathfrak{h}_x$. Since we already know $v_s \in \mathfrak{h}_x$ for all $s > r$, we can show similar to the above that $w = \sum_{s\in(0,r]} v_s \in \mathfrak{h}_x$. By Lemma 6.8, $\psi_r(w) = \psi_r(v_r) \in \mathfrak{h}_x$, and invariance of $\mathfrak{h}_x \cap \mathfrak{h}_p^r$ implies that $v_r \in \mathfrak{h}_x$ as required. ∎

So we have shown that $\mathfrak{h}_x$ satisfies (2)–(4) of Definition 6.1 for any $x \in K'$. Together with Lemma 6.5 this concludes the proof of Proposition 6.2. ∎

7. Stability of conditional measures and product measures

We return to the more general setup of (S, H)-spaces.

Definition 7.1: Let α be a $\mathbb{Z}^k$-action on an (S, H)-space X that preserves the S-leaves. Then the linear functional λ is a **coarse Lyapunov weight** for the (S, H)-space (with respect to α) if there exists $c_2 \geq c_1 > 0$ such that for $x \in X$, $\mathbf{n} \in \mathbb{Z}^k$, $U \in \mathfrak{T}(x)$, $V \in \mathfrak{T}(\alpha^{\mathbf{n}}x)$, and $s, s' \in S$ we have

$$d(\theta_{\mathbf{n},x}^{U,V} s, \theta_{\mathbf{n},x}^{U,V} s') \leq \max(e^{c_1\lambda(\mathbf{n})}, e^{c_2\lambda(\mathbf{n})}) d(s, s') \quad \text{for every } \mathbf{n} \in \mathbb{Z}^k,$$

where $\theta_{\mathbf{n},x}^{U,V}$ is the homeomorphism of S as in (5.5) for the element $\alpha^{\mathbf{n}}$ of the action.

We give some general comments about coarse Lyapunov weights; if $\lambda(\mathbf{n}) < 0$, Definition 7.1 shows that $\alpha^{\mathbf{n}}$ contracts the S-leaves at least by the factor $e^{c_1\lambda(\mathbf{n})}$. If, on the other hand, $\lambda(\mathbf{n}) > 0$, $\alpha^{\mathbf{n}}$ expands the S-leaves at most by the factor $e^{c_2\lambda(\mathbf{n})}$. Using this also for $\alpha^{-\mathbf{n}}$ in both cases, it follows easily that we also have a lower bound, i.e. the above is equivalent to

$$\min(e^{c_1\lambda(\mathbf{n})}, e^{c_2\lambda(\mathbf{n})}) d(s, s') \leq d(\theta_{\mathbf{n},x}^{U,V} s, \theta_{\mathbf{n},x}^{U,V} s') \leq \max(e^{c_1\lambda(\mathbf{n})}, e^{c_2\lambda(\mathbf{n})}) d(s, s').$$

If there exists some $\mathbf{n} \in \mathbb{Z}^k$ with $\lambda(\mathbf{n}) = 0$, it follows that $\alpha^{\mathbf{n}}$ acts isometrically on the (S, H)-leaves. Since we only consider a $\mathbb{Z}^k$-action, there might be no such element.

We can consider nonzero elements $\mathbf{w} \in \mathbb{R}^k$ as asymptotic directions in $\mathbb{Z}^k$ by using sequences $\mathbf{n}_j \in \mathbb{Z}^k$ that stay close to $\mathbb{R}^+\mathbf{w}$. We now extend [6, Prop. 5.1] and [21, Lemma 6.2] accordingly.

Definition 7.2: Let λ be a coarse Lyapunov weight for the $\mathbb{Z}^k$-action α on the (S, H)-space X, and let $\mathbf{w} \in \mathbb{R}^k$ with $\lambda(\mathbf{w}) = 0$. Then x' **w-asymptotically belongs** to the S-leaf of $x \in U$ if there exists some $s_0 \in S$ such that $y = t_U(s_0, x)$ satisfies: for every diverging sequence $\mathbf{n}_j \in \mathbb{Z}^k$ with bounded distance to $\mathbb{R}^+\mathbf{w}$ such that $\alpha^{\mathbf{n}_j} x$ is relatively compact we have $d(\alpha^{\mathbf{n}_j} y, \alpha^{\mathbf{n}_j} x') \to 0$ as $j \to \infty$; see Figure 1.

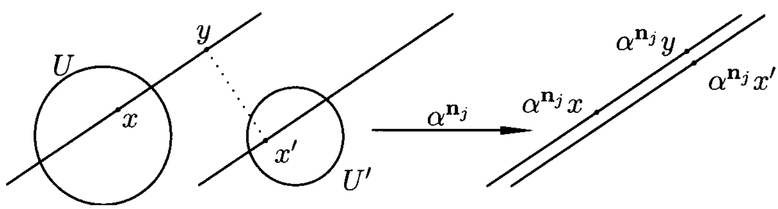

Figure 1. The points y and x' approach each other when $\alpha^{\mathbf{n}_j}$ is applied.

PROPOSITION 7.3: *Let λ be a coarse Lyapunov weight for the $\mathbb{Z}^k$-action α on the (S, H)-space X, and let $\mathbf{w} \in \mathbb{R}^k$ with $\lambda(\mathbf{w}) = 0$. Suppose μ is an α-invariant probability measure with conditional measures $\mu_{x,S}^U$ for $U \in \mathfrak{T}(x)$. Then there exists a null set N such that for $U, U' \in \mathfrak{T}$, $x \in U \setminus N$, and $x' \in U' \setminus N$ the conditional measures for the S-leaves satisfy*

$$\mu_{x,S}^U \propto (\Phi)_* \mu_{x',S}^{U'} \quad \text{for some homeomorphism } \Phi \tag{7.1}$$

whenever x' $\mathbf{w}$-asymptotically belongs to the S-leaf of x. (Here, Φ in general depends on x and x'.)

Additionally, if a homeomorphism $\Psi: S \to S$ satisfies for any $s \in S$ that

$$d(\alpha^{\mathbf{n}_j} \circ t_U(\Psi(s), x), \alpha^{\mathbf{n}_j} \circ t_{U'}(s, x')) \to 0 \quad \text{for } j \to \infty \tag{7.2}$$

along any diverging sequence $\mathbf{n}_j \in \mathbb{Z}^k$ with bounded distance to $\mathbb{R}^+\mathbf{w}$ such that $\alpha^{\mathbf{n}_j} x$ is relatively compact, then $\Phi = \Psi$ satisfies (7.1).

Actually, it is possible to find an isometry $\Phi \in \mathrm{Isom}(S)$ that satisfies (7.1). We will indicate at the end of the proof how to show this extension.

Proof: We first show that there exists for every $\epsilon > 0$ a set X_ϵ with $\mu(X_\epsilon) > 1 - 2\epsilon$ on which the proposition holds. Since $\mu_{x,S}^U$ depends measurably on $x \in U$ (for every $U \in \mathfrak{T}$), there exists a compact set K of measure $\mu(K) > 1 - \epsilon$ on which these functions are continuous by Luzin's theorem. Without loss of generality assume that (5.3), Proposition 5.6 (3), and (5.6) hold for all $x \in K$ and $\alpha^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^k$.

We wish to apply the "ergodic theorem along the direction $\mathbf{w}$". Unless $\mathbb{R}\mathbf{w} \cap \mathbb{Z}^k \neq \{\mathbf{0}\}$ we need to use the suspension flow for this. Let $X_s = X \times [0,1)^k$, $\alpha_s^{\mathbf{v}}(x, \mathbf{u}) = (\alpha^{\mathbf{n}} x, \mathbf{u} + \mathbf{v} - \mathbf{n})$ where $\mathbf{u} \in \mathbb{R}^k$, $(x, \mathbf{u}) \in X_s$, and $\mathbf{n} \in \mathbb{Z}^k$ is chosen such that $\mathbf{u} + \mathbf{v} - \mathbf{n} \in [0,1)^k$. It is easy to check that α_s is a measurable $\mathbb{R}^k$-flow on X_s which preserves $\mu_s = \mu \times \lambda_{[0,1)^k}$. Let $K_s = K \times [0,1)^k$. By the ergodic theorem the function

$$f_s(x, \mathbf{u}) = \lim_{m \to \infty} \frac{1}{m} \sum_{j=0}^{m-1} 1_{K_s}(\alpha_s^{j\mathbf{w}}(x, \mathbf{u}))$$

exists for a.e. $(x, \mathbf{u})$ and satisfies $\int f_s \mathrm{d}\mu_s = \mu(K) > 1 - \epsilon$.

We are going back to the space X. Clearly we can fix some $\mathbf{u} \in [0,1)^k$ such that $f(x) = f_s(x, \mathbf{u})$ exists for a.e. $x \in X$ and $\int f \mathrm{d}\mu > 1 - \epsilon$. Let $\mathbf{n}_j \in \mathbb{Z}^k$ be the unique sequence such that $\mathbf{u} + j\mathbf{w} - \mathbf{n}_j = \mathbf{v}_j \in [0,1)^k$ for all j. From $K_s = K \times [0,1)^k$ and the above we get that

$$f(x) = \lim_{m \to \infty} \frac{1}{m} \sum_{j=0}^{m-1} 1_K(\alpha^{\mathbf{n}_j}(x))$$

exists for a.e. $x \in X$. Furthermore, the sequence $\mathbf{n}_j \in \mathbb{Z}^k$ diverges but does not leave a certain tube around $\mathbb{R}\mathbf{w}$, in fact $\|\mathbf{n}_j - j\mathbf{w}\|_\infty = \|\mathbf{u} - \mathbf{v}_j\|_\infty \leq 1$ and $|\lambda(\mathbf{n}_j)| \leq \|\lambda\|$ for j. Let $X_\epsilon = \{x : f(x) \text{ exists and } f(x) > \frac{1}{2}\}$. Then

$$1 - \epsilon < \int f \mathrm{d}\mu \leq \frac{1}{2}(1 - \mu(X_\epsilon)) + \mu(X_\epsilon) = \frac{1}{2} + \frac{1}{2}\mu(X_\epsilon)$$

and so $\mu(X_\epsilon) > 1 - 2\epsilon$. We can again assume that (5.6) holds for all $x \in X_\epsilon$ and $\alpha^{\mathbf{n}}$ for $\mathbf{n} \in \mathbb{Z}^k$.

Suppose now $x, x' \in X_\epsilon$ satisfy the assumption of the proposition. Since the asymptotic frequencies of the event $\alpha^{\mathbf{n}_j}(x) \in K$ is given by $f(x)$ and since $f(x), f(x') > \frac{1}{2}$, there exists a common subsequence of $\mathbf{n}_j$ (again denoted by $\mathbf{n}_j$) so that $\alpha^{\mathbf{n}_j} x, \alpha^{\mathbf{n}_j} x' \in K$. By compactness we find another subsequence such that

$$x_j = \alpha^{\mathbf{n}_j} x \to z \in K,$$
$$x_j' = \alpha^{\mathbf{n}_j} x' \to z' \in K \quad \text{for } j \to \infty.$$

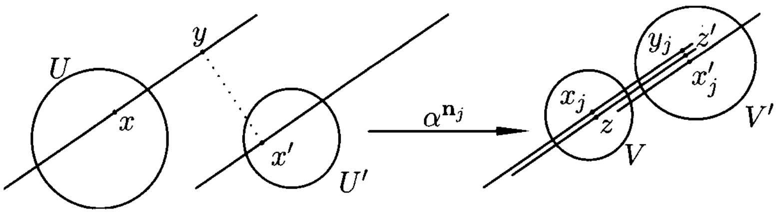

Figure 2. After choosing a subsequence we get two limit points z, z' on the same T-leaf.

Suppose $V \in \mathfrak{T}(z)$ and $V' \in \mathfrak{T}(z')$, then $x_j \in V$, $y_j, x'_j \in V'$ for large enough j as in Figure 2. The construction of K implies that

$$\mu^V_{x_j,S} \to \mu^V_{z,S} \quad \text{for } j \to \infty \tag{7.3}$$

and a similar statement for x'_j and z'. Let $\theta_j = \theta^{U,V}_{\mathbf{n}_j,x}$ and $\theta'_j = \theta^{U',V'}_{\mathbf{n}_j,x'}$ be as in (5.5). By (5.6)

$$\alpha^{\mathbf{n}_j} \circ t_U(\cdot, x) = t_V(\cdot, x_j) \circ \theta_j, \tag{7.4}$$

$$(\theta_j)_* \mu^U_{x,S} \propto \mu^V_{x_j,S}, \tag{7.5}$$

and similarly for x'_j and θ'_j. By Definition 7.1 and our estimate $|\lambda(\mathbf{n}_j)| \leq \|\lambda\|$ the maps θ_j satisfy

$$e^{-c_2\|\lambda\|} d(s_1, s_2) \leq d(\theta_j(s_1), \theta_j(s_2)) \leq e^{c_2\|\lambda\|} d(s_1, s_2).$$

Therefore, we can pass to another subsequence and assume that $\theta_j \to \theta$ and $\theta'_j \to \theta'$ for $j \to \infty$ and two homeomorphisms θ, θ' which satisfy the above estimate as well. (The map θ can be thought of as a map from the S-leaf through x to the S-leaf through z; see Figure 2.) Therefore (7.3) and (7.5) show that

$$\theta_* \mu^U_{x,S} \propto \mu^V_{z,S} \quad \text{and} \quad \theta'_* \mu^{U'}_{x',S} \propto \mu^{V'}_{z',S}, \tag{7.6}$$

since the proportionality constant in (7.5) has to converge to some nonzero real number.

By the assumption on x, x' there exists some $s_0 \in S$ such that $y = t_U(s_0, x)$ satisfies (for the chosen subsequence)

$$y_j = \alpha^{\mathbf{n}_j} y \to z'.$$

Continuity of the map t_V and (7.4) imply

$$y_j = \alpha^{\mathbf{n}_j} y = t_V(\theta_j s_0, x_j) \to t_V(\theta s_0, z),$$

and therefore $z' = t_V(\theta s_0, z)$. By assumption, Proposition 5.6 (3) holds for all points in K. Since $z, z' \in K$ there exists some $\phi \in H_S$ such that

$$t_V(\cdot, z) \circ \phi = t_{V'}(\cdot, z') \quad \text{and} \quad \mu^V_{z,S} = \phi_* \mu^{V'}_{z',S}. \tag{7.7}$$

Together with (7.6) this shows that the main assertion (7.1) of Proposition 7.3 holds for $\Phi = \theta^{-1} \circ \phi \circ \theta'$, which satisfies

$$e^{-2c_2\|\lambda\|} d(s_1, s_2) \le d(\Phi(s_1), \Phi(s_2)) \le e^{2c_2\|\lambda\|} d(s_1, s_2). \tag{7.8}$$

Suppose Ψ satisfies the assumptions stated in the proposition and let $\mathbf{n}_j$ be the sequence constructed above. Using (7.4) we can reformulate (7.2) to

$$d(t_V(\theta_j \circ \Psi(s), x_j), t_{V'}(\theta'_j(s), x'_j)) \to 0 \quad \text{for } j \to \infty.$$

Since $\theta_j \to \theta$ and $\theta'_j \to \theta'$ for $j \to \infty$, we conclude that $t_V(\theta \circ \Psi(s), z) = t_{V'}(\theta'(s), z')$ for any $s \in S$. Note that Φ satisfies the same equation by definition of Φ and ϕ in (7.7). By assumption, (5.3) holds for $z \in K$, and therefore we conclude that $\Phi = \Psi$.

Since the above holds for all $\epsilon > 0$, we can find an increasing sequence $X_{1/n}$ such that $X \setminus \bigcup_n X_{1/n}$ is a null set and the proposition follows.

We now show that (7.1) also holds for an isometry. Let $V \subset \mathbb{R}^k$ be the smallest rational subspace that contains $\mathbf{w}$. Choose a basis of V consisting of elements of $\mathbb{Z}^k \cap V$ that are close to $\mathbb{R}\mathbf{w}$. Then the restriction of α to $\mathbb{Z}^k \cap V$ defines a new $\mathbb{Z}^{k'}$-action α' where $k' = \dim(V)$. It follows from the definition that the restriction λ' of λ to V is a coarse Lyapunov weight for α' (with the same constants). However, by choosing the basis close to $\mathbb{R}\mathbf{w}$ we can achieve that $\|\lambda'\| < 1/\ell$ for some given $\ell \ge 1$. (Here we use the maximum norm on V induced by the chosen basis and the dual norm on the space of linear functions on V.) Applying the above proof to α' gives a homeomorphism Φ_ℓ that satisfies (7.1) and an improved version of (7.8). Varying ℓ and choosing a subsequence we find the isometry $\Phi = \lim_\ell \Phi_\ell$ that satisfies (7.1). ∎

The stability of the conditional measure in Proposition 7.3 implies already that the conditional measures are product measures. For this we will use the following general situation of a foliated space whose leaves are product spaces.

Definition 7.4: Let $S, T, S \times T$ be locally compact second countable metric spaces such that the metric on $S \times T$ induces the product topology and its restriction to $S \times \{e\}$ and $\{e\} \times T$ gives the metric on S and T. Let $H \subseteq \mathrm{Isom}(S \times T)$ be such that all $\phi \in H$ have the form $\phi = \phi_S \times \phi_T$ for homeomorphisms $\phi_S \colon S \to S$ and $\phi_T \colon T \to T$ (which are not assumed to be isometries). Then we say H is a **group of product maps**.

An important case is the case where H is a group with two subgroups S and T such that T is normal in H, $T \cap S = \{e\}$ and $H = ST \simeq S \times T$. If the additional requirements on the metric are satisfied, this gives an example of a group of product maps. In fact, the right translation

$$R_{s_0 t_0}(st) = (st)(s_0 t_0) = (s s_0)(s_0^{-1} t s_0 t_0) \in ST$$

for every $s, s_0 \in S$ and $t, t_0 \in T$ is in this case a product of two maps.

Similar to the case studied in [21, Sect. 6], an $(S \times T, H)$-space where H is a group of product maps induces an $(S, \mathrm{Isom}(S))$-space and an $(T, \mathrm{Isom}(T))$-space in a natural way: we use the same atlas $\mathfrak{T}$ and the restrictions of the parametrization maps $t_U(\cdot, x)$ to S and T respectively.

We say that α **contracts the T-leaves along $\mathbf{w}$** if

$$d(\theta^{U,V}_{x,\mathbf{n}_j} t_1, \theta^{U,V}_{x,\mathbf{n}_j} t_2) \to 0 \quad \text{for } j \to \infty \tag{7.9}$$

for every $t_1, t_2 \in T$ and every diverging sequence $\mathbf{n}_j$ with bounded distance to $\mathbb{R}^+ \mathbf{w}$ (and $U, V, \theta^{U,V}_{x,\mathbf{n}_j}$ as in (5.5)).

THEOREM 7.5: *Let X be an $(S \times T, H)$-space where H is a group of product maps and let α be a $\mathbb{Z}^k$-action on X preserving the $(S \times T, H)$-leaves as well as the two induced foliations. Suppose λ is the Lyapunov weight to the $(S, \mathrm{Isom}(S))$-space and suppose $\mathbf{w} \in \mathbb{R}^k$ satisfies $\lambda(\mathbf{w}) = 0$ and that the $(T, \mathrm{Isom}(T))$-leaves are contracted along $\mathbf{w}$. Then for every α-invariant probability measure μ with conditional measures $\mu^U_{x, S \times T}$ for $U \in \mathfrak{T}(x)$ we have for almost every $x \in X$*

$$\mu^U_{x, S \times T} \propto \mu^U_{x,S} \times \mu^U_{x,T}.$$

Our first step towards Theorem 7.5 is to study the relationship between the conditional measures for the $(S \times T, H)$-leaves and the induced leaf structure.

LEMMA 7.6: *Let $X, S \times T, H$ and μ be as in Theorem 7.5. Let $r_S, r_T > 0$, and choose $r > r_S > 0$ such that $Q = Q_{r_S, r_T} = B^S_{r_S} \times B^T_{r_T} \subseteq B^{S \times T}_r$. Then for a.e. x_0 there exists σ-rings $\mathcal{A}^\times = \mathcal{A}(S \times T)$ and $\mathcal{A}(S)$ with common maximal element*

A such that $\mathcal{A}^\times$ is an $r, S\times T$-flower with open center C containing x_0, $\mathcal{A}(S)$ is countably generated, and the $\mathcal{A}(S)$-atoms are open S-plaques. For some fixed $x\in C$ and any $(s',t')\in Q$ let $f(s',t') = t_U((s',t'),x)$. Then we have

$$[f(s',t')]_{\mathcal{A}(S)} \cap f(Q) = f(B^S_{r_S}\times\{t'\}) \tag{7.10}$$

and

$$\mu_x^{\mathcal{A}^\times}|_{f(Q)} = \int_{f(Q)} (\mu_{x'}^{\mathcal{A}(S)}(f(Q)))^{-1}\mu_{x'}^{\mathcal{A}(S)}|_{f(Q)}\mathrm{d}\mu_x^{\mathcal{A}^\times}(x') \tag{7.11}$$

for a.e. $x\in C$; see Figure 3. Furthermore, this is the decomposition of $\mu_x^{\mathcal{A}^\times}|_{f(Q)}$ into conditional measures with respect to $\mathcal{A}(S)$.

For the proof that the conditional measure is a product measure the above lemma will be useful, since it allows us to replace the atom $[x]_{\mathcal{A}^\times}$, whose shape is in general unknown, by the rectangular set $f(Q)$.

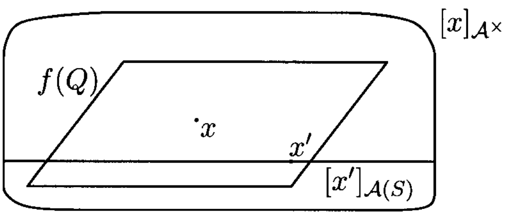

Figure 3. For elements $x\in C$ of the common center we have $f(Q)\subseteq [x]_{\mathcal{A}^\times}\subseteq A$, and for $x' = f(s',t')$ the $\mathcal{A}(S)$-atom $[x]_{\mathcal{A}(S)}$ containing x' intersects $f(Q)$ in $f(B^S_{r_S}\times\{t'\})$.

Proof: Let $U\in\mathfrak{T}$. Recall that we assume that $t_U(\cdot,x_0)$ is injective for a.e. $x_0\in U$ (by our assumption that the conditional measures exist). By [21, Cor. 3.5] there exists an $r, S\times T$-flower $\mathcal{A}^\times = \mathcal{A}(S\times T)$ with maximal element A and center C. In fact the maximal element is $A = t_U(B_r^{S\times T}, B_\epsilon(x_0))$ and the center is $C = B_\epsilon(x_0)$ for any small enough $\epsilon>0$. We choose ϵ small enough such that Lemma 5.2 holds for the $(S\times T, H)$-spaces structure, i.e.

$$t_V(\cdot,y) \text{ is injective on } B_{19r}^{S\times T} \text{ for any } y\in A \text{ and } V\in\mathfrak{T}(y). \tag{7.12}$$

We now construct the countably generated σ-ring $\mathcal{A}(S)$ with the same maximal element A as above. For $y_0\in A$, $V\in\mathfrak{T}(y_0)$, and $\eta>0$ we define

$$D_\eta(y_0) = t_V(B^S_{4r}, B_\eta(y_0)\cap A)\cap A.$$

We claim that

$$B^S_{4r}(z) \cap A \subset D_\eta(y_0) \quad \text{for any } z \in D_\eta(y_0). \tag{7.13}$$

To see this, note first that

$$B^S_{4r}(z) \subset B^S_{8r}(y) \tag{7.14}$$

for some $y \in B_\eta(y_0) \cap A$ by the triangle inequality and the definition of $D_\eta(y_0)$. However, this implies $B^S_{4r}(z) \cap A \subset B^{S\times T}_{4r}(y) \cap A$ by Lemma 5.2 (1). By (7.14) and the injectivity statement in (7.12) this is equivalent to $B^S_{4r}(z) \cap A \subset B^S_{4r}(y) \cap A \subset D_\eta(y_0)$.

Now choose some countable dense sequence $y_m \in A$ and define $\mathcal{A}(S)$ to be the σ-ring generated by the sets $D_{1/n}(y_m)$ for $n, m \geq 1$. Then the maximal element of $\mathcal{A}(S)$ is A. For any $z \in A$ we can choose a sequence y_{m_n} such that $d(z, y_{m_n}) < 1/n$ for all n. Therefore, $D = \bigcap_n D_{1/n}(y_{m_n})$ contains z and by (7.13) also $B^S_{4r}(z) \cap A$. On the other hand, for any $x \in D$ there exist sequences $y'_n \to z$ and $s_n \in B^S_{4r}$ with $x = t_V(s_n, y'_n)$, where we assume $z, y'_n \in V \in \mathfrak{T}$. Choosing a converging subsequence we find $x = t_V(s, z) \in A$ for some $s \in B^S_{5r}$. Applying Lemma 5.2 again, it follows that $x \in B^S_{4r}(z)$ and $D = B^S_{4r}(z) \cap A$. Therefore, the $\mathcal{A}(S)$-atom containing z is

$$[z]_{\mathcal{A}(S)} = B^S_{4r}(z) \cap A = A \cap t_V(B^S_{4r} \times \{e\}, z). \tag{7.15}$$

In fact, $[z]_{\mathcal{A}(S)} \subset D$ but the atom cannot be smaller by (7.13). This shows that $\mathcal{A}(S)$ is a countable generated σ-ring that has open S-plaques as atoms. Replacing $\mathcal{A}(S)$ by $\mathcal{A}(S) \vee \mathcal{A}^\times$ does not affect (7.15) since $[z]_{\mathcal{A}} = B^{S\times T}_{4r}(z) \cap A$ by Definition 5.5 (2), i.e. we can assume that $\mathcal{A}^\times \subset \mathcal{A}(S)$.

Let $x \in C = B_\epsilon(x_0)$ and $(s', t') \in B^{S\times T}_{4r}$. We define $x' = f(s', t') = t_U((s', t'), x)$ and choose some $U' \in \mathfrak{T}(x')$. Suppose $\phi \in H$ satisfies

$$f \circ \phi = t_U(\cdot, x) \circ \phi = t_{U'}(\cdot, x')$$

and $\phi((e, e)) = (s', t')$ as in Definition 5.1 (2). By Definition 7.4 we must have $\phi(S \times \{e\}) = \phi_S(S) \times \phi_T(e) = S \times \{t'\}$. We assume in the following that $x' = f(s', t') \in A$.

By Definition 5.5 (2) and (7.15) we have

$$\begin{aligned}
[x]_{\mathcal{A}^\times} &= A \cap f(B^{S\times T}_{4r}) = [x']_{\mathcal{A}^\times} \quad \text{and} \\
[x']_{\mathcal{A}(S)} &= A \cap t_{U'}(B^S_{4r} \times \{e\}, x') = A \cap t_U(\phi(B^S_{4r} \times \{e\}), x) \\
&= [x']_{\mathcal{A}^\times} \cap f \circ \phi(B^S_{4r} \times \{e\}) = [x]_{\mathcal{A}^\times} \cap f \circ \phi(B^S_{4r} \times \{e\}).
\end{aligned}$$

Assume now additionally that $(s',t') \in Q$. We are going to calculate the intersection of $[x']_{\mathcal{A}(S)} \cap f(Q)$. For this, note first that trivially

$$Q \cap \phi(B_{4r}^S \times \{e\}) \subset Q \cap (S \times \{t'\}) = B_{r_S}^S \times \{t'\}.$$

We claim that in fact

$$Q \cap \phi(B_{4r}^S \times \{e\}) = B_{r_S}^S \times \{t'\}.$$

So let $\tilde{s} \in B_{r_S}^S$; then $(s',t'), (\tilde{s},t') \in Q \subset B_r^{S\times T}$. Therefore, $d((s',t'),(\tilde{s},t')) < 2r$ and since ϕ is an isometry we find $\phi^{-1}(\tilde{s},t') \in B_{2r}^S \times \{e\}$, which shows the claim. Since x is in the center C of the $r, S \times T$-flower $\mathcal{A}^\times$ we also know that $f(Q) \subseteq [x]_{\mathcal{A}^\times}$. Therefore, and by (7.12),

$$\begin{aligned}
[x']_{\mathcal{A}(S)} \cap f(Q) &= f(Q) \cap f \circ \phi(B_{4r}^S \times \{e\}) \\
&= f(Q \cap \phi(B_{4r}^S \times \{e\})) \\
&= f(B_{r_S}^S \times \{t'\}).
\end{aligned}$$

Since $\mathcal{A}^\times \subseteq \mathcal{A}(S)$, the conditional measures $\mu_{x'}^{\mathcal{A}(S)}$ for $x' \in [x]_{\mathcal{A}^\times}$ are the conditional measures for $\mu_x^{\mathcal{A}^\times}$ with respect to $\mathcal{A}(S)$ (for μ-a.e. $x \in A$). In particular

$$\mu_x^{\mathcal{A}^\times} = \int_{[x]_{\mathcal{A}^\times}} \mu_{x'}^{\mathcal{A}(S)} \mathrm{d}\mu_x^{\mathcal{A}^\times}(x').$$

We take the restriction of both sides to $f(Q)$ and get

$$\begin{aligned}
\mu_x^{\mathcal{A}^\times}|_{f(Q)} &= \int_{[x]_{\mathcal{A}}} \mu_{x'}^{\mathcal{A}(S)}|_{f(Q)} \mathrm{d}\mu_x^{\mathcal{A}^\times}(x') \\
&= \int_{[x]_{\mathcal{A}}} \mu_{x'}^{\mathcal{A}(S)}|_{f(Q)} (\mu_{x'}^{\mathcal{A}(S)}(f(Q)))^{-1} \int 1_{f(Q)}(z) \mathrm{d}\mu_{x'}^{\mathcal{A}(S)}(z) \mathrm{d}\mu_x^{\mathcal{A}^\times}(x') \\
&= \int_{[x]_{\mathcal{A}}} \int 1_{f(Q)}(z) (\mu_z^{\mathcal{A}(S)}(f(Q)))^{-1} \mu_z^{\mathcal{A}(S)}|_{f(Q)} \mathrm{d}\mu_{x'}^{\mathcal{A}(S)}(z) \mathrm{d}\mu_x^{\mathcal{A}^\times}(x') \\
&= \int_{f(Q)} (\mu_z^{\mathcal{A}(S)}(f(Q)))^{-1} \mu_z^{\mathcal{A}(S)}|_{f(Q)} \mathrm{d}\mu_x^{\mathcal{A}^\times}(z).
\end{aligned}$$

Note in the second line that the term $(\mu_{x'}^{\mathcal{A}(S)}(f(Q)))^{-1}$ is only undefined when $\mu_{x'}^{\mathcal{A}(S)}|_{f(Q)}$ vanishes anyway. In the next line we used that $\mu_z^{\mathcal{A}(S)} = \mu_{x'}^{\mathcal{A}(S)}$ for $\mu_{x'}^{\mathcal{A}(S)}$-a.e. $z \in [x']_{\mathcal{A}(S)}$ ($\mu_x^{\mathcal{A}^\times}$-a.s.). And finally, we used that $\mu_{x'}^{\mathcal{A}(S)}$ are the conditional measures for $\mu_x^{\mathcal{A}^\times}$ and the σ-algebra $\mathcal{A}(S)$. ∎

The next lemma is the main step towards Theorem 7.5 and shows that the conditional measure $\mu_{x,S\times T}^U$ is the product measure of $\mu_{x,S}^U$ and some second measure that will be specified later.

LEMMA 7.7: *Under the assumptions of Theorem 7.5 there exists for almost every $x \in U$ a locally finite measure ν_x^U on T such that*

$$\mu_{x,S\times T}^U \propto \mu_{x,S}^U \times \nu_x^U.$$

Proof: Fix $r_S, r_T > 0$, and let $r > 0$ and Q be as in Lemma (7.6). Since X is second countable we can find countably many σ-rings as in Lemma (7.6) such that the union of their centers covers almost all of X (with respect to μ). Suppose N is a null set so that Lemma (7.6) holds for all $x \notin N$ and all of the countably many σ-rings constructed above.

By Proposition 5.6 (2)

$$t_U(\cdot,x)_*^{-1}\mu_x^{\mathcal{A}^\times} \propto \mu_{x,S\times T}^U|_{t_U(\cdot,x)^{-1}[x]_{\mathcal{A}^\times}} \quad \text{and} \tag{7.16}$$

$$t_{U'}(\cdot,x')_*^{-1}\mu_{x'}^{\mathcal{A}(S)} \propto \mu_{x',S}^{U'}|_{t_{U'}(\cdot,x')^{-1}[x']_{\mathcal{A}(S)}} \tag{7.17}$$

hold for a.e. $x \in U$, a.e. $x' \in U'$, all $U, U' \in \mathfrak{T}$, and all σ-rings constructed above. Enlarge N to a null set such that the above and Proposition 7.3 hold for $x, x' \in U \setminus N$ and the $(S, \mathrm{Isom}(S))$-space.

Let $\mathcal{A}^\times, \mathcal{A}(S)$ be two of the σ-rings with common center C. Let $x \in U \cap C \setminus N$ be fixed such that $\mu_x^{\mathcal{A}^\times}(N) = 0$ (which holds a.e. by the properties of the conditional measures). We write again $f(s,t) = t_U((s,t),x)$. The idea for the proof of the lemma is to start with (7.11) and use f_*^{-1} as in (7.16) to push this equality to $S \times T$. The difficulty with this is that we have to identify the measure $f_*^{-1}\mu_{x'}^{\mathcal{A}(S)}$ on $S \times T$. (Note that f uses x as the base point, so that we cannot apply (7.17) directly.)

Let $(s',t') \in S \times T$, $x' = f(s',t') = t_U((s',t'),x) \in U'$, and $y = t_U((s',e),x)$. Suppose $\phi \in H$ satisfies

$$f \circ \phi = t_U(\cdot,x) \circ \phi = t_{U'}(\cdot,x') \tag{7.18}$$

and $\phi((e,e)) = (s',t')$ as in Definition 5.1 (2). Then by Definition 7.4 we must have $\phi = \phi_S \times \phi_T$ and $\phi^{-1}(s',e) = (e,t_1)$ for some $t_1 \in T$, and so $y = t_{U'}((e,t_1),x')$. By (7.9) we have $d(\alpha^{\mathbf{n}_j}y, \alpha^{\mathbf{n}_j}x') \to 0$ as $j \to \infty$ for every diverging sequence $\mathbf{n}_j \in \mathbb{Z}^k$ with bounded distance to $\mathbb{R}^+\mathbf{w}$. We conclude that (7.1) holds for some $\Phi \in \mathrm{Isom}(S)$ whenever $x' = f(s',t') \notin N$; see Figure 4.

Let $\Psi = \phi_S\colon S \to S$. It follows similarly that $t_{U'}((s,e),x') = f \circ \phi(s,e) = t_U((\Psi(s),t'),x)$ and $t_U((\Psi(s),e),x)$ are in the same T-leaf. Since the T-leaves are contracted along the direction $\mathbb{R}^+\mathbf{w}$, this shows that Ψ satisfies (7.2) and therefore (7.1) holds for $\Phi = \Psi$. It follows that

$$f_*^{-1}\mu_{x'}^{\mathcal{A}(S)} = \phi_* t_{U'}(\cdot,x')_*^{-1}\mu_{x'}^{\mathcal{A}(S)} \quad \text{by (7.18)}$$

$$\propto \phi_*(\mu^{U'}_{x',S}|_{t_{U'}(\cdot,x')^{-1}[x']_{\mathcal{A}(S)}}) \quad \text{by (7.17)}$$

$$\propto ((\Psi,t')_*\mu^{U'}_{x',S})|_{\phi(t_{U'}(\cdot,x')^{-1}[x']_{\mathcal{A}(S)})}$$

$$\propto (\mu^{U}_{x,S} \times \delta_{t'}))|_{f^{-1}[x']_{\mathcal{A}(S)}} \quad \text{by (7.1)} \tag{7.19}$$

whenever $x' = f(s',t') \notin N$.

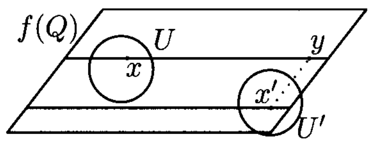

Figure 4. The points $y = f(s',e)$ and $x' = f(s',t')$ belong to the same T-leaf. Proposition 7.3 implies a strong coincidence of the conditional measures for the S-leaves.

We now apply f_*^{-1} to both sides of (7.11); on the left this leads to a measure proportional to $\mu^U_{x,S\times T}|_Q$ by (7.16). For the measure in the integrant on the right we use (7.19) and (7.10) to arrive at a measure proportional to $\mu^U_{x,S}|_{B^S_{r_S}} \times \delta_{t'}$. However, the measure in the integrant of (7.11) (where we include in the measure the normalizing factor) is a probability measure and therefore the proportionality constant is independent of $x' = f(s',t')$. This shows that

$$\mu^U_{x,S\times T}|_Q \propto \int_Q (\mu^U_{x,S}|_{B^S_{r_S}} \times \delta_{t'})\mathrm{d}\mu^U_{x,S\times T}(s',t').$$

Recall that $Q = Q_{r_S,r_T}$. For the measure ν_{r_S,r_T} defined by

$$\nu_{r_S,r_T}(A) = \mu^U_{x,S\times T}(B^S_{r_S} \times A) \quad \text{for a measurable } A \subseteq B^T_{r_T},$$

we have shown that

$$\mu^U_{x,S\times T}|_Q \propto \mu^U_{x,S}|_{B^S_{r_S}} \times \nu_{r_S,r_T}. \tag{7.20}$$

To conclude the proof of the lemma, we need to extend the above to $S \times T$. Suppose N is a null set such that (7.20) holds for all $r_S, r_T \in \mathbb{N}$ and all $x \in U\backslash N$. Suppose $x \in U\backslash N$. We check that the measures ν_{r_S,r_T} can be naturally extended to a measure ν. It is obvious from the above definition that $\nu_{r_S,r'_T} = \nu_{r_S,r_T}|_{B^T_{r'_T}}$ whenever $r'_T < r_T$. Since $\mu^U_{x,S\times T}|_{Q_{r_S,r_T}}$ is a (nonzero) product measure, we see furthermore that $\nu_{r'_S,r_T} \propto \nu_{r_S,r_T}$ for any two positive $r'_S < r_S$. Therefore

$\nu_x^U(A) = (\nu_{r_S,r_T}(B_1^T))^{-1}\nu_{r_S,r_T}(A)$ defines a measure ν_x^U on T independent of r_S, r_T as long as $A \subseteq B_{r_{T'}}^{T'}$. It follows that $\mu_{x,T}^U \propto \mu_{x,S}^U \times \nu_x^U$. ∎

The next lemma finishes the proof of Theorem 7.5; the proof follows closely the second part of the proof of [21, Prop. 6.4].

LEMMA 7.8: *Under the assumptions of Theorem 7.5 the measure ν_x^U in Lemma 7.7 equals $\mu_{x,T}^U$ a.e.*

Proof: We set $r_S = 1$, fix $r_T > 0$, and choose $r > 0$ such that $Q = Q_{1,r_T} = B_1^S \times B_{r_T}^T \subseteq B_r^{S\times T}$. Note that Definition 7.4 and Lemma (7.6) are symmetric in S and T. So we can apply Lemma (7.6) to $S \times T$ and T to find countably many $r, S \times T$-flowers and σ-rings with T-plaques as atoms, such that the centers of the former cover almost all of X. If $\mathcal{A}^\times$ and $\mathcal{A}(T)$ are two such σ-rings with maximal element A, then the decomposition of $\mu_x^{\mathcal{A}^\times}$ into conditional measures with respect to $\mathcal{A}(T)$ is given by

$$\mu_x^{\mathcal{A}^\times}|_{f(Q)} = \int_{f(Q)} (\mu_{x'}^{\mathcal{A}(T)}(f(Q)))^{-1}\mu_{x'}^{\mathcal{A}(T)}|_{f(Q)} \mathrm{d}\mu_x^{\mathcal{A}^\times}(x') \quad \text{for a.e. } x \in C$$

by Lemma (7.6), where as before $f(s,t) = t_U((s,t),x)$ and C is the center of $\mathcal{A}^\times$.

Fix some x satisfying this, (5.3), and (7.16). Note that $Q \subseteq f^{-1}[x]_{\mathcal{A}^\times}$. We conclude from (7.16) that

$$\mu_{x,S\times T}^U|_Q = \int_Q (\mu_{f(s,t)}^{\mathcal{A}(T)}(f(Q)))^{-1} f_*^{-1}\mu_{f(s,t)}^{\mathcal{A}(T)}|_Q \mathrm{d}\mu_{x,S\times T}^U(s,t),$$

where we showed equality by evaluating both sides on the set Q (and using the fact that the measure in the integrant is normalized to be a probability measure). Furthermore, this is the decomposition of $\mu_{x,S\times T}^U|_Q$ into conditional measures with respect to the σ-algebra $\mathcal{B}_{B_1^S} \times \{\emptyset, B_{r_T}^T\}$.

By Lemma 7.7 we have $\mu_{x,S\times T}^U = \mu_{x,S}^U \times \nu_x^U$ a.e. for some measure ν_x^U on T. This shows that

$$\mu_{x,S\times T}^U|_Q = \int_Q (\nu_x^U(B_{r_T}^T))^{-1}\delta_s \times \nu_x^U|_{B_{r_T}^T} \mathrm{d}\mu_{x,S\times T}^U(s,t).$$

Clearly this is also a decomposition of $\mu_{x,S\times T}^U|_Q$ into conditional measures with respect to $\mathcal{B}_{B_1^S} \times \{\emptyset, B_{r_T}^T\}$. Therefore

$$(7.21)\quad \begin{aligned} &(\mu_{f(s,t)}^{\mathcal{A}(T)}(f(Q)))^{-1} f_*^{-1}\mu_{f(s,t)}^{\mathcal{A}(T)}|_Q = \\ &\qquad (\nu_x^U(B_{r_T}^T))^{-1}\delta_s \times \nu_x^U|_{B_{r_T}^T} \quad \text{for } \mu_{x,S\times T}^U\text{-a.e. } (s,t) \in Q. \end{aligned}$$

For the lemma we need to know (7.21) also for $(s,t) = (e,e)$. Here we use Luzin's theorem, for every $\epsilon > 0$ there exists a compact set $K \subseteq C \setminus N$ with $\mu(K) > (1-\epsilon)\mu(C)$ such that $\mu_{x'}^{\mathcal{A}(T)}$ depends continuously on $x' \in K$ (using the weak* topology on the space of probability measures). It is easy to see that $\mu|_K$ has conditional measures proportional to $\mu^U_{x,S\times T}|_{t_U(\cdot,x)^{-1}K}$ with respect to the $(S\times T, H)$-space structure. By Lemma 5.8

$$(e,e) \in \operatorname{supp}(\mu^U_{x,S\times T}|_{t_U(\cdot,x)^{-1}K}) \quad \text{for a.e. } x \in K.$$

Suppose $x \in K$ satisfies the above, (7.21), Proposition 5.6 (2), and Lemma 5.8 for the $(T, \operatorname{Isom}(T))$-space structure. Then $\mu^U_{x,S\times T}(B^{S\times T}_\delta \cap f^{-1}K) > 0$ for every $\delta > 0$, so there exists a sequence $(s_n,t_n) \to (e,e)$ such that $f(s_n,t_n) \in K$ and (7.21) holds for these particular values of (s,t). By continuity

$$\mu_x^{\mathcal{A}(T)} = \lim_{n\to\infty} \mu^{\mathcal{A}(T)}_{f(s_n,t_n)}.$$

Let $g\colon X \to [0,1]$ be continuous with $g(x) = 1$ and $[x]_{\mathcal{A}^\times} \cap \operatorname{supp} g \subseteq f(Q)$. By Lemma 5.8, $\int g \mathrm{d}\mu_x^{\mathcal{A}(T)} > 0$. It follows that $a_n = \mu^{\mathcal{A}(T)}_{f(s_n,t_n)}(f(Q))$ cannot converge to zero. We assume without loss of generality that $a = \lim_{n\to\infty} a_n > 0$ exists. Since f is a homeomorphism between Q and $f(Q)$ it follows from (7.21) that

$$f_*^{-1}\mu_x^{\mathcal{A}(T)}|_Q \propto \delta_e \times \nu_x^U|_{B^T_{r_T}}.$$

However, Proposition 5.6 (2) now shows that

$$\mu^U_{x,T}|_{B^T_{r_T}} \propto \nu^U_x|_{B^T_{r_T}}.$$

Varying r_T shows the lemma and Theorem 7.5. ∎

8. The coarse Lyapunov decomposition and the product structure of the conditional measure

8.1. The abelian case. The results from the last section can be applied for higher rank actions as follows. Although we will not use the particular case of this section in the remainder of this paper, we start with the case where $S_0,\dots,S_\ell$ are second countable, locally compact, abelian groups with translation invariant metrics $d_{S_i}(\cdot,\cdot)$ for $i = 0,\dots,\ell$. We define $T = S_0 \times \cdots \times S_\ell$ and use $d_T(t,t') = \max_i d_{S_i}(t_i,t_i')$ as its metric. We will give conditions which force the conditional measures $\mu_{x,T}$ to be a product measure of conditional measures. This situation appears, for instance, for a $\mathbb{Z}^k$-action on $\mathbb{T}^m$ by commuting automorphisms where the spaces S_i are (certain sums of) common eigenspaces of the defining matrices (see [18] and [8]). This illustrates how we will use Theorem 7.5 in the more complex situation of homogeneous spaces.

Definition 8.1: Let $T = S_0 \times \cdots \times S_\ell$ be as above, let X be a T-space, and let α be a $\mathbb{Z}^k$-action which preserves the T-leaves and the induced S_i-leaves for $i = 0, \dots, \ell$. We say $T = S_0 \times \cdots \times S_\ell$ is the **coarse Lyapunov decomposition of T with central subspace** S_0 and **coarse Lyapunov subspaces** S_i for $i = 1, \dots, \ell$ if there exist linear functionals λ_i for $i = 1, \dots, \ell$ with the following properties:

(1) For $T' = \{e\} \times S_1 \times \cdots \times S_\ell$ the T'-leaves are uniformly contracted by some $\alpha^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^k$.
(2) Every $\alpha^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^k$ acts isometrically on the S_0-leaves
(3) The functionals λ_i and λ_j are linearly independent for $1 \leq i < j \leq \ell$.
(4) For $1 \leq i \leq \ell$ the induced S_i-space has λ_i as its coarse Lyapunov weight (in the sense of Definition 7.1).

Note that Condition (4) implies that $\alpha^{\mathbf{n}_j}$ acts asymptotically isometrically on S_i for any sequence $\mathbf{n}_j$ with $\lambda_i(\mathbf{n}_j) \to 0$ for $j \to \infty$. Unfortunately, this means that the above definition and the theorem below do not apply to the most general case of actions by commuting automorphisms of $\mathbb{T}^m$ together with sums of generalized eigenspaces. A separate argument is needed in this case to move from the eigenspaces to the generalized eigenspaces; see also [16].

We can now formulate our first generalization of [21, Prop. 6.4] to higher rank actions and the coarse Lyapunov decomposition.

THEOREM 8.2: *Suppose $T = S_0 \times \cdots \times S_\ell$ is the product of second countable, locally compact, metric, and abelian groups, and let the metric d_T on T be defined as above. Let X be a T-space. Let α be a $\mathbb{Z}^k$-action that preserves the T-leaves. Suppose T has a coarse Lyapunov decomposition with central subspace S_0 and coarse Lyapunov subspaces S_i for $i = 1, \dots, \ell$. Let μ be an α-invariant measure with conditional measures $\mu_{x,T}$. Then for a.e. x*

$$\mu_{x,T} = \mu_{x,S_0} \times \cdots \times \mu_{x,S_\ell}.$$

By assumption, the leaves corresponding to $T' = \{e\} \times S_1 \times \cdots \times S_\ell$ are contracted by some $\alpha^{\mathbf{n}}$ which acts isometrically on the $(S_0, \mathrm{Isom}(S_0))$-leaves. By Theorem 7.5 we immediately get that $\mu_{x,T} = \mu_{x,S_0} \times \mu_{x,T'}$ a.e. The fact $\mu_{x,T'} = \mu_{x,S_1} \times \cdots \times \mu_{x,S_\ell}$ follows similarly by induction. For this we only need to find some $\mathbf{w} \in \mathbb{R}^k$ satisfying the assumptions of Theorem 7.5 for one of the remaining S_j (with $1 \leq j \leq \ell$).

LEMMA 8.3: *Suppose λ_i for $i = 1, \dots, \ell$ are pairwise linearly independent linear functionals on $\mathbb{R}^k$ such that there exists some $\mathbf{n} \in \mathbb{Z}^k$ with $\lambda_i(\mathbf{n}) < 0$ for $i = 1, \dots, \ell$. Then there exist an index i and a vector $\mathbf{w} \in \mathbb{R}^k$ such that $\lambda_i(\mathbf{w}) = 0$ and $\lambda_j(\mathbf{w}) < 0$ for $j \neq i$.*

Proof: Define the convex set

$$C = \{\lambda \in \sum_{i \in E} \mathbb{R}^+ \lambda_i : \lambda(\mathbf{n}) = -1\}$$

of linear functionals. Then C has at least one extremal point.

Let i be such that $\frac{1}{|\lambda_i(\mathbf{n})|}\lambda_i$ is an extremal point of C. Then there exists a linear functional on C — which we identify with some $\mathbf{w} \in \mathbb{R}^k$ — such that $\lambda(\mathbf{w}) < 0$ for all $\lambda \in C \setminus \{\lambda_i\}$ and $\lambda_i(\mathbf{w}) = 0$. This proves the lemma. ∎

Proof of Theorem 8.2: We already showed that we only have to consider the case of $T' = S_1 \times \cdots \times S_\ell$. By Lemma 8.3, we can find some i and $\mathbf{w} \in \mathbb{R}^k$ with $\lambda_i(\mathbf{w}) = 0$ but $\lambda_j(\mathbf{w}) < 0$ for $j \neq i$. We assume without loss of generality (using that T is a direct product) that $i = 1$. Theorem 7.5 shows $\mu_{x,T'} = \mu_{x,S_1} \times \mu_{x,T''}$ where $T'' = S_2 \times \cdots \times S_\ell$. Induction concludes the proof. ∎

8.2. The homogeneous case and invariance resulting from non-commuting foliations.

In this section we show that the conditional measures with respect to an $\mathbf{m}$-stable subgroup are product measures. This is similar to the result obtained above, but if the different coarse Lyapunov subgroups of H do not commute with each other we are able to obtain some invariance of the conditional measure.

THEOREM 8.4: *Let X be a G_S-space and suppose the $\mathbb{Z}^k$-action α preserves the G_S-leaves. Assume that the adjoint action A on the S-Lie algebra $\mathfrak{g}_s$ is semisimple. Let $H = \exp \mathfrak{h}$ be an $\mathbf{m}$-stable subgroup of G_S, let $H^{\Lambda_1}, \dots, H^{\Lambda_\ell}$ be the different coarse Lyapunov subgroups of H, and let ϕ: $H^{\Lambda_1} \times \cdots \times H^{\Lambda_\ell} \to H$ be defined by $\phi(g_1, \dots, g_\ell) = g_1 \cdots g_\ell$. Then any α-invariant probability measure μ on X satisfies*

$$\mu_x^H \propto \phi_*(\mu_x^{\Lambda_1} \times \cdots \times \mu_x^{\Lambda_\ell}) \quad \text{a.e.,} \tag{8.1}$$

where μ_x^H and $\mu_x^{\Lambda_i}$ are the conditional measures for the H-space and the H^{Λ_i}-space X for $i = 1, \dots, \ell$ respectively.

Notice that we did not give any restrictions on the order of the coarse Lyapunov subgroups. A priori the measure on the right of (8.1) depends on the order. As we will see later this independence is related to the following theorem.

THEOREM 8.5: *Let X, α, and H be as in Theorem 8.4. For any α-invariant probability measure μ there exist for a.e. two subgroups*

$$H_x \subseteq P_x \subseteq H$$

with the following properties:

(1) *μ_x^H is supported by P_x.*
(2) *μ_x^H is left- and right-invariant under multiplication with elements of H_x.*
(3) *H_x and P_x allow a weight decomposition; see Definition 6.1.*
(4) *H_x is a normal subgroup of P_x and any elements $g \in P_x \cap H^{\Lambda_r}$ and $h \in P_x \cap H^{\Lambda_s}$ of different coarse Lyapunov subgroups ($r \neq s$) satisfy that gH_x and hH_x commute with each other in P_x/H_x.*
(5) *$\mu_x^{\Lambda_i}$ is left- and right-invariant under multiplication with elements of $H_x \cap H^{\Lambda_i}$ for $i = 1, \dots, \ell$.*

Note that in the case of commutative coarse Lyapunov subgroups H^{Λ_i} for $i = 1, \dots, \ell$ the statement in (4) is equivalent to P_x/H_x being commutative as well.

8.3. THEOREM 8.4 FOR A PARTICULAR ORDER OF THE SUBGROUPS. In this section we prove by induction the following weaker version of Theorem 8.4.

LEMMA 8.6: *There exists a reordering of $H^{\Lambda_1}, \dots, H^{\Lambda_\ell}$ such that (8.1) holds for that order.*

Proof: Suppose we already showed the lemma for less than ℓ coarse Lyapunov subgroups. Let λ_i be Lyapunov weights with $\Lambda_i = (0, \infty)\lambda_i$ for $i = 1, \dots, \ell$. By Lemma 8.3, we can reorder the weights such that there exists $\mathbf{w} \in \mathbb{R}^k$ with $\lambda_1(\mathbf{w}) = 0$ and $\lambda_j(\mathbf{w}) < 0$ for $j > 1$. Therefore $\mathfrak{h}' = \mathfrak{h}^{\Lambda_2} + \cdots + \mathfrak{h}^{\Lambda_\ell}$ satisfies $[\mathfrak{h}, \mathfrak{h}'] \subseteq \mathfrak{h}'$ by Proposition 4.9, and $H' = \exp \mathfrak{h}'$ is an $\mathbf{m}$-stable normal subgroup of H. Let $\phi'\colon H^{\Lambda_2} \times \cdots \times H^{\Lambda_\ell} \to H'$ be the corresponding product map. By the inductive assumptions we can reorder these weights and obtain (8.1) for $\mu_x^{H'}$.

Let $\psi\colon H^{\Lambda_1} \times H' \to H$ be defined by $\psi(g_1, g') = g_1 g'$. Let d be the metric on H found in Section 4.5. Recall that d is right invariant (as required in Definition 5.3). Lemma 4.14 shows that λ_1 is a coarse Lyapunov weight of the H^{Λ_1}-space with respect to α in the sense of Definition 7.1. Furthermore,

$$d(\theta^{\mathbf{n}_j}(g'), \theta^{\mathbf{n}_j}(h')) \to 0$$

if $g', h' \in H'$ and $\mathbf{n}_j \in \mathbb{Z}^k$ diverges with bounded distance to $\mathbb{R}^+\mathbf{w}$. By Theorem 7.5 this shows that $\mu_x^H \propto \psi_*(\mu_x^{\Lambda_1} \times \mu_x^{H'})$ a.e., and together with the inductive assumptions the lemma follows. ∎

8.4. THE PROOF OF THEOREM 8.5. Note that the order of the subgroups is not important for Theorem 8.5, so that we can use here the order found in Lemma 8.6.

LEMMA 8.7: *For a.e. x and $r \neq s$ we have*

$$(R_{g^{-1}h^{-1}gh})_*\mu_x^H = \mu_x^H \tag{8.2}$$

whenever $g \in \operatorname{supp}\mu_x^{\Lambda_r}$ and $h \in \operatorname{supp}\mu_x^{\Lambda_s}$. Here $R_g\colon H \to H$ is right multiplication $R_g(g') = g'g$.

Recall that Lemma 5.10 states that a.s. the notions of affine invariance and invariance of the conditional μ_x^H are equivalent.

Proof: Let N be a null set such that Proposition 5.6 (3), Lemmas 5.8 and 5.10, Proposition 7.3 (for several vectors $\mathbf{w}$ specified below), and Lemma 8.6 hold for the conditional measures $\mu_x^{\Lambda_i}$ for $i = 1, \ldots, \ell$. Again by Proposition 5.6 (3) we can ensure that

$$\mu_{gx}^H \propto (R_g^{-1})_*\mu_x^H \tag{8.3}$$

holds for any $x, gx \notin N$ where $g \in H$. By Lemma 5.9 we can assume that $\mu_x^{\Lambda_r}(\{g : gx \in N\}) = 0$ for all $x \notin N$ and similarly for $\mu_x^{\Lambda_s}$.

Let $g, h \in H$ be as in the lemma. Then $\mu_x^{\Lambda_s}(B_\epsilon^{\Lambda_s}(h))$ for any $\epsilon > 0$ (where $B_\epsilon^{\Lambda_s}(h)$ denotes the ϵ-ball around $h \in H^{\Lambda_s}$). Therefore, there exists a sequence $h_n \in H^{\Lambda_s}$ with $h_n \to h$ and $h_n x \notin N$. Note that the conditional measure for the H^{Λ_s}-leaves change

$$(R_{h_n}^{-1})_*\mu_x^{\Lambda_s} \propto \mu_{h_n x}^{\Lambda_s}$$

according to Proposition 5.6 (3), while $\mu_{h_n x}^{\Lambda_i} = \mu_x^{\Lambda_i}$ for $i \neq s$. The latter follows from Proposition 7.3 if we use some $\mathbf{w}_i \in \mathbb{R}^k$ satisfying $\lambda_i(\mathbf{w}_i) = 0$ and $\lambda_s(\mathbf{w}_i) < 0$. In particular, $g \in \operatorname{supp}\mu_{h_n x}^{\Lambda_r}$ and again there exists a sequence $g_n \in H^{\Lambda_r}$ with $g_n \to g$ and $y_n = g_n h_n x \notin N$. As before we see that

$$(R_{g_n}^{-1})_*\mu_x^{\Lambda_r} \propto \mu_{y_n}^{\Lambda_r} \quad \text{and} \quad (R_{h_n}^{-1})_*\mu_x^{\Lambda_s} \propto \mu_{y_n}^{\Lambda_s}. \tag{8.4}$$

Since $e \in \operatorname{supp}\mu_x^{\Lambda_r}$ by Lemma 5.8, this implies that $g_n^{-1} \in \operatorname{supp}\mu_{y_n}^{\Lambda_r}$ and similarly $h_n^{-1} \in \operatorname{supp}\mu_{y_n}^{\Lambda_s}$. Just as above we now construct two sequences $g_n' \in$

H^{Λ_r} and $h'_n \in H^{\Lambda_s}$ such that $g'_n \to g^{-1}$, $h'_n \to h^{-1}$, $h'_n y_n \notin N$, and $z_n = g'_n h'_n y_n \notin N$. The analogue to (8.4) is now

$$(8.5) \qquad (R^{-1}_{g'_n})_* \mu^{\Lambda_r}_{y_n} \propto \mu^{\Lambda_r}_{z_n} \quad \text{and} \quad (R^{-1}_{h'_n})_* \mu^{\Lambda_s}_{y_n} \propto \mu^{\Lambda_s}_{z_n}.$$

For $i \neq r, s$ it follows similarly that

$$(8.6) \qquad \mu^{\Lambda_i}_x = \mu^{\Lambda_i}_{h_n x} = \mu^{\Lambda_i}_{y_n} = \mu^{\Lambda_i}_{h'_n y_n} = \mu^{\Lambda_i}_{z_n}.$$

We claim that this implies that $\mu^{\Lambda_i}_{z_n}$ converges (possibly after going to a subsequence) to a measure proportional to $\mu^{\Lambda_i}_x$ for any i. For $i \neq r, s$ this is trivial because of (8.6). For $i = r$ it follows from (8.4)–(8.5) that $\mu^{\Lambda_r}_{z_n}$ is proportional to $(R^{-1}_{g'_n g_n})_* \mu^{\Lambda_r}_x$. By Proposition 5.6 (1) the proportionality constant is just $(\mu^{\Lambda_r}_x(B^{\Lambda_r}_1 g'_n g_n))^{-1}$. Since $g'_n g_n \to e$ we have

$$B^{\Lambda_r}_{1/2} \subseteq B^{\Lambda_r}_1 g'_n g_n \subseteq B^{\Lambda_r}_2$$

for large enough n, and so all of the proportionality constants allow a uniform bound. Therefore, we can choose a subsequence and achieve that $\mu^{\Lambda_i}_{z_n}$ converges to a measure proportional to $\mu^{\Lambda_i}_x$ by Lemma 6.3. The case $i = s$ is similar.

By Lemma 8.6 we know that

$$\mu^H_{z_n} \propto \phi_*(\mu^{\Lambda_1}_{z_n} \times \cdots \times \mu^{\Lambda_\ell}_{z_n}).$$

As before the proportionality constants are bounded, so that $\mu^H_{z_n}$ converges (possibly after going to a subsequence) to a measure proportional to μ^H_x.

However, by (8.3)

$$\mu^H_{z_n} \propto (R^{-1}_{g'_n h'_n g_n h_n})_* \mu^H_x,$$

where the right hand side converges to $(R^{-1}_{g^{-1}h^{-1}gh})_* \mu^H_x$. We conclude that

$$\mu^H_x \propto (R^{-1}_{g^{-1}h^{-1}gh})_* \mu^H_x,$$

but by Lemma 5.10 the proportionality constant has to equal one. ∎

We will now strengthen Lemma 8.7 using the Lie algebra $\mathfrak{h}$. For this it will be convenient to say that $u \in \mathfrak{h}^\Lambda$ is a **support vector (for the coarse Lyapunov weight** Λ**)** if $\exp u \in \operatorname{supp} \mu^\Lambda_x$. Moreover, v is a **weight component (of a support vector for** Λ**)** if $u = \sum_{\lambda_i \in \Lambda} u_i$ with $\exp u \in \operatorname{supp} \mu^\Lambda_x$, $u_i \in \mathfrak{h}^{\lambda_i}$, $\lambda_i \neq \lambda_j$ for $i \neq j$, and $v = u_i$ for some i. We will consider $[\cdot,\cdot]$-monomials w in weight components, and say that w is **mixed** if it is defined using weight components for at least two different coarse Lyapunov weights.

PROPOSITION 8.8: *Let X, α, and H be as in Theorem 8.4. Then for a.e. $x \in X$ the following holds. Suppose w is a mixed $[\cdot,\cdot]$-monomial in weight components. Then*

$$(R_{\exp(w)})_* \mu_x^H = \mu_x^H. \tag{8.7}$$

In particular,

$$(R_{\exp[u,v]})_* \mu_x^H = \mu_x^H$$

whenever u and v are weight components of support vectors for two different coarse Lyapunov weights.

The following will be useful in the proof of Proposition 8.8.

LEMMA 8.9: *Every mixed higher order $[\cdot,\cdot]$-monomial can be expressed as a finite linear combination of $[\cdot,\cdot]$-monomials of the form $w = [v', v''] \in \mathfrak{h}^\zeta$ such that either*

(a) *v' and v'' are both mixed $[\cdot,\cdot]$-monomial, or*

(b) *$v' \in \mathfrak{h}^{\xi'}$ is a single weight component of a support vector, $v'' \in \mathfrak{h}^{\xi''}$ is a mixed $[\cdot,\cdot]$-monomial, and ξ', ξ'' are linearly independent.*

Proof: We will use the Jacobi identity (3.1) in the form

$$[w', [v', v'']] = -[v', [v'', w']] - [v'', [w', v']]. \tag{8.8}$$

For instance, we can use this (repeatedly if necessary) to write every higher order monomial as a finite sum of monomials $w = [w', w'']$ such that w' is a single weight component and w'' is a monomial in weight components. Starting with a mixed monomial it is clear that we actually get a sum of mixed monomials since every term on the right of (8.8) uses all the vectors w', v', v'' of the original expression. We have two cases to study, either w'' is mixed or not.

Suppose $w'' = [v', v'']$ is mixed. Let $w' \in \mathfrak{h}^{\zeta'}$, $w'' \in \mathfrak{h}^{\zeta''}$, $v' \in \mathfrak{h}^{\xi'}$, and $v'' \in \mathfrak{h}^{\xi''}$. If ζ' and ζ'' are linearly independent then $w = [w', w'']$ is as in (b). So assume $\zeta' = c\zeta'' = c(\xi' + \xi'')$ for some $c > 0$. In this case w'' has to be a mixed monomial. If w'' is a higher order monomial itself, we use induction and assume without loss of generality that $w'' = [v', v'']$ is an expression as in (a) or (b). In any case we use (8.8) again. If w'' is as in (a), then it is clear that both expressions on the right of (8.8) are also as in (a). If w'' is as in (b), then ξ' and ξ'' are linearly independent and v'' is mixed. Therefore $[v', [v'', w']]$ is as in (b) since ξ' and $\xi'' + \zeta' = (1+c)\xi'' + c\xi$ are linearly independent, and $[v'', [w', v']]$ is as in (a) since $\zeta' = c\xi' + c\xi''$ and ξ' are linearly independent. If w'' is a degree two monomial, then it follows quite similarly that both terms on the right of (8.8) are as in (b).

Suppose $w'' = [v', v'']$ is not mixed, i.e. w'' is defined using weight components in $\mathfrak{h}^{\mathbb{R}^+\zeta''}$. Then ζ' and ζ'' are linearly independent since $w = [w', w'']$ is mixed and w' is a single weight component. Applying (8.8) we get two expressions of the form $[v, u]$, where v is defined using weight components in $\mathfrak{h}^{\mathbb{R}^+\zeta''}$, $u \in \mathfrak{h}^\lambda$ is a mixed monomial, and ζ'' and λ are linearly independent. If v is a single weight component then $[v, u]$ is as in (b). Otherwise, $v = [v_1, v_2]$ and we can apply (8.8) (repeatedly if necessary) to get a sum of expressions as in (b). ∎

Proof of Proposition 8.8: We first prove (8.7) inductively for all possible choices of weight components for two different coarse Lyapunov weights. We start by describing the inductive argument, which at its heart uses Proposition 6.2 and Lemma 8.7. We fix some inner product on $\mathbb{R}^k$ and use it to identify Lyapunov weights λ with elements of $\mathbb{R}^k$ and coarse Lyapunov weights Λ with rays in $\mathbb{R}^k$. In this sense we can speak about the angle between two coarse Lyapunov weights. Choose some $\mathbf{m} \in \mathbb{Z}^k$ such that $\xi(\mathbf{m}) > 0$ for all weights ξ of H. Our inductive assumption is that (8.7) already holds for mixed $[\cdot,\cdot]$-monomials $w \in \mathfrak{h}^\xi$

(i) whenever $\xi(\mathbf{m}) < t$,

(iii) whenever $\xi(\mathbf{m}) = t$ and w is defined using weight components for two coarse Lyapunov weights such that the angle between them is less than γ.

Since there are only finitely many weights, both assumptions are trivial for small enough t resp. for any t and small enough γ. For the inductive step we assume that ξ satisfies $\xi(\mathbf{m}) = t$ and that $w \in \mathfrak{h}^\xi$ is defined using weight components for two different coarse Lyapunov weights Λ_1 and Λ_1 such that the angle between them is equal to γ.

We first show (8.7) for every higher order $[\cdot,\cdot]$-monomial expression w. By Lemma 8.9 we can write w as a linear combination of $[\cdot,\cdot]$-monomial expressions $v = [v', v'']$ of special natural. By Proposition 6.2 it is enough to show (8.7) for each of these expressions separately. Here we use mainly the inductive assumptions.

Suppose v is as in Lemma 8.9 (a). Then right multiplication by $\exp(v')$ and $\exp(v'')$ fixes μ_x^H by the inductive assumption in (i), and $v = [v', v'']$ satisfies the same by Proposition 6.2.

Assume now v is as in Lemma 8.9 (b). Again by (i), $(R_{\exp(v'')})_* \mu_x^H = \mu_x^H$, and so $\exp(v'') \in \operatorname{supp} \mu_x^H$ (using that $e \in \operatorname{supp} \mu_x^H$ by Lemma 5.8). However, since μ_x^H is a product measure by Lemma 8.6 we conclude from this that $v'' \in \mathfrak{h}^{\zeta''}$

is a support vector. Therefore, $v = [v', v'']$ is a $[\cdot,\cdot]$-monomial expression in the weight components v' and v'' of support vectors. Note that v'' is defined using weight components for Λ_1 and Λ_2 and so $\zeta'' \in \Lambda_1 + \Lambda_2$. Therefore, the angle between $\mathbb{R}^+\zeta''$ and Λ_i for $i = 1, 2$ is less than the angle between Λ_1 and Λ_2. By the assumption in (ii) we conclude that $v = [v', v'']$ satisfies (8.7) .

It remains to consider the quadratic case $w = [u', v'] \in \mathfrak{h}^\xi$ where $u' \in \mathfrak{h}^\lambda$ and $v' \in \mathfrak{h}^\zeta$ are weight components of support vectors u and v for different coarse Lyapunov weights $\mathbb{R}^+\lambda$ and $\mathbb{R}\zeta$. By Lemma 8.7 we already know that $\tilde{w} = (-u) * (-v) * u * v$ satisfies (8.7). By Proposition 6.2 this already shows that the weight component $\tilde{w}_\xi \in \mathfrak{h}^\xi$ of $\tilde{w}$ corresponding to the weight ξ satisfies (8.7). By (3.2)

$$u * v = u + v + \frac{1}{2}[u, v] + \cdots,$$

$$(-u) * (-v) = -u - v + \frac{1}{2}[u, v] + \cdots,$$

and

$$\tilde{w} = (-u) * (-v) * u * v = [u, v] + \cdots, \tag{8.9}$$

where the dots indicate other higher order $[\cdot,\cdot]$-monomial expressions in u and v with both appearing at least once in every expression. We can use this and Proposition 4.9 to find a similar expression for the weight component

$$\tilde{w}_\xi = [u', v'] + \cdots \in \mathfrak{h}_x \cap \mathfrak{h}^\xi, \tag{8.10}$$

where $\xi = \lambda + \zeta$ and the dots indicate various other higher order $[\cdot,\cdot]$-monomial expressions in different weight components of u and v. Note that $w = [u', v']$ is the only quadratic term here, since $\xi = \zeta' + \zeta''$ can only be written in this way as a linear combination of ζ' and ζ''. We already showed that $\tilde{w}_\xi$ and all higher order terms in (8.10) (indicated by the dots) satisfy (8.7). This implies the same for w by Proposition 6.2 and concludes the induction in the case of two coarse Lyapunov weights.

The case where w is defined using weight components to more than two coarse Lyapunov weights follows again by induction. In this case w is necessarily a higher order monomial, so we can use Lemma 8.9. Case (a) again follows from Proposition 6.2 and case (b) follows from the case of two coarse Lyapunov weights considered above. ∎

Our last preparation for the proof of Theorem 8.5 is the next lemma.

LEMMA 8.10: *Suppose ν is a locally finite measure on H, and suppose $H' \subseteq P' \subseteq H$ are subgroups allowing a weight decomposition. If $\operatorname{supp}\nu \subseteq P'$, H' is normal in P', and right multiplication with elements of H' leaves ν invariant, then the same is true for left multiplication. The same implication holds for reversed sides.*

Proof: Note first that the Haar measure m' on H' is bi-invariant since H' is a nilpotent S-Lie group. Fix some $R > 0$ and restrict ν to $B_R^{P'}$. Since this restriction is finite, there exist conditional measures $\nu_g^{\mathcal{A}}$ for the σ-algebra

$$\mathcal{A} = \{A \in \mathcal{B} : A \subseteq B_R^{P'}, AH \setminus A \subseteq P' \setminus B_R^{P'}\}.$$

In other words, $\mathcal{A}$ contains all measurable subsets $A \subseteq B_R^{P'}$ that are intersections of $B_R^{P'}$ and unions of left cosets gH'. Because of right invariance of ν the conditional measures ν_g are almost surely proportional to the restriction of the Haar measure on gH'. Since H' is normal in P', left multiplication by $h \in H'$ maps gH' into itself and leaves the Haar measure on gH' invariant. Therefore $\nu_g(hB) = \nu_g(B)$ almost surely whenever $B, hB \subseteq B_R^{P'}$. This implies $\nu(hB) = \nu(B)$. Since $R > 0$ was arbitrary, it follows that ν is invariant under left multiplication by elements of H'. ∎

Proof of Theorem 8.5: Suppose x satisfies Lemma 8.6 and Proposition 8.8. Let

$$\mathfrak{p}_x = \sum_{\sigma \in S} \langle w_\sigma : w \text{ is a } [\cdot,\cdot]\text{-monomial in weight components}\rangle_{\mathbb{Q}_\sigma},$$

where $w_\sigma \in \mathfrak{h}_\sigma$ for every $\sigma \in S$ and $w = \sum_\sigma w_\sigma$. It is clear that the above linear hull over $\mathbb{Q}_\sigma$ is a Lie algebra over $\mathbb{Q}_\sigma$. Therefore, $P_x = \exp \mathfrak{p}_x$ is a subgroup of H that allows a weight decomposition. Suppose $g = \phi(g_1, \ldots, g_\ell) \in \operatorname{supp}\mu_x^H$. Since $\mu_x^H \propto \phi_*(\mu_x^{\Lambda_1} \times \cdots \times \mu_x^{\Lambda_\ell})$ it follows that $g_i \in \operatorname{supp}\mu_x^{\Lambda_i}$ for $i = 1, \ldots, \ell$. Therefore $\log g_i$ is a support vector and $g_i \in P_x$ for all i, which shows that $g \in P_x$.

We define $\mathfrak{h}_x$ similarly but use only mixed $[\cdot,\cdot]$-monomials w as in Proposition 8.8, i.e. we require that w is defined using support vectors to at least two different Lyapunov weights. Again $H_x = \exp \mathfrak{h}_x$ is a subgroup of P_x which allows a weight decomposition. Moreover, it follows at once that $[\mathfrak{h}_x, \mathfrak{p}_x] \subseteq \mathfrak{h}_x$ and therefore H_x is a normal subgroup of P_x by (3.2). By Proposition 8.8, μ_x^H is invariant under multiplication from the right by elements of H_x; by Lemma 8.10 the same is true for multiplication from the left also.

Suppose now $g \in P_x \cap H^{\Lambda_r}$ and $h \in P_x \cap H^{\Lambda_s}$ with $r \neq s$. Then $[\log(g), \log(h)] \in \mathfrak{h}_x$ and the same holds similarly for all higher order $[\cdot,\cdot]$-monomials. By (8.9) this shows that $g^{-1}h^{-1}gh \in H_x$, in other words that gH_x and hH_x commute with each other in P_x/H_x.

For the last statement suppose first $i = 1$. Since $\mu_x^H \propto \phi_*(\mu_x^{\Lambda_1} \times \cdots \times \mu_x^{\Lambda_\ell})$ is left-invariant by multiplication with elements of $H_x \cap H^{\Lambda_1}$, it follows that the same is true for $\mu_x^{\Lambda_1}$. Furthermore, $\mu_x^{\Lambda_1}$ is supported by $P_x \cap H^{\Lambda_i}$ and $H_x \cap H^{\Lambda_1}$ is normal in this subgroup. Lemma 8.10 implies invariance under multiplication from the right for $i = 1$. Let $H' = H^{\Lambda_2} \cdots H^{\Lambda_\ell}$. Then we show similarly that $\mu_x^{H'}$ is invariant under multiplication from the right by elements of $H_x \cap H'$, and secondly that the same is also true from the left by Lemma 8.10. Induction completes the proof. ∎

8.5. The proof of Theorem 8.4. Recall that we already showed a restricted version of Theorem 8.4 in Lemma 8.6. The first lemma we need for the extension is the following generalization of the fact that the Haar measure on H is the image of the Haar measure on its Lie group $\mathfrak{h} = \log H$ under the exponential map $\exp\colon \mathfrak{h} \to H$.

Lemma 8.11: *Let $H' \subseteq P \subseteq H$ be subgroups that allow weight decompositions. Suppose that H' is a normal subgroup of P. Let ν be a locally finite measure on P and suppose that ν is right-invariant by multiplication with elements of H'. Then $\mathfrak{p} = \tilde{\mathfrak{p}} \oplus \mathfrak{h}'$ for some closed subgroup $\tilde{\mathfrak{p}} \subseteq \mathfrak{p}$ of the Lie algebra $\mathfrak{p} = \log P$ where $\mathfrak{h}' = \log H'$. In that decomposition of $\mathfrak{p}$ we have $\log_* \nu = \tilde{\nu} \times m_{\mathfrak{h}'}$ for some locally finite measure $\tilde{\nu}$ on $\tilde{\mathfrak{p}}$ and the Haar measure $m_{\mathfrak{h}'}$ of $\mathfrak{h}'$.*

Proof: Let $\tilde{\mathfrak{p}} = \sum_{\sigma \in S} \tilde{\mathfrak{p}}_\sigma$ where $\tilde{\mathfrak{p}}_\sigma \subseteq \mathfrak{p}_\sigma$ is a linear complement of $\mathfrak{h}'_\sigma \subseteq \mathfrak{p}_\sigma$. Suppose H is $\mathbf{m}$-unstable.

We claim that $\nu_{\mathfrak{p}} = \log_* \nu$ is invariant under translation by elements of $\mathfrak{h}'$. If we normalize $m_{\mathfrak{h}'}$ such that $m_{\mathfrak{h}'}(B_1^{\mathfrak{h}'}) = 1$, then it is easy to see that the claim implies that $\nu_{\mathfrak{p}} = \tilde{\nu} \times m_{\mathfrak{h}'}$ where $\tilde{\nu}(A) = \nu_{\tilde{\mathfrak{p}}}(A + B_1^{\mathfrak{h}'})$ for any measurable $A \subseteq \tilde{\mathfrak{p}}$.

For any weight λ such that $\lambda(\mathbf{m})$ is maximal, we have $[v, \mathfrak{p}] = 0$ for any $v \in \mathfrak{h}^\lambda$. If additionally $v \in \mathfrak{h}'$, then $w \mapsto w * v = w + v$ preserves $\nu_{\mathfrak{p}}$. Suppose now the claim has been shown for all vectors $v \in \mathfrak{h}' \cap \mathfrak{h}^\xi$ whenever $\xi(\mathbf{m}) > r$, and let $v \in \mathfrak{h}' \cap \mathfrak{h}^\lambda$ with $\lambda(\mathbf{m}) = r$. Let $\mathfrak{h}_{\geq r} = \sum_{\xi:\xi(\mathbf{m}) \geq r} \mathfrak{h}' \cap \mathfrak{h}^\xi$ and $\mathfrak{h}_{<r} = \sum_{\xi:\xi(\mathbf{m})<r} \mathfrak{h}' \cap \mathfrak{h}^\xi$. Then $\mathfrak{p} = (\tilde{\mathfrak{p}} + \mathfrak{h}_{\geq r}) \oplus \mathfrak{h}_{<r}$ and, as above, we already know that $\nu_{\mathfrak{p}} = \nu_{\geq r} \times m_{\mathfrak{h}_{<r}}$ for some measure $\nu_{\geq r}$ on $\mathfrak{p} + \mathfrak{h}_{\geq r}$. By assumption $w \mapsto w * v$ preserves $\nu_{\mathfrak{p}}$. By (3.1), $w * v = w + v + \frac{1}{2}[w, v] + \cdots$ where $[w, v], \ldots \in \mathfrak{h}_{<r}$ since $H' \subseteq P$ is a normal subgroup. Proposition 4.9 shows that the additional terms

can be viewed as a shear along the directions in $\mathfrak{h}_{<r}$. It is easily checked that this does not affect the measure, so that $\nu_{\mathfrak{p}}$ is in fact invariant under translation by v. ∎

Proof of Theorem 8.4: By Lemma 8.6 we already know that the theorem holds for a particular order of the coarse Lyapunov subgroups. So suppose we have them in this order and let π be a permutation of $\{1, \ldots, \ell\}$. Then we wish to show that $\mu_x^H \propto (\phi_\pi)_*(\mu_x^{\Lambda_{\pi(1)}} \times \cdots \times \mu_x^{\Lambda_{\pi(\ell)}})$, where

$$\phi_\pi\colon H^{\Lambda_{\pi(1)}} \times \cdots \times H^{\Lambda_{\pi(\ell)}} \to H$$

is the homeomorphism defined by $\phi_\pi(g_{\pi(1)}, \ldots, g_{\pi(\ell)}) = g_{\pi(1)} \cdots g_{\pi(\ell)}$.

Let

$$\chi\colon H^{\Lambda_1} \times \cdots \times H^{\Lambda_\ell} \to H^{\Lambda_1} \times \cdots \times H^{\Lambda_\ell}$$

be the homeomorphism that satisfies $\chi(g_1, \ldots, g_\ell) = (g_1', \ldots, g_\ell')$ if and only if $g_1' \cdots g_\ell' = g_{\pi(1)} \cdots g_{\pi(\ell)}$. Then the theorem follows if we know that χ preserves $\mu_x^{\Lambda_1} \times \cdots \times \mu_x^{\Lambda_\ell}$. To see this let $\nu = \mu_x^{\Lambda_1} \times \cdots \times \mu_x^{\Lambda_\ell}$, let $\nu_\pi = \mu_x^{\Lambda_{\pi(1)}} \times \cdots \times \mu_x^{\Lambda_{\pi(\ell)}}$, and let

$$\hat{\pi}\colon H^{\Lambda_1} \times \cdots \times H^{\Lambda_\ell} \to H^{\Lambda_{\pi(1)}} \times \cdots \times H^{\Lambda_{\pi(\ell)}}$$

be the map that permutes the coordinates according to π. Then $\hat{\pi}_*\nu = \nu_\pi$ and $\phi_\pi \circ \hat{\pi} = \phi \circ \chi$. So if $\chi_*\nu = \nu$, we conclude that

$$(\phi_\pi)_*\nu_\pi = (\phi_\pi \circ \hat{\pi})_*\nu = (\phi \circ \chi)_*\nu = \phi_*\nu \propto \mu_x^H$$

as claimed in Theorem 8.4.

Suppose $\chi(g_1, \ldots, g_\ell) = (g_1', \ldots, g_\ell')$ where $g_i, g_i' \in P_x \cap H^{\Lambda_i}$ for $i = 1, \ldots, \ell$. By Theorem 8.5 (4) we already know that $g_i H_x = g_i' H_x$ for $i = 1, \ldots, \ell$. Moreover, it follows from (3.2) and Proposition 4.9 that the weight components to any weight λ satisfy

$$(\log(g_i'))^\lambda = (\log(g_i))^\lambda + \cdots,$$

where the dots indicate $[\cdot, \cdot]$-monomials that depend only on the weight components $\log(g_j)^\xi$ for $\xi(\mathbf{m}) < \lambda(\mathbf{m})$ and at least two different j (assuming that H is $\mathbf{m}$-unstable). Therefore all of the additional terms belong to $\mathfrak{h}_x = \log H_x$, so the map induced by χ on $\mathfrak{p}_x$ is just a shear along the subgroup $\mathfrak{h}_x$.

We apply Lemma 8.11 for each of the coarse Lyapunov subgroups H^{Λ_i} and the measure $\mu_x^{\Lambda_i}$ which, by Theorem 8.5 (5), is invariant under multiplication from the right by elements of $H_x \cap H^{\Lambda_i}$. Then the logarithmic image $\log_* \mu_x^{\Lambda_i}$

is invariant under translation by elements of $\mathfrak{h}_x \cap \mathfrak{h}^{\Lambda_i}$. Therefore and from the description of χ above, it follows that χ preserves ν as claimed. ∎

9. Linking conditional measures with entropy, and the high entropy case for rigidity of measures

9.1. Volume decay and entropy. In this section we give a formula relating entropy and conditional measures. We will use and slightly reformulate the results from [25, Sect. 9].

Lemma 9.1: *Let X be a G_S-space for an S-Lie group G_S. Let α be a $\mathbb{Z}^k$-action on X that preserves the G_S-leaves (as in Definition 3.1), and let θ be the corresponding $\mathbb{Z}^k$-action by automorphisms of G_S. Assume that the adjoint action A on the S-Lie algebra $\mathfrak{g}_S$ is semisimple. Let $H \subset G_S$ be an $\mathbf{m}$-stable subgroup for some $\mathbf{m} \in \mathbb{Z}^k$ (as in Definition 4.10). Then for any α-invariant probability measure μ on X the limit*

$$\mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, x) = -\lim_{n\to\infty} \frac{\log \mu_x^H(\theta^{n\mathbf{m}}(B_1^H))}{n}$$

exists for a.e. $x \in X$. If furthermore μ_x^H is supported by a subgroup $P_x \subseteq H$ that allows a weight decomposition, then

$$\mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, x) \le \mathrm{mod}(\alpha^{\mathbf{m}}, P_x) = -\sum_{\sigma\in S}\sum_{\lambda} \lambda(\mathbf{m}) \dim_{\mathbb{Q}_\sigma}(\mathfrak{p}_x \cap \mathfrak{h}_\sigma^\lambda) \tag{9.1}$$

for a.e. $x \in X$. Here $\mathfrak{p}_x = \log P_x$ and λ are all possible weights of H.

We call the expression $\mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, x)$ **volume decay entropy at** x — it can be thought of as a combination of the dimension of μ_x^H at x and the contraction rates of $\theta^{\mathbf{m}}$ (see Ledrappier–Young's entropy formula [20]). In case P_x is invariant under θ, $\mathrm{mod}(\alpha^{\mathbf{m}}, P_x)$ is the negative logarithm of the module of the restriction of $\theta^{\mathbf{m}}$ to P_x. More generally, there exist $c_1, c_2 > 0$ such that

$$c_1 \le m_{P_x}(P_x \cap \theta^{n\mathbf{m}}(B_1^H))e^{n \,\mathrm{mod}(\alpha^{\mathbf{m}}, P_x)} \le c_2$$

where m_{P_x} is the Haar measure of P_x.

Proof: Let $f(x) = \mu_x^H(\theta^{\mathbf{m}}(B_1^H))$. Then (5.6) and Proposition 5.6 (1) show that

$$f(\alpha^{-j\mathbf{m}}x) = \mu^H_{\alpha^{-j\mathbf{m}}x}(\theta^{\mathbf{m}}(B_1^H)) = \frac{\mu_x^H(\theta^{(j+1)\mathbf{m}}(B_1^H))}{\mu_x^H(\theta^{j\mathbf{m}}(B_1^H))}$$

for a.e. x and all $j \in \mathbb{Z}$, and so

$$-\sum_{j=0}^{n-1} \log f(\alpha^{-j\mathbf{m}} x) = -\log \mu_x^H(\theta^{n\mathbf{m}}(B_1^H)).$$

Furthermore, $-\log f(x) \geq 0$ and so

$$-\frac{1}{n} \log \mu_x^H(\theta^{n\mathbf{m}}(B_1^H)) \to \mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, x) \quad \text{a.e.} \tag{9.2}$$

by the ergodic theorem, where $\mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, \cdot)\colon X \to [0,\infty]$ is measurable.

By Luzin's theorem, there exists a compact set K with measure almost equal to one such that $\mathrm{vol}(\alpha^{\mathbf{m}}, H, x)$ in (9.2) exists and is continuous for $x \in K$. Furthermore, we can assume by Proposition 5.6 (3) and Lemma 5.9 that

$$(R_g)_* \mu_{gx}^H \propto \mu_x^H \tag{9.3}$$

holds for $x \in K$ and μ_x^H-a.e. $g \in H$. We claim that every $x \in K$ with $\mu_x^H(\{g \in B_\delta^H : gx \in K\}) > 0$ for every $\delta > 0$ satisfies (9.1) if μ_x^H is supported by P_x. We show this by showing that gx satisfies (9.1) for μ_x^H-a.e. $g \in B_1^H$. The claim then follows by continuity.

Let $\epsilon \geq 0$, let n be a positive integer, and define

$$M_\epsilon = \{g \in B_1^H : \mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, gx) > \mathrm{mod}(\alpha, P_x) + \epsilon\}$$

and

$$M_{\epsilon,n} = \Big\{g \in B_1^H : -\frac{1}{n} \log \mu_x^H(\theta^{n\mathbf{m}}(B_1^H)g) > \mathrm{mod}(\alpha, P_x) + \epsilon\Big\}.$$

Let $g \in M_\epsilon$ such that (9.2) and (9.3) hold for gx. Then $\mu_x^H(Ag) = C\mu_{gx}^H(A)$ for some $C > 0$ and all measurable $A \subseteq H$. Therefore,

$$-\frac{1}{n} \log \mu_x^H(\theta^{n\mathbf{m}} B_1^H g) \to \mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, gx)$$

and $g \in M_{\epsilon,n}$ for large enough n.

Fix for every weight λ a basis $v_1^\lambda, \ldots, v_{d(\lambda)}^\lambda$ of $\mathfrak{p}_x \cap \mathfrak{h}_\infty^\lambda$ and let

$$D_n = \exp\Big(\sum_\lambda \sum_{i=1}^{d(\lambda)} e^{\lambda(n\mathbf{m})}[-\delta, \delta)v_i^\lambda + \sum_\lambda B_{e^{\lambda(n\mathbf{m})}\delta}^{\mathfrak{h}_{S_f}^\lambda}\Big).$$

Then for small enough δ (independently of n) the set $P_x \cap \theta^{n\mathbf{m}}(B_1^H)$ contains $D_n D_n^{-1}$. It is easy to find a partition of P_x consisting of right translates $D_n g$ for g in some index set I. Since the Haar measure of P_x is the image of the Haar measure of $\mathfrak{p}_x$ (see Lemma 8.11), we have $m_{P_x}(D_n) = e^{-n\,\mathrm{mod}(\alpha^{\mathbf{m}},P_x)} m_{P_x}(D_1)$ for all n. Let $I' = \{g \in I : D_n g \subseteq B_2^{P_x}\}$; then $B_1^{P_x} \subseteq \bigcup_{g\in I'} D_n g \subseteq B_2^{P_x}$ for large enough n. Therefore,

$$|I'| \le c e^{n\,\mathrm{mod}(\alpha^{\mathbf{m}},P_x)} \quad \text{with } c = \frac{m_{P_x}(B_2^{P_x})}{m_{P_x}(D_1)}.$$

Furthermore, let $J = \{g \in I' : D_n g \cap M_{\epsilon,n} \neq \emptyset\}$. If $g \in J$ and $h \in D_n g \cap M_{\epsilon,n}$, then

$$\mu_x^H(\theta^{n\mathbf{m}}(B_1^H)h) < e^{-n\,\mathrm{mod}(\alpha^{\mathbf{m}},P_x)} e^{-n\epsilon}$$

by definition of $M_{\epsilon,n}$. Clearly $D_n g \subseteq D_n D_n^{-1} h \subseteq \theta^{n\mathbf{m}}(B_1^H)h$, and using the above estimate for every $g \in J$ we obtain $\mu_x^H(M_{\epsilon,n}) < ce^{-n\epsilon}$. Therefore

$$\mu_x^H(M_\epsilon) \le \mu_x^H\left(\bigcup_{\ell=n}^{\infty} M_{\epsilon,\ell}\right) < \frac{ce^{-n\epsilon}}{1-e^{-\epsilon}},$$

which shows that M_ϵ is a null set for every $\epsilon > 0$. Therefore, $\mu_x^H(M_0) = 0$ as claimed. ∎

Definition 9.2: Let H be as above. A σ-algebra $\mathcal{A}$ of Borel subsets of X is **subordinate to** H if $\mathcal{A}$ is countably generated, for every $x \in X$ the atom $[x]_{\mathcal{A}}$ of x with respect to $\mathcal{A}$ is contained in the leaf Hx, and for a.e. x

$$B_\epsilon^H x \subseteq [x]_{\mathcal{A}} \subseteq B_\rho^H x \quad \text{for some } \epsilon > 0 \text{ and } \rho > 0.$$

A σ-algebra $\mathcal{A}$ is **m-decreasing** if $\alpha^{-\mathbf{m}}\mathcal{A} \subseteq \mathcal{A}$.

LEMMA 9.3: *Assume in addition to the assumptions of Lemma 9.1 that $\mathcal{A}$ is an* $\mathbf{m}$*-decreasing σ-algebra that is subordinate to H. Then*

$$\mathsf{H}_\mu(\mathcal{A}|\alpha^{-\mathbf{m}}\mathcal{A}) = \int \mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, x)\mathrm{d}\mu.$$

Proof: Recall that $\mathsf{H}_\mu(\mathcal{A}|\alpha^{-\mathbf{m}}\mathcal{A}) = \int \mathrm{I}_\mu(\mathcal{A}|\alpha^{-\mathbf{m}}\mathcal{A})(x)\mathrm{d}\mu$ and that

$$\mathrm{I}_\mu(\mathcal{A}|\alpha^{-\mathbf{m}}\mathcal{A})(x) = -\log \mu_x^{\alpha^{-\mathbf{m}}\mathcal{A}}([x]_{\mathcal{A}}) = -\log \frac{\mu_x^H(\{g : gx \in [x]_{\mathcal{A}}\})}{\mu_x^H(\{g : gx \in [x]_{\alpha^{-\mathbf{m}}\mathcal{A}}\})}$$

where we used Proposition 5.6 (2) in the last equation. We define $k(x) = \mu_x^H(\{g : gx \in [x]_{\mathcal{A}}\})$. It is easy to check that k is measurable. By (5.6)

$$\begin{aligned} k(\alpha^{\mathbf{m}}(x)) &= \mu_{\alpha^{\mathbf{m}}(x)}^H(\{g : g\alpha^{\mathbf{m}}(x) \in [\alpha^{\mathbf{m}}(x)]_{\mathcal{A}}\}) \\ &= \mu_x^H(\{\theta^{-\mathbf{m}}g : g\alpha^{\mathbf{m}}(x) \in [\alpha^{\mathbf{m}}(x)]_{\mathcal{A}}\})\mu_{\alpha^{\mathbf{m}}(x)}^H(\theta^{\mathbf{m}}B_1^H) \\ &= \mu_x^H(\{h : \theta^{\mathbf{m}}(h)\alpha^{\mathbf{m}}(x) \in [\alpha^{\mathbf{m}}(x)]_{\mathcal{A}}\})f(\alpha^{\mathbf{m}}(x)) \\ &= \mu_x^H(\{h : hx \in [x]_{\alpha^{-\mathbf{m}}\mathcal{A}}\})f(\alpha^{\mathbf{m}}(x)), \end{aligned}$$

where f is as in the proof of Lemma 9.1. Therefore

$$\mathrm{I}_\mu(\mathcal{A}|\alpha^{-\mathbf{m}}\mathcal{A})(x) = -\log k(x) + \log k(\alpha^{\mathbf{m}}x) - \log f(\alpha^{\mathbf{m}}x)$$

and

$$\frac{1}{n}\sum_{j=0}^{n-1} \mathrm{I}_\mu(\mathcal{A}|\alpha^{-\mathbf{m}}\mathcal{A})(\alpha^{j\mathbf{m}}x) = \frac{1}{n}(\log k(\alpha^{n\mathbf{m}}x) - \log k(x)) - \frac{1}{n}\sum_{j=0}^{n-1}\log f(\alpha^{j\mathbf{m}}x).$$

Here the left hand side converges to a measurable function whose integral is $\mathsf{H}_\mu(\mathcal{A}|\alpha^{-\mathbf{m}}\mathcal{A})$. The sum on the right converges to $\mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, x)$ by (9.2). For the remaining two terms on the right Poincaré recurrence shows that for a.e. x the difference is close to zero for arbitrarily large n. The lemma follows. ∎

PROPOSITION 9.4: *Let G_S be an S-Lie group, let $\Gamma \subset G_S$ be a discrete subgroup, and let $X = G_S/\Gamma$. Let α be an algebraic $\mathbb{Z}^k$-action on X that is either defined by left translation on X or induced by automorphisms of G_S. Assume that the adjoint action on the Lie algebra $\mathfrak{g}_S$ is semisimple. Fix some $\mathbf{m} \in \mathbb{Z}^k$ and let H be the maximal $\mathbf{m}$-stable subgroup of G_S. Then*

$$\mathsf{h}_\mu(\alpha^{\mathbf{m}}) = \int \mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, x)\mathrm{d}\mu$$

for any α-invariant probability measure on X. If additionally μ is α-ergodic, then

$$\mathsf{h}_\mu(\alpha^{\mathbf{m}}) = \mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, x)$$

for a.e. $x \in X$.

Proof: For the second statement it is enough to check that $\mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, x) = \mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, \alpha^{\mathbf{n}}x)$ whenever x satisfies (5.6) for $\alpha^{\mathbf{n}}$. More generally, if $C \subseteq H$ is measurable and $\theta^{\ell_1\mathbf{m}}B_1^H \subseteq C \subseteq \theta^{\ell_2\mathbf{m}}B_1^H$, then it is easy to check that $\mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, x) = \lim_{n\to\infty} -\frac{1}{n}\log\mu_x^H(\theta^{n\mathbf{m}}C)$.

For the first statement, assume first that μ is an $\alpha^{\mathbf{m}}$-invariant and ergodic measure μ. By [25, Prop. 9.2] there exists a σ-algebra $\mathcal{A}$ that is subordinate to H and is $\mathbf{m}$-decreasing. Moreover, $\mathsf{h}_\mu(\alpha^{\mathbf{m}}) = \mathsf{H}_\mu(\mathcal{A}|\alpha^{-\mathbf{m}}\mathcal{A})$. Lemma 9.3 shows the first statement of the proposition in the $\alpha^{\mathbf{m}}$-ergodic case.

Let now μ be any α-invariant measure, and let $\mu = \int \mu_x^{\mathcal{E}}\mathrm{d}\mu$ be the ergodic decomposition of μ with respect to $\alpha^{\mathbf{m}}$. Here $\mathcal{E}$ is the σ-algebra of $\alpha^{\mathbf{m}}$-invariant sets; see also [21, Sect. 5]. Then $\mathsf{h}_\mu(\alpha^{\mathbf{m}}) = \int_W \mathsf{h}_{\mu_x^{\mathcal{E}}}(\alpha^{\mathbf{m}})\mathrm{d}\mu$. On the other hand, the first paragraph above shows that $\mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, x)$ is $\alpha^{\mathbf{m}}$-invariant, and therefore $\mathcal{E}$-measurable. This implies that

$$\mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, x) = \mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, y) \quad \text{for } \mu_x^{\mathcal{E}}\text{-a.e. } y$$

holds for a.e. x. Recall that $\mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, x)$ is defined by the conditional measure μ_x^H. By [21, Cor. 5.4] the conditional measures of the ergodic components satisfy $(\mu_x^{\mathcal{E}})_y^H = \mu_y^H$ for μ-a.e. x and $\mu_x^{\mathcal{E}}$-a.e. y. Suppose x satisfies all of this and that $\mu_x^{\mathcal{E}}$ is an $\alpha^{\mathbf{m}}$-invariant and ergodic measure. Then we have already shown that $\mathsf{h}_{\mu_x^{\mathcal{E}}}(\alpha^{\mathbf{m}}) = \mathrm{vol}_{\mu_x^{\mathcal{E}}}(\alpha^{\mathbf{m}}, H, y) = \mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, y)$ for $\mu_x^{\mathcal{E}}$-a.e. y, which implies that $\mathsf{h}_{\mu_x^{\mathcal{E}}}(\alpha^{\mathbf{m}}) = \mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, x)$ by our assumption on x above. ∎

9.2. Applying Theorem 8.5 in the high entropy case.

THEOREM 9.5: *Let X be a G_S-space for an S-Lie group G_S, and let α be a $\mathbb{Z}^k$-action on X that preserves the G_S-leaves whose adjoint action on the Lie algebra $\mathfrak{g}_S$ is semisimple. Let $H \subseteq G_S$ be an $\mathbf{m}$-stable subgroup for some $\mathbf{m} \in \mathbb{Z}^k$. There exists some $q < 1$ and an S-Lie subgroup $H' \subseteq H$ that is invariant under the induced action θ on G_S and is the image under* exp *of an S-Lie subalgebra $\mathfrak{h}'$ with the following properties for every α-invariant probability measure μ on X.*

(1) *If $\Lambda_1 \neq \Lambda_2$, then $[\log H^{\Lambda_1}, \log H^{\Lambda_2}] \subseteq \log H'$.*
(2) *For a.e. $x \in X$, if*

$$\mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H, x) > q \,\mathrm{mod}(\alpha^{\mathbf{m}}, H), \tag{9.4}$$

then μ_x^H is invariant under left and right multiplication by elements of H'.
(3) *If (9.4) holds a.e., then μ is invariant under left multiplication by elements of H'.*

Note that the above does not assume that the adjoint action has no rank one factors. Because of this, Theorem 9.5 can also be applied in the case of a higher rank action by automorphisms of a nilmanifold.

Proof: Let

$$q = \max_P \frac{\mathrm{mod}(\alpha^{\mathbf{m}}, P)}{\mathrm{mod}(\alpha^{\mathbf{m}}, H)}$$

where the maximum is taken over all proper subgroups $P \subsetneq H$ that allow a weight decomposition. Let $\mathfrak{h}'$ be the S-Lie ideal of $\mathfrak{h}$ that is generated by all $[\mathfrak{h}^{\Lambda_1}, \mathfrak{h}^{\Lambda_2}]$ for different coarse Lyapunov weights $\Lambda_1 \neq \Lambda_2$ of $\mathfrak{h}$. Then $H' = \exp \mathfrak{h}'$ satisfies (1). Suppose x satisfies Theorem 8.5, Lemma 9.1, and (9.4). Then the choice of q shows that $P_x = H$. Since H_x allows a weight decomposition, it follows that $\mathfrak{h}' \subseteq \mathfrak{h}_x = \log H_x$. This shows (2). The last statement follows from Proposition 5.7 applied to the H'-space X. ∎

We conclude the paper by the proofs of our results from Section 2. For this we will need the following lemma.

Lemma 9.6: *Let A be the restriction of an adjoint action to an invariant S-Lie algebra $\mathfrak{h}$, and let V be a subspace of the dual of $\mathbb{R}^k$. Then $\mathfrak{h}' = \langle \mathfrak{h}^\zeta, [\mathfrak{h}^\zeta, \mathfrak{h}^\xi] : \zeta, \xi \notin V \rangle$ is an A-invariant S-Lie ideal in $\mathfrak{h}$.*

Therefore, if there are no rank one factors, then every element of $\mathfrak{h}^\lambda$ is a sum of expressions $[v, w]$ with $v \in \mathfrak{h}^\zeta$, $w \in \mathfrak{h}^\xi$ and $\zeta, \xi \notin \mathbb{R}\lambda$.

Proof: Clearly $\mathfrak{h}' = \sum_{\sigma \in S} \mathfrak{h}' \cap \mathfrak{g}_\sigma$. Moreover, it is clear that $\mathfrak{h}'$ is invariant under A. It remains to show that $[\mathfrak{h}^\eta, \mathfrak{h}'] \subseteq \mathfrak{h}'$ for all η.

Suppose first that $u \in \mathfrak{h}^\eta$ and $v \in \mathfrak{h}^\zeta$ with $\zeta \notin V$. If $\eta \notin V$ then $[u, v] \in \mathfrak{h}'$, and otherwise $[u, v] \in \mathfrak{h}^{\eta+\zeta} \subseteq \mathfrak{h}'$ because $\eta + \zeta \notin V$.

Let $u \in \mathfrak{h}^\eta$, $v \in \mathfrak{h}^\zeta$, and $w \in \mathfrak{h}^\xi$ with $\zeta, \xi \notin V$. Again, if $\eta + \zeta + \xi \notin V$ then there is nothing to show. So assume $\eta + \zeta + \xi \in V$. If $\eta \notin V$ then $\zeta + \xi \notin \mathbb{R}\lambda$ and we are again done. The remaining case is $\eta, \eta + \zeta + \xi \in V$. By the Jacobi identity (3.1), $[u, [v, w]] = -[v, [w, u]] - [w, [u, v]]$, and the two expressions on the right belong to $\mathfrak{h}'$ since $\zeta, \xi + \eta, \xi, \eta + \zeta \notin V$.

For the final statement let $V = \mathbb{R}\lambda$. Then $\mathfrak{h}'$ satisfies (1)–(3) of Definition 2.3. Since there are no rank one factors, we have $\mathfrak{h}' = \mathfrak{h}$ and every $u \in \mathfrak{h}^\lambda$ belongs to $\mathfrak{h}'$. By restricting the sum that expresses u to those terms that belong to $\mathfrak{h}^\lambda$ the lemma follows. ∎

Proof of Theorem 2.4: By Proposition 9.4 we have $\mathrm{vol}_\mu(\alpha^{\mathbf{m}}, H^-, x) = \mathsf{h}_\mu(\alpha^{\mathbf{m}})$ for a.e. x. As in the proof of Theorem 9.5 this implies for large enough $q < 1$ that $P_x = H^-$ for a.e. x, where P_x is as in Theorem 8.5. Let Λ be a coarse Lyapunov weight of H^-. Then, moreover, μ_x^Λ is not supported by any proper subgroup of G_S^Λ that allows a weight decomposition. The same applies similarly for H^+.

We claim for a.e. x that $\mu_x^{\mathbb{R}^+\zeta}$ is left invariant under multiplication by $\exp(w)$ for any $w \in \mathfrak{h}^{\mathbb{R}^+\zeta}$ whenever $\zeta(\mathbf{m}) = 0$ but $\zeta \neq 0$. By definition $\mathfrak{h}$ is the Lie algebra generated by $\mathfrak{h}^-$ and $\mathfrak{h}^+$, so every element of $\mathfrak{h}^{\mathbb{R}\zeta}$ can be written as a sum of $[\cdot,\cdot]$-monomials w in vectors $u \in \mathfrak{h}^- \cup \mathfrak{h}^+$. Moreover, by Lemma 9.6 applied to $V = \{\zeta : \zeta(\mathbf{m}) = 0\}$ it is enough to consider quadratic monomials $w = [u_1, u_2]$. By Proposition 6.2 it is enough to show the claim for every individual such w.

Let $w = [u_1, u_2]$ with $u_i \in \mathfrak{g}_S^{\lambda_i}$, $\lambda_1(\mathbf{m}) < 0$ and $\lambda_2(\mathbf{m}) > 0$. Then $\zeta = \lambda_1 + \lambda_2 \neq 0$, λ_1 and λ_2 are linearly independent, and we can find $\mathbf{n} \in \mathbb{Z}^k$ with $\zeta(\mathbf{n}), \lambda_1(\mathbf{n}), \lambda_2(\mathbf{n}) < 0$. We apply Theorem 8.5 for the maximal $\mathbf{n}$-stable S-Lie subgroup $H_{\mathbf{n}}^-$ that satisfies $\log H_{\mathbf{n}}^- \subseteq \mathfrak{h}_{\mathbf{m}}$. Since $\mu_x^{\mathbb{R}\lambda_i}$ are not supported by any smaller subgroup than $G_S^{\mathbb{R}\lambda_i}$, the claim follows for w.

Now let Λ be a coarse Lyapunov weight such that $\Lambda = \mathbb{R}\lambda$ and $\lambda(\mathbf{m}) < 0$. Let $w \in \mathfrak{h}_{\mathbf{m}}^{\lambda}$. By Lemma 9.6 we can assume that w is (a finite sum of $[\cdot,\cdot]$-binomials) $[v_1, v_2] \in \mathfrak{h}_{\mathbf{m}}^{\lambda}$ where $v_i \in \mathfrak{h}_{\mathbf{m}}^{\xi_i}$ and $\xi_1, \xi_2 \notin \mathbb{R}\lambda$. Clearly ξ_1 and ξ_2 are linearly independent and there exists a $\mathbf{n} \in \mathbb{Z}^k$ with $\xi_1(\mathbf{n}), \xi_2(\mathbf{n}), \lambda(\mathbf{n}) < 0$. We apply Theorem 8.5 for the maximal $\mathbf{n}$-stable S-Lie group $H_{\mathbf{n}}^-$ that satisfies $\log H_{\mathbf{n}}^- \subseteq \mathfrak{h}_{\mathbf{m}}$. If $\xi_1(\mathbf{m}) \neq 0$ then we know $\exp v_1 \in P_x$ a.e., since $\mu_x^{\mathbb{R}\xi_1}$ is not supported by any smaller subgroup. If $\xi_1(\mathbf{m}) = 0$ then we already showed $(L_{\exp(v_1)})_* \mu_x^{\mathbb{R}\xi_1} = \mu_x^{\mathbb{R}\xi_1}$, which implies that $\exp v_1 \in \operatorname{supp} \mu_x^{\mathbb{R}\xi_1} \subseteq P_x$ a.e. The same holds similarly for v_2. It follows that $(L_{\exp(w)})_* \mu_x^{\Lambda} = \mu_x^{\Lambda}$ a.e. Since $w \in \mathfrak{h}_{\mathbf{m}}^{\Lambda}$ and Λ was arbitrary, it follows that μ is left invariant under H^- by Proposition 5.7. ∎

LEMMA 9.7: *The Lie algebra of G_{tw} is $\mathfrak{g}_{\mathrm{tw}} = \mathbb{R}^n \rtimes \mathfrak{g}$ where $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r$ is the Lie algebra of G and $\mathfrak{g}_i$ for $i = 1, \ldots, r$ are the simple factors of $\mathfrak{g}$. The commutator in $\mathfrak{g}_{\mathrm{tw}}$ is given by*

$$[(u,a),(v,b)] = (D\rho(a)(v) - D\rho(b)(u), [a,b]) \tag{9.5}$$

where $D\rho\colon \mathfrak{g} \to \mathrm{SL}(n,\mathbb{R})$ is the derivative of ρ (at the identity element). Therefore, $D\rho(\mathfrak{h} \cap \mathfrak{g})(\mathbb{R}^n) \subset \mathfrak{h}$ for every Lie ideal $\mathfrak{h} \subset \mathfrak{g}_{\mathrm{tw}}$.

Proof: Recall that the multiplication in G_{tw} is defined by

$$(u,g)\cdot(v,h) = (u + \rho(g)(v), gh).$$

From this it is easy to check that (9.5) holds. The last statement follows from (9.5). ∎

Proof of Theorem 2.7: We assume that the twisted Weyl chamber flow α_{tw} on X_{tw} has no local rank one factors and acts without center on the torus fibers (in the sense of Definition 2.5–2.6). Furthermore, let $\mathbf{t} \in \mathbb{R}^k$ be as in the theorem, so that for every simple factor of G there exists a root λ with $\operatorname{Re}\lambda^{(\alpha)}(\mathbf{t}) \neq 0$.

We are going to apply Theorem 2.4 for the restriction $\tilde{\alpha}$ of α to some lattice in $\mathbb{R}^k$ that contains $\mathbf{t}$. Since α embeds $\mathbb{R}^k$ into a Cartan subgroup of G and G is semisimple, it follows easily that the adjoint action to α acts by semisimple elements on $\mathfrak{g}_{\mathrm{tw}}$. Furthermore, notice that $\xi = \operatorname{Re}\lambda^{(\alpha)}$ are the Lyapunov weights of $\tilde{\alpha}$ when λ goes through the roots λ of $\mathfrak{g}$ and the weights λ of the representation ρ.

Now let $\mathfrak{h} \subseteq \mathfrak{g}_{\mathrm{tw}}$ be the Lie algebra generated by the $\mathbf{t}$-stable and $\mathbf{t}$-unstable Lie algebras $\mathfrak{h}^-$ and $\mathfrak{h}^+$. Let $V = \{\xi : \xi(\mathbf{t}) = 0\}$. Then $\mathfrak{h}$ is the Lie algebra generated by all Lyapunov weight spaces $\mathfrak{g}_{\mathrm{tw}}^{\xi}$ with $\xi \notin V$. Lemma 9.6 shows $\mathfrak{h}$ is in fact a Lie ideal in $\mathfrak{g}_{\mathrm{tw}}$. By the choice of $\mathbf{t}$ we have $\mathfrak{g}_i \cap \mathfrak{h} \neq 0$ for all i, and so $\mathfrak{g} \subset \mathfrak{h}$ since $\mathfrak{h} \subseteq \mathfrak{g}_{\mathrm{tw}}$ is an ideal. By Lemma 9.7 we have $W = D\rho(\mathfrak{g})(\mathbb{R}^n) \subset \mathfrak{h}$. Clearly W is invariant under $D\rho(a)$ for any $a \in \mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, this implies that there exists an invariant complement $W^\perp \subset \mathbb{R}^n$ to W. Clearly $D\rho(\mathfrak{g})(W^\perp) \subseteq W^\perp \cap W = \{0\}$ and so $D\rho(a)$ acts trivially on $W^\perp$. This contradicts the fact that α acts without center on the torus fibers unless $W^\perp = 0$. Therefore, we conclude that $\mathfrak{h} = \mathfrak{g}_{\mathrm{tw}}$.

Next we show that the adjoint action on $\mathfrak{g}_{\mathrm{tw}}$ has no rank one factors. So assume that the Lie ideal $\mathfrak{h}' \subset \mathfrak{g}_{\mathrm{tw}}$ is as in Definition 2.3. If for some i the simple Lie algebra $\mathfrak{g}_i$ is not part of $\mathfrak{h}'$, then our assumption on α contradicts Definition 2.3 (3). Therefore, $\mathfrak{g} \subset \mathfrak{h}'$ and just as above $\mathbb{R}^n \subset \mathfrak{h}'$ as well, which shows that there are no rank one factors.

Finally, let μ be a measure on X_{tw} as in Theorem 2.7. Then μ is invariant under $\tilde{\alpha}$ but possibly not ergodic. However, it is easy to see that almost all of its ergodic components ν have the same entropy for the element $\alpha^{\mathbf{t}}$, i.e. $\mathsf{h}_\nu(\alpha^{\mathbf{t}}) = \mathsf{h}_\mu(\alpha^{\mathbf{t}})$. We choose q as in Theorem 2.4, and conclude that ν is invariant under $H^- = \exp\mathfrak{h}^-$ and $H^+ = \exp\mathfrak{h}^+$. Let $H_\nu \subseteq G_{\mathrm{tw}}$ be the maximal subgroup such that μ is left invariant under all of its elements. Clearly H_ν is closed, and since $H^-, H^+ \subset H_\nu$ it follows that the Lie algebra to H_ν is $\mathfrak{g}_{\mathrm{tw}}$. Since G_{tw} is connected, $H_\nu = G_{\mathrm{tw}}$ and ν is the Haar measure of X_{tw}. Since this holds for almost all ergodic components ν of μ, we conclude that μ must be the Haar measure of X_{tw}. ∎

Proof of Theorem 2.8: For $\mathbf{m} \in \mathbb{Z}^k$ we define $m_{k+1} = -(m_1 + \cdots + m_k)$. Suppose $\sigma = p$ is a rational prime. Then $\alpha_p^{\mathbf{m}}$ is the diagonal matrix with entries $p^{m_1}, \ldots, p^{m_{k+1}}$ and it is easy to check that

$$\alpha_p^{\mathbf{m}}(I_{k+1} + tE_{ab})\alpha_p^{-\mathbf{m}} = I_{k+1} + p^{m_a - m_b} tE_{ab} \tag{9.6}$$

for every $t \in \mathbb{Q}_p$ and every pair $a \neq b$ of indices between 1 and $k+1$. Similarly, $\alpha_\infty^{\mathbf{m}}$ is the diagonal matrix with entries $e^{m_1}, \ldots, e^{m_{k+1}}$ and

$$\alpha_\infty^{\mathbf{m}}(I_{k+1} + tE_{ab})\alpha_\infty^{-\mathbf{m}} = I_{k+1} + e^{m_a - m_b} tE_{ab} \tag{9.7}$$

for every $t \in \mathbb{R}$.

It follows from (9.6) resp. (9.7) that $H_\sigma^{(i,j)}$ for $1 \leq i, j \leq k+1$ and $i \neq j$ are the coarse Lyapunov subgroups for α_σ. We will show that the corresponding conditional measures $\mu_x^{\sigma,(i,j)}$ are almost surely invariant under $H_\sigma^{(i,j)}$. Then the well-known Proposition 5.7 will show that μ is invariant under $H_\sigma^{(i,j)}$, and the theorem will follow since $\mathrm{SL}(k+1, \mathbb{Q}_\sigma)$ is generated by these subgroups.

Let ℓ be another index between 1 and $k+1$ (by assumption $k \geq 2$) and choose $\mathbf{m} \in \mathbb{Z}^k$ such that $m_i > m_\ell > m_j$. Then (9.6) and (9.7) show that $H = H_\sigma^{(i,j)} H_\sigma^{(i,\ell)} H_\sigma^{(\ell,j)}$ is an $\mathbf{m}$-stable subgroup if $\sigma = p$ and is an $\mathbf{m}$-unstable subgroup if $\sigma = \infty$. By assumption, there exists a null set N such that for $x \notin N$ none of the conditional measures $\mu_x^{\sigma,(i,j)}$, $\mu_x^{\sigma,(i,\ell)}$ and $\mu_x^{\sigma,(\ell,j)}$ are supported by the identity element alone. If $x \notin N$ satisfies in addition Theorems 8.4 and 8.5, then $P_x = H$, H_x contains $H_\sigma^{(i,j)}$, and $\mu_x^{\sigma,(i,j)}$ is invariant under $H_\sigma^{(i,j)}$ as claimed. ∎

Proof of Corollary 2.9: By Theorem 2.8 the first condition implies that μ is invariant under $\mathrm{SL}(k+1, \mathbb{Q}_\sigma)$. So does the second condition; this can be seen just as in the real case [6, Thm. 4.1(iv)].

Since μ is ergodic under α_σ and $\alpha_\sigma^{\mathbf{m}} \in \mathrm{SL}(k+1, \mathbb{Q}_\sigma)$ for $\mathbf{m} \in \mathbb{Z}^k$, μ is in fact invariant and ergodic under $\mathrm{SL}(k+1, \mathbb{Q}_\sigma)$. By [25, Thm. 2] or [31, Thm. 1] it follows that μ is algebraic. ∎

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