Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings

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1 Introduction

1.1 Background and statement of results

An (L, C) quasi-isometry is a map $\Phi : X \longrightarrow X'$ between metric spaces such that for all $x_1, x_2 \in X$ we have

$$L^{-1}d(x_1, x_2) - C \leq d(\Phi(x_1), \Phi(x_2)) \leq Ld(x_1, x_2) + C \tag{1}$$

and

$$d(x', Im(\Phi)) < C \tag{2}$$

for all $x' \in X'$. Quasi-isometries occur naturally in the study of the geometry of discrete groups since the length spaces on which a given finitely generated group acts cocompactly and properly discontinuously by isometries are quasi-isometric to one another [Gro]. Quasi-isometries also play a crucial role in Mostow's proof of his rigidity theorem: the theorem is proved by showing that equivariant quasi-isometries are within bounded distance of isometries.

This paper is concerned with the structure of quasi-isometries between products of symmetric spaces and Euclidean buildings. We recall that Euclidean space, hyperbolic space, and complex hyperbolic space each admit an abundance of self-quasi-isometries [Pan]. For example we get quasi-isometries $\mathbb{E}^2 \longrightarrow \mathbb{E}^2$ by taking shears in rectangular $(x_1, x_2) \mapsto (x_1, x_2 + f(x_1))$ or polar $(r, \theta) \mapsto (r, \theta + \frac{f(r)}{r})$ coordinates, where $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : [0, \infty) \longrightarrow \mathbb{R}$ are Lipschitz. Any diffeomorphism[1] $\Phi : \partial\mathbb{H}^n \longrightarrow \partial\mathbb{H}^n$ of the ideal boundary can be extended continuously to a quasi-isometry $\Phi : \mathbb{H}^n \longrightarrow \mathbb{H}^n$. Likewise any contact diffeomorphism[2] $\partial\Phi : \partial\mathbb{C}\mathbb{H}^n \longrightarrow \partial\mathbb{C}\mathbb{H}^n$) can be extended continuously to a quasi-isometry $\Phi : \mathbb{C}\mathbb{H}^n \longrightarrow \mathbb{C}\mathbb{H}^n$) [Pan]. Quasi-isometries of the remaining rank 1 symmetric spaces of noncompact type, on the other hand, are very special. They are essentially isometries:

Theorem 1.1.1 ([Pan]) *Let X be either a quaternionic hyperbolic space $\mathbb{H}\mathbb{H}^n$, $n > 1$, or the Cayley hyperbolic plane $\mathbb{C}a\mathbb{H}^2$. Then any quasi-isometry of X lies within bounded distance of an isometry.*

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[1]Any quasi-conformal homeomorphism arises as the boundary homeomorphism of a quasi-isometry by [Tuk].

[2]The boundary of $\mathbb{C}\mathbb{H}^n$ can be endowed with an $Isom(\mathbb{C}\mathbb{H}^n)$ invariant contact structure by projecting the contact structure from a unit tangent sphere $S^{2n-1}_p\mathbb{C}\mathbb{H}^n$ to $\partial\mathbb{C}\mathbb{H}^n$ using the exponential map.

Note that Pansu's theorem is a strengthening of Mostow's rigidity theorem for these rank one symmetric spaces X, as it applies to all quasi-isometries of X, whereas Mostow's argument only treats those quasi-isometries which are equivariant with respect to lattice actions. The main results of this paper are the following higher rank analogs of Pansu's theorem.

Theorem 1.1.2 (Splitting) *For $1 \le i \le k$, $1 \le j \le k'$ let each X_i, X'_j be either a nonflat irreducible symmetric space of noncompact type or an irreducible thick Euclidean Tits building with cocompact affine Weyl group (see section 4.1 for the precise definition). Let $X = \mathbb{E}^n \times \prod_{i=1}^k X_i$, $X' = \mathbb{E}^{n'} \times \prod_{j=1}^{k'} X'_j$ be metric products.[3] Then for every L, C there are constants $\bar{L}$, $\bar{C}$ and $\bar{D}$ such that the following holds. If $\Phi : X \longrightarrow X'$ is an (L, C) quasi-isometry, then $n = n'$, $k = k'$, and after reindexing the factors of X' there are $(\bar{L}, \bar{C})$ quasi-isometries $\Phi_i : X_i \longrightarrow X'_i$ so that $d(p' \circ \Phi, \prod \Phi_i \circ p) < \bar{D}$, where $p : X \longrightarrow \prod_{i=1}^k X_i$ and $p' : X' \longrightarrow \prod_{i=1}^k X'_i$ are the projections.*

A more general theorem about quasi-isometries of products is proved in [KKL].

Theorem 1.1.3 (Rigidity) *Let X and X' be as in theorem 1.1.2, but assume in addition that X is either a nonflat irreducible symmetric space of noncompact type of rank at least 2, or a thick irreducible Euclidean building of rank at least 2 with cocompact affine Weyl group and Moufang Tits boundary. Then any (L, C) quasi-isometry $\Phi : X \longrightarrow X'$ lies at distance $< D$ from a homothety $\Phi_0 : X \longrightarrow X'$, where D depends only on (L, C).*

Theorem 1.1.3 settles a conjecture made by Margulis in the late 1970's, see [Gro, p. 179] and [GrPa, p. 73]. We will show in [KlLe] that the Moufang condition on the Tits boundary of X can be dropped.

As an immediate consequence of theorems 1.1.2 1.1.3, and [Mos] we have:

Corollary 1.1.4 (Quasi-isometric classification of symmetric spaces) *Let X, X' be symmetric spaces of noncompact type. If X and X' are quasi-isometric, then they become isometric after the metrics on their de Rham factors are suitably renormalized.*

Mostow's work [Mos] implies that two quasi-isometric rank 1 symmetric spaces of noncompact type are actually isometric (up to a scale factor); and it was known by [AS] that two quasi-isometric symmetric spaces of noncompact type have the same rank.

We will discuss other applications of theorems 1.1.2 and 1.1.3 in a separate paper.

1.2 Commentary on the proof

Our approach to theorems 1.1.2 and 1.1.3 is based on the fact that if one scales the metrics on X and X' by a factor λ, then (L, C) quasi-isometries become $(L, \lambda C)$ quasi-isometries. Starting with a sequence $\lambda_i \to 0$ we apply the ultralimit construction of [DW, Gro] to take a limit of the sequence $\Phi : \lambda_i X \longrightarrow \lambda_i X'$, getting an $(L, 0)$ quasi-isometry (i.e. a biLipschitz homeomorphism) $\Phi_\omega : X_\omega \longrightarrow X'_\omega$ between the limit spaces. The first step is to determine the geometric structure of these limit spaces:

Theorem 1.2.1 *X_ω and X'_ω are thick (generalized) Euclidean Tits buildings (cf. section 4.1).*

[3] The distance function on the product space is given by the Pythagorean formula.

The second step is to study the topology of the Euclidean buildings X_ω, X'_ω. We establish rigidity results for homeomorphisms of Euclidean buildings which are topological analogs of theorems 1.1.2 and 1.1.3:

Theorem 1.2.2 *Let Y_i, Y'_i be thick irreducible Euclidean buildings with topologically transitive affine Weyl group (cf. section 4.1.1), and let $Y = \mathbb{E}^n \times \prod_{i=1}^k Y_i$, $Y' = \mathbb{E}^{n'} \times \prod_{j=1}^{k'} Y'_j$. If $\Psi : Y \to Y'$ is a homeomorphism, then $n = n'$, $k = k'$, and after reindexing factors there are homeomorphisms $\Psi_i : Y_i \to Y'_i$ so that $p' \circ \Psi = \prod \Psi_i \circ p$ where $p : Y \to \prod_{i=1}^k Y_i$ and $p' : Y' \to \prod_{i=1}^k Y'_i$ are the projections.*

Theorem 1.2.3 *Let Y be an irreducible thick Euclidean building with topologically transitive affine Weyl group and rank ≥ 2. Then any homeomorphism from Y to a Euclidean building is a homothety.*

For comparison we remark that if Y and Y' are thick irreducible Euclidean buildings with crystallographic (i.e. discrete cocompact) affine Weyl group, then one can use local homology groups to see that any homeomorphism carries simplices to simplices. In particular, the homeomorphism induces an incidence preserving bijection of the simplices of Y with the simplices of Y', which easily implies that the homeomorphism coincides with a homothety on the 0-skeleton. In contrast to this, homeomorphisms of rank 1 Euclidean buildings with nondiscrete affine Weyl group (i.e. $\mathbb{R}$-trees) can be quite arbitrary: there are examples of $\mathbb{R}$-trees T for which every homeomorphism $A \to A$ of an apartment $A \subset T$ can be extended to a homemorphism of T. However, we have always:

Proposition 1.2.4 *If X, X' are Euclidean buildings, then any homeomorphism $\Psi : X \to X'$ carries apartments to apartments.*

In the third step, we deduce theorems 1.1.2 and 1.1.3 from their topological analogs. By using a scaling argument and proposition 1.2.4 we show that if X and X' are as in theorem 1.1.2, and $\Phi : X \to X'$ is an (L, C) quasi-isometry, then the image of a maximal flat in X under Φ lies within uniform Hausdorff distance of a maximal flat in X'; the Hausdorff distance can be bounded uniformly by (L, C). In the case of theorem 1.1.2 we use this to deduce that the quasi-isometry respects the product structure, and in the case of theorem 1.1.3 we use it to show that Φ induces a well-defined homeomorphism $\partial\Phi : \partial X \to \partial X'$ of the geometric boundaries which is an isometry of Tits metrics. We conclude using Tits' work [Ti1] (as in [Mos]) that $\partial\Phi$ is also induced by an isometry $\Phi_0 : X \to X'$, and $d(\Phi, \Phi_0)$ is bounded uniformly by (L, C).

The reader may wonder about the relation between theorems 1.1.2 and 1.1.3 and Mostow's argument in the higher rank case. An important step in Mostow's proof shows that if Γ acts discretely and cocompactly on symmetric spaces X and X', then any Γ-equivariant quasi-isometry $\Phi : X \to X'$ carries maximal flats in X to within uniform distance of maximal flats in X'. The proof in [Mos] exploits the dense collection of maximal flats with cocompact Γ-stabilizer[4]. One can then ask if there is a "direct" argument showing that maximal flats in X are carried to within uniform distance of maximal flats in X' by any quasi-isometry[5]; for instance, by analogy with the rank 1 case one may ask whether *any* r-quasi-flat[6] in a symmetric space of rank r must lie within bounded

[4]If $\mathbb{Z}^r \subset \Gamma$ acts cocompactly on a maximal flat $F \subset X$, then $\mathbb{Z}^r$ will stabilize $\Phi(F)$ and a flat F' in X'. One can then get a uniform estimate on the Hausdorff distance between $\Phi(F)$ and F'.

[5]Obviously this statement is true by theorems 1.1.2 and 1.1.3.

[6]An r-quasi-flat is a quasi-isometric embedding $\phi : \mathbb{E}^r \to X$; a quasi-isometric embedding is a map satisfying condition (1), but not necessarily (2).

distance of a maximal flat. The answer is no. If X is a rank 2 symmetric space, then the geodesic cone $\cup_{s\in S}\overline{ps}$ over any embedded circle S in the Tits boundary $\partial_{Tits}X$ is a 2-quasi-flat. Similar constructions produce nontrivial r-quasi-flats in symmetric spaces of $rank \geq 2$. But in fact this is the only way to produce quasiflats, by

Theorem 1.2.5 (Structure of quasi-flats) *Let X be as in theorem 1.1.2, and let $r = rank(X)$. Given L, C there are $D, D' \in \mathbb{Z}$ such that every (L, C) r-quasi-flat $Q \subset X$ lies within the D-tubular neighborhood $N_D(\cup_{F\in\mathcal{F}}F)$ of a union of at most D maximal flats. Moreover, the limit set of Q is the union of at most D' closed Weyl chambers in the Tits boundary $\partial_{Tits}X$.*

It follows easily that if L is sufficiently close to 1 (in terms of the geometry of the spherical Coxeter complex (S, W) associated to X) then any (L, C) r-quasi-flat in X is uniformly close to a maximal flat. In the special case that X is a symmetric space, theorem 1.2.5 was proved independently by Eskin and Farb, approximately one year after we had obtained the main results of this paper for symmetric spaces.

We would like to mention that related rigidity results for quasi-isometries have been proved in [Sch].

1.3 Organization of the paper

Section 2 contains background material which will be familiar to many readers; we recommend starting with section 3, and using section 2 as a reference when needed. We provide the straightforward generalisation of some well-known facts about Hadamard spaces to the non-locally-compact case. This is needed when we study the limit spaces X_ω which are non-locally compact Hadamard spaces.

Sections 3 and 4 give a self-contained exposition of the building theory used elsewhere in the paper. This exposition has several aims. First, we hope that it will make building theory more accessible to geometers since it is presented using the language of metric geometry, and we do not require any knowledge of algebraic groups. Second, it introduces a new definition of buildings (spherical and Euclidean) which is based on metric geometry rather than a combinatorial structure such as a polysimplicial complex. Tits' original definition of a building was motivated by applications to algebraic groups, whereas the objectives of this paper are primarily geometric. Here buildings (spherical and Euclidean) arise as geometric limits of symmetric spaces, and we found that the geometric definition in sections 3 and 4 could be verified more directly than the standard one; moreover, the Euclidean buildings that arise as limits are "nondiscrete", and do not admit a natural polysimplicial structure. Finally, sections 3 and 4 contain a number of new results, and reformulations of standard results tailored to our needs.

Section 5 shows that the asymptotic cone of a symmetric space or Euclidean building is a Euclidean building.

Section 6 discusses the topology of Euclidean buildings, proving theorems 1.2.2, 1.2.3, 1.2.4.

Section 7 proves that if X, X' and Φ are as in theorem 1.1.2, then the image of a maximal flat under Φ is uniformly Hausdorff close to a flat (actually the hypotheses on X and X' can be weakened somewhat, see corollary 7.1.5). General quasiflats are also studied in section 7; we prove there theorem 1.2.5.

Section 8 contains the proofs of theorems 1.1.2 and 1.1.3, building on section 7. There is considerable overlap in the final step of the argument with [Mos] in the symmetric space case.

1.4 Suggestions to the reader

Readers who are already familiar with building theory will probably find it useful to read sections 3.1, 3.2 and 4.1, to normalize definitions and terminology.

The special case of theorem 1.1.2 when $X = X' = \mathbb{H}^2 \times \mathbb{H}^2$ already contains most of the conceptual difficulties of the general case, but one can understand the argument in this case with a minimum of background. To readers who are unfamiliar with asymptotic cones, and readers who would like to quickly understand the proof in a special case, we recommend an abbreviated itinerary, see appendix 9. In general, when the burden of axioms and geometric minutae seems overwhelming, the reader may read with the Rank 1× Rank 1 case in mind without losing much of the mathematical content.

Contents

2 Preliminaries

2.1 Spaces with curvature bounded above

General references for this section are [ABN, Ba, BGS].

2.1.1 Definition

If $\kappa \in \mathbb{R}$, let M^2_κ be the two dimensional model space with constant curvature κ; let $D(\kappa) = Diam(M^2_\kappa)$. A complete metric space $(X, |\cdot|)$ is a $CAT(\kappa)$ space if

1. Every pair $x_1, x_2 \in X$ with $|x_1x_2| < D(\kappa)$ is joined by a geodesic segment.

2. **Triangle** or **Distance Comparison.**

   Every geodesic triangle in X with perimeter $< 2D(\kappa)$ is at least as thin as the corresponding triangle in M^2_κ. More precisely: for each geodesic triangle Δ in X with sides $\sigma_1, \sigma_2, \sigma_3$ with $Perimeter(\Delta) = |\sigma_1| + |\sigma_2| + |\sigma_3| < 2D(\kappa)$ we construct a comparison triangle $\tilde{\Delta}$ in M^2_κ with sides $\tilde{\sigma}_i$ satisfying $|\tilde{\sigma}_i| = |\sigma_i|$. Each point x on Δ corresponds to a unique point $\tilde{x}$ on $\tilde{\Delta}$ which divides the corresponding side in the same ratio. We require that for all $x_1, x_2 \in \Delta$ we have $|x_1x_2| \leq |\tilde{x}_1\tilde{x}_2|$.

Remark 2.1.1 *Note that we do not require X to be locally compact. Also, X needn't be path connected when $\kappa > 0$. This is slightly more general than some other definitions in the literature.*

Example 2.1.2 *A complete 1-connected Riemannian manifold with sectional curvature $\leq \kappa \leq 0$ and all its closed convex subsets are is a $CAT(\kappa)$ spaces.*

In particular, Hadamard manifolds are CAT(0)-spaces. This is why we will also call CAT(0)-spaces *Hadamard spaces.*

Example 2.1.3 (Berestovski) *Any simplicial complex admits a piecewise spherical $CAT(1)$ metric.*

Condition 2 implies that any two points x_1, x_2 with $|x_1x_2| < D(\kappa)$ are connected by precisely one geodesic; hence we may speak unambiguously of $\overline{x_1x_2}$ as *the* geodesic segment joining x_1 to x_2. $CAT(\kappa)$ spaces for $\kappa \le 0$ are contractible geodesic spaces.

To see that upper curvature bounds behave well under limiting operations, it is convenient to use an equivalent definition of $CAT(\kappa)$ spaces which only refers to finite configurations of points rather than geodesic triangles. If $v, x, y, p \in X$, and $\tilde{v}, \tilde{x}, \tilde{y}, \tilde{p} \in M^2_\kappa$ we say that $\tilde{v}, \tilde{x}, \tilde{y}, \tilde{p}$ form a *δ-comparison quadruple* if

1. $\tilde{p}$ lies on $\overline{\tilde{x}, \tilde{y}}$.
2. $||vx| - |\tilde{v}\tilde{x}|| < \delta$, $||vy| - |\tilde{v}\tilde{y}|| < \delta$, $||xy| - |\tilde{x}\tilde{y}|| < \delta$, $||xp| - |\tilde{x}\tilde{p}|| < \delta$, $||py| - |\tilde{p}\tilde{y}|| < \delta$

By a compactness argument, we note that there exists a function $\delta_\kappa(P, \epsilon) > 0$ such that for every $\epsilon > 0$, and every quadruple of points v, x, y, p in a $CAT(\kappa)$ space X satisfying $|vx| + |xy| + |yv| < P < 2D(\kappa)$, each $\delta_\kappa(P, \epsilon)$-comparison quadruple $\tilde{v}, \tilde{x}, \tilde{y}, \tilde{p}$ satisfies $|vp| \le |\tilde{v}\tilde{p}| + \epsilon$. We will refer to this condition as the *δ_κ-four-point condition.* It is a closed condition on four point metric spaces with respect to the Hausdorff topology. A complete metric space X is a $CAT(\kappa)$ space if and only if it satisfies the δ_κ-four-point condition and every pair of points $x, y \in X$ with $|xy| < D(\kappa)$ has approximate midpoints, i.e. for every $\epsilon' > 0$ there is a $m \in X$ with $|xm|, |my| < \frac{|xy|}{2} + \epsilon'$. To see this, note that in the presence of the δ_κ-four-point condition approximate midpoints are close to one another, so one may produce a genuine midpoint by taking limits. By taking successive midpoints, one can produce a geodesic segment.

2.1.2 Coning

Let Σ be a metric space with $Diam(\Sigma) \le \pi$. The *metric cone $C(\Sigma)$ over* Σ is defined as follows. The underlying set will be $\Sigma \times [0, \infty)/\sim$ where $\sim$ collapses $\Sigma \times \{0\}$ to a point. Given $v_1, v_2 \in \Sigma$, we consider embeddings $\rho : \{v_1, v_2\} \times [0, \infty) \to \mathbb{E}^2$ such that $|\rho(v_i, t) = |t|$ and $\angle_0(\rho(v_1, t_1), \rho(v_2, t_2)) = |v_1v_2|$, and we equip $C(\Sigma)$ with the unique metric for which these embeddings are isometric. $C(\Sigma)$ is $CAT(0)$ iff Σ is $CAT(1)$.

2.1.3 Angles and the space of directions of a $CAT(\kappa)$ space

Henceforth we will say that a triple v, x, y defines a triangle $\Delta(v, x, y)$ provided $|vx| + |xy| + |yv| < 2Diam(M^2_\kappa)$. $\tilde{\angle}_v(x, y)$ will denote the angle of the comparison triangle at the vertex $\tilde{v}$. If x', y' are interior points on the segments $\overline{vx}, \overline{vy}$, then $\tilde{\angle}_v(x', y') \le \tilde{\angle}_v(x, y)$. From this monotonicity it follows that $\lim_{x', y' \to v} \tilde{\angle}_v(x', y')$ exists, and we denote it by $\angle_v(x, y)$. This definition of angle coincides with the notion of the angle between two segments in the Riemannian case. One checks that one obtains the same limit if only one of the points x', y' approaches v:

$$\angle_v(x, y) = \lim_{x' \to v} \tilde{\angle}_v(x', y) \tag{3}$$

$\angle_v$ satisfies the triangle inequality. Note that from the definition we have

$$\angle_v(x, y) \le \tilde{\angle}_v(x, y). \tag{4}$$

In the equality case a basic rigidity phenomenon occurs:

Triangle Filling Lemma 2.1.4 *Let x, y, v be as before. If $\angle_v(x,y) = \tilde{\angle}_v(x,y)$, then also the other angles of the triangle $\Delta(v,x,y)$ coincide with the corresponding comparison angles; moreover the region in M^2_κ bounded by the comparison triangle can be isometrically embedded into X so that corresponding vertices are identified.*

The angles of a triangle depend upper-semicontinuously on the vertices:

Lemma 2.1.5 *Suppose $v, x, y \in X$ define a triangle, $v \neq x, y$, and $v_k \to v$, $x_k \to x$, $y_k \to y$. Then v_k, x_k, y_k define a triangle for almost all k and*

$$\limsup_{k\to\infty} \angle_{v_k}(x_k, y_k) \leq \angle_v(x,y).$$

In the special case that $v_k \in \overline{vx_k} - \{v\}$ holds $\lim_{k\to\infty} \angle_{v_k}(x_k, y_k) = \angle_v(x,y)$ and $\lim_{k\to\infty} \angle_{v_k}(v, y_k) = \pi - \angle_v(x,y)$.

Proof. For $x' \in \overline{vx} - \{v\}$ and $y' \in \overline{vy} - \{v\}$ we can choose sequences of points $x'_k \in \overline{v_k x_k}$, $y'_k \in \overline{v_k y_k}$ with $x'_k \to x'$ and $y'_k \to y'$. Then $\angle_{v_k}(x_k, y_k) \leq \tilde{\angle}_{v_k}(x'_k, y'_k) \to \tilde{\angle}_v(x', y')$ and the first assertion follows by letting $x', y' \to v$. If $v_k \in \overline{vx_k} - \{v, x_k\}$ then $\angle_v(x_k, y_k) \leq anglesum(\Delta(v, v_k, y_k)) - \angle_{v_k}(v, y_k)$ and $\pi - \angle_{v_k}(v, y_k) \leq \tilde{\angle}_{v_k}(x_k, y_k)$ while $\limsup anglesum(\Delta(v, v_k, y_k)) \leq \pi$. Sending k to infinity, we get $\angle_v(x,y) \leq \pi - \liminf \angle_{v_k}(v, y_k) \leq \liminf \angle_{v_k}(x_k, y_k)$ and hence the second assertion. □

The condition that two geodesic segments with initial point $v \in X$ have angle zero at v is an equivalence relation; we denote the set of equivalence classes by $\Sigma^*_v X$. The angle defines a metric on $\Sigma^*_v X$, and we let $\Sigma_v X$ be the completion of $\Sigma^*_v X$ with respect to this metric. We call elements of $\Sigma_v X$ *directions at v* (or simply directions), and $\overrightarrow{vx}$ denotes the direction represented by $\overline{vx}$. We define the *logarithm map* as the map $\log_v = \log_{\Sigma_v X} : B_v(D(\kappa)) \setminus v \to \Sigma_v X$ which carries x to the direction $\overrightarrow{vx}$. The *tangent cone* of X at v, denoted $C_v X$, is the metric cone $C(\Sigma_v X)$; we have a logarithm map $\log_v = \log_{C_v X} : B_v(D(\kappa)) \to C_v X$.

Given a basepoint $v \in X$, $x \in X$ with $d(v,x) < D(\kappa)$, and $\lambda \in [0,1]$, let $\lambda x \in X$ be the point on $\overline{vx}$ satisfying $\frac{|v(\lambda x)|}{|vx|} = \lambda$. We define a family of pseudo-metrics on $B_v(D(\kappa))$ by $d_\epsilon(x,y) = \frac{1}{\epsilon} d(\epsilon x, \epsilon y)$. They converge to a limit pseudo-metric d_0. The pseudo-metric space $(B_v(D(\kappa)), d_\epsilon)$ satisfies the $\delta_{\epsilon^2\kappa}$-four-point condition, so the limit pseudo-metric space $(B_v(D(\kappa)), d_0)$ satisfies the δ_0-four-point condition. But $d_0(x,y) = d(\log_v x, \log_v y)$ where $\log_v : B_v(D(\kappa)) \longrightarrow C_v X$ is the logarithm defined above, so we see that the tangent cone $C_v X$ satisfies the δ_0-four-point condition ($C(\Sigma^*_v X)$ is dense in $C_v X$, and every four-tuple in $C(\Sigma^*_v X)$ is homothetic to a four-tuple in $\log_v(B_v(D(\kappa)))$. If z_λ is the midpoint of the segment $\overline{(\lambda x)(\lambda y)}$, then

$$d(\log_v x, \log_v y) = \lim_{\epsilon\to 0} \frac{1}{\epsilon} d(\epsilon x, \epsilon y)$$

$$= \lim_{\epsilon\to 0} \frac{2}{\epsilon} d(\epsilon x, z_\epsilon) = \lim_{\epsilon\to 0} \frac{2}{\epsilon} d(z_\epsilon, \epsilon y)$$

$$\geq \max(\lim_{\epsilon\to 0} 2d(\log_v x, \frac{1}{\epsilon}\log_v z_\epsilon), \lim_{\epsilon\to 0} 2d(\log_v x, \frac{1}{\epsilon}\log_v z_\epsilon)).$$

So $C_v X$ also has approximate midpoints. Since $C_v X$ is complete, it is a $CAT(0)$ space; consequently $\Sigma_v X$ is a $CAT(1)$ space. This fact is due to Nikolaev [Nik].

2.2 CAT(1)-spaces

CAT(1)-spaces are of special importance to us, because they will turn up as spaces of directions and Tits boundaries of Hadamard spaces.

2.2.1 Spherical join

Let B_1 and B_2 be CAT(1)-spaces with diameter $Diam(B_i) \leq \pi$. Their spherical join $B_1 \circ B_2$ is defined as follows. The underlying set will be $B_1 \times [0, \frac{\pi}{2}] \times B_2 / \sim$ where "$\sim$" collapses the subsets $\{b_1\} \times \{0\} \times B_2$ and $B_1 \times \{\frac{\pi}{2}\} \times \{b_2\}$ to points. Given $b_i, b'_i \in B_i$ $(i = 1, 2)$, we consider embeddings $\rho : \{b_1, b'_1\} \times [0, \frac{\pi}{2}] \times \{b_2, b'_2\} \to S^3$. We think of S^3 as the unit sphere in $\mathbb{C}^2$ and require that $t \mapsto \rho(b_1, t, b_2)$ and $t' \mapsto \rho(b'_1, t', b'_2)$ are unit speed geodesic segments whose initial (resp. end) points lie on the great circle $S^1 \times \{0\}$ (resp. $\{0\} \times S^1$) and have distance $d_{B_1}(b_1, b'_1)$ (resp. $d_{B_2}(b_2, b'_2)$). The distance of the points in $B_1 \circ B_2$ represented by (b_1, t, b_2) and (b'_1, t', b'_2) is then defined as the (spherical) distance of their ρ-images in S^3; it is independent of the choice of ρ. To see that $B_1 \circ B_2$ is again a CAT(1)-space and that the spherical join operation is associative, observe that the metric cone $C(B_1 \circ B_2)$ is canonically isometric to $C(B_1) \times C(B_2)$ and that the product of CAT(0)-spaces is CAT(0).

The *metric suspension* of a CAT(1)-space with diameter $\leq \pi$ is defined as its spherical join with the CAT(1)-space $\{south, north\}$ consisting of two points with distance π.

Lemma 2.2.1 *Let B_1 and B_2 be CAT(1)-spaces with diameter π and suppose s is an isometrically embedded unit sphere in the spherical join $B = B_1 \circ B_2$. Then there are isometrically embedded unit spheres s_i in B_i so that $s_1 \circ s_2$ contains s.*

Proof. We apply lemma 2.3.8 to the metric cone $C(B) \cong C(B_1) \times C(B_2)$. $C(s)$ is a flat in $C(B)$ and hence contained in the product of flats $F_i \subseteq C(B_i)$. $s_i := \partial_{Tits} F_i$ is a unit sphere in B_i and $s_1 \circ s_2 = \partial_{Tits}(F_1 \times F_2) \supseteq \partial_{Tits} C(s) = s$. □

2.2.2 Convex subsets and their poles

We call a subset C of a CAT(1)-space B *convex* iff for any two points $p, q \in C$ of distance $d(p, q) < \pi$ the unique geodesic segment $\overline{pq}$ is contained in C. Closed convex subsets of B are CAT(1)-spaces with respect to the subspace metric inherited from B. Basic examples of convex subsets are tubular neighborhoods with radius $\leq \frac{\pi}{2}$ of convex subsets, e.g. balls of radius $\leq \frac{\pi}{2}$.

Suppose that $C \subset B$ is a closed convex subset with *radius* $Rad(C) \geq \pi$, i.e. for each $p \in C$ exists $q \in C$ with $d(p, q) \geq \pi$. We define the set of *poles* for C as

$$Poles(C) := \Big\{\eta \in B : d(\eta, \cdot)\big|_C \equiv \frac{\pi}{2}\Big\}.$$

If $Diam(C) > \pi$ then C has no pole. If $Diam(C) = Rad(C) = \pi$ then $Poles(C)$ is closed and convex, because it can be written as an intersection $Poles(C) = \bigcap_{\xi \in C} B_{\frac{\pi}{2}}(\xi)$ of convex balls. The convex hull of C and $Poles(C)$ is the union of all segments joining points in C to points in $Poles(C)$, and is canonically isometric to $C \circ Poles(C)$. This follows, for instance, when one applies the discussion in section 2.3.3 to the parallel sets of $C(C)$ in the metric cone $C(B)$.

Consider the special case that C consists of two *antipodes*, i.e. points with distance π, ξ and $\hat{\xi}$. Then the convex hull of $\{\xi, \hat{\xi}\}$ and $Poles(\{\xi, \hat{\xi}\})$ is exactly the union of minimizing geodesic segments connecting $\xi, \hat{\xi}$ and it is canonically isometric to the metric suspension of $Poles(\{\xi, \hat{\xi}\})$.

2.3 Hadamard spaces

We will call CAT(0)-spaces also Hadamard spaces, because they are the synthetic analog of (closed convex subsets in) Hadamard manifolds, i.e. simply connected complete manifolds of nonpositive curvature, cf. example 2.1.3.

2.3.1 The geometric boundary

Let X be a Hadamard space. Two geodesic rays are *asymptotic* if they remain at bounded distance from one another, i.e. if their Hausdorff distance is finite. Asymptoticity is an equivalence relation, and we let $\partial_\infty X$ be the set of equivalence classes of asymptotic rays; we sometimes refer to elements of $\partial_\infty X$ as ideal points or ideal boundary points. For any point $x \in X$ and any ideal boundary point $\xi \in \partial_\infty X$ there exists a unique ray $\overline{x\xi}$ starting at x which represents ξ. The pointed Hausdorff topology on rays emanating from $x \in X$ induces a topology on $\partial_\infty X$. This topology does not depend on the base point x and is called the *cone topology* on $\partial_\infty X$. $\partial_\infty X$ with the cone topology is called the *geometric boundary*. The cone topology naturally extends to $X \cup \partial_\infty X$. If X is locally compact, then $\partial_\infty X$ and $\bar{X} := X \cup \partial_\infty X$ are compact and $\bar{X}$ is called the *geometric compactification* of X.

2.3.2 The Tits metric

Earlier we defined the angle between two geodesics $\overline{vx}$, $\overline{vy}$ at $v \in X$ by using the monotonicity of comparison angles $\tilde{\angle}_v(x', y')$ as $x' \to v$, $y' \to v$. Now we consider a pair of rays $\overline{v\xi}$, $\overline{v\eta}$, and define their *Tits angle* (or *angle at infinity*) by

$$\angle_{Tits}(\xi, \eta) := \lim_{x' \to \xi, y' \to \eta} \tilde{\angle}_v(x', y') \tag{5}$$

where $x' \in \overline{v\xi}$ and $y' \in \overline{v\eta}$. $\angle_{Tits}$ defines a metric on $\partial_\infty X$ which is independent of the basepoint v chosen. We call the metric space $\partial_{Tits} X := (\partial_\infty X, \angle_{Tits})$ the *Tits boundary* of X and $\angle_{Tits}$ the *Tits (angle) metric*. The estimate

$$\tilde{\angle}_v(x', y') = \underbrace{\pi - \tilde{\angle}_{x'}(v, y')}_{\leq \underbrace{\angle_{x'}(\xi, y')}_{\overset{y' \to \eta}{\longrightarrow} \angle_{x'}(\xi, \eta)}} - \underbrace{\tilde{\angle}_{y'}(v, x')}_{\overset{y' \to \eta}{\longrightarrow} 0}$$

implies, combined with (4):

$$\angle_v(\xi, \eta) \leq \tilde{\angle}_v(x', y') \leq \angle_{x'}(\xi, \eta)$$

Consequently, the Tits angle can be expressed as

$$\angle_{Tits}(\xi, \eta) = \lim_{t \to \infty} \angle_{r(t)}(\xi, \eta) \tag{6}$$

for any geodesic ray $r : \mathbb{R}^+ \to X$ asymptotic to ξ or η, and also as:

$$\angle_{Tits}(\xi, \eta) = \sup_{x \in X} \angle_x(\xi, \eta) \tag{7}$$

Still another possibility (the last one which we will state) to define the Tits angle is as follows: If $r_i : \mathbb{R}^+ \to X$ are geodesic rays asymptotic to ξ_i then

$$2 \sin \frac{\angle_{Tits}(\xi_1, \xi_2)}{2} = \lim_{t \to \infty} \frac{d(r_1(t), r_2(t))}{t}. \tag{8}$$

The next lemma relates the cone topology on $\partial_\infty X$ to the Tits topology. Fix $v \in X$ and consider the comparison angle

$$\tilde{\angle}_v : (X \setminus \{v\}) \times (X \setminus \{v\}) \to [0, \pi].$$

By monotonicity, it can be extended to a function

$$\tilde{\angle}_v : (\bar{X} \setminus \{v\}) \times (\bar{X} \setminus \{v\}) \to [0, \pi].$$

Note that for $\xi, \eta \in \partial_\infty X$, we have $\tilde{\angle}_v(\xi, \eta) = \angle_{Tits}(\xi, \eta)$.

Lemma 2.3.1 (Semicontinuity of comparison angle) *$\tilde{\angle}_v$ is lower semicontinuous with respect to the cone topology: If $x_k, y_k, \xi, \eta \in \bar{X} - \{v\}$ such that $\xi = \lim_{k\to\infty} x_k$ and $\eta = \lim_{k\to\infty} y_k$ then $\tilde{\angle}_v(\xi, \eta) \le \liminf_{k\to\infty} \tilde{\angle}_v(x_k, y_k)$.*

Proof. We treat the case $\xi, \eta \in \partial_\infty X$, the other cases are similar or easier. Since the segments (or rays) $\overline{vx_k}, \overline{vy_k}$ are converging to the rays $\overline{v\xi}, \overline{v\eta}$ respectively, we may choose $x'_k \in \overline{vx_k}$ and $y'_k \in \overline{vy_k}$ such that $|x'_k v|, |y'_k v| \to \infty$ and $d(x'_k, \overline{v\xi}) \to 0, d(y'_k, \overline{v\eta}) \to 0$. Hence by triangle comparison we have

$$\tilde{\angle}_v(x_k, y_k) \ge \tilde{\angle}_v(x'_k, y'_k) \to \angle_{Tits}(\xi, \eta).$$

□

Lemma 2.3.2 *Every pair $\xi, \eta \in \partial_\infty X$ with $\angle_{Tits}(\xi, \eta) < \pi$ has a midpoint.*

Proof. Pick $v \in X$. Take sequences $x_i \in \overline{v\xi}, y_i \in \overline{v\eta}$ with $|x_i| = |y_i| \to \infty$. Let m_i be the midpoint of $\overline{x_i y_i}$. Since $\Delta(v, x_i, y_i)$ is isosceles, $\tilde{\angle}_v(x_i, m_i) = \tilde{\angle}_v(m_i, y_i) \le \frac{1}{2}\tilde{\angle}_v(x_i, y_i)$, by lemma 2.3.1 it suffices to show that $\overline{vm_i}$ converges to a ray $\overline{v\mu}$, for some $\mu \in \partial_\infty X$.

For $i < j$, set $\lambda_{ij} := \frac{|vx_i|}{|vx_j|}$. By triangle comparison, we have the following inequalities:

$$|x_i(\lambda_{ij} m_j)| \le \lambda_{ij} |x_j m_j| = \frac{\lambda_{ij}}{2} |x_j y_j|$$

$$|y_i(\lambda_{ij} m_j)| \le \lambda_{ij} |y_j m_j| = \frac{\lambda_{ij}}{2} |x_j y_j|$$

$$|x_i(\lambda_{ij} m_j)| + |y_i(\lambda_{ij} m_j)| \ge |x_i y_i|$$

Since $\lambda_{ij} \frac{|x_j y_j|}{|x_i y_i|} \to 1$ as $i, j \to \infty$, we have

$$\frac{|x_i(\lambda_{ij} m_j)|}{|x_i m_i|} \to 1, \frac{|y_i(\lambda_{ij} m_j)|}{|y_i m_i|} \to 1 \Longrightarrow \frac{|m_i(\lambda_{ij} m_j)|}{|x_i m_i|} \to 0$$

and, since $\angle_{Tits}(\xi, \eta) < \pi$, this in turn implies:

$$\frac{|m_i(\lambda_{ij} m_j)|}{|v m_i|} \to 0.$$

Fixing $t > 0$, if we set $\frac{t}{|vm_i|} = \aleph_i$, then $|(\aleph_i m_i)(\aleph_i \lambda_{ij} m_j)| \to 0$ as $i, j \to \infty$. Since $|v(\aleph_i m_i)| = t$, this shows that the segments $\overline{vm_i}$ converge in the pointed Hausdorff topology to a ray $\overline{v\mu}$ as desired. □

The completeness of X implies that $(\partial_\infty X, \angle_{Tits})$ is complete. The metric cone $C(\partial_\infty X, \angle_{Tits})$ (the Tits cone) is complete and has midpoints. Moreover, since every quadruple in $C(\partial_\infty X, \angle_{Tits})$ is approximated metrically (up to rescaling) by quadruples in X, $C(\partial_\infty X, \angle_{Tits})$ satisfies the δ_0-four-point condition and is therefore a $CAT(0)$ space. By section 2.1.1 we conclude:

Proposition 2.3.3 *The Tits boundary of a Hadamard space is a $CAT(1)$ space.*

There is a natural 1-Lipschitz exponential map $\exp_p : C(\partial_{Tits} X) \to X$ defined as follows: For $[(\xi, t)] \in C(\partial_{Tits} X) = \partial_{Tits} X \times [0, \infty)/\sim$ let $\exp_p[(\xi, t)]$ be the point on $\overline{p\xi}$ at distance t from p. The logarithm map $\log_p : X - \{p\} \to \Sigma_p X$ extends contiuously to the geometric boundary and induces there a 1-Lipschitz map $\log_p : \partial_{Tits} X \to \Sigma_p X$. The Triangle Filling Lemma 2.1.4 implies the following rigidity statement:

Flat Sector Lemma 2.3.4 *Suppose the restriction of $\log_p : \partial_{Tits} X \to \Sigma_p X$ to the subset $A \subseteq \partial_{Tits} X$ is distance-preserving. Then the restriction of $\exp_p : C(\partial_{Tits} X) \to X$ to $C(A) \subseteq C(\partial_{Tits} X)$ is an isometric embedding.*

2.3.3 Convex subsets and parallel sets

A subset of a Hadamard space is *convex* if, with any two points, it contains the unique geodesic segment connecting them. Closed convex subsets of Hadamard spaces are Hadamard themselves with respect to the subspace metric. Important examples of convex sets are tubular neighborhoods of convex sets and horoballs. We will denote by $HB_\xi(x)$ the horoball centered at the point $\xi \in \partial_\infty X$ and containing $x \in X$ in its boundary.

Let C_1 and C_2 be closed convex subsets of a Hadamard space X. Then by (4), the distance function $d(\cdot, C_2)\big|_{C_1} = d_{C_2}\big|_{C_1} : C_1 \to \mathbb{R}_{\geq 0}$ is convex and the nearest point projection $\pi_{C_2}\big|_{C_1} : C_1 \to C_2$ is distance-nonincreasing. $d_{C_2}\big|_{C_1}$ is constant iff $\pi_{C_2}\big|_{C_1}$ is an isometric embedding. In this situation, we have the following rigidity statement:

Flat Strip Lemma 2.3.5 *Let C_1 and C_2 be closed convex subsets in the Hadamard space X. If $d_{C_2}\big|_{C_1} \equiv d$ then there exists an isometric embedding $\psi : C_1 \times [0, d] \to X$ such that $\psi(\cdot, o) = id_{C_1}$ and $\psi(\cdot, d) = \pi_{C_2}\big|_{C_1}$.*

This is easily derived from the Triangle Filling Lemma 2.1.4, respectively from the following direct consequence of it:

Flat Rectangle Lemma 2.3.6 *Let $x_i \in X$, $i \in \mathbb{Z}/4\mathbb{Z}$, be points so that for all i holds $\angle_{x_i}(x_{i-1}, x_{i+1}) \geq \frac{\pi}{2}$. Then there exists an embedding of the flat rectangular region $[0, |x_0 x_1|] \times [0, |x_1 x_2|] \subset \mathbb{E}^2$ into X carrying the vertices to the points x_i.*

We call the closed convex sets $C_1, C_2 \subseteq X$ *parallel*, $C_1 \| C_2$, iff $d_{C_2}\big|_{C_1}$ and $d_{C_1}\big|_{C_2}$ are constant, or equivalently, $\pi_{C_2}\big|_{C_1}$ and $\pi_{C_1}\big|_{C_2}$ are isometries inverse to each other. Being parallel is no equivalence relation for arbitrary closed convex subsets. However, it is an equivalence relation for closed convex sets with extendible geodesics, because two such subsets are parallel iff they have finite Hausdorff distance. (A Hadamard space is said to have *extendible geodesics* if each geodesic segment is contained in a complete geodesic.)

Let $Y \subseteq X$ be a closed convex subset with extendible geodesics. Then $Rad(\partial_{Tits} Y) = \pi$. The *parallel set* P_Y of Y is defined as the union of all convex subsets parallel to Y. P_Y is closed, convex and splits canonically as a metric product

$$P_Y \cong Y \times N_Y. \tag{9}$$

Here N_Y is a Hadamard space (not necessarily with extendible geodesics) and the subsets $Y \times \{pt\}$ are the convex subsets parallel to Y. The cross sections of P_Y orthogonal to these convex subsets can be constructed as intersections of horoballs:

$$\{y\} \times N_Y = P_Y \cap \bigcap_{\xi \in \partial_{Tits} Y} HB_\xi(y) \qquad \forall y \in Y. \tag{10}$$

Applying the Flat Sector Lemma 2.3.4 one sees furthermore that $\partial_{Tits} N_Y$ is canonically identified with $Poles(\partial_{Tits} Y) \subset \partial_{Tits} X$; $\partial_{Tits} P_Y$ is the convex hull in $\partial_{Tits} X$ of $\partial_{Tits} Y$ and $Poles(\partial_{Tits} Y)$ and we have the canonical decomposition:

$$\partial_{Tits} P_Y \cong \partial_{Tits} Y \circ Poles(\partial_{Tits} Y) \tag{11}$$

2.3.4 Products

The metric product of Hadamard spaces X_i is defined as usual using the Pythagorean law. It is again Hadamard and its Tits boundary and spaces of directions decompose canonically:

$$\partial_{Tits}(X_1 \times \cdots \times X_n) = \partial_{Tits} X_1 \circ \cdots \circ \partial_{Tits} X_n \tag{12}$$

$$\Sigma_{(x_1,\ldots,x_n)}(X_1 \times \cdots \times X_n) = \Sigma_{x_1} X_1 \circ \cdots \circ \Sigma_{x_n} X_n \tag{13}$$

Proposition 2.3.7 *If X is a Hadamard space with extendible geodesics then all join decompositions of $\partial_{Tits} X$ are induced by product decompositions of X.*

Proof. Assume that the Tits boundary decomposes as a spherical join $\partial_{Tits} X = B_1 \circ B_{-1}$ and consider, for $x \in X$ and $i = \pm 1$, the convex subsets $C_i(x) := \bigcap_{\xi \in B_{-i}} HB_\xi(x)$ obtained from intersecting horoballs. Using extendability of geodesics, i.e. $Rad \Sigma_x X = \pi$, one verifies that $\partial_{Tits} C_i = B_i$, C_i has extendible geodesics and $C_{\pm 1}(x)$ are orthogonal in the sense that $\Sigma_x C_i(x) = Poles(\Sigma_x C_{-i}(x))$. Furthermore any two sets $C_1(x)$ and $C_{-1}(x')$ intersect in the point $\pi_{C_1(x)}(x') = \pi_{C_{-1}}(x)$. The assertion follows by applying the Flat Rectangle Lemma 2.3.6. $\square$

Lemma 2.3.8 *Let X_1 and X_2 be Hadamard spaces and suppose that F is a flat in the product space $X = X_1 \times X_2$. Then there are flats $F_i \subseteq X_i$ so that $F_1 \times F_2 \supseteq F$.*

Proof. Consider unit speed parametrizations $c, c' : \mathbb{R} \to F$ for two parallel geodesics γ, γ' in F. Then $c_i := \pi_{X_i} \circ c$ and $c'_i := \pi_{X_i} \circ c'$ are constant speed parametrizations for geodesics γ_i, γ'_i in X_i. Since the distance functions $d := d_X(c, c')$ and $d_i := d_{X_i}(c_i, c'_i)$ are convex, satisfy $d^2 = d_1^2 + d_2^2$ and d is constant, it follows that the d_i are constant, i.e. γ_i and γ'_i are parallel. Since this works for any pair of parallel geodesics contained in F, it follows that $\pi_{X_i} F$ is a flat in F_i. $\square$

2.3.5 Induced isomorphisms of Tits boundaries

We now show that any $(1, A)$-quasi-isometric embedding of one Hadamard space into another induces a well-defined topological embedding of geometric boundaries which preserves the Tits distance.

Proposition 2.3.9 *Let X_1 and X_2 be Hadamard spaces and suppose that $\Phi : X_1 \to X_2$ is a $(1, A)$-quasi-isometric embedding. Then there is a unique extension $\bar{\Phi} : \bar{X}_1 \to \bar{X}_2$ such that*

1. *$\bar{\Phi}(\partial_\infty X_1) \subseteq \partial_\infty X_2$,*
2. *$\bar{\Phi}$ is continuous at boundary points.*
3. *$\bar{\Phi}\big|_{\partial_\infty X_1}$ is a topological embedding which preserves the Tits distance.*

We let $\partial_\infty \Phi \stackrel{def}{=} \bar{\Phi}\big|_{\partial_\infty X}$.

Proof. We first observe that there is a function $\epsilon(R)$ (depending on A but not on the spaces X_1 and X_2) with $\epsilon(R) \to 0$ as $R \to \infty$ such that if $p, x, y \in X_1$ and $d(p, x), d(p, y) > R$ then

$$|\tilde{\angle}_p(x, y) - \tilde{\angle}_{\Phi(p)}(\Phi(x), \Phi(y)| < \epsilon(R). \tag{14}$$

Lemma 2.3.10 *Suppose that x_i is a sequence of points in X_1 which converges to a boundary point ξ_1. Then $\Phi(x_i) \in X_2$ converges to a boundary point ξ_2.*

Proof of lemma: Pick a base point p. There are points $y_i \in \overline{px_i}$ such that $d(p, y_i) \to \infty$ and $\lim_{i,j\to\infty} \tilde{\angle}_p(y_i, y_j) = 0$. By (14), the points $\Phi(y_i)$ converge to a boundary point ξ_2. Applying (14) again, we conclude that $\Phi(x_i)$ converges to ξ_2 as well. □

Proof of Proposition continued: From the previous lemma we see that if x_i and x_i' are sequences in X_1 converging to the same point in $\partial_\infty X_1$ then the sequences $\Phi(x_i)$ and $\Phi(x_i')$ converge to the same point in $\partial_\infty X_2$. This allows us to extend Φ to a well-defined map $\bar{\Phi} : \bar{X}_1 \to \bar{X}_2$.

We now prove that $\bar{\Phi}$ is continuous at every boundary point ξ. Let $x_i \in \bar{X}_1$ be a sequence of points converging to $\xi \in \partial_\infty X_1$. By the lemma, we may choose $y_i \in X_1$ with $y_i \in \overline{px_i}$ so that for every R the Hausdorff distance between $\overline{\Phi(p)\Phi(y_i)} \cap B_R(\Phi(p))$ and $\overline{\Phi(p)\bar{\Phi}(x_i)} \cap B_R(\Phi(p))$ tends to zero as $R \to \infty$. Hence $\lim_{R\to\infty} \bar{\Phi}(x_i) = \lim_{R\to\infty} \Phi(y_i) = \bar{\Phi}(\xi)$ by the lemma.

Another consequence of the lemma is that the image ray $\Phi(\overline{p\xi})$ diverges sublinearily from the ray $\overline{\Phi(p)\bar{\Phi}(\xi)}$ in the sense that

$$\lim_{R\to\infty} \frac{1}{R} \cdot d_H(\Phi(\overline{p\xi} \cap B_R(p)), \overline{\phi(p)\bar{\Phi}(\xi)} \cap B_R(\Phi(p))) = 0$$

where d_H denotes the Hausdorff distance. This implies that $\partial_\infty \Phi \stackrel{def}{=} \bar{\Phi}\big|_{\partial_\infty X_1}$ preserves the Tits distance and is a homeomorphism onto its image. □

2.4 Ultralimits and Asymptotic cones

The presentation here is a slight modification of [Gro], see also [KaLe].

2.4.1 Ultrafilters and ultralimits

Definition 2.4.1 *A nonprincipal ultrafilter is a finitely additive probability measure* ω *on the subsets of the natural numbers* $\mathbb{N}$ *such that*

1. $\omega(S) = 0$ *or* 1 *for every* $S \subset \mathbb{N}$.
2. $\omega(S) = 0$ *for every finite subset* $S \subset \mathbb{N}$.

Given a compact metric space X and a map $a : \mathbb{N} \longrightarrow X$, there is a unique element $\omega\text{-}\lim a \in X$ such that for every neighborhood U of $\omega\text{-}\lim a$, $a^{-1}(U) \subset \mathbb{N}$ has full measure. In particular, given any bounded sequence $a : \mathbb{N} \longrightarrow \mathbb{R}$, $\omega\text{-}\lim a$ (or a_ω) is a limit point selected by ω.

2.4.2 Ultralimits of sequences of pointed metric spaces

Let $(X_i, d_i, \star_i)$ be a sequence of metric spaces with basepoints $\star_i$. Consider $X_\infty = \{x \in \prod_{i\in\mathbb{N}} X_i \big| d_i(x_i, \star_i)$ is bounded $\}$. Since $d_i(x_i, y_i)$ is a bounded sequence we may define $\tilde{d}_\omega : X_\infty \times X_\infty \longrightarrow \mathbb{R}$ by $\tilde{d}_\omega(x, y) = \omega\text{-}\lim d_i(x_i, y_i)$. $\tilde{d}_\omega$ is a pseudo-distance. We define the ultralimit of the sequence $(X_i, d_i, \star_i)$ to be the quotient metric space (X_ω, d_ω). $x_\omega \in X_\omega$ denotes the element corresponding to $x = (x_i) \in X_\infty$. $\star_\omega := (\star_i)$ is the basepoint of (X_ω, d_ω).

Lemma 2.4.2 *If* $(X_i, d_i, \star_i)$ *is a sequence of pointed metric spaces, then* $(X_\omega, d_\omega, \star_\omega)$ *is complete.*

Proof. Let x^j_ω be a Cauchy sequence in X_ω, where $x^j_\omega = \omega\text{-}\lim x^j_i$. Let $N_1 = \mathbb{N}$. Inductively, there is an ω-full measure subset $N_j \subseteq N_{j-1}$ such that $i \in N_j$ implies $|d_i(x^k_i, x^l_i) - d_\omega(x^k, x^l)| < \frac{1}{2^j}$, for $1 \le k, l \le j$. For $i \in N_j - N_{j-1}$, define $y_i = x^j_i$. Then $x^j_\omega \to y_\omega$. □

The concept of ultralimits is an extension of Hausdorff limits.

Lemma 2.4.3 *If* $(X_i, d_i, \star_i)$ *form a Hausdorff precompact family of pointed metric spaces, then* $(X_\omega, d_\omega, \star_\omega)$ *is a limit point of the sequence* $(X_i, d_i, \star_i)$ *with respect to the pointed Hausdorff topology.*

Proof. To see this, pick ϵ, R, and note that there is an N such that we can find an N element sequence $\{x^j_i\}^N_{j=1} \subset X_i$ which is ϵ-dense in X_i. The N sequences x^j_i for $1 \le j \le N$ give us N elements in $x^j_\omega \in X_\omega$. If $y_\omega \in X_\omega$, $y_\omega \in B_{\star_\omega}(R)$, then for ω-a.e. i, $d_i(y_i, \star_i) < R$. Consider $d_\omega(y_\omega, x^j_\omega)$. Given $\epsilon > 0$, $|d_\omega(y_\omega, x^j_\omega) - d_i(y_i, x^j_i)| < \epsilon$ for ω-a.e. i, which implies that $d_\omega(y_\omega, x^j_\omega) < \epsilon$ for some $1 \le j \le N$. Hence we've seen that $B_{\star_\omega}(R)$ is totally bounded, and for all $\epsilon > 0$ there is an ϵ-net in $B_{\star_\omega}(R)$ which is a Hausdorff limit point of ϵ-nets in the X_i's. It follows that $(X_i, d_i, \star_i)$ subconverges to $(X_\omega, d_\omega, \star_\omega)$ in the pointed Hausdorff topology. □

In general, the ultralimit X_ω is not Hausdorff close to the spaces X_i in the "approximating" sequence. However, the Hausdorff limits of any precompact sequence of subspaces $Y_i \subset X_i$ canonically embed into X_ω.

The importance of ultralimits for the study of the large-scale geometry from the following fact: If for each i, $f_i : X_i \to Y_i$ is a (L, C)-quasi-isometry with $d_i(f_i(\star_i), \star_i)$ bounded then the f_i induce an (L, C)-quasi-isometry $f_\omega : X_\omega \to Y_\omega$.

It follows that if for each i, and every pair of points $a_i, b_i \in X_i$ the distance $d_i(a_i, b_i)$ is the infimum of lengths of paths joining a_i to b_i then every pair of points $a_\omega, b_\omega \in X_\omega$ is joined by a geodesic segment.

Lemma 2.4.4 *If $(X_i, d_i, \star_i)$ is a $CAT(\kappa)$ space for each i, then so is $(X_\omega, d_\omega, \star_\omega)$. If $d_\omega(a_\omega, b_\omega) < D(\kappa)$, then the geodesic segment $\overline{a_\omega b_\omega}$ is an ultralimit of geodesic segments. If $\kappa \leq 0$ and each X_i has extendible geodesics then each ray (respectively complete geodesic) in X_ω is an ultralimit of rays (respectively complete geodesics) in the X_i's.*

Proof. If each $(X_i, d_i, \star_i)$ is a $CAT(\kappa)$ length space, then clearly $(X_\omega, d_\omega, \star_\omega)$ satisfies the δ_κ-four-point condition since this is a closed condition. Hence $(X_\omega, d_\omega, \star_\omega)$ is a $CAT(\kappa)$ length space since it is a geodesic space satisfying the δ_κ-four-point condition.

If $a_\omega, b_\omega \in X_\omega$ with $|a_\omega b_\omega| < D(\kappa)$, then there is a unique geodesic segment joining a_ω to b_ω. On the other hand, if $a_\omega = \omega\text{-}\lim a_i$, $b_\omega = \omega\text{-}\lim b_i$, then the ultralimit of the geodesic segments $\overline{a_i b_i}$ is a such a geodesic segment.

Now suppose $a_\omega^0, a_\omega^1, ...$ determine a ray, in the sense that $d_\omega(a_\omega^i, a_\omega^k) = d_\omega(a_\omega^i, a_\omega^j) + d_\omega(a_\omega^j, a_\omega^k)$ for $i \leq j \leq k$. Let $N_1 = \mathbb{N}$. Inductively, there is an ω-full measure $N_j \subseteq N_{j-1}$ such that $\overline{a_i^0 a_i^l}$ is within a $\frac{1}{2^j}$ neighborhood of the segment $\overline{a_i^0 a_i^j}$ for $i \in N_j$, $0 \leq l \leq j$. For $i \in N_j - N_{j-1}$ extend the segment $\overline{a_i^0 a_i^j}$ to a ray $\overline{a_i^0 \xi_i}$ with initial point a_i^0. Then the ultralimit of the sequence $\overline{a_i^0 \xi_i}$ is the ray we started with. The case of complete geodesics follows from similar reasoning. □

Lemma 2.4.5 *Suppose that there is a $D > 0$ such that for each i, $Isom(X_i)$ has an orbit which is D-dense in X_i. If $\lambda_i > 0$ and $\lambda_i \to 0$, then the ultralimit of $(X_i, \lambda_i d_i, \star_i)$ is independent of the choice of basepoints $\star_i$, and has a transitive isometry group.*

2.4.3 Asymptotic cones

Let X be a metric space and let $\star_n \in X$ be a sequence of basepoints. We define the *asymptotic cone* $\text{Cone}(X)$ of X with respect to the non-principal ultrafilter ω, the sequence of scale factors λ_n with $\omega\text{-}\lim \lambda_n = \infty$ and basepoints $\star_n$, as the ultralimit of the sequence of rescaled spaces $(X_n, d_n, \star_n) := (X, \frac{1}{\lambda_n} \cdot d, \star_n)$. When the sequence $\star_n \equiv \star$ is constant , then $\text{Cone}(X)$ does not depend on the basepoint $\star$ and has a canonical basepoint $\star_\omega$ which is represented by any sequence $(x_n) \subset X$ satisfying $\omega\text{-}\lim_n \frac{1}{\lambda_n} \cdot d(x_n, \star) = 0$, for instance, by any constant sequence (x).

Proposition 2.4.6

- *If X is a geodesic metric space, then* $\text{Cone}(X)$ *is a geodesic metric space.*
- *If X is a Hadamard space, then* $\text{Cone}(X)$ *is a Hadamard space.*
- *If X is a $CAT(\kappa)$-space for some $\kappa < 0$, then* $\text{Cone}(X)$ *is a metric tree.*
- *If the orbits of $Isom(X)$ are at bounded Hausdorff distance from X, then* $\text{Cone}(X)$ *is a homogeneous metric space.*
- *A (L, C) quasi-isometry of metric spaces $\phi : X \to Y$ induces a bilipschitz map* $\text{Cone}(\phi) : \text{Cone}(X) \to \text{Cone}(Y)$ *of asymptotic cones.*

If we're given an (L, C) quasi-isometry $\Phi : X \longrightarrow Y$, then

Assume now that X is a Hadamard space. Let $(F_n)_{n \in \mathbb{N}}$ be a sequence of k-flats in X and suppose that $\omega\text{-}\lim_n \frac{1}{\lambda_n} d(F_n, \star) < \infty$. Then the ultralimit of the embeddings of pointed metric spaces

$$\underbrace{\left(F_n, \frac{1}{\lambda_n} \cdot d_{F_n}, \pi_{F_n}(\star)\right)}_{\cong \mathbb{R}^k} \hookrightarrow \left(X, \frac{1}{\lambda_n} \cdot d_X, \pi_{F_n}(\star)\right)$$

is a k-flat

$$\mathbb{R}^k \hookrightarrow \text{Cone}(X)$$

in the asymptotic cone. We denote the family of all k-flats in Cone(X) arising in this way by $\mathcal{F}(k)$.

3 Spherical buildings

Our viewpoint on spherical buildings is slightly different from the standard one: for us a spherical building is a $CAT(1)$ space equipped with extra structure. This viewpoint is well adapted to the needs of this paper, because the spherical buildings which we consider arise as Tits boundaries and spaces of directions of Hadamard spaces. Apart from the choice of definitions and the viewpoint, this section does not contain anything new; the same results and many more can be found (often in slightly different form) in [Ti1, Ron, Brbk, Brn1, Brn2].

3.1 Spherical Coxeter complexes

Let S be a Euclidean unit sphere. By a *reflection* on S we mean an involutive isometry whose fixed point set, its *wall*, is a subsphere of codimension one. If $W \subset Isom(S)$ is a finite subgroup generated by reflections, we call the pair (S, W) a *spherical Coxeter complex* and W its *Weyl group*.

The finite collection of walls belonging to reflections in W divide S into isometric open convex sets. The closure of any of these sets is called a chamber, and is a fundamental domain for the action of W. Chambers are convex spherical polyhedra, i.e. finite intersections of hemispheres. A *face* of a chamber is an intersection of the chamber with some walls.

A *face* (resp. *open face*) of S is a face (resp. open face) of a chamber of S. Two faces of S are *opposite* or *antipodal* if they are exchanged by the canonical involution of S; two faces are opposite iff they contain a pair of antipodal points in their interiors. A *panel* is a codimension 1 face, a *singular sphere* is an intersection of walls, a *half-apartment* or *root* is a hemisphere bounded by a wall and a *regular* point in S is an interior point of a chamber. The regular points form a dense subset. The orbit space

$$\Delta_{mod} := S/W$$

with the orbital distance metric is a spherical polyhedron isometric to each chamber. The quotient map

$$\theta = \theta_S : S \longrightarrow \Delta_{mod} \tag{15}$$

is 1-Lipschitz and its restriction to each chamber is distance preserving. For $\delta, \delta' \in \Delta_{mod}$, we set

$$D(\delta, \delta') := \{d_S(x, x') | x, x' \in S, \theta x = \delta, \theta x' = \delta'\}$$

and

$$D^+(\delta) := D(\delta, \delta) \setminus \{0\}$$

Note that D^+ is continuous on each open face of Δ_{mod}.

An *isomorphism* of spherical Coxeter complexes $(S, W), (S', W')$ is an isometry $\alpha : S \to S'$ carrying W to W'. We have an exact sequence

$$1 \to W \to Aut(S, W) \to Isom(\Delta_{mod}) \to 1.$$

Lemma 3.1.1 *If $g \in W$, then $Fix(g) \subseteq S$ is a singular sphere. If $Z \subset S$ then the subgroup of W fixing Z pointwise is generated by the reflections in W which fix Z pointwise.*

Proof. Every W-orbit intersects each closed chamber precisely once. Therefore the stabiliser of a face $\sigma \subset S$ fixes σ pointwise. So for all $g \in W$, $Fix(g)$ is a subsphere and a subcomplex, i.e. it is a singular sphere.

By the above, without loss of generality we may assume that Z is a singular sphere. Let W_Z be the group generated by reflections fixing Z pointwise. If σ is a top-dimensional face of the singular sphere Z then each W-chamber containing σ is contained in a unique W_Z-chamber; therefore W_Z acts (simply) transitively on the W-chambers containing σ. Since W acts simply (transitively) on W-chambers, it follows that $Fixator(Z) = Fixator(\sigma) = W_Z$. □

3.2 Definition of spherical buildings

Let (S, W) be a spherical Coxeter complex. A *spherical building modelled on* (S, W) is a $CAT(1)$-space B together with a collection $\mathcal{A}$ of isometric embeddings $\iota : S \to B$, called *charts*, which satisfies properties SB1-2 described below and which is closed under precomposition with isometries in W. An *apartment* in B is the image of a chart $\iota : S \to B$; ι is a chart of the apartment $\iota(S)$. $\mathcal{A}$ is called the atlas of the spherical building.

SB1: Plenty of apartments. Any two points in B are contained in a common apartment.

Let ι_{A_1}, ι_{A_2} be charts for apartments A_1, A_2, and let $C = A_1 \cap A_2$, $C' = \iota_{A_2}^{-1}(C) \subset S$. The charts ι_{A_i} are *W-compatible* if $\iota_{A_1}^{-1} \circ \iota_{A_2}\big|_{C'}$ is the restriction of an isometry in W.

SB2: Compatible apartments. The charts are W-compatible.

It will be a consequence of corollary 3.9.2 below that the atlas $\mathcal{A}$ is maximal among collections of charts satisfying axioms SB1 and SB2.

We define walls, singular spheres, half-apartments, chambers, faces, antipodal points, antipodal faces, and regular points to be the images of corresponding objects in the spherical Coxeter complex. The building is called *thick* if each wall belongs to at least 3 half-apartments. The axioms yield a well-defined 1-Lipschitz *anisotropy map* [7]

$$\theta_B : B \longrightarrow S/W =: \Delta_{mod} \tag{16}$$

satisfying the *discreteness condition*:

$$d_B(x_1, x_2) \in D(\theta_B(x_1), \theta_B(x_2)) \qquad \forall x_1, x_2 \in B \tag{17}$$

If $\alpha : S \longrightarrow S$ is an automorphism of the spherical Coxeter complex, then we modify the atlas by precomposing with α; the atlases obtained this way correspond to symmetries of Δ_{mod}.

If $\mathcal{A}'$ is an atlas of charts $\iota : S' \longrightarrow B$ giving a (S', W') building structure on B, then this spherical building is *equivalent* to $(B, \mathcal{A})$ if there is an isomorphism of spherical Coxeter complexes $\alpha : (S', W') \longrightarrow (S, W)$ so that $\mathcal{A}' = \{\iota \circ \alpha | \iota \in \mathcal{A}\}$.

If B and B' are spherical buildings modelled on a Coxeter complex (S, W), with atlases $\mathcal{A}$ and $\mathcal{A}'$, an *isomorphism* is an isometry $\phi : B \to B'$ such that the correspondence $\iota \mapsto \phi \circ \iota$ defines a bijection $\mathcal{A} \to \mathcal{A}'$.

[7]The motivation for this terminology comes from the role θ_B plays in the structure of symmetric spaces and Euclidean buildings.

3.3 Join products and decompositions

Let B_i, $i = 1, \ldots, n$, be spherical buildings modelled on spherical Coxeter complexes (S_i, W_i) with atlases $\mathcal{A}_i$ and spherical model polyhedra Δ^i_{mod}. Then $W := W_1 \times \cdots \times W_n$ acts canonically as a reflection group on the sphere $S = S_1 \circ \cdots \circ S_n$. We call the Coxeter complex (S, W) the spherical join of the Coxeter complexes (S_i, W_i) and write

$$(S, W) = (S_1, W_1) \circ \cdots \circ (S_n, W_n) \tag{18}$$

The model polyhedron Δ_{mod} of (S, W) decomposes canonically as

$$\Delta_{mod} = \Delta^1_{mod} \circ \cdots \circ \Delta^n_{mod}. \tag{19}$$

The CAT(1)-space

$$B = B_1 \circ \cdots \circ B_n \tag{20}$$

carries a natural spherical building structure modelled on (S, W). The charts ι for its atlas $\mathcal{A}$ are the spherical joins $\iota = \iota_1 \circ \cdots \circ \iota_n$ of charts $\iota_i \in \mathcal{A}_i$. We call B equipped with this building structure the *spherical (building) join* of the buildings B_i.

Proposition 3.3.1 *Let B be a spherical building modelled on the Coxeter complex (S, W) with atlas $\mathcal{A}$ and assume that there is a decomposition (19) of its model polyhedron. Then:*

1. *There is a decomposition (18) of (S, W) as a join of spherical Coxeter complexes so that $S_i = \theta_S^{-1}(\Delta^i_{mod})$.*
2. *There is a decomposition (20) of B as a join of spherical buildings so that $B_i = \theta_B^{-1}(\Delta^i_{mod})$.*

Proof. 1. We identify Δ_{mod} with a W-chamber in S and define S_i to be the minimal geodesic subsphere containing Δ^i_{mod}. Then $S_i \subseteq Poles(S_j)$ for all $i \neq j$ and hence $S = S_1 \circ \cdots \circ S_n$ by dimension reasons. Each wall containing a codimension-one face of Δ_{mod} is orthogonal to one of the spheres S_i and contains the others. Hence $W = W_1 \times \cdots \times W_n$ where W_i is generated by the reflections in W at walls orthogonal to S_i. W_i acts as a reflection group on S_i and the claim follows.

2. Since any two points in B are contained in an apartment, one sees by applying the first assertion that the B_i are convex subsets and B is canonically isometric to the join of CAT(1)-spaces $B = B_1 \circ \cdots \circ B_n$. The collection of charts $\iota\big|_{S_i}$, $\iota \in \mathcal{A}$, forms an atlas for a spherical building structure on B_i and B is canonically isomorphic to the spherical building join of the B_i. □

We call a spherical polyhedron *irreducible* if it is a spherical *simplex* with diameter $< \pi/2$ and dihedral angles $\leq \pi/2$ or if it is a sphere or a point. Accordingly, we call a spherical Coxeter complex[8] or a spherical building *irreducible* if its model polyhedron is irreducible. The spherical model polyhedron Δ_{mod} has dihedral angles $\leq \frac{\pi}{2}$. A polyhedron of this sort has a unique minimal decomposition as the spherical join (19) of irreducible spherical simplices (which may be single points) and, if non-empty, the unique maximal unit sphere contained in Δ_{mod}. By Proposition 3.3.1, (19) corresponds to unique minimal decompositions (18) of the Coxeter complex (S, W) as a join of Coxeter complexes and (20) of B as a spherical building join. We call these decompositions the *de Rham decompositions* of (S, W) and B. The sphere factor in (19) occurs iff the fixed point

[8]This definition is slightly different form the usual one, which corresponds to irreducibility of linear representations.

set of the Weyl group is non-empty. We call the corresponding factor in the de Rham decomposition the *spherical de Rham factor*.

If W acts without fixed point, then Δ_{mod} is a spherical simplex [9] and the collection of chambers in S and B give rise to simplicial complexes.

Lemma 3.3.2 *Let (S, W) be an irreducible spherical Coxeter complex with non-trivial Weyl group W. Then for each chamber σ there is a wall which is disjoint from the closure $\bar{\sigma}$.*

Proof. Let τ' be a wall and $p \in S$ be a point at maximal distance $\frac{\pi}{2}$ from τ'. Pick a chamber σ' containing p in its closure. Then $\bar{\sigma'} \cap \tau' = \emptyset$, because $Diam(\sigma') < \frac{\pi}{2}$ due to irreducibility. Since W acts transitively on chambers, the claim follows. □

Proposition 3.3.3 *Assume that B_1 and B_2 are CAT(1)-spaces and that their join $B = B_1 \circ B_2$ admits a spherical building structure. Then the B_i inherit natural spherical building structures from B. In particular, the spherical building B cannot be thick irreducible with non-trivial Weyl group.*

Proof. Applying lemma 2.2.1 to apartments in B, we see that there exist $d_1, d_2 \in \mathbb{N}$ so that every apartment $A \subseteq B$ splits as $A = A_1 \circ A_2$ where A_i is a d_i-dimensional unit sphere in B_i. Fix a chart ι_0 in the atlas $\mathcal{A}$ for the given spherical building structure on B. Denote by S_2 the d_2-sphere $\iota_0^{-1} B_2$ in the model Coxeter complex (S, W) and by $S_1 := Poles(S_2)$ the complementary d_1-sphere. The subgroup $W_1 \subseteq W$ generated by reflections at walls containing S_2 acts as a reflection group on S_1. Consider all charts $\iota \in \mathcal{A}$ with $\iota\big|_{S_2} = \iota_0\big|_{S_2}$. The collection $\mathcal{A}_1$ of their restrictions $\iota\big|_{S_1}$ forms an atlas for a spherical building structure on B_1 with model Coxeter complex (S_1, W_1).

If B is thick, then its chambers are precisely the (closures of the) connected components of the subset of manifold points. Hence the joins $\sigma_1 \circ \sigma_2$ of chambers $\sigma_i \subset B_i$ are contained in chambers of B. So the chambers of B have diameter $\geq \frac{\pi}{2}$ and B cannot be irreducible with non-trivial Weyl group. □

3.4 Polyhedral structure

Let Δ' be a face of Δ_{mod} and let $\sigma : \Delta' \to B$ be the chart for a face in B, i.e. an isometric embedding so that $\theta_B \circ \sigma = id|_{\Delta'}$.

Sublemma 3.4.1 *$\sigma(Int\Delta')$ is an open subset of $\theta_B^{-1}(\Delta')$.*

Proof. Let x be a point in $\sigma(Int\Delta')$ and assume that there exists a sequence (x_n) in $\theta_B^{-1}(\Delta' \setminus \sigma(Int(\Delta'))$ which converges to x. There are points $x'_n \in Im(\sigma)$ with $\theta_B(x'_n) = \theta_B(x_n)$ Since θ_B has Lipschitz constant 1 and σ is distance-preserving, we have

$$d_B(x_n, x) \geq d_{\Delta_{mod}}(\theta_B(x_n), \theta_B(x)) = d_B(x'_n, x)$$

and by the triangle inequality

$$2 \cdot \underbrace{d_B(x_n, x)}_{\to 0} \geq d_B(x'_n, x_n) \geq D^+(\theta_B(x_n)).$$

[9] By [GrBe][theorem 4.2.4], Δ_{mod} is a simplex if W acts fixed point freely. Observe that having distance less than $\pi/2$ is an equivalence relation on the vertices. This implies the decomposition (19).

Since D^+ is continuous on $Int\Delta'$, the right-hand side has a positive limit:

$$\lim_{n\to\infty} D^+(\theta_B(x_n)) = D^+(\theta_B(x)) > 0,$$

a contradiction. □

Lemma 3.4.2 *Any two faces of with a common interior point coincide. Consequently, the intersection of faces in B is a face in B.*

Proof. To verify the first assertion, consider two face charts $\sigma_1, \sigma_2 : \Delta' \to B$ of the same type. By Sublemma 3.4.1, $\{\delta \in \Delta' | \sigma_1(\delta) = \sigma_2(\delta)\} \cap Int\Delta'$ is an open subset of $Int\Delta'$. It is also closed, and hence empty or all of $Int\Delta'$ if Δ' is connected. If Δ' is disconnected, it must be the maximal sphere factor of Δ_{mod} and all apartment charts agree on Δ'. Hence $\sigma_1|_{\Delta'} = \sigma_2|_{\Delta'}$ also in this case.

The intersections of two faces is a union of faces by the above; since it is convex, it is a face. □

As a consequence, the collection of finite unions of faces of B is a lattice under the binary operations of union and intersection; we will denote this lattice by $\mathcal{K}B$. In the case that the Weyl group acts without fixed point, the chambers of B are simplices, and $\mathcal{K}B$ is the lattice of finite subcomplexes of a simplicial complex. In general the polyhedron of this simplicial complex is not homeomorphic to B since it has the weak topology.

3.5 Recognizing spherical buildings

The following proposition gives an easily verified criterion for the existence of a spherical building structure on a CAT(1)-space.

Proposition 3.5.1 *Let (S, W) be a spherical Coxeter complex, and let B be a CAT(1)-space of diameter π equipped with a 1-Lipschitz anisotropy map θ_B as in (16) satisfying the discreteness condition (17). Suppose moreover that each point and each pair of antipodal regular points is contained in a subset isometric to S. Then there is a unique atlas $\mathcal{A}$ of charts $\iota : S \longrightarrow B$ forming a spherical building structure on B modelled on (S, W), with associated anisotropy map θ_B.*

Proof. The discreteness condition (17) implies that, for any face Δ' of Δ_{mod}, the restriction of θ_B to $\theta_B^{-1}(Int\Delta')$ is locally distance preserving and distance preserving on minimizing geodesic segments contained in $\theta_B^{-1}(Int\Delta')$. Therefore, if $A \subset B$ is a subset isometric to S, the restriction of θ_B to $A^{reg} := A \cap \theta_B^{-1}(Int\Delta_{mod})$ is locally isometric and the components of A^{reg} are open convex polyhedra which project via θ_B isometrically onto $Int\Delta_{mod}$. (17) implies moreover that A^{reg} is dense in A. Hence A is tesselated by isometric copies of Δ_{mod} and there is an isometry ι_A with $\theta_B \circ \iota_A = \theta_S$ which is unique up to precomposition with elements in W. If A_1 and A_2 are subsets isometric to S, and $\iota_{A_1}, \iota_{A_2} : S \longrightarrow B$ are isometries as above then $A_1 \cap A_2$ is convex, and we see that ι_{A_1} and ι_{A_2} are W-compatible. We now refer to the isometries $\iota_A : S \longrightarrow B$ as charts and to their images as apartments. The collection $\mathcal{A}$ of all charts will be the atlas for our spherical building structure.

Since any point lies in some apartment, it lies in particular in a face, i.e. in the image of an isometric embedding $\sigma : \Delta' \to B$ of a face $\Delta' \subseteq \Delta_{mod}$ satisfying $\theta_B \circ \sigma = id|_{\Delta'}$. Lemma 3.4.2 applies and the faces fit together to form a polyhedral structure on B. The apartments are subcomplexes.

It remains to verify that any two points with distance less than π lie in a common apartment. It suffices to check this for any regular points x_1, x_2, since any point lies in a chamber and an apartment containing an interior point of a chamber contains the whole chamber (lemma 3.4.2). There is an apartment A_1 containing x_1. Consider a minimizing geodesic c joining x_1 and x_2. By sublemma 3.4.1, A_1 is a neighborhood of x_1. Hence near its endpoint x_1, c is a geodesic in the sphere A_1. Since B is a CAT(1)-space, we can extend c beyond x_1 inside A_1 to a minimizing geodesic $\bar{c}$ of length π joining x_2 through x_1 to a point $\hat{x}_2 \in A_1$. By our assumption, the points $x_2, \hat{x}_2$ are contained in an apartment A_2. A_2 contains all minimizing geodesics connecting x_2 and $\hat{x}_2$, because x_2 is regular. In particular $\bar{c}$ and therefore both points x_1, x_2 lie in A_2. □

From the proof of proposition 3.5.1 we have:

Corollary 3.5.2 *Let B be a spherical building of dimension d, and let $T \subseteq B$ be a subset isometric to the Euclidean unit sphere of dimension d. Then T is an apartment in B.*

3.6 Local conicality, projectivity classes and spherical building structure on the spaces of directions

Suppose that the spherical building B has dimension at least 1.

Lemma 3.6.1 *Let (B, θ_B) be a spherical building modelled on Δ_{mod}, and let $p, \hat{p} \in B$ be antipodal points, i.e. $d(p, \hat{p}) = \pi$. Then the union of the geodesic segments of length π from p to $\hat{p}$ is a metric suspension which contains a neighborhood of $\{p, \hat{p}\}$.*

Proof. By the discussion in section 2.2.1, the union of the geodesic segment of length π from p to $\hat{p}$ is a metric suspension. By (17) we can choose $\rho > 0$ such that $\{q \in B_{2\rho}(p) \big| \theta_B(q) = \theta_B(p)\} = \{p\}$, $\{q \in B_{2\rho}(\hat{p}) \big| \theta_B(q) = \theta_B(\hat{p})\} = \{\hat{p}\}$. If $q \in B_\rho(\hat{p})$, then any extension of $\overline{pq}$ to a segment $\overline{pqr}$ of length π will satisfy $\theta(r) = \theta(\hat{p})$, forcing $r = \hat{p}$ by the choice of ρ. Likewise, if we extend $\overline{\hat{p}q}$ to a segment of length π, where $q \in B_\rho(p)$, then it will terminate at p. Hence the lemma. □

As a consequence, for sufficiently small positive ϵ, the ball $B_\epsilon(p)$ is canonically isometric to a truncated spherical cone of height ϵ over $\Sigma_p B$, the isometry given by the "logarithm map" at p. In particular, $\Sigma_p^* B = \Sigma_p B$. Any face intersecting $B_\epsilon(p)$ contains p and the face σ_p spanned by p.

The lemma implies furthermore that for any pair of antipodes $p, \hat{p} \in B$ there is a canonical isometry

$$persp_{p,\hat{p}} : \Sigma_p B \to \Sigma_{\hat{p}} B \tag{21}$$

determined by the property that all geodesics c of length π joining p and $\hat{p}$ satisfy $persp_{p,\hat{p}}(\Sigma_p c) = \Sigma_{\hat{p}} c$.

Two points in B are *antipodal* iff they have distance π. Two faces σ_1 and σ_2 are antipodal or *opposite* if there are antipodal points ξ_1 and ξ_2 so that ξ_i lies in the interior of σ_i; in this case each point in σ_1 has a unique antipode in σ_2.

Definition 3.6.2 *The relation of being antipodal generates an equivalence relation and we call the equivalence classes* projectivity classes.

Lemma 3.6.3 *Suppose that the spherical building B is thick. Then every projectivity class intersects every chamber.*

Proof. Let C_1 and C_2 be adjacent chambers, i.e. $\pi = C_1 \cap C_2$ is a panel. It suffices to show that for each point in C_1, C_2 contains a point in the same projectivity class. To see this, pick an apartment $A \supseteq C_1 \cup C_2$ and let $\hat{\pi}$ be the panel in A opposite to π ($\hat{\pi} = \pi$ is possible). Since B is thick there is a chamber C with $C \cap A = \hat{\pi}$. C is opposite to both C_1 and C_2 and our claim follows. □

Pick $p_0 \in S$ so that $\theta_S(p_0) = \theta_B(p)$. Now consider the collection of all apartment charts $\iota_A : S \longrightarrow B$ where $\iota_A(p_0) = p$. These induce isometric embeddings $\Sigma_{p_0}\iota_A : \Sigma_{p_0}S \longrightarrow \Sigma_p B$. Let $W_{p_0} \subseteq Isom(\Sigma_{p_0}S)$ be the finite group generated by the reflections in walls passing through p_0.

Proposition 3.6.4 *$\Sigma_p B$ together with the collection of embeddings $\Sigma_{p_0}\iota_A : \Sigma_{p_0}S \longrightarrow \Sigma_p B$ as above is a spherical building modelled on $(\Sigma_{p_0}S, W_p)$. If $\hat{p} \in B$ is an antipode of B, then we have a 1-1 correspondence between apartments (respectively half-apartments) in B containing $\{p, \hat{p}\}$ and apartments (respectively half-apartments) in $\Sigma_p B$. $\Sigma_p B$ is thick provided B is thick.*

Proof. Any two points $\overrightarrow{pq_1}, \overrightarrow{pq_2} \in \Sigma_p B$ lie in an apartment; namely choose q_1, q_2 close to p, then any apartment A containing q_1, q_2 will contain p and $\overrightarrow{pq_i} \in \Sigma_p A$. So SB1 holds. $\Sigma_p B$ satisfies SB2 since we are only using charts $\iota_A : S \longrightarrow B$ with $\iota_A(p_0) = p$ and B itself satisfies SB2. The remaining assertions follow immediately from the definition of the spherical building structure on $\Sigma_p B$. □

3.7 Reducing to a thick building structure

A *reduction* of the spherical building structure on B consists of a reflection subgroup $W' \subset W$ and a subset $\mathcal{A}' \subset \mathcal{A}$ which defines a spherical building structure modelled on (S, W'). The Δ_{mod}-direction map θ_B can then be factored as $\pi \circ \theta'_B$ where

$$\theta'_B : B \longrightarrow W' \backslash S =: \Delta'_{mod}$$

is the Δ'_{mod}-direction map for the building modelled on (S, W'), and $\pi : W' \backslash S = \Delta'_{mod} \longrightarrow \Delta_{mod} = W \backslash S$ is the canonical surjection.

Proposition 3.7.1 *Let B be a spherical building modelled on the spherical Coxeter complex (S, W), with anisotropy polyhedron $\Delta_{mod} = W \backslash S$. Then there exists a reduction $(W, \mathcal{A}')$ which is a* thick *building structure on B. W' is unique up to conjugacy in W; $\mathcal{A}'$ is determined by W'. In particular, the thick reduction is unique up to equivalence, so the polyhedral structure is defined by the $CAT(1)$ space itself.*

The proof will occupy the remainder of this paper.

We set $d = dim(B)$, $R_B = \{p \in B \big| \Sigma_p B \text{ is isometric to a standard } S^{d-1}\}$, and $S_B = B \setminus R_B$. If $p \in B$ and $\rho > 0$ is small enough that $B_\rho(p)$ is a (spherical) conical neighborhood of p, then $S_B \cap B_\rho(p) \setminus \{p\}$ corresponds to the cone over $S_{\Sigma_p B}$. It then follows by induction on $dim(B)$ that $S_B \cap A$ is a union of Δ_{mod}-walls for each apartment $A \subset B$.

Consider an apartment $A \subset B$, and a pair of walls $H_1, H_2 \subset A$ contained in S_B.

Lemma 3.7.2 *If H'_2 is the image of H_2 under reflection in the wall H_1 (inside the apartment A), then H'_2 is contained in S_B.*

Proof. To see this, consider an interior point p of a codimension 2 face σ of $H_1 \cap H_2$. $\Sigma_p B$ decomposes as a metric join $\Sigma_p \sigma \circ B_p$ where B_p is a 1-dimensional spherical building, and the walls H_1, H_2, and H_2' correspond to walls $\bar{H}_1$, $\bar{H}_2$, and $\bar{H}_2'$ in B_p; A corresponds to an apartment $\bar{A}$ in B_p. The wall $\bar{H}_1$ is just a pair of points in B_p, and this pair of points is joined by at least three differents semi-circles of length π. These three semi-circles can be glued in pairs to form three different apartments in B_p. Using the fact that an antipode of a point in S_{B_p} also lies in S_{B_p}, it is clear that the image of $\bar{H}_2$ under reflection in $\bar{H}_1$ is also in S_{B_p}. Hence the wall $\Sigma_p H_2' \subset \Sigma_p B$ is contained in three half-apartments, and proposition 3.6.4 then implies that H_2' lies in three half-apartments. □

The reflections in the walls in $A \cap S_B$ generate a group G_A, and by [Hum, p. 24] the only reflections in G_A are reflections in walls in $A \cap S_B$; also, the closures of connected components of $A \setminus S_B$ are fundamental domains for the action of G_A on A.

Sublemma 3.7.3 *Let $U \subseteq B$ be a connected component of $B \setminus S_B$, and suppose $U \cap A \neq \emptyset$ for some apartment A. Then $U \subseteq A$.*

Proof. $U \cap A$ is an open and closed subset of U, so $U \cap A = U$. □

We claim that the isomorphism class of G_A is independent of A. To show this, it suffices to show that the isometry type of a chamber Δ^A_{mod} is independent of A. For $i = 1, 2$ let A_i be an apartment, and let $\Delta^{A_i}_{mod}$ be a chamber for G_{A_i}. If $A_3 \subset B$ is an apartment containing an interior point from each $\Delta^{A_i}_{mod}$, then the sublemma gives $\Delta^{A_i}_{mod} \subset A_3$. But then the $\Delta^{A_i}_{mod}$ are both chambers for G_{A_3}, so they are isometric. Hence each pair (A, G_A) is isomorphic to a fixed spherical Coxeter complex (S, W^{th}) for some reflection subgroup $W^{th} \subseteq W$. We denote the quotient map and model polyhedron by

$$\theta^{th}_S : S \to S/W^{th} =: \Delta^{th}_{mod}.$$

We call the closure of components of $B \setminus S_B$, Δ^{th}_{mod}-chambers. We can identify the Δ^{th}_{mod}-chambers with Δ^{th}_{mod} in a consistent way by the following construction: Let $A_0 \subseteq B$ be an apartment and $p_0 \in A_0 \cap R_B$ be a smooth point. We define the retraction $\rho : B \to A_0$ by assigning to each point p in the open ball $B_\pi(p_0)$ the unique point $\rho(p) \in A_0$ so for which the segments $\overline{p_0 p}$ and $\overline{p_0 \rho(p)}$ have same length and direction $\overrightarrow{p_0 p} = \overrightarrow{p_o \rho(p)}$ at p_0. ρ extends continuously to the discrete set $B \setminus B_\pi(p_0)$ which maps to the antipode of p_0 in A_0. If A is an apartment passing through p_0 then $A \cap A_0$ contains the Δ^{th}_{mod}-chamber spanned by p_0 and $\rho|_A : A \to A_0$ is an isometry which preserves the tesselations by chambers. Composing ρ with the quotient map $A_0 \to A_0/G_{A_0}$ we obtain a 1-Lipschitz map

$$\theta^{th}_B : B \to \Delta^{th}_{mod} \tag{22}$$

which restricts to an isometry on each chamber. Applying proposition 3.5.1 we see that B is a spherical building modelled on (S, W^{th}). B is a thick building since we already verified in lemma 3.7.2 above that if $H \subset S_B$ is a wall, then it lies in at least three half-apartments.

Corollary 3.7.4 *For $i = 1, 2$ let B_i be a thick spherical building modelled on (S_i, W_i) with atlas $\mathcal{A}_i$. If $\phi : B_1 \to B_2$ is an isometry then we may identify the spherical Coxeter complexes by an isometry $\alpha : (S_1, W_1) \to (S_2, W_2)$ so that ϕ becomes an isomorphism of spherical buildings.*

3.8 Combinatorial and geometric equivalences

We recall (section 3.4) that for any building B, $\mathcal{K}B$ is the lattice of finite unions of faces of B.

Proposition 3.8.1 *Let B_1, B_2 be spherical buildings of equal dimension. Then any lattice isomorphism $\mathcal{K}B_1 \to \mathcal{K}B_2$ is induced by an isometry $B_1 \to B_2$ of CAT(1) spaces. This isometry is unique if the buildings B_i do not have a spherical deRham factor.*

Proof. First recall that lattice isomorphisms preserve the partial ordering by inclusion since $C_1 \subset C_2 \iff C_1 \cup C_2 = C_2$.

We first assume that the buildings B_i have no deRham factor and hence the $\mathcal{K}B_i$ come from simplicial complexes. In this case the lattice isomorphism $\mathcal{K}B_1 \longrightarrow \mathcal{K}B_2$ carries k-dimensional faces of B_1 to k-dimensional faces of B_2. To see this, note that vertices of B_i are the minimal elements of the lattice $\mathcal{K}B_i$ and k-simplices are characterized (inductively) as precisely those subcomplexes which contain $k+1$ vertices and are not contained in the union of lower dimensional simplices.

Consider a codimension-2 face σ of a chamber C in B_i. For an interior point $s \in \sigma$, $\Sigma_s B_i$ is isometric to the metric join $\Sigma_s\sigma \circ B_i^\sigma$ where B_i^σ is a 1-dimensional spherical building. The dihedral angle of C along σ equals the length of a chamber in the 1-dimensional building B_i^σ.

Sublemma 3.8.2 *The chamber length of a 1-dimensional spherical building is determined combinatorially as $2\pi/l$ where l is the combinatorial length of a minimal circuit.*

Proof. Combinatorial paths in a 1-dimensional spherical building determine geodesics. Closed geodesics in a CAT(1) space have length at least 2π since points at distance $< \pi$ are joined by a unique geodesic segment. The closed paths of length 2π are the apartments. □

Proof of proposition 3.8.1 cont. As a consequence of the sublemma, the lattice isomorphism $\mathcal{K}B_1 \to \mathcal{K}B_2$ induces a correspondence between chambers which preserves dihedral angles. Since the dihedral angles determine the isometry type of a spherical simplex [GrBe][theorem 5.1.2], there is a unique map of CAT(1)-spaces $B_1 \to B_2$ which is isometric on chambers and induces the given combinatorial isomorphism. Since the metric on each B_i is characterized as the largest metric for which the chamber inclusions are 1-Lipschitz maps, we conclude that our map $B_1 \to B_2$ is an isometry. In the general case, the buildings B_i may have a spherical deRham factor S_i and split as $B_i = S_i \circ B_i'$. The lattices $\mathcal{K}B_i$ and $\mathcal{K}B_i'$ are isomorphic: to a subcomplex C_i' of $\mathcal{K}B_i'$ corresponds the subcomplex $S_i \circ C_i'$ of $\mathcal{K}B_i$. The lattice isomorphism $\mathcal{K}B_1' \cong \mathcal{K}B_1 \to \mathcal{K}B_2 \cong \mathcal{K}B_2'$ is induced by a unique isometry $B_1' \to B_2'$ by the discussion above. It follows that $Dim B_1' = Dim B_2'$ and $Dim S_1 = Dim S_2$. Any isometry $S_1 \to S_2$ gives rise to an isometry $B_1 \to B_2$ which induces the isomorphism $\mathcal{K}B_1 \to \mathcal{K}B_2$. □

3.9 Geodesics, spheres, convex spherical subsets

We call a subset of a CAT(1)-space *convex* if with every pair of points with distance less than π it contains the minimal geodesic segment joining them. The following generalises corollary 3.5.2.

Proposition 3.9.1 *Let $C \subset B$ a convex subset which is isometric to a convex subset of a unit sphere. Then C is contained in an apartment.*

Proof. We proceed by induction on the dimension of B. The claim is trivial if $dim(B) = 0$. We assume therefore that $dim(B) > 0$ and that our claim holds for buildings of smaller dimension than B.

Let A be an apartment so that the number of open faces in A which have non-empty intersection with C is maximal. Suppose $C \nsubseteq A$. Let $p \in C \cap A$ and $q \in C \setminus A$ be points with $\overrightarrow{pq} \notin \Sigma_p A$. Denote by V the union of all minimizing geodesics in A which connect p to its antipode $\hat{p}$ and intersect $C - \{p, \hat{p}\}$. V is a convex subset of A and canonically isometric to the suspension of $\Sigma_p(C \cap A) = \Sigma_p C \cap \Sigma_p A$. By induction assumption, there is an apartment A' through p such that $\Sigma_p C \subseteq \Sigma_p A'$. A' can be chosen to contain $\hat{p}$. Then $C \cap A \subseteq V \subseteq A'$ and $\overrightarrow{pq} \in \Sigma_p A'$. Hence the number of open sectors in A' intersecting C is strictly bigger than the number of such sectors in A, a contradiction. Therefore $C \subseteq A$. □

Corollary 3.9.2 *Any minimizing geodesic in a spherical building B is contained in an apartment. Any isometrically embedded unit sphere $K \subseteq B$ is contained in an apartment. In particular $dim(K) \leq rank(B) - 1$.*

3.10 Convex sets and subbuildings

A *subbuilding* is a subset $B' \subseteq B$ so that $\{\iota \in \mathcal{A} | \iota(S) \subseteq B'\}$ forms an atlas for a spherical building structure; in particular B' is closed and convex.

Lemma 3.10.1 *Let $s \subset B$ be a subset isometric to a standard sphere. Then the union $B(s)$ of the apartments containing s is a subbuilding. There is a canonical reduction $(W', \mathcal{A}')$ of the spherical building structure on $B(s)$; its walls are precisely the W-walls of $B(s)$ which contain s. When equipped with this building structure, $B(s)$ decomposes as a join of s and another spherical building which we call $Link(s)$. If $p \in s$ then log_p maps $Link(s)$ isometrically to the join complement of $\Sigma_p s$ in $\Sigma_p B(s)$. Furthermore, if $p \in s$ lies in a W-face σ of maximal possible dimension, then there is a bijective correspondence between W-chambers containing σ, W'-chambers of $B(s)$, chambers of $Link(s)$, and W_p-chambers in $\Sigma_p B$.*

Proof. Let ξ and $\hat{\xi}$ be interior points of faces in s with maximal dimension. Then $B(s)$ is the union of all geodesic segments of length π from ξ to $\hat{\xi}$. Proposition 3.6.4 implies that every pair of points in $B(s)$ is containined in an apartment $A \subset B(s)$.

Pick $\iota_0 \in \mathcal{A}$ with $s \subseteq \iota_0(S)$, and set $\mathcal{A}' = \{\iota \in \mathcal{A} | \iota\big|_{s_0} = \iota_0\big|_{s_0}\}$. Let $W' \subseteq W$ be the subgroup generated by reflections fixing s_0 pointwise. According to lemma 3.1.1, the coordinate changes for the charts in $\mathcal{A}'$ are restrictions of elements of W'. Therefore $\mathcal{A}'$ is an atlas for a spherical building structure on $B(s)$ modelled on (S, W').

Since $s_0 \subseteq S$ is a join factor of the spherical Coxeter complex (S, W'), $B(s)$ decomposes as a join of spherical buildings $B(s) = s \circ Link(s)$ by section 3.3. Any two points in $Link(s)$ lie in an apartment $s \subseteq A \subset B(s)$, so log_p maps $Link(s)$ isometrically to the join complement of $\Sigma_p s$ in $\Sigma_p B(s)$. The remaining statements follow. □

The building $B(s)$ splits as a spherical join of the singular sphere s and a spherical building which we denote by $Link(s)$:

$$B(s) = s \circ Link(s)$$

Lemma 3.10.2 *If $\xi \in B$ and η lies in the apartment $A \subseteq B$, then there is a $\hat{\xi} \in A$ with $\pi = d(\xi, \hat{\xi}) = d(\xi, \eta) + d(\eta, \hat{\xi})$. If $d(\xi, \eta) \geq \frac{\pi}{2}$ then ξ has an antipode in every top-dimensional hemisphere $H \subset A$.*

Proof. When $DimB = 0$ the lemma is immediate. If $d(\xi, \eta) < \pi$ then by induction $\overrightarrow{\eta\xi} \in \Sigma_\eta B$ has an antipode in $\Sigma_\eta A$. Therefore we may extend $\overline{\xi\eta}$ to a geodesic segment $\overline{\xi\eta\hat{\xi}}$ with $\overline{\eta\hat{\xi}} \subset A$ of length π. The second statement follows by letting η be the pole of the hemisphere. □

Proposition 3.10.3 *Let C be a convex subset in the spherical building B. If C contains an apartment then C is a subbuilding of full rank.*

Proof. By the lemma, any point $\xi \in C$ has an antipode $\hat{\xi}$ in C. By lemma 3.6.1, the union $C_{\xi,\hat{\xi}}$ of all minimizing geodesics from ξ to $\hat{\xi}$ which intersect $C - \{\xi, \hat{\xi}\}$ is a neighborhood of ξ in C. In particular, for sufficiently small $\epsilon > 0$, $C \cap B_\epsilon(\xi)$ is a cone over $\Sigma_\xi C$. Since $\hat{\xi}$ can be chosen to lie in an apartment $A_0 \subseteq C$ by our assumption, and since the apartment $\Sigma_{\hat{\xi}} A_0$ in $\Sigma_{\hat{\xi}} C$ corresponds to an apartment in $C_{\xi,\hat{\xi}}$, we see that C is a union of apartments. It remains to check that any two points $\xi, \eta \in C$ lie in an apartment contained in C. Choose an apartment A with $\eta \in A \subseteq C$. For $\overrightarrow{\eta\xi} \in \Sigma_\eta C$ there exists an antipodal direction in $\Sigma_\eta A$ and we can extend $\overline{\xi\eta}$ into A to a geodesic $\overline{\xi\eta\hat{\xi}}$ of length π. To the apartment $\Sigma_{\hat{\xi}} A$ in $\Sigma_{\hat{\xi}} C$ corresponds an apartment $A' \subseteq C_{\xi,\hat{\xi}}$ containing $\overline{\xi\eta\hat{\xi}}$. □

3.11 Building morphisms

We call a map $\phi : B \to B'$ between buildings of equal dimension a *building morphism* if it is isometric on chambers. Later, when looking at Euclidean buildings, we will encounter natural examples of building morphisms, namely the canonical maps from the Tits boundary to the spaces of directions.

A building morphism ϕ has Lipschitz constant 1. ϕ maps sufficiently short segments emanating from a point p isometrically to geodesic segments. Therefore it induces well-defined maps

$$\Sigma_p \phi : \Sigma_p B \to \Sigma_{\phi(p)} B' \tag{23}$$

Since the chambers in B containing p correspond to the chambers in $\Sigma_p B$ (with respect to its natural induced building structure, cf. Proposition 3.6.4), and similarly for B', the maps (23) are building morphisms, as well. We call the morphism ϕ *spreading* if there is an apartment $A_0 \subseteq B$ so that $\phi\big|_{A_0}$ is an isometry.

Lemma 3.11.1 *Let $\phi : B \to B'$ be a spreading building morphism. Then, if $\xi_1, \xi_2 \in B$ are points with $\phi(\xi_1) = \phi(\xi_2) =: \xi'$, the images of $\Sigma_{\xi_1} \phi$ and $\Sigma_{\xi_2} \phi$ in $\Sigma_{\xi'} B'$ coincide.*

Proof. If ϕ is spreading then each point $\xi' \in \phi(B)$ has an antipode $\hat{\xi}' \in \phi(B)$. Any points $\xi \in \phi^{-1}(\xi')$ and $\hat{\xi} \in \phi^{-1}(\hat{\xi}')$ are antipodes and minimizing geodesics connecting ξ and $\hat{\xi}$ are mapped isometrically to geodesics connecting ξ' and $\hat{\xi}'$, i.e. $\phi\big|_{B(\xi,\hat{\xi})} : B(\xi, \hat{\xi}) \to B'(\xi', \hat{\xi}')$ is the spherical suspension of the morphism $\Sigma_\xi \phi$. There are canonical isometries $persp_{\xi,\hat{\xi}} : \Sigma_\xi B \to \Sigma_{\hat{\xi}} B$ and $persp_{\xi',\hat{\xi}'} : \Sigma_{\xi'} B' \to \Sigma_{\hat{\xi}'} B'$, cf. 3.6.1, and we have:

$$\Sigma_{\hat{\xi}} \phi \circ persp_{\xi,\hat{\xi}} = persp_{\xi',\hat{\xi}'} \circ \Sigma_\xi \phi \tag{24}$$

The assertion follows. □

Lemma 3.11.2 *Let $\phi : B \to B'$ be a spreading building morphism. Suppose $\xi_1 \in B$, $\xi_2' \in B'$ and set $\xi_1' := \phi\xi_1$.*

Then there is an apartment $A \subseteq B$ containing ξ_1 such that $\phi\big|_A$ is an isometry and the apartment $A' := \phi A \subseteq B'$ contains ξ_2'.

Proof. Let us first assume that $\xi_2' \in A_2' = \phi A_2$ where A_2 is an apartment in B such that $\phi\big|_{A_2}$ is an isometry. Then there is a geodesic segment $\overline{\xi_1'\xi_2'\hat{\xi}_1'}$ of length π such that $\overline{\xi_2'\hat{\xi}_1'} \subset A_2'$ (lemma 3.10.2). Let $\hat{\xi}_1 \in A_2$ be the lift of $\hat{\xi}_1'$. By proposition 3.6.4, the subbuilding $B(\xi_1, \hat{\xi}_1)$ contains an apartment A with $\Sigma_{\hat{\xi}_1} A = \Sigma_{\hat{\xi}_1} A_2$. $\phi\big|_A$ is an isometry, because it is an isometry near $\hat{\xi}_1$. By construction, $\xi_2' \in \phi A$.

The above argument implies that, since ϕ is spreading by assumption, that each point $\xi_1 \in B$ lies in an apartment A_1 so that $\phi\big|_{A_1}$ is an isometry. Therefore the assumption in the beginning of the proof is always satisfied and the proof is complete. □

Corollary 3.11.3 *Let ϕ be as in lemma 3.11.2. Then:*

1. *$\phi(B)$ is a subbuilding in B'.*
2. *The induced morphisms $\Sigma_\xi\phi$ are spreading.*
3. *For all $\xi_1 \in B$, $\xi_2' \in \phi(B)$ exists $\xi_2 \in \phi^{-1}\xi_2'$ such that*
$$d_B(\xi_1, \xi_2) = d_{B'}(\phi\xi_1, \xi_2'). \tag{25}$$
4. *If ξ_2 satisfies (25) then there exists an apartment $A \subseteq B$ containing ξ_1, ξ_2 such that $\phi\big|_A$ is an isometry.*

Proof. The first three assertions follow immediately from the lemma. We prove the fourth assertion:

By 1. we find a geodesic segment $\overline{\xi_1'\xi_2'\hat{\xi}_1'}$ of length π contained in $\phi(B)$. By 3. there exists a lift $\hat{\xi}_1$ of $\hat{\xi}_1'$ such that $d_B(\xi_2, \hat{\xi}_1) = d_{B'}(\xi_2', \hat{\xi}_1')$. Applying the previous lemma to the morphism $\Sigma_{\xi_1}\phi$, which is spreading by2., we find an apartment $A \subseteq B(\xi_1, \hat{\xi}_1)$ containing the geodesic segment $\overline{\xi_1\xi_2\hat{\xi}_1}$ and so that $\Sigma_{\xi_1}\phi\big|_{\Sigma_{\xi_1}A}$, and therefore also $\phi\big|_A$, is an isometry. □

Proposition 3.11.4 *Let B and B' be spherical buildings modelled on Δ_{mod}, and let $\phi : B \longrightarrow B'$ be a surjective morphism of spherical buildings so that $\theta_B = \theta_{B'} \circ \phi$. Suppose τ is a face of B and σ' is a face of B' contained in $\phi(B)$ so that $\phi\tau \subseteq \sigma'$. Then there exists a face σ of B with $\tau \subseteq \sigma$ and $\phi\sigma = \sigma'$.*

Proof. Let ξ be an interior point of τ and let σ_1 be a face of B with $\phi\sigma_1 = \sigma'$. σ_1 contains (in its boundary) a point ξ_1 with $\phi\xi_1 = \phi\xi$, and by lemma 3.11.1 there exists a face σ containing ξ (and therefore τ) with $\phi\sigma = \phi\sigma_1 = \sigma'$. □

Corollary 3.11.5 *Let B, B' and ϕ be as in proposition 3.11.4. If $h' \subset B'$ is a half-apartment with wall m', and $m \subset B$ lifts m', then there is a half-apartment $h \subset B$ containing m which lifts h'.*

Proof. Let $\tau' \subset h'$ be a chamber with a panel $\sigma' \subset m'$, and let $\sigma \subset m$ be the lift of σ' in m. Applying proposition 3.11.4 we get a chamber $\tau \subset B$ so that the half-apartment h spanned by $\tau \cup m$ lifts h'. □

3.12 Root groups and Moufang spherical buildings

A good reference for the material in this section is [Ron]

Definition 3.12.1 *([Ron, p. 66]) Let (B, Δ_{mod}) be a spherical building, and let $a \subset B$ be a root. The* **root group** *U_a of a is defined as the subgroup of $Aut(B, \Delta_{mod})$ consisting of all automorphisms g which fix every chamber $C \subset B$ with the property that $C \cap a$ contains a panel $\pi \not\subset \partial a$.*

We let $G_B \subset Aut(B, \Delta_{mod})$ be the subgroup generated by all the root groups of B.

Proposition 3.12.2 (Properties of root groups) *Let B be a thick spherical building.*

1. *If U_a acts transitively on the apartments containing a for every root a contained in some apartment A_0, then the group generated by these root groups acts transitively on pairs (C, A) where C is a chamber in an apartment $A \subseteq B$.*

2. *Suppose (B, Δ_{mod}) is irreducible and has dimension at least 1. Then the only root group element $g \in U_a$ which fixes an apartment containing a is the identity.*

Lemma 3.12.3 *Let A and A' be apartments in the spherical building B. Then there exist apartments $A_0 = A, A_1, \ldots, A_k = A'$ so that $A_{i-1} \cap A_i$ is a half-apartment containing $A \cap A'$ for all i.*

Proof. Suppose that A and A' are apartments which do not satisfy the conclusion of the lemma and so that the complex $A \cap A'$ has the maximal possible number of faces. We derive a contradiction by constructing an apartment A'' whose intersection with A respectively A' strictly contains $A \cap A'$.

If $A \cap A'$ is empty, we choose A'' to be any apartment which has non-empty intersection with both A and A'. If $A \cap A'$ is contained in a singular sphere s of dimension $dim(A \cap A') < dim(B)$ we pick a chambers $\sigma \subset A$ and $\sigma' \subset A'$ with $dim(\sigma \cap s) = dim(\sigma' \cap s) = dims$. The subbuilding $B(s)$ contains an apartment A'' with $s \cup \sigma \cup \sigma' \subset A''$ and A'' has the desired property. It remains to consider the case that $A \cap A'$ contains chambers and is strictly contained in a half-apartment. Then there is a half-apartment $h \subset A$ containing $A \cap A'$ and so that $\partial h \cap A \cap A'$ contains a panel π. Let $\sigma' \subset A'$ be a chamber with $\sigma' \cap A \cap A' = \pi$. The convex hull A'' of $h \cup \sigma'$ is an apartment with the desired property. ☐

Proof of proposition: 1. Let G_A be the group generated by the root groups U_a where a runs through all roots contained in an apartment $A \subset B$. If $g \in U_a$ then $G_A = G_{gA}$ because $U_{gA} = gU_a g^{-1}$ for all roots $a \subset A$. By lemma 3.12.3, given any apartment A' there is a sequence $A_0, \ldots, A_k = A'$ such that $A_{i-1} \cap A_i$ is a root. Hence $G_{A_0} = G_{A_1} = \ldots = G_{A'}$ and it follows that $G_B = G_{A'}$ for all apartments A'.

Let σ_1 and σ_2 be chambers in B which share a panel $\pi = \sigma_1 \cap \sigma_2$. Since B is thick, there is a third chamber σ with $\sigma \cap \sigma_i = \pi$. Pick apartments A_i containing $\sigma \cup \sigma_i$. Applying lemma 3.12.3 again, we see that there is a $g \in G_B$ so that $g(A_1) = A_2$, and g fixes σ_3. Hence $g\sigma_1 = \sigma_2$ and we conclude by induction that G_B acts transitively on chambers.

Let A_1, A_2 be apartments and σ_1, σ_2 be chambers such that $\sigma_i \subseteq A_i$. By the above argument, there exists $g \in G_B$ with $g\sigma_1 = \sigma_2$. By lemma 3.12.3 there is a $g' \in G_B$ with $g'(gA_1) = A_2$ and $g'\sigma_2 = \sigma_2$. Hence G_B acts transitively on pairs $C \subset A$ as claimed.

2. Since B is irreducible, there is a chamber σ contained in the interior of a (see lemma 3.3.2). Since the convex set $B' = Fix(g)$ contains the apartment A it is a subbuilding by proposition 3.10.3. Moreover, B' contains an open neighborhood of σ by the definition of U_a. Note that if π and π' are

opposite panels in B', then B' contains every chamber containing π iff it contains every chamber containing π' (lemma 3.6.1). Since for each panel π there is a panel $\pi_1 \subset \partial\sigma$ in the same projectivity class (see definition 3.6.2 and lemma 3.6.3) we see that B' contains every chamber in B with a panel in B'. When $Dim(B) = Dim(B') = 1$ this implies that B' is open in B, forcing $B' = B$; in general we show by induction that $\forall p \in B'$ we have $\Sigma_p B' = \Sigma_p B$, which implies that $B' \subset B$ is open and consequently $B' = B$. □

Definition 3.12.4 *A spherical building (B, Δ_{mod}) is* **Moufang** *if for each root $a \subset B$ the root group U_a acts transitively on the apartments containing the root a. When B is irreducible and has rank at least 2, then by 2 above, U_a acts simply transitively on apartments containing a.*

The spherical building associated with a reductive algebraic group ([Ti1, chapter 5] is Moufang. In particular, irreducible spherical buildings of dimension at least 2 are Moufang.

4 Euclidean buildings

There are many different ways to axiomatize Euclidean buildings. For us, the key geometric ingredient is an assignment of Δ_{mod}-*directions* to geodesics segments in a Hadamard space. Just as with symmetric spaces, Δ_{mod}-directions capture the anisotropy of the space, and they behave nicely with respect to geometric limiting operations such as ultralimits, Tits boundaries, and spaces of directions.

4.1 Definition of Euclidean buildings

4.1.1 Euclidean Coxeter complexes

Let E be a finite-dimensional Euclidean space. Its Tits boundary is a round sphere and there is a canonical homomorphism

$$\rho : Isom(E) \to Isom(\partial_{Tits} E) \tag{26}$$

which assigns to each affine isometry its rotational part. We call a subgroup $W_{aff} \subset Isom(E)$ an *affine Weyl group* if it is generated by reflections and if the reflection group $W := \rho(W_{aff}) \subset Isom(\partial_{Tits} E)$ is finite. The pair (E, W_{aff}) is said to be a *Euclidean Coxeter complex* and

$$\partial_{Tits}(E, W_{aff}) := (\partial_{Tits} E, W) \tag{27}$$

is called its *spherical Coxeter complex at infinity.* Its *anisotropy polyhedron* is the spherical polyhedron

$$\Delta_{mod} := (\partial_{Tits} E)/W.$$

An oriented geodesic segment $\overline{xy}$ in a E determines a point in $\partial_{Tits} E$ and we call its projection to Δ_{mod} the Δ_{mod}-*direction* of $\overline{xy}$.

A *wall* is a hyperplane which occurs as the fixed point set of a reflection in W_{aff} and *singular subspaces* are defined as intersections of walls. A half-space bounded by a wall is called singular or a *half-apartment.* An intersection of half-apartments is a *Weyl-polyhedron. Weyl cones with tip at a point* p are complete cones with tip at p for which the boundary at infinity is a single face in $\partial_{Tits} E$.

Fix a point $p \in E$. By $W(p)$, we denote the subgroup of W_{aff} which is generated by reflections in the walls passing through p. $W(p)$ embeds via ρ as a subgroup of W. A *Weyl sector with tip at*

p is a Weyl polyhedron for the Euclidean Coxeter complex $(E, W(p))$; note that a Weyl sector need not be a Weyl cone, and a Weyl cone need not be a Weyl sector. A subsector of a sector σ is a sector $\sigma' \subset \sigma$ with $\partial_{Tits}\sigma' = \partial_{Tits}\sigma$; σ lies in a finite tubular neighborhood of σ'. A *Weyl chamber* is a Weyl polyhedron for which the boundary at infinity is a Δ_{mod} chamber; Weyl chambers are necessarily Weyl cones. The Coxeter group $W(p)$ acts on $\Sigma_p E$, so we have a Coxeter complex

$$\Sigma_p(E, W_{aff}) := (\Sigma_p E, W(p))$$

with anisotropy map by

$$\theta_p : \Sigma_p E \longrightarrow \Sigma_p E / W(p) =: \Delta_{mod}(p).$$

The faces in $(\Sigma_p E, W(p))$ correspond to the Weyl sectors of E with tip at p.

We call the Coxeter complex (E, W_{aff}) *irreducible* iff its anisotropy polyhedron, or equivalently, its spherical Coxeter complex at infinity is irreducible. In this case, the action of W on the translation subgroup $T \triangleleft W_{aff}$ forces T to be trivial, a lattice, or a dense subgroup. In the latter case we say that W_{aff} is *topologically transitive*.

4.1.2 The Euclidean building axioms

Let (E, W_{aff}) be a Euclidean Coxeter complex. A *Euclidean building modelled on* (E, W_{aff}) is a Hadamard space X endowed with the structure described in the following axioms.

EB1: Directions. *To each nontrivial oriented segment $\overline{xy} \subset X$ is assigned a Δ_{mod}-direction $\theta(\overline{xy}) \in \Delta_{mod}$. The difference in Δ_{mod}-directions of two segments emanating from the same point is less than their comparison angle, i.e.*

$$d(\theta(\overline{xy}), \theta(\overline{xz})) \leq \tilde{\angle}_x(y, z) \tag{28}$$

Recall that given $\delta_1, \delta_2 \in \Delta_{mod}$, $D(\delta_1, \delta_2)$ is the finite set of possible distances between points in the Weyl group orbits $\theta^{-1}_{\partial_{Tits}E}(\delta_1)$ and $\theta^{-1}_{\partial_{Tits}E}(\delta_2)$.

EB2: Angle rigidity. *The angle between two geodesic segments $\overline{xy}$ and $\overline{xz}$ lies in the finite set $D(\theta(\overline{xy}), \theta(\overline{xz}))$.*

We assume that there is given a collection $\mathcal{A}$ of isometric embeddings $\iota : E \longrightarrow X$ which preserve Δ_{mod}-directions and which is closed under precomposition with isometries in W_{aff}. These isometric embeddings are called *charts*, their images *apartments*, and A is called the *atlas* of the Euclidean building.

EB3: Plenty of apartments. *Each segment, ray and geodesic is contained in an apartment.*

The Euclidean coordinate chart ι_A for an apartment A is well-defined up to precomposition with an isometry $\alpha \in \rho^{-1}(W)$. Two charts ι_{A_1}, ι_{A_2} for apartments A_1, A_2 are said to be *compatible* if $\iota_{A_1}^{-1} \circ \iota_{A_2}$ is the restriction of an isometry in W_{aff}. This holds automatically when $W_{aff} = \rho^{-1}(W)$.

EB4: Compatibility of apartments. *The Euclidean coordinate charts for the apartments in X are compatible.*

It will be a consequence of Corollary 4.6.2 below that the atlas $\mathcal{A}$ is maximal among collections of charts satisfying axioms EB3 and EB4.

We define walls, singular flats, half-apartments, Weyl cones, Weyl sectors, and Weyl polyhedra in the Euclidean building to be the images of the corresponding objects in the Euclidean Coxeter complex under charts. The set of Weyl cones with tip at a point x will be denoted by $\mathcal{W}_x$. The *rank* of the Euclidean building X is defined to be the dimension of its apartments. X is *thick* if each wall bounds at least 3 half-apartments with disjoint interiors. We call X a *Euclidean ruin* if its underlying set or the atlas $\mathcal{A}$ is empty.

4.1.3 Some immediate consequences of the axioms

Axiom EB1 implies the following compatibility properties for the Δ_{mod}-directions of geodesic segments.

Lemma 4.1.1 *Let x, y, z be points in X.*

1. *If y lies on $\overline{xz}$, then $\theta(\overline{xz}) = \theta(\overline{xy}) = \theta(\overline{yz})$.*
2. *If $\overrightarrow{xy}, \overrightarrow{xz} \in \Sigma_x X$ coincide, then $\theta(\overline{xy}) = \theta(\overline{xz})$.*
3. *Asymptotic geodesic rays in X have the same Δ_{mod}-direction.*

We call a segment, ray or geodesic in X *regular* if its Δ_{mod}-direction is an interior point of Δ_{mod}.

Lemma 4.1.2 1. *If $p \in X$ and $x_i \in \overline{X} - p$, then the $\overline{px_i}$ initially span a flat triangle if $\angle_p(x_1, x_2) > 0$, and they initially coincide if $\angle_p(x_1, x_2) = 0$.*

2. *If $p_i \in X$ and $\xi_i \in \partial_{Tits} X$, then the rays $p_i\xi_i$ are asymptotic to the edges of a flat sector*

Proof. 1. After extending the segments $\overline{px_i}$ to rays if necessary, we may assume without loss of generality that $x_i \in \partial_{Tits} X$. If $z \in \overline{px_1}$, then $\theta(\overline{zx_i}) = \theta(x_i)$ so $\angle_z(x_1, x_2) \in D(\theta(x_1), \theta(x_2))$ which is a finite set. But $\angle_z(x_1, x_2) \to \angle_p(x_1, x_2)$ monotonically as $z \to p$, which implies that $\angle_z(x_1, x_2) = \angle_p(x_1, x_2)$, $\angle_z(p, x_2) = \pi - \angle_p(x_1, x_2)$ when z is sufficiently close to p. Therefore $\Delta(p, z, x_2)$ is a flat triangle (with a vertex at ∞) when z is sufficiently close to p.

2. follows from similar reasoning and the property (6) of the Tits distance. □

4.2 Associated spherical building structures

4.2.1 The Tits boundary

The Tits boundary $\partial_{Tits} X$ is a $CAT(1)$-space, see 2.3.2. Lemma 4.1.1 implies that there is a well-defined Δ_{mod}-direction map

$$\theta_{\partial_{Tits} X} : \partial_{Tits} X \longrightarrow \Delta_{mod} \qquad (29)$$

which is 1-Lipschitz by (28).

Proposition 4.2.1 *$\partial_{Tits} X$ carries a spherical building structure modelled on the spherical Coxeter complex $(\partial_{Tits} E, W)$ with Δ_{mod}-direction map (29).*

Proof. We verify that the assumptions of proposition 3.5.1 are satisfied. Axiom EB2 implies that (29) satisfies the discreteness condition (17). If A is a Euclidean apartment in X then $\partial_{Tits} A$ is a standard sphere in $\partial_{Tits} X$. Clearly, any point $\xi \in \partial_{Tits} X$ lies in a standard sphere. It remains to check that any two points ξ_1 and ξ_2 in $\partial_{Tits} X$ with Tits distance π are ideal endpoints of a geodesic in X. To see this, pick $p \in X$ and note that the angle $\angle_z(\xi_1, \xi_2)$ increases monotonically as z moves along the ray $\overline{p\xi_1}$ towards ξ_1. But by EB2 $\angle_z(\xi_1, \xi_2)$ assumes only finitely many values, so when z is sufficiently far out we have $\angle_z(\xi_1, \xi_2) = \angle_{Tits}(\xi_1, \xi_2) = \pi$, and the rays $\overline{z\xi_i}$ fit together to form a geodesic with ideal endpoints ξ_1 and ξ_2. □

4.2.2 The space of directions

The space of directions $\Sigma_x X$ is a $CAT(1)$-space (see section 2.1.3). Lemma 4.1.1 implies that there is a well-defined 1-Lipschitz map from the space of germs of segments in a point $x \in X$:

$$\theta_{\Sigma_x X} : \Sigma_x^* X \longrightarrow \Delta_{mod} \tag{30}$$

In this section we check that this map induces a spherical building structure on $\Sigma_x X$. By axiom EB2, $\theta = \theta_{\Sigma_x X}$ satisfies the discreteness condition (17).

Lemma 4.2.2 *$\Sigma_x^* X$ is complete, so $\Sigma_x^* X = \Sigma_x X$.*

Proof. Let (x_k) be a sequence in $X - \{x\}$ such that $(\overrightarrow{xx_k})$ is Cauchy in $\Sigma_x^* X$. Then $\theta(\overrightarrow{xx_k})$ is Cauchy in Δ_{mod} and we denote its limit by δ. If $A_k \subset X$ is an apartment containing $\overline{xx_k}$ then $\overrightarrow{xx_k} \in \Sigma_x A_k \subset \Sigma_x^* X$ and $\Sigma_x A_k$ contains a spherical polyhedron σ_k such that $\overrightarrow{xx_k} \in \sigma_k$ and $\theta|_{\sigma_k}; \sigma_k \to \Delta_{mod}$ is an isometry. There is a unique $\xi_k \in \sigma_k$ with $\theta(\xi_k) = \delta$ and we have $d(\xi_k, \overrightarrow{xx_k}) = d_{\Delta_{mod}}(\delta, \theta(\overrightarrow{xx_k})) \to 0$. Hence (ξ_k) is Cauchy with $\theta(\xi_k) \equiv \delta$ and $\lim \overrightarrow{xx_k} = \lim \xi_k$ in $\Sigma_x X$. The discreteness condition (17) implies that (ξ_k) is eventually constant and therefore $(\overrightarrow{xx_k})$ has a limit in $\Sigma_x^* X$. □

We now apply proposition 3.5.1 to verify that $\Sigma_x X$ carries a natural structure as a spherical building modelled on $(\partial_{Tits} E, W)$. The only condition which remains to be checked is that antipodal points $\overrightarrow{xx_1}$ and $\overrightarrow{xx_2}$ in $\Sigma_x X$ lie in a subset isometric to $S = \partial_{Tits} E$. But $\angle_x(x_1, x_2) = \pi$ implies that $\overline{x_1 x_2} = \overline{xx_1} \cup \overline{xx_2}$ and if $A \subset X$ is an apartment containing $\overline{x_1 x_2}$ then $\Sigma_x A \subset \Sigma_x X$ is a spherical apartment containing $\overrightarrow{xx_1}$ and $\overrightarrow{xx_2}$.

Lemma 4.2.3 *All standard spheres in $\Sigma_x X$ are of the form $\Sigma_x A$ where A is an apartment in X passing through x.*

Proof. By corollary 3.9.2, standard spheres are Δ_{mod}-apartments, so we can find antipodal regular points $\xi_1, \xi_2 \in \alpha$. Then there is a segment $\overline{x_1 x_2}$ through x with $\overrightarrow{xx_i} = \xi_i$. If $A \subseteq X$ is an apartment containing $\overline{x_1 x_2}$ then $\Sigma_x A \cap \alpha \supseteq \{\xi_1, \xi_2\}$ and the spherical apartments α and $\Sigma_x A$ coincide because they share a pair of regular antipodes (lemma 3.6.1). □

There are two natural reductions of the Weyl group which we shall consider. First, according to section 3.7 there is a thick spherical building structure with atlas $\mathcal{A}^{th}(x)$ and anisotropy map

$$\theta_x^{th} : \Sigma_x X \longrightarrow \Delta_{mod}^{th}(x); \tag{31}$$

This structure is unique up to equivalence. The second reduction is analogous to the structure constructed in proposition 3.6.4. We postpone discussion of this structure until 4.4.1 because we don't have an analog of lemma 3.1.1 in the case of nondiscrete Euclidean Coxeter complexes.

4.3 Product(-decomposition)s

Let X_i, $i = 1, \ldots, n$, be Euclidean buildings modelled on Coxeter complexes (E_i, W^i_{aff}) with atlases $\mathcal{A}_i$ and anisotropy polyhedra Δ^i_{mod}. Then $W_{aff} := W^1_{aff} \times \cdots \times W^n_{aff}$ acts canonically as a reflection group on $E := E_1 \times \cdots \times E_n$. We call the Coxeter complex (E, W_{aff}) the *product* of the Coxeter complexes (E_i, W^i_{aff}) and write

$$(E, W_{aff}) = (E_1, W^1_{aff}) \times \cdots \times (E_n, W^n_{aff}). \tag{32}$$

There are corresponding join decompositions

$$(\partial_{Tits} E, W) = (\partial_{Tits} E_1, W_1) \circ \cdots \circ (\partial_{Tits} E_n, W_n) \tag{33}$$

of the spherical Coxeter complex at infinity and

$$\Delta_{mod} = \Delta^1_{mod} \circ \cdots \circ \Delta^n_{mod} \tag{34}$$

of the anisotropy polyhedron. The Hadamard space

$$X = X_1 \times \cdots \times X_n \tag{35}$$

carries a natural Euclidean building struture modelled on (E, W_{aff}). The charts for its atlas $\mathcal{A}$ are the products $\iota = \iota_1 \times \cdots \times \iota_n$ of charts $\iota_i \in \mathcal{A}_i$. We call X equipped with this building structure the *Euclidean building product* of the buildings X_i.

Proposition 4.3.1 *Let X be a Euclidean building modelled on the Coxeter complex (E, W_{aff}) with atlas $\mathcal{A}$ and assume that there is a join decomposition (34) of its anisotropy polyhedron. Then*

1. *There is a decomposition (32) of (E, W_{aff}) as a product of Euclidean Coxeter complexes so that a segment $\overline{xy} \subset E$ is parallel to the factor E_i iff its Δ_{mod}-direction $\theta(\overline{xy})$ lies in Δ^i_{mod}.*

2. *There is a decomposition (35) of X as a product of Euclidean buildings so that a segment $\overline{xy} \subset E$ is parallel to the factor E_i iff its Δ_{mod}-direction $\theta(\overline{xy})$ lies in Δ^i_{mod}.*

Proof. 1. Proposition 3.3.1 implies that the spherical Coxeter complex at infinity decomposes as a join

$$(\partial_{Tits} E, W) = (S_1, W_1) \circ \cdots \circ (S_n, W_n) \tag{36}$$

of spherical Coxeter complexes. By proposition 2.3.7, this decomposition is induced by a metric product decomposition $E = E_1 \times \cdots \times E_n$ so that $\partial_{Tits} E_i$ is canonically identified with S_i and, hence, a segment $\overline{xy} \subset E$ is parallel to the factor E_i iff $\theta(\overline{xy}) \in \Delta^i_{mod}$. (36) implies that W_{aff} decomposes as the product $W_{aff} = W^1_{aff} \times \cdots \times W^n_{aff}$ of reflection groups W^i_{aff} acting on E_i, thus establishing the desired decomposition (32).

2. Arguing as in the proof of the first part, we obtain a metric decomposition (35) as a product of Hadamard spaces so that $\overline{xy} \subset X$ is parallel to the factor X_i iff $\theta(\overline{xy}) \in \Delta^i_{mod}$. Furthermore, the $\partial_{Tits} X_i$ carry spherical building structures modelled on $(\partial_{Tits} E_i, W_i)$ so that the spherical building $\partial_{Tits} X$ decomposes as the spherical building join of the $\partial_{Tits} X_i$. Each chart $\iota : E \to X$, $\iota \in \mathcal{A}$, decomposes as a product of Δ^i_{mod}-direction preserving isometric embeddings $\iota_i : E_i \to X_i$. The

collection $\mathcal{A}_i$ of all ι_i arising in this way forms an atlas for a Euclidean building structure on X_i and (35) becomes a decomposition as a product of Euclidean buildings. □

We call a Euclidean building *irreducible* if its anisotropy polyhedron is irreducible, compare section 3.3. According to the previous proposition, the unique minimal join decomposition of the anisotropy polyhedron Δ_{mod} into irreducible factors corresponds to unique minimal product decompositions of the Euclidean Coxeter complex (E, W_{aff}) and the Euclidean building X into irreducible factors. We call these decompositions the de Rham decompositions and the maximal Euclidean factors with trivial affine Weyl group the Euclidean de Rham factors.

4.4 The local behavior of Weyl-cones

In this section we study the set $\mathcal{W}_p$ of Weyl cones with tip at p. The main result (corollary 4.4.3) is that in a sufficiently small neighborhood of p, a finite union of these cones is isometric to the metric cone over the corresponding finite union of Δ_{mod} faces in $\Sigma_p X$. This proposition plays an important role in section 6.

Let W_1 and W_2 be Weyl cones in X with tip at p. The Weyl cone W_i determines a face $\Sigma_p W_i$ in the spherical building $(\Sigma_p X, \Delta_{mod})$.

Sublemma 4.4.1 *Suppose that $\Sigma_p W_1 = \Sigma_p W_2$ in $\Sigma_p X$. Then $W_1 \cap W_2$ is a neighborhood of p in W_1 and W_2.*

Proof. According to lemma 4.1.2 each point in the face $\Sigma_p W_1 = \Sigma_p W_2$ is the direction of a segment in $W_1 \cap W_2$ which starts at p. We can pick finitely many points in $\Sigma_p W_1 = \Sigma_p W_2$ whose convex hull is the whole face. The convex hull of the corresponding segments is contained in the convex set $W_1 \cap W_2$ and is a neighborhood of p in W_1 and W_2. □

Locally the intersection of Weyl cones with tip at a point p is given by their infinitesimal intersection in the space of directions $\Sigma_p X$:

Lemma 4.4.2 *If $W_1, W_2 \in \mathcal{W}_p$, then there is a Weyl cone $W \in \mathcal{W}_p$ with $\Sigma_p W = \Sigma_p W_1 \cap \Sigma_p W_2$. For every such W there is an $\epsilon > 0$ so that:*

$$W_1 \cap W_2 \cap B_\epsilon(p) = W \cap B_\epsilon(p)$$

Hence the intersection of Weyl cones with tip at the same point is locally a Weyl cone.

Proof. By lemma 3.4.2 the intersection $\Sigma_p W_1 \cap \Sigma_p W_2$ is a Δ_{mod}-face and hence there is a $W \in \mathcal{W}_p$ such that $\Sigma_p W = \Sigma_p W_1 \cap \Sigma_p W_2$. By the previous sublemma, there are $W_i' \in \mathcal{W}_p$ with $W_i' \subseteq W_i$ and a positive ϵ so that

$$W_1' \cap B_\epsilon(p) = W_2' \cap B_\epsilon(p) = W \cap B_\epsilon(p)$$

for any such W. If x is a point in $W_1 \cap W_2$ different from p then $\overrightarrow{px} \in \Sigma_p W$, so $\overline{px} \subset W_1' \cap W_2'$. Therefore

$$W_1 \cap W_2 \cap B_\epsilon(p) = W_1' \cap W_2' \cap B_\epsilon(p) = W \cap B_\epsilon(p).$$

□

Corollary 4.4.3 *If $W_1, \ldots, W_k \in \mathcal{W}_p$, then there is an $\epsilon > 0$ such that $(\cup_i W_i) \cap B_p(\epsilon)$ maps isometrically to $(\cup_i C_p W_i) \cap B(\epsilon) \subset C_p X$ via $\log_p$.*

Proof. Let $\mathcal{C}$ denote the finite subcomplex of $\Sigma_p X$ detemined by $\cup_i \Sigma_p W_i$. Pick $\sigma_1, \sigma_2 \in \mathcal{C}$. By lemma 4.2.3 these lie in an apartment $\Sigma_p A_{\sigma_1 \sigma_2} \subseteq \Sigma_p X$ for some apartment $A_{\sigma_1 \sigma_2} \subset X$ passing through p. If σ_1 is a face of $\Sigma_p W_i$ and σ_2 is a face of $\Sigma_p W_j$, then by the sublemma above we may assume without loss of generality that $(W_i^{\sigma_1} \cup W_j^{\sigma_2}) \cap B_p(\epsilon) \subseteq A_{\sigma_1 \sigma_2}$ where $W_i^{\sigma_1}$ (resp. $W_j^{\sigma_2}$) is the subcone of W_i (resp. W_j) with $\Sigma_p W_i^{\sigma_1} = \sigma_1$ (resp. $\Sigma_p W_j^{\sigma_2} = \sigma_2$). Since there are only finitely many such pairs $\sigma_1, \sigma_2 \in \mathcal{C}$, for sufficiently small $\epsilon > 0$, every pair of segments $\overline{px_1}, \overline{px_2} \subseteq \cup_i W_i$ bounds a flat triangle provided $|px_i| < \epsilon$. $\square$

4.4.1 Another building structure on $\Sigma_p X$, and the local behavior of Weyl sectors.

Let $\alpha \subset \Sigma_p X$ be a Δ_{mod}-apartment. By lemma 4.2.3 there is an apartment $A \subset X$ with $\Sigma_p X = \alpha$, and by corollary 4.4.3 any two such apartments coincide near p. Hence the walls in A which pass through p define a reflection group $W_\alpha \subset Isom(\alpha)$.

Lemma 4.4.4 *The reflection group W_α contains the reflection group W_α^{th} coming from the thick spherical building structure on $\Sigma_p X$.*

Proof. Let $m \subset \alpha$ be a wall for the $\Delta_{mod}^{th}(p)$ structure. There are apartments $A_i \subset X$ through p, $i = 1, 2, 3$, so that $\Sigma_p A_1 = \alpha$ and the $\Sigma_p A_i$ intersect in half-apartments with boundary wall m. By corollary 4.4.3 the pairwise intersections of the A_i are half-spaces near p. Choose charts $\iota_{A_1}, \iota_{A_2}, \iota_{A_3} \in \mathcal{A}$ and let $\phi_{ij} \in W_{aff}$ be the unique isometry inducing $\iota_{A_i}^{-1} \circ \iota_{A_j}$. Then $\phi_{12} \circ \phi_{23} \circ \phi_{31}$ is a reflection at a wall w passing through $x = \iota_{A_1}^{-1}(p)$ and satisfying $\Sigma_p \iota_{A_1} w = m$. $\square$

Fixing one apartment $\alpha \subset \Sigma_p X$, we take a chart $\iota : S \longrightarrow \alpha$ from the atlas $\mathcal{A}^{th}(p)$, and enlarge $\mathcal{A}^{th}(p)$ by precomposing each chart $\iota' \in \mathcal{A}^{th}(p)$ with elements of $\iota_*^{-1}(W_\alpha) \subset Isom(S)$. Clearly this defines an atlas $\mathcal{A}(p)$ for a spherical building structure modelled on $\Delta_{mod}(p) \stackrel{def}{=} \alpha / W_\alpha$.

Let $A, A_1 \subset X$ be apartments so that $\Sigma_p A = \alpha$, $\Sigma_p A_1 = \alpha_1$, and $\alpha \cap \alpha_1$ contains a chamber $C \subset \alpha$. If $\iota_A, \iota_{A_1} : E \longrightarrow X$ are charts from the atlas $\mathcal{A}$, then since $A \cap A_1$ is a cone near p by lemma 4.4.3, it follows that $\Sigma_p(\iota_{A_1} \circ \iota_A^{-1}) : \Sigma_p A = \alpha \longrightarrow \alpha_1 = \Sigma_p A_1$ carries W_α faces in α to W_{α_1} faces in α_1, while at the same time it carries $\Delta_{mod}(p)$ faces of α to $\Delta_{mod}(p)$ faces of α_1. So every $\Delta_{mod}(p)$ face $\sigma \subset \Sigma_p X$ is a $W_{\alpha'}$ face for every apartment α' containing σ. Since the $W_{\alpha'}$'s are all isomorphic, this clearly implies that $\Sigma_p W$ is a $\Delta_{mod}(p)$ face for every Weyl sector with tip at p. So we have shown:

Proposition 4.4.5 *There is a spherical building structure $(\Sigma_p X, \mathcal{A}(p))$ modelled on $(S, \Delta_{mod}(p))$ so that $\Delta_{mod}(p)$-faces in $\Sigma_p X$ correspond bijectively to the spaces of directions of Weyl sectors with tip at p. In particular, if $A \subset X$ is any apartment passing through p, then there is a 1-1 correspondence between walls $m \subset A$ passing through p and $\Delta_{mod}(p)$-walls in the apartment $\Sigma_p A$, given by $m \mapsto \Sigma_p m$. When X is a thick building, then $\mathcal{A}(p)$ coincides with $\mathcal{A}^{th}(p)$ for every $p \in X$.*

Corollary 4.4.6 *Corollary 4.4.3 holds when the W_i are Weyl sectors with tip at p. If A_1 and A_2 be apartments in X then $A_1 \cap A_2$ is either empty or a Weyl polyhedron. In particular, if $A_1 \cap A_2$ contains a complete regular geodesic then $A_1 = A_2$.*

Proof. Each Weyl sector with tip at p is a finite union of Weyl cones with tip at p. Hence a finite union of Weyl sectors with tip at p is a finite union of Weyl cones with tip at p, and the first statement follows.

If $A_1, A_2 \subset X$ are apartments and $p \in A_1 \cap A_2$, then $\Sigma_p Q_1 \cap \Sigma_p A_2$ is a convex $\Delta_{mod}(p)$ subcomplex of $\Sigma_p A_i$. Hence there are $\Delta_{mod}(p)$ half apartments $h_1, \dots, h_k \subset \Sigma_p A_1$ so that $\cap_i h_i = \Sigma_p A_1 \cap \Sigma_p A_2$. By proposition 4.4.5, for each i there is a half-apartment $H_i \subset A$ with $\Sigma_p H_i = h_i$. Therefore $A_1 \cap A_2 \cap B_p(\epsilon) = (\cap H_i) \cap B_p(\epsilon)$ and so $A_1 \cap A_2$ is a Weyl polyhedron near p. Consequently $A_1 \cap A_2$ is a Weyl polyhedron. □

4.5 Discrete Euclidean buildings

We call the Euclidean building X *discrete* if the affine Weyl group W_{aff} is discrete or, equivalently, if the collection of walls in the Euclidean Coxeter complex E is locally finite.

If p is a point in E then σ_p denotes the intersection of all closed half-apartments containing p, i.e. the smallest Weyl polyhedron containing p. By corollary 4.4.6, each affine coordinate chart $\iota_A : E \to X$ maps σ_p to the minimal Weyl polyhedron in X which contains $\iota_A(p)$. Hence for any point $x \in X$ there is a minimal Weyl polyhedron σ_x containing it. We say that x *spans* σ_x. σ_x is the intersection of all half-apartments containing x and, if X is thick, the intersection of all such apartments. The lattice of Weyl polyhedra σ_y with $x \in \sigma_y$ is isomorphic to the polyhedral complex $\mathcal{K}\Sigma_x X$.

Proposition 4.5.1 *In a discrete Euclidean building X each point x has a neighborhood $B_\epsilon(x)$ which is canonically isometric to the truncated Euclidean cone of height ϵ over $\Sigma_x X$.*

Proof. Let $\iota_A : E \to X$ be a chart with $x = \iota_A(p)$ and choose $\epsilon > 0$ so that any wall intersecting $B_\epsilon(p)$ contains p. Then for any point $y \in B_\epsilon(p)$, the polyhedron σ_y contains x and any apartment intersecting $B_\epsilon(p)$ passes through x. Hence any two segments $\overline{xy}$ and $\overline{xz}$ of length $< \epsilon$ lie in a common apartment and it follows that $B_\epsilon(p)$ is isometric to a truncated cone. □

Assume now that W_{aff} is discrete and cocompact. Then the walls partition E into polysimplices which are fundamental domains for the action of W_{aff}. This induces on X a structure as a polysimplicial complex. The polysimplices are spanned by their interior points. If X is moreover irreducible, then this complex is a simplicial complex.

4.6 Flats and apartments

Proposition 4.6.1 *Any flat F in X is contained in an apartment. In particular, the dimension of a flat is less or equal to the rank of X.*

Proof. Among the faces in $\partial_{Tits} X$ which intersect the sphere $\partial_{Tits} F$ we pick a face σ of maximal dimension. Then $\sigma \cap \partial_{Tits} F$ is open in $\partial_{Tits} F$. Let c be a geodesic in F with $c(\infty) \in Int(\sigma)$ and let A be an apartment containing c. Then $\partial_{Tits} A$ contains σ and $c(-\infty)$ and convexity implies $\partial_{Tits} F \subseteq \partial_{Tits} A$. Since $F \cap A \neq \emptyset$, it follows that F is contained in the apartment A. □

As a consequence, we obtain the following geometric characterization of apartments in Euclidean buildings:

Corollary 4.6.2 *The r-flats in X are precisely the apartments.*

The next lemma says that a regular ray which stays at finite Hausdorff distance from an apartment approaches this apartment at a certain minimal rate given by the extent of its regularity.

Lemma 4.6.3 *Suppose $\xi \in \partial_{Tits}X$ is regular and that the ray $\overline{p\xi}$ remains at bounded distance from an apartment F. Then every point $x \in \overline{p\xi}$ with*

$$d(x,p) \geq \frac{d(p,F)}{\sin(d_{\Delta_{mod}}(\theta\xi, \partial\Delta_{mod}))}$$

lies in F.

Proof. Let y be a point on the ray $\overline{\pi_A(p)\xi}$, and let $z \in \overline{py}$ be the point where the segment $\overline{py}$ enters A (we may have $z = y$). By lemma 4.1.2 $\angle_z(p,A) > 0$, and by lemma 3.4.1 we have $\angle_z(p,A) \geq d_{\Delta_{mod}}(\theta(\overline{pz}), \partial\Delta_{mod})$. The comparison triangle $\Delta(a,b,c)$ in the Euclidean plane for the triangle $\Delta(p, \pi_A(p), z)$ satisfies $\angle_b(a,c) \geq \frac{\pi}{2}$ and $\angle_c(a,b) \geq d_{\Delta_{mod}}(\theta(\overline{pz}), \partial\Delta_{mod})$. Hence $d(p,A) \geq d(p,z)\sin(d_{\Delta_{mod}}(\theta(\overline{pz}), \partial\Delta_{mod}))$. Since $\theta(\overline{pz}) = \theta(\overline{py}) \to \theta(\overline{p\xi})$ as $y \in \overline{p\xi}$ tends to ∞, the claim follows. □

Corollary 4.6.4 *Each complete regular geodesic which lies in a tubular neighborhood of an apartment A must be contained in A. If A_1 and A_2 are apartments in X and A_2 lies in a tubular neighborhood of A_1, then $A_1 = A_2$.*

Another implication of the previous lemma is the following analogue of lemma 4.4.2 at infinity.

Lemma 4.6.5 *If $C_1, C_2 \subset X$ are Weyl chambers with $\partial_{Tits}C_1 = \partial_{Tits}C_2$, then there is a chamber $C \subseteq C_1 \cap C_2$.*

Proof. It is enough to consider the case that the building X is irreducible. The claim is trivial if the affine Weyl group is finite and we can hence assume that W_{aff} is cocompact. If ρ is a regular geodesic ray in C_1 then, by the previous lemma, it enters C_2 in some point p and $C_1 \cap C_2$ contains the metric cone K centered at p with ideal boundary $\partial_{Tits}K = \partial_{Tits}C_i$. Since W_{aff} is cocompact, K clearly contains a Weyl chamber. □

Proposition 4.6.6 *There is a bijective correspondence between apartments in X and $\partial_{Tits}X$ given by:*

$$A \subseteq X \leftrightarrow \partial_{Tits}A \subseteq \partial_{Tits}X$$

Proof. We have to show that every apartment K in $\partial_{Tits}X$ is the boundary of a unique apartment in X. Since K contains a pair of regular antipodal points, there is a regular geodesic c whose ideal endpoints lie in K. c is contained in an apartment A. Since the apartments $\partial_{Tits}A$ and K have antipodal regular points in common, they coincide as a consequence of lemma 3.6.1. A is unique by corollary 4.6.4. □

Lemma 4.6.7 *Let A be an apartment in X. If c is a geodesic arriving at $p \in A$, it can be extended into A.*

Proof. If η is the direction of c at p then, by lemma 3.10.2, η has an antipode in the spherical apartment $\Sigma_p A$. Hence c has an extension into A. □

Corollary 4.6.8 *For any point x and any apartment A in X the geodesic cone over A at x lies in the cone over $\partial_{Tits}A$. In particular, it is contained in a finite union of apartments passing through x.*

Sublemma 4.6.9 *Let Y be a Euclidean building with associated admissible spherical polyhedron Δ_{mod}. Then for each direction $\delta \in int(\Delta_{mod})$ the subset $\theta^{-1}(\delta)$ in the geometric boundary $\partial_\infty Y$ is totally disconnected with respect to the cone topology.*

Proof. Suppose that $y, y', y'' \in Y$ so that $\theta(\overline{yy'}) = \theta(\overline{yy''}) = \delta$. Define the point z by $\overline{yy'} \cap \overline{yy''} = \overline{yz}$. If $z \neq y', y''$ then the angle rigidity axiom EB2 implies that $\angle_z(y', y'') \geq \alpha_0 := 2cdotd_{\Delta_{mod}}(\delta, \partial\Delta_{mod})$ and by triangle comparison we obtain:

$$|y'z| \leq \frac{1}{\sin\alpha_0} \cdot d(y', \overline{yy''})$$

As a consequence, for each $z \in Y$ the closed subset $\{\xi \in \partial_\infty Y | \theta(\xi) = \delta \text{ and } z \in \overline{y\xi}\}$ of $\theta^{-1}(\delta)$ is also open and we see that each point in $\theta^{-1}(\delta)$ has a neighborhood basis consisting of open and closed sets. □

4.7 Subbuildings

A *subbuilding* $X' \subseteq X$ is by definition a metric subspace which admits a Euclidean building structure. This implies that X' is closed and convex and that $\partial_{Tits}X'$ is a spherical subbuilding of $\partial_{Tits}X$ which is closed with respect to the cone topology. We consider a partial converse:

Proposition 4.7.1 *Let X be a Euclidean building and $B \subseteq \partial_{Tits}X$ a subbuilding of full rank. Then the union X' of all apartments A with $\partial_{Tits}A \subseteq B$ has the following properties:*

- *If X' is closed then it is a subbuilding of full rank and the subbuilding $\partial_{Tits}X' \subseteq \partial_{Tits}X$ is the closure $\bar{B}$ of B with respect to the cone topology. Furthermore, X' is the unique subbuilding with $\partial_{Tits}X' = \bar{B}$.*
- *If X is discrete or locally compact then X' is closed.*

Proof. Observe that

$$X' \cup \{A \text{ apartment} | \partial_{Tits}A \subseteq B\} = \cup\{A \text{ apartment} | \partial_{Tits}A \subseteq \bar{B}\}.$$

We first show that X' is a convex subset. Consider points $x_1, x_2 \in X'$. There are apartments A_i with $x_1 \in A_i \subseteq X'$. By lemma 3.10.2, there exist $\xi_i \in \partial_{Tits}A_i$ with $\angle_{x_i}(x_{3-i}, \xi_i) = \pi$. The canonical map $\psi : \partial_{Tits}X \to \Sigma_{x_1}X$ is a building morphism and satisfies the assumption of proposition 3.11.2. Thus, since $\angle_{x_1}(\xi_1, \xi_2) = \pi$, there is an apartment $\partial_{Tits}A \subseteq X'$ which contains ξ_1, ξ_2 and projects isometrically to $\Sigma_{x_1}X$ via ψ. This means that $x_1 \in A$. Consequently $\overline{x_1x_2} \subset A$ and X' is convex. Similarly, one shows that any ray and geodesic in X' lies in an apartment A which is limit of apartments A_n with $\partial_{Tits}A_n \subseteq B$, i.e. $\partial_{Tits}A \subseteq \bar{B}$ and $A \subseteq X'$. The building axioms are inherited from X and if X' is a closed subset then it is complete and a Hadamard space. This proves assertion (i).

(ii) Assume that X is discrete and $x \in \bar{X}'$. Any point $x' \in X'$ lies in an apartment $A \subseteq X'$, and if x' is sufficiently close to x then A contains x. Hence X' is close in this case.

Assume now that X is locally compact and that $(x_n) \subset X'$ is Cauchy with limit $x \in X$. Let $p \in X'$ be some base point. Any segment $\overline{px_n}$ lies in some apartment $A_n \subseteq X'$ and we can pick rays $\overline{px'_n\xi_n}$ in A_n so that $\lim x'_n = x$ and $\theta\xi_n = \theta\overline{px}$. After passing to a subsequence, we may assume that (ξ_n) converges to a point $\xi \in \bar{B}$. Since $\theta\xi_n = \theta\xi$, lemma 4.1.2 implies that the segments $\overline{p\xi_n} \cap \overline{p\xi} \subset X' \cap \overline{p\xi}$ converge to $\overline{p\xi}$. Hence $\overline{p\xi}$ contains x lies in X'. $\square$

4.8 Families of parallel flats

Let X be a Euclidean building and $F \subseteq X$ a flat. If another flat F' has finite Hausdorff distance from F then F and F' bound a flat strip, i.e. an isometrically embedded subset of the form $F \times I$ with a compact interval $I \subset \mathbb{R}$. In this case, the flats F and F' are called *parallel*. Consider the union P_F of all flats parallel to F. P_F is a closed convex subset of X and splits isometrically as

$$P_F \cong F \times Y.$$

Proposition 4.8.1 *P_F is a subbuilding of X and Y admits a Euclidean building structure.*

Proof. By proposition 4.6.1, P_F is the union of all apartments which contain F in a tubular neighborhood, and $\partial_{Tits}P_F$ is the union of all apartments in $\partial_{Tits}X$ which contain the sphere $\partial_{Tits}F$. The subset $\partial_{Tits}P_F \subseteq \partial_{Tits}X$ is convex by lemma 4.1.2 and a subbuilding by proposition 3.10.3. Proposition 4.7.1 implies that P_F is a subbuilding of X. As a consequence, the Hadamard space Y inherits a Euclidean building structure. $\square$

If $dim(F) = rank(X) - 1$, then Y is a building of rank one, i.e. a metric tree. Since $\Sigma_y Y$ is in this case a zero-dimensional spherical building, any two rays $\overline{y\eta_1}$ and $\overline{y\eta_2}$ in Y either initially coincide or their union is a geodesic. This implies:

Lemma 4.8.2 *(i) Let H_1 and H_2 be two flat half-spaces of dimension $rank(X)$ whose intersection $H_1 \cap H_2$ coincides with their boundary flats. Then $H_1 \cup H_2$ is an apartment.*

(ii) If $A_1, A_2, A_3 \subseteq X$ are apartments, and for each $i \neq j$ the intersection $A_i \cap A_j$ is a half-apartment, then $A_1 \cap A_2 \cap A_3$ is a wall in X.

Lemma 4.8.3 *Let $C_1, C_2, C_3 \subset \partial_{Tits}X$ be distinct adjacent chambers, with $\pi = C_1 \cap C_2 \cap C_3$ their common panel. Then there is a $p \in X$ so that if $Cone(p, \pi) = \cup\{\overline{p\xi} | \xi \in \pi\}$, then $log_{p'}(C_i) \subset \Sigma_{p'}X$ are distinct chambers for every $p' \in Cone(p, \pi)$ and any apartment $A \subset X$ such that $\partial_{Tits}A$ contains two of the C_i must intersect $Cone(p, \pi)$.*

Proof. Let $m \subset \partial_{Tits}X$ be a wall containing the panel π. Then each chamber C_i lies in a unique half-apartment h_i bounded by m, and pairs of these half-apartments form apartments. Let A_{ij} be the apartment in X with $\partial_{Tits}A_{ij} = h_i \cup h_j$. By lemma 4.8.2, $\cap A_{ij}$ is a wall $M \subset X$, and we clearly have $\partial_{Tits}M = m$. If $p \in M$, then the half-apartments $\log_p h_i \subset \Sigma_p X$ are bounded by $\log_p m = \Sigma_p M$, so they are distinct; otherwise $\cap A_{ij} \neq M$. Hence the chambers $\log_p C_i \subset \log_p h_i$ are distinct chambers.

If $A \subset X$ is an apartment with $C_i \cup C_j \subset \partial_{Tits}A$, $i \neq j$, then there are chambers $\hat{C}_i, \hat{C}_j \subset A \cap A_{ij}$ with $\partial_{Tits}\hat{C}_i = C_i$, $\partial_{Tits}\hat{C}_j = C_j$. The Tits boundary of the Weyl polyhedron $P = A_{ij} \cap A$ contains $C_i \cup C_j$, so it intersects $Cone(p, \pi)$. $\square$

4.9 Reducing to a thick Euclidean building structure

This subsection is the Euclidean analog of section 3.7.

Definition 4.9.1 *Let X be a Euclidean building modelled on the Euclidean Coxeter complex (E, W_{aff}), with atlas $\mathcal{A}'$. The affine Weyl group may be* reduced *to a reflection subgroup $W'_{aff} \subset W_{aff}$ if there is a W'_{aff} compatible subset $\mathcal{A}' \subset \mathcal{A}$ forming an atlas for a Euclidean building modelled on (E, W'_{aff}).*

In constrast to the spherical building case, the affine Weyl group of a Euclidean building does not necessarily have a canonical reduction with respect to which it becomes thick. For example, a metric tree with variable edge lengths does not admit a thick Euclidean building structure. However, there is always a canonical minimal reduction, and this is thick when it has no tree factors.

Proposition 4.9.2 *Let X be a Euclidean building modelled on (E, W_{aff}). Then there is a unique minimal reduction $W'_{aff} \subset W_{aff}$ so that (X, E, W'_{aff}) splits as a product $\prod X_i$ where each X_i is either a thick irreducible Euclidean building or a 1-dimensional Euclidean building. The thick irreducible factors are either metric cones over their Tits boundary (when the affine Weyl group has a fixed point) or their affine Weyl group is cocompact.*

Proof. We first treat the case when $(\partial_{Tits} X, \Delta_{mod})$ is a thick irreducible spherical building of dimension at least 1.

Step 1: Each apartment $A \subset X$ has a canonical affine Weyl group G_A. If $A \subset X$ is an apartment, a wall $M \subset A$ is *strongly singular* if there is an apartment $A' \subset X$ so that $A \cap A'$ is a half apartment bounded by M. Since $\partial_{Tits} X$ is thick and irreducible, for every wall $m \subset \partial_{Tits} A$ there is a strongly singular wall $M \subset A$ with $\partial_{Tits} M = m$.

Sublemma 4.9.3 *The collection $\mathcal{M}_A$ of strongly singular walls in A is invariant under reflection in any strongly singular wall in A.*

Proof. Note that a wall $M \subset A$ is strongly singular iff $\Sigma_p M \subset \Sigma_p X$ is a wall with respect to the thick building structure $(\Sigma_p X, \Delta_{mod}^{th}(p))$; this is because any half-apartment $h \subset \Sigma_p X$ with boundary $\Sigma_p M$ can be lifted to a half-apartment $H \subset X$ with boundary M, $\Sigma_p H = h$ by applying proposition 3.11.4 to the surjective spherical building morphism $\log_p : \partial_{Tits} X \longrightarrow \Sigma_p X$.

If $M_1, M_2 \subset A$ are strongly singular walls intersecting at $p \in A$, then $\Sigma_p M_i$ is a $\Delta_{mod}^{th}(p)$ wall in $\Sigma_p A \subset \Sigma_p X$, and so if we reflect $\Sigma_p M_2$ in $\Sigma_p M_1$ (inside the apartment $\Sigma_p A$), we get another $\Delta_{mod}^{th}(p)$ wall which is then the space of directions of the desired strongly singular wall M_3.

Now suppose that $M_1, M_2 \in \mathcal{M}_A$ are parallel. Δ_{mod} is irreducible so there is a strongly singular wall M_3 intersecting both M_i at an acute angle. Reflect M_2 in M_3 to get M_4, reflect M_3 in M_1 to get M_5, and M_4 in M_1 to get M_6, and finally reflect M_6 in M_5 to get a wall which is the image of M_2 under reflection in M_1. The walls M_i are all in $\mathcal{M}_A$, so we're done. □

Proof of proposition 4.9.2 continued. Hence for every apartment $A \subset X$ the collection of strongly singular walls in A gives us a group $G_A \subset Isom(A)$ which is generated by reflections.

Step 2: The group G_A is independent of A. Since $\partial_{Tits} G_A \subset Isom(\partial_{Tits} A)$ is an irreducible Coxeter group, it follows that G_A is either a discrete group of isometries or it has a dense orbit. When G_A is discrete, it is generated by the reflections in the strongly singular walls which intersect a given

G_A-chamber in codimension 1 faces. When G_A has a dense orbit, it is generated by all the reflections in strongly singular walls passing through any open set. If two apartments A_1 and A_2 intersect in an open set, it follows that G_{A_1} is isomorphic to G_{A_2}; therefore G_A is independent of A. So there is a well-defined Coxeter complex (E, W'_{aff}) attached to X.

Step 3: Finding (E, W'_{aff}) apartment charts. If Z is a convex domain in an apartment $A \subset X$ and $\iota : U \longrightarrow Z$ is an isometry of an open set $U \subset E$ onto an open set in Z, then there is a unique extension of ι to an isometry of a convex set $\hat{Z} \subset E$ onto Z.

Pick an apartment $A_0 \subset X$ and an isometry $\iota_0 : E \longrightarrow A_0$ which carries $W'_{aff} \subset Isom(E)$ to G_{A_0}. Then restrict to a W'_{aff} chamber $\hat{C}_0 \subset E$ and its image $C_0 \stackrel{def}{=} \iota_0(\hat{C}_0) \subset A_0$. Given any chamber $C \subset X$, there is an apartment A_1 containing subchambers of C and C_0. There is a unique isometry $\iota_1 : E \longrightarrow A_1$ so that ι_1^{-1} and ι_0^{-1} agree on the subchambers $C_0 \cap A_0 \subset A_1$, and a unique isometry $\iota_C : E \supset \hat{C} \longrightarrow C$ so that ι_C^{-1} and ι_1^{-1} agree on the subchamber $C \cap A_1$. If A_2 is another apartment with $\partial_{Tits} C_0, \partial_{Tits} C \subset \partial_{Tits} A_2$, we get another isometry $\iota_2 : E \longrightarrow A_2$; but the convex set $A_1 \cap A_2$ contains subchambers of C_0 and C, so ι_1^{-1} and ι_2^{-1} agree on a subchamber of C. Therefore ι_C is independent of the choice of apartment asymptotic to $C_0 \cup C$.

Sublemma 4.9.4 *Let $A \subset X$ be an apartment, and let $C_1, C_2 \subset \partial_{Tits} A$ be adjacent Δ_{mod}-chambers ($C_1 \cap C_2$ is a panel). For $i = 1, 2$ we let $\iota_{C_1}(A) : E \longrightarrow A$ be the unique isometric extension of $\iota_{\bar{C}_i}$ where $\bar{C}_i \subset A$ is a W'_{aff}-chamber with $\partial_{Tits} \bar{C}_i = C_i$. Then $\iota_{C_2}^{-1}(A) \circ \iota_{C_1}(A) \in W'_{aff}$.*

Proof. For $i = 1, 2$ let $A_{ij} \subset X$ be an apartment with $C_0 \cup C_i \subset \partial_{Tits} A_i$. If C_1 is contained in the convex hull of $C_0 \cup C_2$ (or $C_2 \subset ConvexHull(C_0 \cup C_1)$) then $C_1 \cup C_2 \subset \partial_{Tits}(A \cap A_2)$, so the sublemma follows from the fact that $\iota_{C_i}^{-1}(A)$ restricted to $A \cap A_2$ coincides with $\iota_2^{-1}\big|_{A \cap A_2}$. So we may assume that there is a chamber $C_3 \subset \partial_{Tits} A_1 \cap \partial_{Tits} A_2$ which meets C_1 and C_2 in the panel $\pi = C_1 \cap C_2$. By lemma 4.8.3 (applied to the original Euclidean building (X, E, W_{aff})), there is a point $p \in A_1 \cap A_2$ so that $Cone(p, \pi) \subset A_1 \cap A_2$ and $\log_p(C_i) \subset \Sigma_p X$ are distinct chambers for $i = 1, 2, 3$. Therefore ι_1^{-1} and ι_2^{-1} agree on $Cone(p, \pi)$. Hence the isometries $\iota_{C_1}^{-1}(A), \iota_{C_2}^{-1}$ agree on $Cone(p, \pi)$, which means that $\iota_{C_2}^{-1}(A) \circ \iota_{C_1}(A) : E \longrightarrow E$ is a reflection. But since $\Sigma_p(Cone(p, \pi)) = \log_p(C_1) \cap \log_p(C_2) \cap \log_p(C_3)$, $Cone(p, \pi)$ spans a strongly singular wall in A and so the reflection $\iota_{C_2}^{-1}(A) \circ \iota_{C_1}(A) \in W'_{aff}$. □

Proof of proposition 4.9.2 continued. By sublemma 4.9.4, we see that for each apartment $A \subset X$, there is a canonical collection of isometries $\iota : E \longrightarrow A$ which are mutally W'_{aff} compatible, and which are compatible with the $\iota_C : \hat{C} \longrightarrow C$ for every chamber $C \subset A$. We refer to such isometries as W'_{aff}-charts, and to the collection of W'_{aff}-charts (for all apartments) as the (E, W'_{aff}) atlas $\mathcal{A}'$.

Sublemma 4.9.5 *Let $A_1, A_2 \subset X$ be apartments with d-dimensional intersection $P = A_1 \cap A_2$. If $p \in P$ is an interior point of the Weyl polyhedron P, then there is an apartment $A_3 \subset X$ so that A_3 contains a neighborhood of $p \in P$, and $A_3 \cap A_i$ contains a Weyl chamber.*

Proof. We have $\Sigma_p A_1 \cap \Sigma_p A_2 = \Sigma_p P$ by lemma 4.4.3. Let $\sigma_1 \subset \Sigma_p P$ be a $d-1$-dimensional face of $\Sigma_p P$, and let σ_2 be the opposite face in $\Sigma_p P$. If $\tau_1 \subset \Sigma_p A_1$ is a chamber containing σ_1, then we may find an opposite chamber $\tau_2 \subset \Sigma_p A_2$. But then τ_2 contains a face opposite σ_1, and this must be σ_2 since each face in an apartment has a unique opposite face in that apartment. Let $C_i \subset \partial_{Tits} A_i$ be the chamber such that $\log_p C_i = \tau_i$. Then there is a unique apartment $A_3 \subset X$ with $C_1 \cup C_2 \subset \partial_{Tits} A_3$. $\Sigma_p P \subset \Sigma_p A_3$, so A_3 has the properties claimed. □

Proof of proposition 4.9.2 continued. If $A_1, A_2 \subset X$ are apartments with $A_1 \cap A_2 \neq \emptyset$, then any W'_{aff} charts $\iota_i : E \longrightarrow A_i$ are W'_{aff} compatible since by sublemma 4.9.5 we have a third apartment $A_3 \subset X$ so that ι_1 and ι_2 are both W'_{aff} compatible with $\iota_3 : E \longrightarrow A_3$ on an open set $U \subset A_1 \cap A_2$. Hence $\mathcal{A}'$ gives X the structure of a Euclidean building modelled on (E, W'_{aff}). From the construction of W'_{aff} it is clear that $(X, \mathcal{A}')$ is thick.

Step 4: The case when X is a 1-dimensional Euclidean building, i.e. a metric tree. Let $A_0 \subset X$ be an apartment, $\partial_{Tits} A_0 = \{\eta_1, \eta_2\}$. For each $p \in X$ let $\pi_{A_0}(p) \in A_0$ be the nearest point in A_0, and $p_{A_0} \in A_0$ be a point (there are at most two) with $d(p_{A_0}, \pi_{A_0}(p)) = d(p, A_0)$. Let $\mathcal{M} \subset A_0$ be the set of points p_{A_0} where $p \in X$ is a branch point: $|\Sigma_p X| \geq 3$; let $G \subset Isom(A_0)$ be the group generated by reflections at points in $\mathcal{M}$. For each $\xi \in \partial_{Tits} X \setminus \eta_1$ there is a unique isometry ι_ξ from the apartment $A_0 = \overline{\eta_1 \eta_2}$ to the apartment $\overline{\eta_1 \xi}$ which is the identity on the half-apartment $\overline{\eta_1 \eta_2} \cap \overline{\eta_1 \xi}$. If $\xi_1 \neq \xi_2$, then we have two isometries $\iota_1, \iota_2 : A_0 \longrightarrow \overline{\xi_1 \xi_2}$ where ι_i^{-1} agrees with ι_{ξ_i} on $\overline{\eta_1 \xi_i} \cap \overline{\xi_1 \xi_2}$. By inspection $\iota_2^{-1} \circ \iota_1 \in G$. Hence for each apartment $A \subset X$ we have a well-defined set of isometries $A_0 \longrightarrow A$. As in step 3 it follows that these isometries are G-compatible, so they define an atlas $\mathcal{A}'$ for a Euclidean building structure on X.

Step 5: X is an arbitrary Euclidean building modelled on (E, W_{aff}). Let $W \stackrel{def}{=} \partial_{Tits} W_{aff}$, and let $W' \subset W \subset Isom(\partial_{Tits} E)$ be the canonical reduced Weyl group of $\partial_{Tits} X$ given by section 3.7. Let $\bar{W}_{aff} \subset Isom(E)$ be the inverse image of W' under the canonical homomorphism $Isom(E) \longrightarrow Isom(\partial_{Tits} E)$. Let $\theta' : \partial_{Tits} X \longrightarrow \Delta'_{mod} \stackrel{def}{=} S/W'$ be the Δ'_{mod}-anisotropy map. We may define Δ'_{mod}-directions for rays $\overline{x\xi} \subset X$ by the formula $\theta'(\overline{x\xi}) = \theta'(\xi) \in \Delta'_{mod}$. We define the Δ'_{mod}-direction of a geodesic segment $\overline{xy} \subset X$ by setting $\theta'(\overline{xy}) = \theta'(\overline{x\xi_1})$ for any ray $\overline{x\xi_1}$ extending $\overline{xy}$; if $\overline{x\xi_2}$ is another ray extending $\overline{xy}$ then $\xi_1 \in \partial_{Tits} X$ and $\xi_2 \in \partial_{Tits} X$ are both antipodes of $\eta \in \partial_{Tits} X$ where $\overline{y\eta}$ is a ray extending $\overline{yx}$, so $\theta'(\overline{xy})$ is well-defined. The remaining Euclidean building axioms follow easily from the fact that any two segments $\overline{px}$, $\overline{py}$ initially lie in an apartment $A \subset X$ (corollary 4.4.3) and for our compatible $(E, \bar{W}_{aff})$ apartment charts we may take all isometric embeddings $i : E \longrightarrow X$ for which $\partial_{Tits} i : \partial_{Tits} E \longrightarrow \partial_{Tits} X$ is an apartment chart for $(\partial_{Tits} X, \Delta'_{mod})$.

We may now apply proposition 4.3.1 to see that $(X, E, \bar{W}_{aff})$ splits as a product of Euclidean buildings $(X, E, \bar{W}_{aff}) = (\prod X_i, \prod E_i, \prod W^i_{aff})$ so that each $\partial_{Tits} X_i$ is irreducible. Let $(W^i_{aff})' \subset W^i_{aff}$, $\mathcal{A}_i$ be the canonical subgroup and atlas constructed in steps 1-4, and set $W'_{aff} = \prod (W^i_{aff})' \subset Isom(E)$, $\mathcal{A}' = \prod \mathcal{A}_i$. Then $(X_i, E_i, (W^i_{aff})', \mathcal{A}_i)$ has the properties claimed in the proposition. Fix an apartment $A_0 \subset X$ and a chart $\iota_{A_0} \in \mathcal{A}$. If $A_0, \ldots, A_k = A_0$ is a sequence of apartments so that $A_{i-1} \cap A_i$ is a half-apartment for each i, then there is a unique isometry $g_i : A_{i-1} \longrightarrow A_i$ so that g_i is the identity on $A_{i-1} \cap A_i$. Axiom EB4 implies that $g_i \circ \ldots \circ g_1 \circ \iota_{A_0} \in \mathcal{A}$ for each i, so in particular $g = g_k \circ \ldots \circ g_1 \in \iota_{A_0 *}(W_{aff})$. From the construction of $(W^i_{aff})'$ it is clear that the group of all such isometries $g : A_0 \longrightarrow A_0$ contains $\iota'_{A_0}(W'_{aff}) \subset Isom(A_0)$ where $\iota'_{A_0} \in \mathcal{A}'$. So $W'_{aff} \subset W_{aff}$ is a minimal reduction of W_{aff}. $\square$

4.10 Euclidean buildings with Moufang boundary

This is a continuation of section 3.12.

Proposition 4.10.1 (More properties of root groups) *Let B be a thick irreducible spherical building of dimension at least 1, and let X be a Euclidean building with Tits boundary B.*

1. *For every root group* $U_a \subset Aut(B, \Delta_{mod})$ *and every* $g \in U_a$ *there is a unique automorphism* $g_X : X \longrightarrow X$ *so that* $\partial_{Tits} g_X = g$*. In other words, if* G *is the group generated by the root groups, then the action of* G *on* $\partial_{Tits}X$ *"extends" to an action on* X *by building automorphisms. Henceforth we will use the same notation to denote this extended action.*

2. *Suppose* $g \in U_a$ *is nontrivial. If* $A \subseteq X$ *is an apartment such that* $\partial_{Tits}A \supset a$*, then* $g(A) \cap A$ *is a half-apartment; moreover* $Fix(g) \cap A = g(A) \cap A$*.*

Proof. See [Ron, Affine buildings II, esp. prop. 10.8], or [Ti2, p. 168].

For the remainder of this section X will be a thick, nonflat irreducible Euclidean building of rank ≥ 2. Therefore Δ_{mod} is a spherical simplex with diameter $< \frac{\pi}{2}$ and the faces of $\partial_{Tits}X$ define a simplicial complex.

Lemma 4.10.2 *Let* $A \subset X$ *be an apartment,* $p_0 \in X$*,* $p \in A$ *the nearest point in* A*, and* $a \subset \partial A$ *a root. Then the stabilizer of* p_0 *in the root group* U_a *fixes* p*.*

Proof. Using lemma 3.10.2 extend the geodesic segment $\overline{p_0 p}$ to a geodesic ray $\overline{p_0 \xi} = \overline{p_0 p} \cup \overline{p\xi}$ so that the ray $\overline{p\xi}$ lies in the half apartment $Cone(p, a) \subset A$. If $g \in U_a$ fixes p_0, then it fixes the ray $\overline{p_0 \xi}$, and hence the half-apartment $Cone(p, a)$. □

We now assume that the spherical building $(\partial_{Tits}X, \Delta_{mod})$ is Moufang. Pick $p \in X$, and let $(\Sigma_p X, \Delta^{th}_{mod}(p))$ denote the thick spherical building defined by the space of directions $\Sigma_p X$ with its reduced Weyl group (see section 3.7). Suppose $H_+ \subset X$ is a half-apartment whose boundary wall passes through p, $h_+ \stackrel{def}{=} \Sigma_p H_+ \subset \Sigma_p X$ is a $\Delta^{th}_{mod}(p)$ root, and let $a_+ = \partial_{Tits} H_+ \subset \partial_{Tits} X$. If U_{a_+} is the root group associated to a_+, and $V_{a_+} \subset U_{a_+}$ is the subgroup fixing p, then we have a homomorphism $\Sigma_p : V_{a_+} \longrightarrow Aut(\Sigma_p X, \Delta^{th}_{mod}(p))$.

Lemma 4.10.3 *The image of* V_{a_+} *is the root group* U_{h_+} *associated with* h_+*, and this acts transitively on apartments in* $\Sigma_p X$ *containing* h_+*. In particular,* $(\Sigma_p X, \Delta^{th}_{mod}(p))$ *is a thick Moufang spherical building.*

Proof. By corollary 3.11.5, if $h_- \subset \Sigma_p X$ is a $\Delta^{th}_{mod}(p)$ root with $\partial h_- = \partial h_+ = \Sigma_p(\partial H_+)$, then there is a half-apartment $H_- \subset X$ so that H_- and H_+ have the same boundary and $\Sigma_p H_- = h_-$. Given two such $\Delta_{mod}(p)$ roots $h^1_-, h^2_- \subset \Sigma_p X$ so that $h^i_- \cup h_+$ forms an apartment in $\Sigma_p X$, we get two half apartments H^i_- so that $H^i_- \cup H_+$ forms an apartment in X. Since $(\partial_{Tits}X, \Delta_{mod})$ is Moufang, the root group $U_{a_+} \subset Aut(\partial_{Tits}X, \Delta_{mod})$ contains an element which carries H^1_- to H^2_-. By 3.12.2, g "extends" uniquely to an isometry $g : X \longrightarrow X$ which carries the apartment $H^1_- \cup H_+$ to the apartment $H^2_- \cup H_+$, fixing H_+ (see 4.10.1). It remains only to show that the isometry $\Sigma_p g : \Sigma_p X \longrightarrow \Sigma_p X$ is contained in the root group $U_{h_+} \subset Aut(\Sigma_p X, \Delta^{th}_{mod}(p))$. Clearly $\Sigma_p g$ fixes h_+. Let $C \subset \Sigma_p X$ be a $\Delta^{th}_{mod}(p)$ chamber such that $C \cap h_+$ contains a panel π with $\pi \not\subset \partial h_+$. Using proposition 3.11.4 we may lift C to a (subcomplex) $\tilde{C} \subset \partial_{Tits}X$ so that $\tilde{C} \cap \partial a_+$ maps isometrically to $C \cap \partial h_+$ under $log_p : \partial_{Tits}X \longrightarrow \Sigma_p X$. g fixes an interior point of $\tilde{C}$, so $\Sigma_p g$ fixes an interior point of C, which implies that $\Sigma_p g$ fixes C as desired. □

Definition 4.10.4 *A point* $s \in X$ *is a* **spot** *if either*
1. The affine Weyl group W_{aff} *has a dense orbit or*
2. W_{aff} *is discrete and* s *corresponds to a 0-simplex in the complex associated with* X*.*

If $A \subseteq X$, then $Spot(A)$ is the set of spots in A.

Lemma 4.10.5 *If $A \subset X$ is an apartment, $p_0 \in A$ is a spot, then for every $p \neq p_0$ there is a root $a \subset \partial_{Tits} A$ and a $g \in U_a$ so that g fixes p_0 but not p.*

Proof. For each $\Delta_{mod}^{th}(p_0)$ root $h_+ \subset \Sigma_{p_0} X$ we have a singular half-apartment $H_+ \subset A$ with $\Sigma_{p_0} H_+ = h_+$, and this gives us a root $a_+ = \partial_{Tits} H_+ \subset \partial_{Tits} X$, the root group U_{a+}, and the subgroup $V_{a+} \subset U_{a_+}$ fixing p_0. By lemma 4.10.3, the image of V_{a_+} in $Aut(\Sigma_{p_0} X, \Delta_{mod}^{th}(p_0))$ is the root group U_{h_+}. Since $(\Sigma_{p_0} X, \Delta_{mod}^{th}(p_0))$ is Moufang, the group G_{p_0} generated by the V_{h_+}'s as h_+ runs over all $\Delta_{mod}^{th}(p_0)$ roots in $\Sigma_p A$ acts transitively on $\Delta_{mod}^{th}(p_0)$ chambers in $\Sigma_{p_0} X$ (see 3.12.2). If $p \in X - p_0$ is fixed by every V_{a_+}, then $\overrightarrow{p_0 p} \in \Sigma_{p_0} X$ is fixed by G_{p_0}, which means that it lies in every $\Delta_{mod}^{th}(p_0)$ chamber of $\Sigma_{p_0} X$, forcing $\overrightarrow{p_0 p} \in \Sigma_{p_0} A$. Hence the point $q \in A$ nearest p is different from p_0, so we may find a singular half-apartment $H_+ \subset A$ containing p_0 but not q (because p_0 is a spot), and use the root group $U_{\partial_{Tits} H_+}$ to move q while fixing H_+. This contradicts the assumption that p is fixed by every V_{a_+}. □

Proposition 4.10.6 *Let X be a thick, nonflat Euclidean building of rank at least two, and suppose $\partial_{Tits} X$ is an irreducible Moufang spherical building. Let $G \subset Aut(\partial_{Tits} X, \Delta_{mod})$ be the subgroup generated by the root groups of $\partial_{Tits} X$, and consider the isometric action of G on X.*

1. *The fixed point set of a maximal bounded subgroup $M \subset G$ is a spot, and the stabilizer of a spot is a maximal bounded subgroup.*
2. *A spot $p \in X$ lies in the apartment $A \subset X$ iff p is the unique spot in X which is fixed by the stabilizer of p in U_a for every root $a \subset \partial_{Tits} A$.*
3. *If $A \subset X$ is an apartment, and $a \subset \partial_{Tits} A$ is a root, then as g runs through all non-trivial elements of U_a, we obtain all singular half-apartments $H \subset A$ with $\partial_{Tits} H = a$ as subsets $A \cap Fix(g)$.*

Proof. Let $M \subseteq G$ be a maximal bounded subgroup. By the Bruhat-Tits fixed point theorem [BT], M has a nonempty fixed-point set $Fix(M)$. $Fix(M)$ contains a spot since when W_{aff} is discrete the fixed point set of a group of building automorphisms is a subcomplex. By lemma 4.10.5, we see that if $p_0 \in Fix(M)$, then maximality of M forces $Fix(M) = \{p_0\}$. Conversely, if $p_0 \in X$ is a spot, then the stabilizer of p_0 has fixed point set $\{p_0\}$ by lemma 4.10.5, and by the Bruhat-Tits fixed point theorem, the stabilizer is a maximal bounded subgroup.

For every $p \in X$ and every apartment $A \subset X$, let $G(p, A)$ be the group generated by the stabilizers of p in the root groups U_a, where $a \subset \partial_{Tits} A$ is a root. If $p \in A \subset X$ is a spot, then by lemma 4.10.5 we have $Fix(G(p, A)) = \{p\}$. If $p \notin A \subset X$, then the nearest point $p_0 \in A$ to p is contained in $Fix(G(p, A))$ by lemma 4.10.2; hence $Fix(G(p, A))$ contains a spot other than p_0.

Claim 3 follows from property 2 of proposition 4.10.1, the fact that $\partial_{Tits} X$ is Moufang, and the fact that every singular half-apartment is the intersection of two apartments.

□

Definition 4.10.7 *If $A \subset X$ is an apartment, then the* **half-apartment topology** *on $Spot(A)$ is the topology generated by open singular half-apartments contained in A.*

With the half-apartment topology, $Spot(A)$ is discrete when W_{aff} is discrete and coincides with the metric topology when W_{aff} has dense orbit.

5 Asymptotic cones of symmetric spaces and Euclidean buildings

In this section we arrive at the heart of the geometric part in the proof of our main results. We show that asymptotic cones of symmetric spaces and ultralimits of sequences of Euclidean buildings (of bounded rank) are Euclidean buildings.

Our main motivation for choosing the Euclidean building axiomatisation EB1-4 is that these axioms behave well with respect to ultralimits. Indeed, the Euclidean building axioms EB1, EB3 and EB4 which are also satisfied by symmetric spaces, i.e. the existence of Δ_{mod}-directions and an apartment atlas, pass directly to ultralimits. However, unlike Euclidean buildings, symmetric spaces do not satisfy the angle rigidity axiom EB2. The verification of EB2 for ultralimits of symmetric spaces (lemma 5.2.2) is the only technical point and, as opposed to the building case (lemma 5.1.2), non-trivial. Symmetric spaces satisfy angle rigidity merely at infinity; their Tits boundaries are spherical buildings. Intuitively speaking, the rescaling process involved in forming ultralimits pulls the spherical building structure (the missing angle rigidity property) from infinity to the spaces of directions.

5.1 Ultralimits of Euclidean buildings are Euclidean buildings

Theorem 5.1.1 *Let X_n, $n \in \mathbb{N}$, be Euclidean buildings with the same anisotropy polyhedron Δ_{mod}. Then, for any sequence of basepoints $\star_n \in X_n$, the ultralimit $(X_\omega, \star_\omega) = \omega\text{-}lim(X_n, \star_n)$ admits a Euclidean building structure with anisotropy polyhedron Δ_{mod}.*

Proof. X_ω is a Hadamard space (lemma 2.4.4). A Euclidean building structure on X_ω consists of an assignment of Δ_{mod}-directions for segments (axioms EB1+EB2) and of an atlas of compatible charts for apartments (axioms EB3+EB4), cf. section 4.1.2. We assume that X has no Euclidean deRham factor. The general case allowing a Euclidean deRham factor is a trivial consequence.

EB1: We can assign a Δ_{mod}-direction to an oriented geodesic segment in X_ω as follows. A segment $\overline{x_\omega y_\omega}$ arises as ultralimit of a sequence of segments $\overline{x_n y_n}$ in X, and we define the direction as:

$$\theta(\overline{x_\omega y_\omega}) := \omega\text{-}\lim_n \theta(\overline{x_n y_n}) \in \Delta_{mod} \tag{37}$$

The ultralimit (37) exists because Δ_{mod} is compact. Inequality (28) in EB1 passes to the ultralimit:

$$d_{\Delta_{mod}}(\omega\text{-}\lim \theta(\overline{x_n y_n}), \omega\text{-}\lim \theta(\overline{x_n z_n})) \leq \tilde{\angle}_{x_\omega}(y_\omega, z_\omega).$$

This implies that the left-hand side of (37) is well-defined and

$$d_{\Delta_{mod}}(\theta(\overline{x_\omega y_\omega}), \theta(\overline{x_\omega z_\omega})) \leq \tilde{\angle}_{x_\omega}(y_\omega, z_\omega).$$

Thus axiom EB1 holds. EB1 implies lemma 4.1.1. Therefore, segments which contain a given segment have the same Δ_{mod}-direction and we can assign Δ_{mod}-directions to geodesic rays.

EB2: Since geodesics are extendible in X_ω, it suffices to show:

Lemma 5.1.2 *If $x_\omega \in X_\omega$ and $\xi_\omega, \eta_\omega \in \partial_{Tits} X_\omega$ then $\angle_{x_\omega}(\xi_\omega, \eta_\omega)$ is contained in $D := D(\theta(\overline{x_\omega \xi_\omega}), \theta(\overline{x_\omega \eta_\omega}))$.*

Proof. The rays $\overline{x_\omega\xi_\omega}$ and $\overline{x_\omega\eta_\omega}$ are ultralimits of sequences of rays $\overline{x_n\xi_n}$ and $\overline{x_n\eta_n}$ in X_n and we can choose $\xi_n, \eta_n \in \partial_{Tits}X_n$ so that $\theta(\xi_n) = \theta(\overline{x_\omega\xi_\omega})$ and $\theta(\eta_n) = \theta(\overline{x_\omega\eta_\omega})$. Let $\rho_n : [0,\infty) \to X_n$ be a unit speed parametrisation for the geodesic ray $\overline{x_n\xi_n}$. The angle $\angle_{\rho_n(t)}(\xi_n, \eta_n)$ is non-decreasing and continuous from the right in t (lemma 2.1.5) and, since X_n satisfies EB2, takes values in the finite set D. For $d \in D$ set $t_n(d) := min\{t \geq 0 : \angle_{\rho_n(t)}(\xi_n, \eta_n) \geq d\} \in [0,\infty]$ and $t_\omega(d) := \omega\text{-lim}\, t_n(d)$. Then there exist $d_0 \in D$ and $T > 0$ with $t_\omega(d_0) = 0$ and $2T \leq t_\omega(d)$ for all $d > d_0$. The points $x'_n := \rho_n(t_n(d_0))$ and $x''_n := \rho_n(T)$ satisfy for ω-all n: $x'_\omega := \omega\text{-lim}\, x'_n = x_\omega$, $x''_\omega := \omega\text{-lim}\, x''_n \neq x_\omega$ and the ideal triangle $\Delta(x'_n, x''_n, \eta_n)$ has angle sum π. By a version of the Triangle Filling Lemma 2.1.4 for ideal triangles in Hadamard spaces, $\Delta(x'_n, x''_n, \eta_n)$ can be filled in by a semi-infinite flat strip S_n. The ultralimit $\omega\text{-lim}\, S_n$ is a semi-infinite flat strip filling in the ideal triangle $\Delta(x_\omega, x''_\omega, \eta_\omega)$ and therefore $\angle_{x_\omega}(\xi_\omega, \eta_\omega) = \omega\text{-lim}\, \angle_{x'_n}(\xi_n, \eta_n) = d_0 \in D$, as desired.

EB3 and EB4: After enlarging the affine Weyl groups of the model Coxeter complexes of the buildings X_n, we may assume that the X_n are modelled on the same Euclidean Coxeter complex (E, W_{aff}) whose affine Weyl group W_{aff} contains the full translation subgroup of $Isom(E)$, i.e. $\rho^{-1}(W) = W_{aff}$ where $\rho : Isom(E) \to Isom(\partial_{Tits}E)$ is the canonical homomorphism (26) associating to an affine isometry its rotational part. (Here we use that the X_n don't have Euclidean factors.)

The atlases $\mathcal{A}_n$ for the building structures on X_n give rise to an atlas for a building structure on X_ω as follows: If $\iota_n \in \mathcal{A}_n$ are charts for apartments in X_n so that $\omega\text{-lim}\, d(\iota_n(e), \star_n) < \infty$ for (one and hence) each point $e \in E$, then the ultralimit $\iota_\omega := \omega\text{-lim}\, \iota_n : E \to X_\omega$ is an isometric embedding which parametrises a flat in X_ω. The collection $\mathcal{A}_\omega$ of all such embeddings ι_ω satisfies axiom EB3 in view of lemma 2.4.4. Axiom EB4 holds trivially, because coordinate changes $\iota_\omega^{-1} \circ \iota'_\omega$ between charts $\iota_\omega, \iota'_\omega \in \mathcal{A}_\omega$ are Δ_{mod}-direction preserving isometries between convex subsets of E and such isometries are induced by isometries in $\rho^{-1}(W) = W_{aff}$. Hence $\mathcal{A}_\omega$ is an atlas for a Euclidean building struture on X_ω with model Coxeter complex (E, W_{aff}), and the proof of the theorem is complete. □

Corollary 5.1.3 *Let X be a Euclidean building modelled on the Coxeter complex (E, W_{aff}) and denote by $\hat{W}_{aff}$ the subgroup of $Isom(E)$ generated by W_{aff} and all translations which preserve the de Rham decomposition of (E, W_{aff}) and act trivially on the Euclidean de Rham factor. Then any asymptotic cone X_ω inherits a Euclidean building structure modelled on $(E, \hat{W}_{aff})$. The building X_ω is thick if X is thick and the affine Weyl group W_{aff} is cocompact.*

Proof. $X_\omega = \omega\text{-lim}(X_n, *_n)$ where the λ_n are scale factors with $\omega\text{-lim}\, \lambda_n = 0$, X_n is the rescaled building $\lambda_n X_n$ and $*_n \in X_n$ are base points. X_ω inherits the Euclidean building structure modelled on $(E, \hat{W}_{aff})$ which was constructed in the proof of the previous theorem.

Suppose now in addition that X is thick and W_{aff} is cocompact. Then any wall $w_n \subset X_n$ *branches*, i.e. there are half-apartments $H_{ni} \subset X_n$, $i = 1, 2, 3$, so that the intersection of any two of them equals w_n and the union of any two of them is an apartment (lemma 4.8.2). If a sequence of walls w_n satisfies $\omega\text{-lim}\, d(w_n, \star_n) < \infty$, it follows that the ultralimit of the sequence (w_n) is a branching wall in X_ω. Since W_{aff} is cocompact by assumption, there is a positive number d so that any flat in X, whose ideal boundary is a wall in $\partial_{Tits}X$, lies within distance at most d from a branching wall in X. In view of $\omega\text{-lim}\, \lambda_n = 0$, this implies that any flat in X_ω, whose ideal boundary is a wall in $\partial_{Tits}X_\omega$, is a branching wall. Thus, the Euclidean building structure on X_ω is thick. □

5.2 Asymptotic cones of symmetric spaces are Euclidean buildings

We start by recalling some well-known facts from the geometry of symmetric spaces which will be needed later; as references for this material may serve [BGS, Eb].

Let X be a symmetric space of noncompact type. In particular, X is a Hadamard manifold, i.e. a complete simply-connected Riemannian manifold of nonpositive sectional curvature. To simplify language, we assume that X has no Euclidean factor. The identity component G of the isometry group of X is a semisimple Lie group and acts transitively on X. A *k-flat* in X is a totally geodesic submanifold isometric to Euclidean k-space. We recall that G acts transitively on the family of maximal flats. In particular, any two maximal flats in X have the same dimension r; it is called the *rank* of X. We will call the maximal flats also *apartments*. Pick an apartment E in X and let W_{aff} be the quotient of the set-wise stabiliser $Stab_G(E)$ by the point-wise stabiliser $Fix_G(E)$. Then W_{aff} can be identified with a subgroup of $Isom(E)$. This subgroup is generated by reflections at hyperplanes and contains the full translation group. We call (E, W_{aff}) the Euclidean Coxeter complex associated to X. Its isomorphism type does not depend on the choice of E, because G acts transitively on apartments. Consider the collection of all isometric embeddings $\iota : E \to X$ so that W_{aff} is identified with $Stab_G(\iota(E))/Norm_G(\iota(E))$. Walls, singular flats, Weyl chambers et cetera are defined as images of corresponding objects in E via the maps ι. Note that the singular flats are precisely the intersections of apartments. The induced isometric embeddings $\partial_{Tits}\iota : \partial_{Tits}E \to \partial_{Tits}X$ form an atlas for a thick spherical building structure on $\partial_{Tits}X$ modelled on the spherical Coxeter complex $(\partial_{Tits}E, W) = \partial_{Tits}(E, W_{aff})$. W is isomorphic to the Weyl group of the symmetric space X. Composing the anisotropy map $\theta_{\partial_{Tits}X} : \partial_{Tits}X \to \Delta_{mod}$ with the map $SX \to \partial_{Tits}X$ which assigns to every unit vector v the ideal endpoint of the geodesic ray $t \mapsto exp(tv)$ one obtains a natural map

$$\theta : SX \to \Delta_{mod} \tag{38}$$

from the unit sphere bundle of X to the anisotropy polyhedron Δ_{mod}. We will call $\theta(v)$ the *Δ_{mod}-direction* of $v \in SX$; Δ_{mod}-directions of oriented segments, rays and geodesics are defined as the Δ_{mod}-direction of the velocity vectors for a unit speed parametrisation. The orbits for the natural G-action on SX are precisely the inverse images under θ of points. Let S_pX be the unit sphere at $p \in X$, equipped with the angular metric, and let G_p be the isotropy group of p. Then θ induces a canonical isometry $S_p/G_p \simeq \Delta_{mod}$ where S_p/G_p is equipped with the orbital distance metric. The quotient map $S_pX \to \Delta_{mod}$ is 1-Lipschitz and, for any $x, y \in X$ we have the following counterpart to inequality (28):

$$d_{\Delta_{mod}}(\theta(\overrightarrow{px}), \theta(\overrightarrow{py})) \le \angle_p(x, y) \le \tilde{\angle}_p(x, y) \tag{39}$$

The goal of this section is to prove the following theorem.

Theorem 5.2.1 *Let X be a non-empty symmetric space with associated Euclidean Coxeter complex (E, W_{aff}). Then, for any sequence of base points $*_n \in X$ and scale factors λ_n with ω-lim $\lambda_n = 0$, the asymptotic cone $X_\omega = \omega$-lim$(\lambda_n X, *_n)$ is a thick Euclidean building modelled on (E, W_{aff}). Moreover, X_ω is homogeneous, i.e. has transitive isometry group.*

Proof. EB1: Let Δ_{mod} be the anisotropy polyhedron for (E, W_{aff}). The construction of Δ_{mod}-directions for segments in X_ω is the same as in the building case. We define directions by (37) and (39) implies that the definition is good and that EB1 holds.

EB3 and EB4: The Euclidean Coxeter complex (E, W_{aff}) is invariant under rescaling, because $W_{aff} \subset Isom(E)$ contains all translations. Apartments in X_ω and their charts arise as ultralimits of sequences of apartments and charts in X, and axioms EB3 and EB4 follow as in the building case, cf. section 5.1.

EB2: The only nontrivial task is to verify the angle rigidity axiom EB2. This will be done in the following lemma.

Lemma 5.2.2 *If $p \in X_\omega$ and $x_1, x_2 \in X_\omega - \{p\}$, then $\angle_p(x_1, x_2) \in D(\theta(\overline{px_1}), \theta(\overline{px_2}))$.*

Proof. If $z'_k \in \overline{px_1} - p$ and $z'_k \to p$, then $\angle_{z'_k}(x_1, x_2) \to \angle_p(x_1, x_2)$ and $\angle_{z'_k}(p, x_2) \to \pi - \angle_p(x_1, x_2)$ by lemma 2.1.5. Since $\theta(\overline{z'_k x_2}) \to \theta(\overline{px_2})$ we can find $x'_{1k} \in \overline{z'_k x_1}$, $x'_{2k} \in \overline{z'_k x_2}$, and $p'_k \in \overline{z'_k p}$ such that $\tilde{\angle}_{z'_k}(x'_{1k}, x'_{2k}) \to \angle_p(x_1, x_2)$, $\tilde{\angle}_{z'_k}(p'_k, x'_{2k}) \to \pi - \angle_p(x_1, x_2)$, and $\theta(\overline{z'_k x'_{2k}}) = \theta(\overline{z'_k x_2}) \to \theta(\overline{px_2})$. Since geodesic segments in X_ω are ultralimits of geodesic segments in $\lambda_n X$, we can find sequences $p_k, x_{1k}, x_{2k}, z_k \in X$ such that $z_k \in \overline{p_k x_{1k}}$, $\tilde{\angle}_{z_k}(x_{1k}, x_{2k}) \to \angle_p(x_1, x_2)$, $\tilde{\angle}_{z_k}(p_k, x_{2k}) \to \pi - \angle_p(x_1, x_2)$, $\theta(\overline{z_k x_{2k}}) \to \theta(\overline{px_2})$, $\theta(\overline{p_k x_{1k}}) \to \theta(\overline{px_1})$, and finally $|z_k x_{1k}|, |z_k x_{2k}|, |z_k p_k| \to \infty$. Applying a sequence of elements $g_k \in G = (Isom(X))^o$ we may assume in addition that z_k is a constant sequence, $z_k \equiv o$. Hence the sequences of segments $\overline{ox_{1k}}, \overline{ox_{2k}}, \overline{op_k}$ subconverge to rays $\overline{o\xi_1}, \overline{o\xi_2}$, and $\overline{o\eta}$ respectively, which satisfy the following properties:

1. $\theta_{\partial_{Tits} X}(\xi_i) = \theta(\overline{o\xi_i}) = \theta(\overline{px_i})$
2. $\angle_{Tits}(\xi_1, \xi_2) \le \angle_p(x_1, x_2)$, $\angle_{Tits}(\eta, \xi_2) \le \pi - \angle_p(x_1, x_2)$ by lemma 2.3.1.
3. $\overline{o\xi_1} \cup \overline{o\eta}$ is a geodesic, so $\angle_{Tits}(\xi_1, \eta) = \pi$.

We conclude that

$$\angle_p(x_1, x_2) = \angle_{Tits}(\xi_1, \xi_2) \in D(\theta(\xi_1), \theta(\xi_2)) = D(\theta(\overline{px_1}), \theta(\overline{px_2}))$$

as desired. ☐

Hence we have constructed a Euclidean building struture on X_ω. Since G acts transitively on Weyl chambers in X, it follows that the isometry group of X_ω acts transitively on Weyl chambers in X_ω; in particular, X_ω is homogeneous. To see that the building structure on X_ω is thick it is therefore enough to check that the induced spherical building structure of $\Sigma_{*_\omega} X_\omega$ modelled on $(\partial_{Tits} E, W)$ is thick. One way to see this is to construct a canonical isometric embedding α of the thick spherical building $\partial_{Tits} X$ modelled on $(\partial_{Tits} E, W)$ into $\Sigma_{*_\omega} X_\omega$ by assigning to $\xi \in \partial_{Tits} X$ the initial direction in $*_\omega$ of the geodesic ray ω-lim $\overline{*_n \xi}$ in X_ω. That α is isometric follows, for instance, from the definition (8) of the Tits distance. This finishes the proof of the theorem. ☐

6 The topology of Euclidean buildings

In this section, X will denote a rank r Euclidean building. The main goal in this section is to understand homeomorphisms of X. As motivation for the approach taken here, consider a closed interval I topologically embedded in an $\mathbb{R}$-tree T. Because every interior point $p \in I - \partial I$ of the interval disconnects T, every path $c : [0, 1] \longrightarrow T$ joining the endpoints of I must pass through p, i.e. $c([0, 1]) \supseteq I$. A similar phenomenon occurs in X if we consider topological embeddings of closed balls $B \subset X$ of dimension equal to $rank(X)$: if $[c] \in H_r(X, \partial B)$ and $[\partial c] \in H_{r-1}(\partial B)$ is the fundamental class of ∂B, then the image of the chain c contains B. By using 4.6.8, we can construct such c so that $Image(c) - U$ is contained in finitely many flats, where U is any given neighborhood of ∂B. It follows that any $b \in B - \partial B$ has a neighborhood V_b in X such that $B \cap V_b$ is contained in finitely many flats.

6.1 Straightening simplices

If Z is a Hadamard space, there is a natural way to "straighten" singular simplices $\sigma : \Delta_k \longrightarrow Z$ (cf. [Thu]). Using the usual ordering on the vertices of the standard simplex, we define the straightened simplex $Str(\sigma)$ by "coning": if $Str(\sigma\big|_{\Delta_{k-1}})$ has been defined, then $Str(\sigma)$ is fixed by the requirement that on each segment joining $p \in \Delta_{k-1}$ with the vertex opposite Δ_{k-1} in Δ_k, $Str(\sigma)$ restricts to a constant speed geodesic. $Str(\sigma)$ lies in the convex hull of the vertices of σ. This straightening operation induces a chain equivalence on $C_*(Z)$. By using the geodesic homotopy between $Str(\sigma)$ and σ, one constructs a chain homotopy H from the chain map Str to the identity with the property that $Image(H(\sigma)) \subseteq ConvexHull(Image(\sigma))$ for any singular simplex σ.

When Z is the Euclidean building X, then it follows from lemma 4.6.8 that for every singular chain $c \in C_k(\text{Cone}(X))$, $Image(Str(c))$ is contained in finitely many apartments.

Corollary 6.1.1 *If $V \subseteq U \subseteq X$ are open sets, then $H_k(U, V) = 0$ for every $k > r = rank(X)$.*

Proof. If $[c] \in H_k(U, V)$, then after barycentrically subdividing if necessary, we may assume that the convex hull of every singular simplex in c (respectively ∂c) lies in U (respectively V). The straightened chain $Str(c)$ determines the same relative class as c since $Image(H(c)) \subset U$, $Image(H(\partial c)) \subset V$ and

$$Str(c) - c = \partial H(c) + H(\partial c).$$

But the straightened chain is carried by a finite union of apartments (corollary 4.6.8), which is a polyhedron of dimension $rank(X)$, so $[Str(c)] = [c] = 0$. □

Lemma 6.1.2 *Let Z be a regular topological space, and assume that $H_k(U_1, U_2) = 0$ for every pair of open subsets $U_2 \subseteq U_1 \subseteq Z$, $k > r$. If $Y \subseteq Z$ is a closed neighborhood retract and $U \subset Z$ is open, then the homomorphism $H_r(Y, Y \cap U) \longrightarrow H_r(Z, U)$ induced by the inclusion is a monomorphism. In particular, the inclusion $Y \longrightarrow Z$ induces a monomorphism $H_r(Y, Y - y) \to H_r(Z, Z - y)$ of local homology groups for every $y \in Y$.*

Proof. If $[c_1] \in H_r(Y, Y \cap U)$, then there is a compact pair $(K_1, K_2) \subseteq (Y, Y \cap U)$ and $[c_2] \in H_r(K_1, K_2)$ so that $i_*([c_2]) = [c_1]$ where $i : (K_1, K_2) \longrightarrow (Y, Y \cap U)$ is the inclusion. If $[c_1]$ is in the kernel of $H_r(Y, Y \cap U) \longrightarrow H_r(Z, U)$ then there is a compact pair $(K_1, K_2) \subseteq (K_3, K_4) \subseteq (Z, U)$ such that $j_*([c_2]) = 0$, where $j : (K_1, K_2) \longrightarrow (K_3, K_4)$ is the inclusion.

Let $r : V \longrightarrow Y$ be a retraction, where V is an open neighborhood of Y in Z. Choose disjoint open sets $W_1, W_2 \subset Z$ such that $Y - U \subseteq W_1$, $K_4 \subseteq W_2$; this is possible since $Y - U$ is closed, K_4 is compact, and Z is regular. Shrink V if necessary so that $r^{-1}(Y - U) \subset W_1$. We now have: $H_r(Y, Y \cap U) \longrightarrow H_r(V, r^{-1}(Y \cap U))$ is a monomorphism since r is a retraction; $H_r(V, r^{-1}(Y \cap U)) \longrightarrow H_r(V \cup W_2, r^{-1}(Y \cap U) \cup W_2)$ is an isomorphism by excision; $H_r(V \cup W_2, r^{-1}(Y \cap U) \cup W_2) \longrightarrow H_r(Z, r^{-1}(Y \cap U) \cup W_2)$ is a monomorphism by the exact sequence of the triple $(Z, V \cup W_2, r^{-1}(Y \cap U) \cup W_2)$ and $H_{r+1}(Z, V \cup W_2) = 0$. It follows that $[c_1] = 0$. □

6.2 The Local structure of support sets

Recall that X denotes a rank r Euclidean building. Let Y be a subset of a topological space Z. If $[c] \in H_k(Z, Y)$, then we define $Support(Z, Y, [c]) \subset Z - Y$ to be the set of points $z \in Z - Y$ such that the image of $[c]$ in the local homology group $H_k(Z, Z - \{z\})$ is nonzero. $Support(Z, Y, [c])$ is a closed subset in $Z - Y$, and contained in the image of the chain c.

Lemma 6.2.1 *Let B be a topologically embedded closed r-ball in X, Y a subset containing ∂B, and denote by μ the image of a generator of $H_r(B, \partial B)$ induced by the inclusion $(B, \partial B) \to (X, Y)$. Then $Support(X, Y, \mu) = B - Y$.*

Proof. We may apply lemma 6.1.2 since B is a closed (absolute) neighborhood retract. Therefore $Support(X, Y, \mu)$ coincides with $Support(B, B \cap Y, [B]) = B - Y$ where $[B]$ denotes the generator of $H_r(B, \partial B)$ which is mapped to μ. □

Now let U be an open subset of X and consider $[c] \in H_r(X, U)$. After subdividing the chain c if necessary, we may assume that the convex hull of each simplex of ∂c is contained in U, so that $[Str(c)] = [c]$. By 6.1, $c_1 = Str(c)$ is carried by a finite union of apartments $\mathcal{P}$, so $[c]$ is the image of $[c_1] \in H_r(\mathcal{P}, \mathcal{P} \cap U)$ under the inclusion $H_r(\mathcal{P}, \mathcal{P} \cap U) \longrightarrow H_r(X, U)$. Applying lemma 6.1.2 to the neighborhood retract $\mathcal{P}$, we find that the inclusion $Support(\mathcal{P}, \mathcal{P} \cap U, [c_1])$ in X coincides with $Support(X, U, [c])$. Hence we have reduced the problem of understanding $Support(X, U, [c])$ to a problem about supports in the finite polyhedron $\mathcal{P}$.

Recall that $\Sigma_p X$ has a thick spherical building structure with anisotropy polyhedron $\Delta_{mod}^{th}(p)$ (see section 4.2.2).

Lemma 6.2.2 *Pick $p \in \mathcal{P} \setminus \bar{U}$. When $\epsilon > 0$ is sufficiently small, $\log_p$ maps $Support(\mathcal{P}, \mathcal{P} \cap U, [c_1]) \cap B_p(\epsilon)$ isometrically to $(\cup_i C(C_i)) \cap B(\epsilon) \subset C(\Sigma_p X) = C_p X$, where the $C_i \subset \Sigma_p X$ are $\Delta_{mod}^{th}(p)$ chambers and $C(C_i) \subset C_p X$ is the cone over C_i.*

Proof. $\mathcal{P}$ is a finite union of apartments, so by corollary 4.4.3 when $\epsilon > 0$ is sufficiently small $\log_p$ maps $\mathcal{P} \cap B_p(\epsilon)$ isometrically to $(\cup_i C_p A_i) \cap B(\epsilon) \subset C_p X$, where the $A_i \subset \mathcal{P}$ are the apartments passing through p. We may assume that $U \subset X \setminus \overline{B_p(\epsilon)}$. Then $[c_1]$ determines a class $[c_2] \in H_r(\mathcal{P} \cap \overline{B_p(\epsilon)}, \mathcal{P} \cap \partial B_p(\epsilon))$. $\cup_i \Sigma_p A_i \subset \Sigma_p X$ has a polyhedral structure induced by the thick building atlas $\mathcal{A}^{th}(p)$, and this induces a polyhedral structure on the pair $(\mathcal{P} \cap \overline{B_p(\epsilon)}, \mathcal{P} \cap \partial B_p(\epsilon))$. The r-dimensional faces of this polyhedron are (truncated) cones over $\Delta_{mod}^{th}(p)$ chambers in the $\Delta_{mod}^{th}(p)$ subcomplex $\cup_i \Sigma_p A_i \subset \Sigma_p X$. Hence the lemma follows from elementary homology theory. □

Corollary 6.2.3 *If B is a topologically embedded r-ball in X, then for every $p \in X \setminus \partial B$ there are finitely many $\Delta_{mod}^{th}(p)$ chambers $C_i \subset \Sigma_p X$ so that $\log_p$ maps $B \cap B_p(\epsilon)$ isometrically to $(\cup_i C(C_i)) \cap B(\epsilon) \subset C_p X$ for sufficiently small $\epsilon > 0$.*

Proof. Let $\mu \in H_r(B, \partial B)$ be the relative fundamental class. Then $Support(X, \partial B, [\mu]) = B \setminus \partial B$ by lemma 6.2.1, and the corollary follows from lemma 6.2.2. □

6.3 The topological characterization of the link

If Z is a topological space and $z \in Z$, then we say that two subsets $S_1, S_2 \subset Z$ *have the same germ at z* if $S_1 \cap N = S_2 \cap N$ for some neighborhood N of z. The equivalence classes of subsets with the same germ at z will be denoted $Germ_z(Z)$.

Pick a point x in the rank r Euclidean building X. Consider the collection $\mathcal{S}_1(x)$ of germs of topological embeddings of $\mathbb{R}^r$ passing through $x \in X$. Let $\mathcal{S}_2(x)$ be the lattice of germs generated by $\mathcal{S}_1(x)$ under finite intersection and union.

Lemma 6.3.1 *The lattice $\mathcal{S}_2(x)$ is naturally isomorphic to the lattice $\mathcal{K}\Sigma_x X$ generated by the $\Delta^{th}_{mod}(x)$ faces of $\Sigma_x X$ under finite intersection and union.*

Proof. By lemma 6.2.2 we know that elements of $\mathcal{S}_1(x)$ correspond to finite unions of $\Delta^{th}_{mod}(x)$ chambers in $\Sigma_x X$. Intersections of $\Delta^{th}_{mod}(x)$ chambers yield $\Delta^{th}_{mod}(x)$ faces of $\Sigma_x X$, so we have a well defined map of lattices $\Xi : \mathcal{S}_2(x) \longrightarrow \mathcal{K}\Sigma_x X$ by taking each element of $\mathcal{S}_2(x)$ to its space of directions at x (which is a finite union of $\Delta^{th}_{mod}(x)$ faces). Ξ is injective by Corollary 4.4.3. The image of Ξ contains the apartments in $\mathcal{K}\Sigma_x X$, and since $(\Sigma_x X, \mathcal{A}^{th})$ is a thick spherical building every $\Delta^{th}_{mod}(x)$ face of $\Sigma_x X$ is an intersection of apartments, and hence Ξ is onto. □

6.4 Rigidity of homeomorphisms

In this section we prove the following results about homeomorphisms of Euclidean buildings:

Proposition 6.4.1 *A homeomorphism of Euclidean buildings carries apartments to apartments.*

Note that homeomorphic Euclidean buildings must have the same rank since the rank is the highest dimension where local homology groups don't vanish.

Theorem 6.4.2 *Let X, X' be thick Euclidean buildings with topologically transitive affine Weyl group and $\phi : Y = X \times \mathbb{E}^n \to Y' = X' \times \mathbb{E}^{n'}$ a homeomorphism. Then $n = n'$, and ϕ carries fibers of the projection $Y \to X$ to fibers of the projection $Y' \to X'$ inducing a homeomorphism $\bar{\phi} : X \to X'$.*

Theorem 6.4.3 *Let $X = \prod_{i=1}^k X_i$, $X' = \prod_{i=1}^l X'_i$ be thick Euclidean buildings with topologically transitive affine Weyl groups, and irreducible factors X_i, X'_j. Then a homeomorphism $\phi : X \to X'$ preserves the product structure.*

Theorem 6.4.4 *Let X, X' be irreducible thick Euclidean buildings with topologically transitive affine Weyl group, and suppose $rank(X) \geq 2$. Then any homeomorphism $X \longrightarrow X'$ is a homothety.*

6.4.1 The induced action on links

Let X, X' be Euclidean buildings, and let $\phi : X \longrightarrow X'$ be a homeomorphism. Pick a point x in X, and set $x' = \phi(x) \in X'$. The homeomorphism ϕ induces an isomorphism of lattices $\mathcal{S}_2(x) \to \mathcal{S}_2(x')$ (see section 6.3) and therefore a dimension preserving isomorphism $\mathcal{K}\phi_x : \mathcal{K}\Sigma_x X \longrightarrow \mathcal{K}\Sigma_{x'} X'$ of lattices. By proposition 3.8.1 the lattice isomorphism $\mathcal{K}\phi_x$ is induced by an isometry $\Sigma_x \phi : \Sigma_x X \to \Sigma_{x'} X'$.

6.4.2 Preservation of flats

Consider a singular k-flat F. Its germ at a point $x \in F$ is a subcomplex of $\mathcal{K}\Sigma_x X$. The image of this subcomplex L under $\mathcal{K}\phi_x$ is the subcomplex L' associated to the germ of $\phi(F)$ in $\mathcal{K}\Sigma_{\phi(x)} X$. L determines a standard $(k-1)$-sphere in $\Sigma_x X$. Since $\mathcal{K}\phi_x$ is induced by an isometry $\Sigma_x \phi : \Sigma_x X \to \Sigma_{\phi(x)} X$, L' determines a standard $(k-1)$-sphere in $\Sigma_{\phi(x)} X$. This sphere is the space of directions of a singular k-flat F'. $\phi(F)$ and F' coincide locally, because their germs coincide. Hence $\phi(F)$ is a complete simply-connected metric space which is locally isometric to Euclidean k-space $\mathbb{E}^k$. Therefore, $\phi(F)$ is isometric to $\mathbb{E}^k$.

6.4.3 Homeomorphisms preserve the product structure

Let X, X' be Euclidean buildings which decompose as products

$$X = \prod_{i=1}^{k} X_i, \qquad X' = \prod_{i=1}^{l} X'_i$$

of thick irreducible Euclidean buildings X_i, X'_j with almost transitive affine Weyl group. We have a corresponding decomposition of the spherical buildings $\Sigma_x X$ and $\Sigma_{x'} X'$ into joins of irreducible spherical buildings:

$$\Sigma_x X = \circ \Sigma_{x_i} X_i, \qquad \Sigma_{x'} X' = \circ \Sigma_{x'_i} X'_i$$

We recall that this metric join decomposition is unique, cfproposition 3.3.3, and therefore for each $x \in X$ the isometry $\Sigma_x \phi : \Sigma_x X \to \Sigma_{\phi(x)} X'$ decomposes as a join $\Sigma_x \phi = \circ \Sigma_x \phi_i$ of isometries $\Sigma_x \phi_i : \Sigma_{x_i} X_i \to \Sigma_{(\phi(x))_i} X'_{\sigma(i)}$ where σ is a permutation of $\{1, \dots, k\}$. In particular, X and X' have the same number of irreducible factors. We claim that the permutation σ is independent of the point x. To see this, note that any two points $y, z \in X$ lie in an apartment A and consider the map between apartments $\phi|_A : A \to \phi(A)$ (compare section 6.4.2). A parallel family of singular flats in A is carried by $\phi|_A$ to a continuous family of singular flats in $\phi(A)$; since there are only finitely many parallel families of singular subspaces, we conclude by continuity that $\phi\big|_A$ carries parallel singular flats to parallel singular flats. Consequently the permutation σ is independent of x as claimed. We assume without loss of generality that σ is the identity. Our discussion implies that a singular flat contained in a fiber of the projection $p_i : X \to X_i$ is carried by ϕ to a flat in a fiber of the projection $p'_i : X' \to X'_i$. Therefore each fiber of the projection $p_i : X \to X_i$ is carried by ϕ to a fiber of the projection $p'_i : X' \to X'_i$. Hence for each i there is a homeomorphism $\phi_i : X_i \to X'_i$ such that $\phi_i \circ p_i = p'_i \circ \phi$, and it follows that $\phi = \prod_i \phi_i$.

6.4.4 Homeomorphisms are homotheties in the irreducible higher rank case

Let X, X' be as in theorem 6.4.4. Let A be an apartment in X and consider the foliations of A by parallel singular hyperplanes. Since X is irreducible of rank r, we can pick out $r+1$ of these foliations $\mathcal{H}_0, \dots, \mathcal{H}_r$ such that the corresponding collection of roots is r-independent (i.e. every subset of r elements is linearly independent) (compare section 3.1). In fact, this property of the root system is equivalent to irreducibility.

The image of A under ϕ is an apartment A' and the foliations $\mathcal{H}_i$ are carried to foliations $\mathcal{H}'_i$ of A' by parallel singular hyperplanes. Note that these are also r-independent, since any r-fold intersection of mutually non-parallel hyperplanes belonging to these foliations is a point. Choose affine coordinates $x_1, \dots, x_r$ for A such that the leaves of $\mathcal{H}_0$ are level sets of $x_1 + \cdots + x_r$ and the leaves of the foliation $\mathcal{H}_i$ for $i \geq 1$ are level sets of x_i. Choose similar coordinates $x'_1, \dots, x'_r$ on the target A' so that $\phi(\{x_i = 0\}) = \{x'_i = 0\}$ and $\phi(\{\sum x_i = 1\}) = \{\sum x'_i = 1\}$. Consider those leaves in A which contain lattice points. Since ϕ maps leaves to leaves one sees by taking successive intersections of these leaves that ϕ carries lattice points to lattice points by a homomorphism. By the same reason ϕ induces a homomorphism on rational points and hence, by continuity, an $\mathbb{R}$-linear isomorphism.

We now know that $\phi\,|_A : A \to A'$ is an affine map preserving singular subspaces. Angles between singular subspaces are preserved, because the isomorphisms of simplicial complexes $\mathcal{K}\phi_x$ are induced by isometries. Hence the simplices $\{x_i \geq 0, \sum x_i \leq 1\}$ and $\{x'_i \geq 0, \sum x'_i \leq 1\}$ are homothetic and

ϕ is a homothety on A. By considering intersections of apartments one sees that the homothety factors are the same for all apartments. We conclude that ϕ is a homothety.

6.4.5 The case of Euclidean deRham factors

We now consider Hadamard spaces $X = Y \times \mathbb{E}^n$ where Y is a thick Euclidean building of rank $r - n$ with almost transitive affine Weyl group. Clearly lemma 4.6.7 continues to hold for X, and so do lemma 4.6.8 and the homological statements in section 6.1. Applying the reasoning from section 6.2 we conclude:

Lemma 6.4.5 *Every topologically embedded r-ball in X is locally a finite union $\cup_i C_i \times \mathbb{E}^n$ where the $C_i \subset Y$ are Weyl chambers.*

It follows that every closed subset of X which is homeomorphic to $\mathbb{E}^n$ is a union of deRham fibers, since its intersection with each fiber of $p : X \to Y$ is open and closed in this fiber. If $x \in X$, we may characterize the fiber of $p : X \to Y$ passing through x as the intersection of all closed subsets homeomorphic to $\mathbb{E}^n$ which contain x.

Now let $X' = Y' \times \mathbb{E}^{n'}$, where Y' is a thick building of rank $r' - n'$. If $\phi : X \to X'$ is a homeomorphism, then we have $r = r'$ by comparing local homology groups. Since the fibers of the projection maps $p : X \to Y$, $p' : X' \to Y'$are characterized topologically as above, we conclude that ϕ maps fibers of p homeomorphically onto fibers of p'; therefore $n = n'$ and ϕ induces a homeomorphism $\bar{\phi} : Y \to Y'$ of quotient spaces.

7 Quasiflats in symmetric spaces and Euclidean buildings

In this section, X will be a Hadamard space which is a finite product of symmetric spaces and Euclidean buildings. We have a unique decomposition

$$X = \mathbb{E}^n \times \prod_i X_i \tag{40}$$

where $n \in \mathbb{N}_0$ and the X_i are non-flat irreducible symmetric spaces or Euclidean buildings. The maximal Euclidean factor $\mathbb{E}^n$ is called the Euclidean deRham factor. An *apartment* is by definition a maximal flat and splits as a product of apartments in the factors. All apartments in X have equal dimension and it is called the *rank* of X. *Singular flats* are defined as products of singular flats in the factors. If the building factors are thick, then singular flats can be characterized as finite intersections of apartments. Note that the only singular flat in $\mathbb{E}^n$ is $\mathbb{E}^n$ itself and hence every singular flat in X is a union of deRham fibers

7.1 Asymptotic apartments are close to apartments

Proposition 7.1.1 *Let $\mathcal{Q}$ be a family of subsets in X with the property that for any sequence of sets $Q_n \in \mathcal{Q}$, base points $q_n \in Q_n$, and scale factors d_n with ω-$\lim d_n = \infty$, the ultralimit ω-$\lim_n(\frac{1}{d_n}Q_n, q_n)$ is an apartment in the asymptotic cone ω-$\lim_n(\frac{1}{d_n}X, q_n)$. Then there is a positive constant D so that any set $Q \in \mathcal{Q}$ is a D Hausdorff approximation of a maximal flat $F(Q)$ in X.*

Proof. Let us consider a single set Q in $\mathcal{Q}$ and choose a base point $q \in Q$. The ultralimit ω-lim$(\frac{1}{n}Q, q)$ is an apartment in the asymptotic cone ω-lim$(\frac{1}{n}X, q)$ which contains the base point $* := (q)$.

Step 1. We first show that Q is, in a sense to be made precise, quasi-convex in regular directions. Let $\overline{x_\omega y_\omega}$ be a regular segment in ω-lim$(\frac{1}{n}Q, q)$ which contains $*$ as interior point. $\overline{x_\omega y_\omega}$ is the ultralimit of a sequence of segments $\overline{x_n y_n}$ in X with endpoints $x_n, y_n \in Q$. There is a compact set $A \subset Int(\Delta_{mod})$ which contains the directions of ω-all segments $\overline{x_n y_n}$. Let F_n be a maximal flat containing the segment $\overline{x_n y_n}$. (F_n is unique for ω-all n.) Pick $\epsilon > 0$ so that $d(A, \partial\Delta_{mod}) > \epsilon$. Denote by D_n the diamond-shaped subset of all points $p \in F_n$ so that $\angle_{x_n}(p, y_n) \leq \epsilon$ and $\angle_{y_n}(p, x_n) \leq \epsilon$.

Sublemma 7.1.2 *There exists $r > 0$ so that for ω-all n the sets D_n are contained in the tubular r-neighborhood of Q.*

Proof. We prove this by contradiction: Choose a point $z_n \in D_n$ at maximal distance d_n from Q and assume that ω-lim $d_n = \infty$. Then the asymptotic cone ω-lim$(\frac{1}{d_n}X, z_n) = \text{Cone}(X)$ contains the apartments $F' := \omega\text{-lim}_n \frac{1}{d_n}F_n$ and $F'' := \omega\text{-lim}_n \frac{1}{d_n}Q$. The point $z_\omega = (z_n)$ is contained in F' but not in F'' and therefore F' and F'' are distinct apartments in Cone(X). Let $\overline{z_\omega x'_\omega}$ (respectively $\overline{z_\omega y'_\omega}$) be the ultralimits of the sequences of segments $\overline{z_n x_n}$ (respectively $\overline{z_n y_n}$). By the choice of the points z_n, the points x'_ω and y'_ω are contained in $F'' \cup \partial_\infty F''$. Since we can extend incoming geodesic segments in apartments according to 4.6.7, we may assume without loss of generality that $x'_\omega, y'_\omega \in \partial_\infty F''$. Let W_1 and W_2 be the Weyl chambers in Cone(X) centered at z_ω which are spanned by the rays $r_1 := \overline{z_\omega x'_\omega}$ and $r_2 := \overline{z_\omega y'_\omega}$. By the choice of ϵ and the definition of D_n, the rays r_1 and r_2 yield in the space of directions Σ_{z_ω} Cone(X) interior points of antipodal chambers. Consequently, the union $W_1 \cup W_2$ contains a regular geodesic c passing through z_ω. Since $\partial_\infty W_i \cap \partial_\infty F''$ contains the regular point $r_i(\infty)$, the chamber $\partial_\infty W_i$ is entirely contained in $\partial_\infty F''$. Thus the ideal endpoints $c(\pm\infty)$ of c are contained in $\partial_\infty F''$ and we conclude by 4.6.4 that $c \subset F''$ and hence $z_\omega \in F''$, a contradiction. □

Step 2. Suppose $q_n \in Q$ and ω-lim $\frac{1}{n}d(q, q_n) = 0$.

Sublemma 7.1.3 *ω-lim $d(q_n, D_n) < \infty$.*

Proof. The constant sequence q and the sequence q_n yield the same point in the ultralimit ω-lim $\left(\frac{1}{n}X, q\right)$, which is an interior point of ω-lim $\left(\frac{1}{n}D_n, q\right)$. Therefore

$$\omega\text{-lim} \frac{d(q_n, D_n)}{d(q_n, F_n \setminus D_n)} = 0. \tag{41}$$

If ω-lim $d(q_n, D_n) = \infty$, then $\dot{F} := \omega\text{-lim}\left(\frac{1}{d(q_n,D_n)}D_n, q_n\right) \subseteq \omega\text{-lim}\left(\frac{1}{d(q_n,D_n)}F_n, q_n\right)$ is a complete apartment in $\omega\text{-lim}\left(\frac{1}{d(q_n,D_n)}X, q_n\right)$ (by (41)) which lies at unit distance from ω-lim $q_n \in \left(\frac{1}{d(q_n,D_n)}Q, q_n\right)$, which is also an apartment in $\omega\text{-lim}\left(\frac{1}{d(q_n,D_n)}X, q_n\right)$. This contradicts corollary 4.6.4. □

We now know that there is a $r_1 > 0$ such that for every $R > 0$, $Q \cap B_q(R) \subset N_{r'}(D_n)$ for ω-all n, for otherwise we could produce a sequence contradicting sublemma 7.1.3[10].

Step 3. By steps 1 and 2, we know that there is an r_2 such that for every R, $Q \cap B_q(R)$ and $D_n \cap B_q(R)$ are r_2-Hausdorff close to one another for ω-all n.

[10] EXPLANATION OF THIS STATEMENT

Sublemma 7.1.4 *For every $R > 0$, $D_n \cap B_q(R)$ form an ω-Cauchy sequence*[11] *with respect to the Hausdorff metric.*

Proof. Suppose X is a symmetric space. Since for ω-all n the sets $D_n \cap B_q(R)$ have mutual Hausdorff distance $\leq 2r_2$, if the sublemma were false we could find Hausdorff convergent subsequences of $\{D_n\}$ with distinct limits. The limits would be distinct maximal flats lying at finite Hausdorff distance from one another, which is a contradiction.

If X is a Euclidean building, then failure of the sublemma would give sequences $k_n, l_n \to \infty$ and a radius R so that the Hausdorff distance between $D_{k_n} \cap B_q(R)$ and $D_{l_n} \cap B_q(R)$ remains bounded away from zero. Then the $\omega\text{-lim}(D_{k_n}, q)$ and $\omega\text{-lim}(D_{l_n}, q)$ are distinct apartments in the Euclidean building $\omega\text{-lim}(X, q)$ lying at finite Hausdorff distance from one another, contradicting corollary 4.6.4. $\square$

By the sublemma, $\omega\text{-lim}\, D_n \cap B_q(R)$ exists for all R (as an ω-limit of a sequence in the metric space of subsets of $B_q(R)$ endowed with the Hausdorff metric) and so we obtain a maximal flat $F \subset X$ with Hausdorff distance $\leq r_2$ from Q.

Step 4. We saw that each set Q in $\mathcal{Q}$ is the Hausdorff approximation of a maximal flat $F(Q)$. Denote by $d(Q)$ the Hausdorff distance of Q and $F(Q)$. Assume that there is a sequence of sets $Q_n \in \mathcal{Q}$ with $\lim d(Q_n) = \infty$. Choose base points $u_n \in X$ so that u_n is contained in one of the sets Q_n or $F(Q_n)$ but not in the tubular $d(Q_n)/2$-neighborhood of the other. Then the apartments $\omega\text{-lim}\, \frac{1}{d(Q_n)} Q_n$ and $\omega\text{-lim}\, \frac{1}{d(Q_n)} F(Q_n)$ have finite non-zero Hausdorff distance in the asymptotic cone $\omega\text{-lim}(\frac{1}{d(Q_n)} X, u_n)$. This contradicts 4.6.4. The proof of the proposition is now complete. $\square$

Corollary 7.1.5 *There is a positive constant $D_0 = D_0(L, C, X, X')$ such that for any (L, C)-quasi-isometry $\phi : X \to X'$ and any apartment A in X, the image $\phi(A)$ is a D_0-Hausdorff approximation of an apartment A' in X'.*

Proof. According to proposition 6.4.1, for any sequence of basepoints and any sequence of scale factors λ_k, the asymptotic cone Φ_ω of Φ carries apartments to apartments. We can apply proposition 7.1.1 to the collection $\mathcal{Q}$ of all images $\phi(A) \subseteq X'$ of apartments A in X. $\square$

7.2 The structure of quasi-flats

In this section X will be a symmetric space or a locally compact Euclidean building of rank r, with model polyhedron Δ_{mod}. Y will be an arbitrary Euclidean building with model polyhedron Δ_{mod}.

The goals of this section are:

Theorem 7.2.1 *For each (L, C) there is a ρ such that every (L, C) r-quasiflat $Q \subset X$ is contained in a ρ-tubular neighborhood of a finite union of maximal flats, $Q \subset N_\rho(\cup_{F \in \mathcal{F}} F)$ where $card(\mathcal{F}) < \rho$.*

and

Corollary 7.2.2 *The limit set of an (L, C) r-quasiflat $Q \subset X$ consists of finitely many Weyl chambers in $\partial_{Tits} X$; the number of chambers can be bounded by L and C.*

[11] A sequence x_n in a metric space X is ω-Cauchy if a subsequence with full ω-measure is Cauchy. If X is complete, then we define $\omega\text{-lim}\, x_n$ to be the limit of this subsequence.

Lemma 7.2.3 *Let $P \subset Y$ be a closed subset homeomorphic to $\mathbb{R}^r$. P is locally conical (by corollary 6.2.3), so it has a well-defined space of directions $\Sigma_p P$ for every $p \in P$. We have:*

1. *If $p \in P$ then every $v \in \Sigma_p Y$ has an antipode in $\Sigma_p P$.*

2. *If $w \in \Sigma_p P$, then there is a ray $\overline{p\xi} \subset P$, $\xi \in \partial_{Tits} Y$ such that $\overrightarrow{p\xi} = w$.*

Proof. Since P is locally a cone over a $\Sigma_p P$, we have $H_{r-1}(\Sigma_p P) \simeq \mathbb{Z}$, and the inclusion $\Sigma_p P \to \Sigma_p Y$ induces a monomorphism $H_{r-1}(\Sigma_p P) \to H_{r-1}(\Sigma_p Y)$ since $\Sigma_p Y$ is an $r-1$-dimensional spherical building. Now if the first claim weren't true, then $\Sigma_p P \subset \Sigma_p Y$ would lie inside the contractible open ball $B_v(\pi) \subset \Sigma_p Y$, making $H_{r-1}(\Sigma_p P) \to H_{r-1}(\Sigma_p Y)$ trivial.

The second claim now follows from the first by a continuity argument: w is the direction of a geodesic segment contained in P since P is locally conical, and a maximal extension of this segment must be a ray. □

Although we won't need the following corollary, we include it because its proof is similar in spirit to – but more transparent than – the proof of theorem 7.2.1.

Corollary 7.2.4 *If $P \subset Y$ is is bilipschitz to $\mathbb{E}^r$ then P is contained in a finite number of apartments. The number of apartments is bounded by the biLipschitz constant of P.*

Proof. Let $\alpha \in \Delta_{mod}$ be the barycenter of Δ_{mod}, and consider the collection of rays with Δ_{mod}-direction α contained in P. Since P is biLipschitz to $\mathbb{E}^r$, a packing argument bounds the number of equivalence classes of such rays (we know that the Tits distance between distinct classes of rays is bounded away from zero (cf. 4.1.2)). Let $\mathcal{S} \subset \partial_{Tits} Y$ be the (finite) set of Weyl chambers determined by this set of rays, and let $\mathcal{T}$ be the finite collection of flats in Y which are determined by pairs of antipodal Weyl chambers in $\mathcal{S}$. We claim that P is contained in $\cup_{F \in \mathcal{T}} F$. To see this, note that if $p \in P$ then by lemma 7.2.3 we can find a geodesic contained in P with Δ_{mod}-direction α which starts at p. This geodesic has ideal boundary points in $\mathcal{S}$, so by 4.6.3 the geodesic lies in $\cup_{F \in \mathcal{T}} F$. □

Another consequence of lemma 7.2.3 is

Corollary 7.2.5 *Pick $\alpha \in \Delta_{mod}$ and $L, C, \epsilon > 0$. Then there is a D such that if $Q \subset X$ is an (L, C) r-quasiflat, $y \in Q$, and $R > D$, then there is a $z \in Q$ with $\angle(\theta(\overline{yz}), \alpha) < \epsilon$, $|d(y,z) - R| < \epsilon R$.*

Proof. If not, then there is a sequence Q_k of quasiflats, $y_k \in Q_k$, and $R_k \to \infty$ such that for every $z_k \in Q_k$ with $|d(y_k, z_k) - R_k| < \epsilon R_k$ we have $\angle(\theta(\overline{y_k z_k}), \alpha) \geq \epsilon$. Taking the ultralimit of $\frac{1}{R_k} Q_k \subset \frac{1}{R_k} X$ we get $y_\omega \in Q_\omega \subset X_\omega$ and for every $z_\omega \in Q_\omega$ with $|d(y_\omega, z_\omega) - 1| < \epsilon$ we have $\angle(\theta(y_\omega z_\omega), \alpha) \geq \epsilon$. But this contradicts lemma 7.2.3 since Q_ω is biLipschitz to $\mathbb{E}^r$: we can pick $v \in \Sigma_{y_\omega} Q_\omega$ with $\theta(v) = \alpha$ and find a geodesic segment $\overline{y_\omega z_\omega} \subset Q_\omega$ with $\overrightarrow{y_\omega z_\omega} = v$, and for $\omega - all\ k$ z_k satisfies the conditions of the lemma. □

Lemma 7.2.3 implies that quasi-flats "spread out": a pair of points y_0, z_0 lying in a quasi-flat $Q \subset X$ can be extended to an almost collinear quadruple y_1, y_0, z_0, z_1 while maintaining the regularity of Δ_{mod}-directions. To deduce this we first prove a precise statement for Euclidean buildings.

Lemma 7.2.6 *Let $\alpha_1 \in \Delta_{mod}$ be a regular point, and let $\epsilon_1 > 0$. Then there is a $\delta_1 \in (0, \epsilon_1)$ with the following property. If $P \subset Y$ is a closed subset homeomorphic to $\mathbb{R}^r$ and $y_0, z_0 \in P$ satisfy $\angle(\theta(\overline{y_0 z_0}), \alpha_1) \leq \delta_1$, then there are points $y_1, z_1 \in P$ so that*

$$d(z_0, z_1) = d(y_0, y_1) = d(y_0, z_0) \tag{42}$$

$$\tilde{\angle}_{y_0}(y_1, z_0), \tilde{\angle}_{z_0}(y_0, z_1) > \pi - \epsilon_1 \tag{43}$$

$$\angle(\theta(\overline{y_1 z_1}), \alpha_1) < \delta_1 \tag{44}$$

The proof requires:

Sublemma 7.2.7 *Suppose $x, y, z \in Y$ and $\angle_x(y, z) = \max(D(\theta(\overline{xy}), \theta(\overline{xz}))$ (cf. 3.1). Then x, y, z are the vertices of a flat (convex) triangle and $\vec{yz} \in \Sigma_y Y$ lies on the segment joining $\vec{yx}$ to a point $v \in \Sigma_y Y$, where $\theta(v) = \theta(\overline{xz})$ and v and $\vec{yx}$ lie in a single chamber.*

Proof of sublemma 7.2.7: Extend the geodesic segments $\overline{xy}, \overline{xz}$ to geodesic rays $\overline{x\xi_1}$ and $\overline{x\xi_2}$, $\xi_i \in \partial_{Tits} X$. By hypothesis

$$\angle_x(y, z) = \max(D(\theta(\overline{xy}), \theta(\overline{xz}))) = \max(D(\theta(\xi_1), \theta(\xi_2))) = \angle_{Tits}(\xi_1, \xi_2).$$

So $x\xi_1\xi_2$ determine a flat convex sector S. Note that $\vec{yx}$ and $\vec{y\xi_2}$ lie in a single chamber of $\Sigma_y X$ since $\angle_y(x, \xi_2) = \pi - \angle_y(\xi_1, \xi_2) = \pi - \max D(\theta(\xi_1), \theta(\xi_2)) = \min D(Ant(\theta(\xi_1)), \theta(\xi_2)) = \min D(\theta(\vec{yx}), \theta(\xi_2))$. Hence Δxyz bounds a flat convex triangle $T \subset S$, and so $\vec{yz}$ lies on the geodesic segment which has endpoints $\vec{yx}$ and $\vec{y\xi_2}$. □

Proof of lemma 7.2.6: Pick $z_1 \in P$ so that $\overline{z_0 z_1} \subset P$, $d(z_0, z_1) = d(y_0, z_0)$, $\theta(\overline{z_0 z_1}) = \alpha$, and $\vec{z_0 z_1} \in \Sigma_{z_0} Y$ lies in a chamber antipodal to $\vec{z_0 y_0}$; similarly choose $y_1 \in P$ so that $\overline{y_0 y_1} \subset P$, $d(y_0, y_1) = d(y_0, z_0)$, $\theta(\overline{y_0 y_1}) = Ant(\alpha)$, and $\vec{y_0 y_1} \in \Sigma_{y_0} Y$ lies in a chamber antipodal to $\vec{y_0 z_0}$. Applying sublemma 7.2.7 we conclude that z_0, y_0, z_1, are the vertices of a flat convex triangle, and $\vec{y_0 z_1} \in \Sigma_{y_0} Y$ lies on the segment joining $\vec{y_0 z_0}$ to $v \in \Sigma_{y_0} Y$ where $\theta(v) = \theta(\overline{z_0 z_1}) = \alpha$ and v and $\vec{y_0 z_0}$ lie in the same chamber. In particular $\vec{y_0 z_1}$ and $\vec{y_0 y_1}$ lie in antipodal chambers of $\Sigma_{y_0} Y$, so applying lemma 7.2.7 again, we find that $\theta(\vec{y_1 z_1})$ lies on the segment joining $\theta(\overline{y_0 z_1})$ to $\theta(\overline{y_1 y_0}) = \alpha$. y_1 and z_1 clearly satisfy the stated conditions since $\tilde{\angle}_{y_0}(y_1, z_0) \geq \angle_{y_0}(y_1, z_0) = \pi - \angle(\theta(\overline{y_0 z_0}, \alpha) \geq \pi - \delta_1 > \pi - \epsilon_1$ and $\tilde{\angle}_{z_0}(y_0, z_1) \geq \angle_{z_0}(y_0, z_1) = \pi - \angle(\theta(\overline{y_0 z_0}, \alpha) \geq \pi - \delta_1 > \pi - \epsilon_1$. □

Corollary 7.2.8 *Let $\alpha_2 \in \Delta_{mod}$ be a regular point, and let $L, C, \epsilon_2 > 0$ be given. Then there are $D_2 > 0$, $\delta_2 \in (0, \epsilon_2)$ with the following property. If $Q \subseteq X$ is an (L, C) r-quasiflat, and $y_0, z_0 \in Q$ satisfy*

$$d(y_0, z_0) > D_2, \angle(\theta(\overline{y_0 z_0}), \alpha_2) \leq \delta_2 \tag{45}$$

then there are points $y_1, z_1 \in Q$ so that

$$|d(z_0, z_1) - d(y_0, z_0)|, |d(y_0, y_1) - d(y_0, z_0)| < \epsilon_2 d(y_0, z_0) \tag{46}$$

$$\tilde{\angle}_{y_0}(y_1, z_0), \tilde{\angle}_{z_0}(y_0, z_1) > \pi - \epsilon_2 \tag{47}$$

$$\angle(\theta(\overline{y_1 z_1}), \alpha_2) < \delta_2 \tag{48}$$

Proof. Let δ_2, λ_2 be the constants produced by the previous lemma with $\alpha_1 = \alpha_2$, $\epsilon_1 = \epsilon_2$. We claim that when $y_0, z_0 \in Q$ and $\angle(\theta(\overline{y_0 z_0}), \alpha_2) < \delta_2$ and $d(y_0, z_0)$ is sufficiently large, then there will exist points y_1, z_1 satisfying (46), (47), (48). But this follows immediately from the previous lemma by taking ultralimits. □

By applying corollary 7.2.8 inductively we get

Corollary 7.2.9 *With notation as in corollary 7.2.8, there are sequences $y_i, z_i \in Q$, $i \geq 1$ such that the inequalities (45), (46), (47), (48) hold when we increment all the indices on the y's and z's by i.*

Lemma 7.2.10 *Fix $\mu > 0$, and consider all configurations (y, z, F) where $y, z \in X$, $\angle(\theta(\overline{yz}), \partial\Delta_{mod}) \geq \mu$, and $F \subset X$ is a maximal flat. Then there is a D_3 such that the fraction of the segment $\overline{yz}$ lying outside the tubular neighborhood $N_{D_3}(F)$ tends to zero with $\nu(y, z, F) \stackrel{def}{=} \max\left(\frac{d(y,F)}{d(y,z)}, \frac{d(z,F)}{d(y,z)}\right)$.*

Proof. Recall that the distance function $d(F, \cdot)$ is convex, so if the lemma were false there would be sequences y_k, z_k, $w_k \in X$, $F_k \subset X$, with $\angle(\theta(\overline{y_k z_k}), \partial\Delta_{mod}) \geq \mu$, $d(y, z) \to \infty$, $w_k \in \overline{y_k z_k}$ with $d(w_k, y_k), d(w_k, z_k) > \epsilon d(y_k, z_k)$, $\nu_k(y_k, z_k, F_k) \to 0$ but $d(w_k, F_k) \to \infty$. Let p_k, q_k, $r_k \in F_k$ be the points nearest y_k, w_k, z_k respectively. By various triangle inequalities and property (39) from section 5.2 we have $\tilde{\angle}_{q_k}(p_k, y_k), \tilde{\angle}_{q_k}(r_k, z_k) \to 0$ and $\angle(\theta(\overline{p_k q_k}), \theta(\overline{y_k z_k}), \angle(\theta(\overline{q_k r_k}), \theta(\overline{y_k z_k})) \to 0$. Therefore if we set $R_k = d(q_k, w_k)$ and take the ultralimit of $(\frac{1}{R_k}X, q_k)$ we will get a configuration $q_\omega, w_\omega \in X_\omega$, an apartment $F_\omega \subset X_\omega$, and $\xi_1, \xi_2 \in \partial_{Tits} X_\omega$ so that q_ω is the point in F_ω nearest to w_ω, $(\overline{q_\omega \xi_1}, q_\omega) = \omega\text{-lim}(\overline{q_k p_k}, q_k)$, $(\overline{q_\omega \xi_2}, q_\omega) = \omega\text{-lim}(\overline{q_k r_k}, q_k)$, $(\overline{w_\omega \xi_1}, q_\omega) = \omega\text{-lim}(\overline{w_k y_k}, q_k)$, $(\overline{w_\omega \xi_2}, q_\omega) = \omega\text{-lim}(\overline{w_k z_k}, q_k)$. In particular, the rays $\overline{w_k \xi_1}$ and $\overline{w_k \xi_2}$ fit together to form the geodesic $\omega\text{-lim}\,\overline{y_k z_k}$ and $\angle(\theta(\xi_i), \partial\Delta_{mod}) \geq \mu$. But this contradicts corollary 4.6.4. □

Corollary 7.2.11 *Fix $\alpha_3 \in \Delta_{mod}$. Then there are constants ϵ_4, ν_4, D_4 such that if*

1. *$y_i, z_i \in X$, $i \geq 0$ are sequences which satisfy (45), (46), (47), (48) (when subscripts are incremented by i) with $\epsilon_2 < \epsilon_4$, $d(y_0, z_0) > D_4$.*
2. *A maximal flat $F \subset X$ satisfies $d(y_k, F), d(z_k, F) < \nu_4 d(y_k, z_k)$ for some k.*

Then $d(y_i, F), d(z_i, F) < \nu_4 d(y_i, z_i)$ for all $0 \leq i \leq k$.

Proof. If ν_4 is sufficiently small, then the trisection points $\tilde{y}, \tilde{z}$ of any sufficiently long segment $\overline{yz} \subset X$ with $\angle(\theta(\overline{yz}), \partial\Delta_{mod}) \geq \mu$, $\max\left(\frac{d(y,F)}{d(y,z)}, \frac{d(z,F)}{d(y,z)}\right) < \nu_4$ will satisfy $\max\left(\frac{d(\tilde{y},F)}{d(\tilde{y},\tilde{z})}, \frac{d(\tilde{z},F)}{d(\tilde{y},\tilde{z})}\right) \ll \nu_4$ by lemma 7.2.10. If we take $\epsilon_4 \ll \nu_4$ then $\angle(\theta(\overline{y_i z_i}), \partial\Delta_{mod})$ will be bounded away from zero and y_{i-1}, z_{i-1} will lie close to the trisection points of $\overline{y_i z_i}$ so corollary 7.2.11 follows by induction on $k - i$. □

Proof of theorem 7.2.1:
Step 1: Fix $\alpha_4 \in \Delta_{mod}$, and let ϵ_5, ν_5, D_5 be the constants produced by corollary 7.2.11 with $\alpha_3 = \alpha_4$. Let D_6, δ_6 be the constants given by corollary 7.2.8 with $\alpha_2 = \alpha_4$, $\epsilon_2 = \epsilon_5$. Finally, let D_7 be the constant produced by corollary 7.2.5 with $\alpha = \alpha_4$, $\epsilon = \min(\delta_6, \frac{1}{2})$. Setting $D_8 = \max(D_5, D_6, D_7)$, for each $y_0 \in Q$ we may find a $z_0 \in Q$ with $D_8 < d(y_0, z_0) < 2D_8$ so that $\angle(\theta(y_0, z_0), \alpha_4) < \delta_6$ (by corollary 7.2.5). By corollary 7.2.9 we may extend the pair $y_0, z_0 \in Q$ to a pair of sequences y_i, z_i

satisfying (45)-(48) with $\alpha_2 = \alpha_4$, $\epsilon_2 = \epsilon_5$. Then any maximal flat $F \subset X$ with $d(y_k, F), d(z_k, F) < \nu_5 d(y_k, z_k)$ for some $0 \leq k < \infty$ satisfies $d(y_i, F), d(z_i, F) < \nu_5 d(y_i, z_i)$ for all $0 \leq i \leq k$ by corollary 7.2.11; in particular

$$d(y_0, F) < \nu_5 d(y_0, z_0) < 2\nu_5 D_8 \tag{49}$$

We may assume in addition that ϵ_5 is small enough that

$$2d(y_{i-1}, z_{i-1}) < d(y_i, z_i) < 4d(y_{i-1}, z_{i-1}) \tag{50}$$

$$\text{and } d(y_i, y_{i-1}), d(z_i, z_{i-1}) < 2d(y_{i-1}, z_{i-1}). \tag{51}$$

It follows that

$$\max(d(y_i, y_0), d(z_i, y_0)) < 2d(y_i, z_i) \tag{52}$$

for all i.

Step 2: Fix $q \in Q$ and set $\nu_6 = \frac{\nu_5}{16}$. For each R pick a covering of $B_q(R) \cap Q$ by $\nu_6 R$-balls $\{B_{p_i}(\nu_6 R)\}$ with minimal cardinality; the cardinality of this covering can be bounded by r and the quasiflat constants (L, C). For each pair p_i, p_j of centers pick a maximal flat containing them, and denote the resulting collection of maximal flats by $\mathcal{F}_R$.

Claim: If $y_0 \in Q$, then $d(y_0, \cup_{F \in \mathcal{F}_R} F) < 2\nu_5 D_8$ for sufficiently large R.
Proof of claim: We will use the sequences y_i, z_i constructed in step 1 and estimate (49). Take the maximal i such that $y_i, z_i \in B_q(R)$. Then

$$\max(d(y_{i+1}, q), d(z_{i+1}, q)) > R$$

$$\Longrightarrow \max(d(y_{i+1}, y_0), d(z_{i+1}, y_0)) > R - d(q, y_0)$$

$$\Longrightarrow d(y_{i+1}, z_{i+1}) \geq \frac{1}{2}(R - d(q, y_0)) \text{ by (52)}$$

$$\Longrightarrow d(y_i, z_i) \geq \frac{1}{8}(R - d(q, y_0)) \text{ by (50).}$$

Since $\mathcal{F}_R$ contains a maximal flat F with

$$d(y_i, F), d(z_i, F) < \nu_6 R = \left(\frac{8\nu_6 R}{R - d(q, y_0)}\right) \cdot \frac{1}{8}(R - d(q, y_0))$$

$$\leq 8\nu_6 \left(\frac{R}{R - d(q, y_0)}\right) d(y_i, z_i)$$

$$\leq \frac{\nu_5}{2} \left(\frac{R}{R - d(q, y_0)}\right) d(y_i, z_i).$$

Therefore for sufficiently large R there is an $F \in \mathcal{F}_R$ and k such that $d(y_k, F), d(z_k, F) < \nu_5 d(y_k, z_k)$, so $d(y_0, F) < 2\nu_5 D_8$ as claimed. □
Proof of theorem 7.2.1 concluded: We may now take a convergent subsequence of the $\mathcal{F}_R$'s, and the limit collection $\mathcal{F}$ satisfies $Q \subset N_{2D_8}(\cup_{F \in \mathcal{F}} F)$ and $card(\mathcal{F}) \leq \limsup card(\mathcal{F}_R)$ which is bounded by r and (L, C). □

Proof of corollary 7.2.2: By theorem 7.2.1 there is a finite collection $\mathcal{F}$ of maximal flats so that Q lies in a finite tubular neighborhood of $\cup_{F \in \mathcal{F}} F$. The limit set of each $F \in \mathcal{F}$ is its Tits boundary

$\partial_{Tits}F$, which is an apartment of $\partial_{Tits}X$. The union of these apartments gives us a finite subcomplex $\mathcal{G} \subset \partial_{Tits}X$ which is a union of closed Weyl chambers.

Clearly $LimSet(Q) \subseteq \mathcal{G}$; we will show that if $\xi \in LimSet(Q)$ then ξ lies in a closed Weyl chamber $C \subset LimSet(Q)$. We have $q_k \in Q$ such that $\overline{\star q_k} \to \overline{\star\xi}$ in the pointed Hausdorff topology.

Consider $\cup_{F\in\mathcal{F}}F$. Any ultralimit $\omega\text{-lim}\left(\frac{1}{R_k}(\cup_{F\in\mathcal{F}}F),\star\right)$ is canonically isometric to the Euclidean cone over $\mathcal{G}$. $\omega\text{-lim}(\frac{1}{R_k}Q,\star)$ embeds in $\omega\text{-lim}\left(\frac{1}{R_k}(\cup_{F\in\mathcal{F}}F),\star\right)$ as a biLipschitz copy of $\mathbb{E}^r$; by the discussion in section 6.2 $\omega\text{-lim}(\frac{1}{R_k}Q,\star))$ is the cone over a collection of closed Weyl chambers in $\mathcal{G}$. In particular $\omega\text{-lim}\,\overline{\star q_k} = \overline{\star_\omega q_\omega}$ lies in a closed Weyl chamber contained in $\omega\text{-lim}(\frac{1}{R_k}Q,\star))$, so the corresponding Weyl chamber of $\mathcal{G}$ is contained in $LimSet(Q)$, and it contains ξ. $\square$

8 Quasi-isometries of symmetric spaces and Euclidean buildings

In this section our goal is to prove theorems 1.1.2 and 1.1.3 stated in the introduction.

Let X, X', and Φ be as in theorem 1.1.2. By corollary 7.1.5, Φ carries apartments close to apartments; in particular, X and X' have the same rank r.

8.1 Singular flats go close to singular flats

Lemma 8.1.1 *For any $R > 0$ there is an $D(R) > 0$ such that if F is a singular flat in X and $\mathcal{A}(F)$ is the collection of apartments containing F, then $\cap_{A\in\mathcal{A}(F)}N_R(A) \subset N_{D(R)}(F)$.*

Proof. It suffices to verify the assertion for irreducible non-flat spaces X.

Consider first the case that X is a symmetric space. The transvections along geodesics in F preserve all the flats containing F. Hence, if there is a sequence $x_n \in \cap_{A\in\mathcal{A}(F)}N_R(A)$ with $d(x_n, F)$ tending to infinity, then we may assume without loss of generality that the nearest point to x_n on F is a given point p. The segments $\overline{px_n}$ subconverge to a ray $\overline{p\xi}$ which lies in $\cap_{A\in\mathcal{A}(F)}N_R(A)$ and is orthogonal to F. Since for each apartment $A \in \mathcal{A}(F)$, we have $p \in A$ and the ray $\overline{p\xi}$ remains in a bounded neighborhood of A, it follows that $\overline{p\xi} \subset \cap_{A\in\mathcal{A}(F)}F$. Hence $\cap_{A\in\mathcal{A}(F)}F$ contains a $(k+1)$-flat, which is a contradiction.

Assume now that X is an irreducible thick Euclidean building with cocompact affine Weyl group. Consider a point $x \in X \setminus F$ and let $p \in F$ be the nearest point in F. Then $u := \overrightarrow{px} \in \Sigma_pX$ satisfies $\angle_p(u, \Sigma_pF) \geq \frac{\pi}{2}$. We pick a chamber C in Σ_pX containing u and choose a face σ of C at maximum distance from u. Denote by v the vertex of C opposite to σ. By our assumption, $diam(\Delta_{mod}) < \frac{\pi}{2}$ and therefore $v \notin \Sigma_pF$. Since F is a finite intersection of apartments, lemma 4.1.2 implies $\Sigma_pF = \cap_{A\in\mathcal{A}(F)}\Sigma_pA$ and there is an apartment A with $F \subset A \subset X$ and $v \notin \Sigma_pA$. Σ_pA is then disjoint from the open star of v, and so $d(u, \Sigma_pA) \geq d(u,\sigma) \geq \alpha_0 > 0$ where α_0 depends only on the geometry of Δ_{mod}. If $x \in N_R(A)$ then angle comparison implies that $d(x, F) \leq \frac{R}{\sin\alpha_0}$ and our claim holds with $D(R) = \frac{R}{\sin\alpha_0}$. This completes the proof of the lemma. $\square$

Proposition 8.1.2 *For every apartment $A \subset X$, let $A' \subset X'$ denote the unique apartment at finite Hausdorff distance from $\Phi(A)$. There are constants $D_0(L, C, X, X')$ and $D(L, C, X, X')$ so that if $F = \cap_{A\supseteq F}A \subset X$ is a singular flat, then*

1. *$\Phi(F) \subset \cap_{A\supset F} N_{D_0}(A')$,*

2. *The Hausdorff distance $d_H(\Phi(F), \cap_{A\supset F} N_{D_0}(A')) < D$,*

3. *There is a singular flat $F' \subset \cap_{A\supset F} N_{D_0}(A')$ with $d_H(\Phi(F), F') < D$.*

In particular, two quasi-isometries $\Phi_1, \Phi_2 : X \longrightarrow X'$ inducing the same bijection on apartments induce the same map of singular flats up to $2D$-Hausdorff approximation.

Proof. Let F and $\mathcal{A}(F)$ be as in the previous lemma. By corollary 7.1.5, for every apartment $A \subseteq X$, $\phi(A)$ is D_0-Hausdorff close to an apartment in X' which we denote by A'. Thus $\phi(F) \subset \cap_{A\in\mathcal{A}(F)} N_{D_0}(A')$.

Sublemma 8.1.3 *For each $d \geq D_0$ there exists a constant $D_1 = D_1(L, C, d) > 0$ with the property that $\cap_{A\in\mathcal{A}(F)} N_d(A')$ lies within Hausdorff distance D_1 from $\phi(F)$.*

Proof. Pick a quasi-inverse ϕ^{-1} of ϕ. For each point $y \in \cap_{A\in\mathcal{A}(F)} N_d(A')$ and each $A \in \mathcal{A}(F)$, $\phi^{-1}y$ is uniformly close to $\phi^{-1}A'$. But $\phi^{-1}A'$ is uniformly Hausdorff close to $\phi^{-1}\phi A$ and therefore to A. Lemma 8.1.1 implies that Y has uniformly bounded distance from F. □

Proof of proposition 8.1.2 continued. Fixing $A_0 \in \mathcal{A}(F)$, we conclude that $C := (\cap_{A\in\mathcal{A}(F)} N_{2D_0}(A')) \cap A'_0$ is a convex Hausdorff approximation of $\phi(F)$.

Sublemma 8.1.4 *Let $C \subset \mathbb{E}^l$ be a convex subset which is quasi-isometric to $\mathbb{E}^k$. Then C contains a k-dimensional affine subspace.*

Proof. Fix $q \in C$ and let $\hat{C} \subseteq C$ be the convex cone consisting of all complete rays starting in q and contained in C. For any sequence $\lambda_n \to 0$ of scale factors, the ultralimit ω-lim$(\lambda_n \cdot C, q)$ is isometric to $\hat{C}$. Therefore $\hat{C}$ is homeomorphic to $\mathbb{E}^k$ and hence isometric to $\mathbb{E}^k$. □

Proof of proposition 8.1.2 continued. It follows that $\phi(F)$ is uniformly close to a flat $\bar{F}$ in X'. Since ϕ_ω carries singular flats to singular flats, $\partial_{Tits}\bar{F}$ is a singular sphere in $\partial_{Tits}X'$. X' has cocompact affine Weyl group, so $\bar{F}$ lies within uniform Hausdorff distance from a singular flat F'. □

8.2 Rigidity of product decomposition and Euclidean deRham factors

We now prove theorem 1.1.2. The product decompositions of X and X' correspond to a decompositions of asymptotic cones

$$X_\omega = \mathbb{E}^n \times \prod_i X_{i\omega}, \qquad X'_\omega = \mathbb{E}^{n'} \times \prod_j X'_{j\omega} \tag{53}$$

where the $X_{i\omega}, X'_{j\omega}$ are irreducible thick Euclidean buildings. They have the property that every point is a vertex and their affine Weyl group contains the full translation subgroup, in particular the translation subgroup is transitive. We are in a position to apply theorems 6.4.2 and 6.4.3: The Euclidean deRham factors of X and X' have equal dimension, $n = n'$, and X, X' have the same number of irreducible factors. After renumbering the factors if necessary, there are homeomorphisms $(\phi_\omega)_i : X_{i\omega} \to X'_{i\omega}$ such that

$$(\phi_\omega)_i \circ p_{i\omega} = p'_{i\omega} \circ \phi_\omega$$

where $p_i : X \to X_i$ and $p'_i : X' \to X'_i$ are the projections onto factors. Now let F be a singular flat which is contained in a fiber of p_i. By proposition 8.1.2, $\phi(F)$ is uniformly Hausdorff close to a flat $F' \subset X'$. Since $F'_\omega \subset X'_\omega$ is contained in a fiber of $p'_{i\omega}$, F' must be contained in a fiber of p'_i. Any two points in a fiber $p_i^{-1}(x_i)$, $x_i \in X_i$, are contained in some singular flat $F \subset p_i^{-1}(x_i)$ and consequently ϕ carries fibers of p_i into uniform neighborhoods of fibers of p'_i. Since an analoguous statement holds for a quasi-inverse of ϕ, we conclude that ϕ carries p_i-fibers uniformly Hausdorff close to p'_i-fibers and so there are quasi-isometries $\phi_i : X_i \to X'_i$ so that

$$\phi \circ p_i = p'_i \circ \phi$$

holds up to bounded error. This concludes the proof of Theorem 1.1.2.

8.3 The irreducible case

In this section we prove theorem 1.1.3. Note that theorem 1.1.2 implies that X' is also ireducible, with $rank(X) = rank(X')$.

8.3.1 Quasi-isometries are approximate homotheties

We recall from proposition 7.1.5 that Φ carries each apartment A in X uniformly close to a unique apartment in X' which we denote by A'. We prove next that in our irreducible higher-rank situation the restriction of Φ to A can be approximated by a homothety. As a consequence, the quasi-isometry Φ is an almost homothety. This parallels the topological result in section 6.4.4.

Proposition 8.3.1 *There are positive constants $a = a(\Phi)$ and $b = b(L, C, X, X')$ such that for every apartment $A \subset X$ exists a homothety $\Psi_A : A \to A'$ with scale factor a which approximates $\Phi|_A$ up to pointwise error b.*

Proof. If we compose $\Phi\big|_A$ with the projection $X' \to A'$, we get a map $\Psi'_A : A \to A'$ which, according to proposition 8.1.2, carries walls to within bounded distance of walls. Parallel walls in A are carried to Hausdorff approximations of parallel walls in A'. Moreover, due to our assumption of cocompact affine Weyl group, each hyperplane parallel to a wall is carried to within bounded distance of a wall. By lemma 3.3.2 exist $r + 1$ singular half-spaces in A which intersect in a bounded affine r-simplex with non-empty interior. Consider the collection $\mathcal{C}$ of hyperplanes in A which are parallel to the boundary wall of one of these half-spaces. Any r pairwise non-parallel hyperplanes in $\mathcal{C}$ lie in general position, i.e. intersect in one point. Hence we may apply lemma 8.3.3 below to the collection $\mathcal{C}$ and conclude that Ψ'_A is within uniform finite distance of an affine transformation $\Psi_A : A \to A'$. Since Φ_ω is a homothety on asymptotic cones by the discussion in section 6.4.4, it follows that Ψ_A is a homothety: For suitable positive constants a_A and b we therefore have

$$|d(\Psi_A(x_1), \Psi_A(x_2)) - a_A d(x_1, x_2)| \leq b \quad \forall\, x_1, x_2 \in A$$

and b depends on L, C, X, X' but not on the apartment A. To see that the constant a_A is independent of the apartment A note that for any other apartment $A_1 \subset X$ there is a geodesic asymptotic to both A and A_1. It follows that $a_{A_1} = a_A$. □

Corollary 8.3.2 *There are positive constants $a = a(\Phi)$ and $b = b(L, C, X, X')$ such that the quasi-isometry $\Phi : X \longrightarrow X'$ satisfies*

$$|d(\Phi(x_1), \Phi(x_2)) - a \cdot d(x_1, x_2)| \leq b \qquad \forall\, x_1, x_2 \in X.$$

Here $L^{-1} \leq a \leq L$.

Proof. This follows from the previous proposition, because any two points in X lie in a common apartment. □

Lemma 8.3.3 *For $n \geq 2$, let $\alpha_0, \ldots, \alpha_n \in (\mathbb{R}^n)^*$ be a collection of linear functionals any n of which are linearly independent, and let $\mathcal{H}_i$ be the collection of affine hyperplanes $\{\alpha_i^{-1}(c)\}_{c\in\mathbb{R}}$. There is a function $D(C)$ with $\lim_{C\to 0} D(C) = 0$ satisfying the following: If $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a locally bounded map such that for all $H \in \mathcal{H}_j$, $\phi(H) \subset N_C(H')$ for some $H' \in \mathcal{H}_j$, then there is a an affine transformation ϕ_0 with scalar linear part which preserves the hyperplane families $\mathcal{H}_j$ such that $d(\phi, \phi_0) < D(C)$.*

Proof. After applying an affine transformation if necessary we may assume that $\alpha_0 = \sum_{i=1}^n x_i$, $\alpha_j = x_j$ for $1 \leq j \leq n$, and $\phi(0) = 0$. There is a $C_2 \in \mathbb{R}$ such that the image of each k-fold intersection of hyperplanes from $\cup_i \mathcal{H}_i$ lies within the C_2 neighborhood an intersection of the same type. In particular, for each $1 \leq j \leq n$, ϕ induces a (C_3, ϵ_3) quasi-isometry ϕ_j of the j^{th} coordinate axis, with $\phi_j(0) = 0$. It suffices to verify that each ϕ_j lies at uniform distance from a linear map since ϕ lies at uniform distance from $\prod_{j=1}^n \phi_j$. Also, it suffices to treat the case $n = 2$ since for each $1 \leq j \leq n$ we may consider the (C_4, ϵ_4)-quasi-isometry that ϕ induces on the $x_i x_j$ coordinate plane, and this satisfies the hypotheses of the lemma (with somewhat different constants).

We claim there is a C_5 such that for y, z in the first coordinate axis, we have $|\phi_1(y+z) - (\phi_1(y) + \phi_1(z))| < C_5$. To see this first note that when C equals zero the additivity can be deduced from a geometric construction involving 6 lines and 6 of their intersection points. When $C > 0$, the same construction can be performed with uniformly bounded error at each step.

By lemma 8.3.4 below, ϕ_1 and analgously ϕ_j lies at uniform distance from a linear map. □

Lemma 8.3.4 *Suppose $\psi : \mathbb{R} \to \mathbb{R}$ is a locally bounded function satisfying $|\psi(y+z) - \psi(y) - \psi(z)| \leq D$ for all $y, z \in \mathbb{R}$. Then $|\psi(x) - ax| \leq D$ for some $a \in \mathbb{R}$.*

Proof. Since $|\psi(2^n) - 2\psi(2^{n-1})| \leq D$, the sequence $(\frac{\psi(2^n)}{2^n})$ is Cauchy and converges to a real number a. Let $x > 0$ and choose numbers $q_n \in \mathbb{N}$ and $r_n \in \mathbb{R}$ with $|r_n| \leq x$ such that $2^n = q_n x + r_n$. Then

$$|\psi(2^n) - q_n\psi(x) - \psi(r_n)| \leq (q_n + 1)D$$

and hence, using that ψ is locally bounded,

$$|\underbrace{\frac{\psi(2^n)}{2^n}}_{\to a} x - \underbrace{\frac{q_n x}{2^n}}_{\to 1} \psi(x) - \underbrace{\frac{\psi(r_n)x}{2^n}}_{\to 0}| \leq \underbrace{\frac{(q_n+1)x}{2^n}}_{\to 1} D.$$

When n tends to infinity, we obtain in the limit

$$|ax - \psi(x)| \leq D.$$

Similarly, there is a real number a_- such that for all $x < 0$ we have $|a_-x - \psi(x)| \leq D$. Since $|\psi(x) + \psi(-x)| \leq D + |\psi(0)|$, it follows that $a = a_-$. □

Proof of theorem 1.1.3 concluded. By corollary 8.3.2 we may scale the metric on X' by the factor $\frac{1}{a}$ so that Φ becomes a $(1, \frac{A}{a})$ quasi-isometry. Applying proposition 2.3.9 we conclude that Φ induces a map $\partial_\infty \Phi : \partial_\infty X \longrightarrow \partial_\infty X'$ which is a homeomorphism of geometric boundaries preserving the Tits metric. By the main result of 3.7, $\partial_\infty \Phi$ gives an isomorphism of spherical buildings $\partial_\infty \Phi : (\partial_{Tits} X, \Delta_{mod}) \longrightarrow (\partial_{Tits} X', \Delta'_{mod})$, after possibly changing to an equivalent spherical building structure on $\partial_{Tits} X'$. Consequently, for every $\delta \in \Delta_{mod}$, $\partial_\infty \Phi$ maps the set $\theta^{-1}(\delta) \subset \partial_{Tits} X$ to the corresponding set $\theta'^{-1}(\delta) \subset \partial_{Tits} X'$, and $\Phi\big|_{\theta^{-1}(\delta)}$ is a cone topology homeomorphism. When δ is a regular point, the subsets $\theta^{-1}(\delta) \subset \partial_{Tits} X$ and $\theta'^{-1}(\delta) \subset \partial_{Tits} X'$ are either manifolds of dimension at least 1 or totally disconnected spaces by sublemma 4.6.9, depending on whether X and X' are symmetric spaces or Euclidean buildings. Therefore either X and X' are both symmetric spaces of noncompact type, or they are both irreducible Euclidean buildings with Moufang boundary. In the latter case we are done by theorem 8.3.9; when X and X' are both symmetric spaces we apply proposition 8.3.8 to get a homothety $\Phi_0 : X \longrightarrow X'$ with $\partial_\infty \Phi_0 = \partial_\infty \Phi$. By proposition 8.1.2, $d(\Phi(v), \Phi_0(v)) < D$ for every vertex $v \in X$, and since the affine Weyl group of X is cocompact the vertices are uniform in X, and so we have $d(\Phi, \Phi_0) < D'$. Hence Φ_0 is an isometry. □

8.3.2 Inducing isometries of ideal boundaries of symmetric spaces

We consider a symmetric space X of non-compact type and denote by G the identity component of its isometry group.

Sublemma 8.3.5 *Let $F \subset X$ be a maximal flat and let $\pi_F : X \to F$ be the nearest point retraction. Given a compact set $K \subset Int(\Delta_{mod})$ and $\epsilon > 0$, there is a $\delta > 0$ such that if $p \in X$, $x \in F$, $\theta(\overline{px}) \in K$, and $\angle_p(x, \pi_F(p)) > \frac{\pi}{2} - \delta$, then $d(p, F) < \epsilon$.*

Proof. Note that as q moves from p to $\pi_F(p)$ along the segment $\overline{\pi_F(p)p}$, $\angle_q(x, \pi_F(p))$ increases monotonically. If the sublemma were false, we could find a sequence $p_k \in X$, $x_k \in F$ so that $\angle_{p_k}(x_k, \pi_F(p_k)) \to \frac{\pi}{2}$ and $d(p_k, F) \geq \epsilon$. Since $\angle_{\pi_F(p_k)}(x_k, p_k) = \frac{\pi}{2}$, triangle comparison implies that $\frac{|p_k \pi_F(p_k)|}{|p_k x_k|} \to 0$. Hence by taking $q_k \in \overline{p_k \pi_F(p_k)}$ with $d(q_k, F) = \epsilon$ we have $\angle_{x_k}(p_k, q_k) \to 0$, so $d_{\Delta_{mod}}(\theta(\overline{q_k x_k}), K) \to 0$. Modulo the group G, we may extract a convergent subsequence of the configurations $(F, \overline{q_k x_k})$ getting a maximal flat F, a point q_∞ with $d(q_\infty, F) = \epsilon$, and $x_\infty \in \partial_\infty F$ such that $\angle_{q_\infty}(x_\infty, \pi_F(q_\infty)) = \frac{\pi}{2}$, and $\theta(x_\infty) \in K$. This is absurd. □

Sublemma 8.3.6 *Let F_i be a sequence of maximal flats in X so that $\partial_\infty F_i \to \partial_\infty F$ where F is a maximal flat, i.e. for each open neighborhood U of $\partial_\infty F$ in $\partial_\infty X$ with respect to the cone topology, $\partial_\infty F_i$ is contained in U for sufficiently large i. Then $F_i \to F$ in the pointed Hausdorff topology.*

Proof. Let $\xi, \eta \in \partial_\infty F$ be antipodal regular points and choose points $\xi_i, \eta_i \in \partial_\infty F_i$ so that $\xi_i \to \xi$ and $\eta_i \to \eta$. Then for $x \in F$ we have $\angle_x(\xi_i, \eta_i) \to \pi$ and consequently $\angle_x(\pi_{F_i} x, \xi_i) \to \frac{\pi}{2}$, $\angle_x(\pi_{F_i} x, \eta_i) \to \frac{\pi}{2}$. Applying sublemma 8.3.5, we conclude that $d(x, F_i) \to 0$. The claim follows since this holds for all $x \in F$. □

Lemma 8.3.7 *Let $\partial_\infty : G \to Homeo(\partial_\infty X)$ be the homomorphism which takes each isometry to its induced boundary homeomorphism. Then ∂_∞ is a topological embedding when $Homeo(\partial_\infty X)$ is given the compact-open topology.*

Proof. ∂_∞ is continuous, because the natural action of G on $\partial_\infty X$ is continuous. To see that ∂_∞ is a topological embedding, it suffices to show that if $g_i \in G$ is a sequence with $\partial_\infty(g_i) \to e \in Homeo(\partial_\infty X)$, then $g_i \to e \in G$. Let x be a point in X and choose finitely many (e.g. two) maximal flats $F_1, \ldots, F_k$ with $F_1 \cap \cdots \cap F_k = \{x\}$. Since $\partial_\infty(g_i) \to e \in Homeo(\partial_\infty X)$, $\partial_\infty g_i F_j$ converges to $\partial_\infty F_j$ in the sense that for each open neighborhood U_j of $\partial_\infty F_j$ in $\partial_\infty X$ with respect to the cone topology, $\partial_\infty g_i F_j$ is contained in U_j for sufficiently large i. By the previous sublemma we know that $g_i F_j \to F_j$ in the pointed Hausdorff topology. □

Proposition 8.3.8 *Let X and X' be irreducible symmetric spaces of rank at least 2. Then any cone topology continuous Tits isometry*

$$\psi : \partial_{Tits} X \to \partial_{Tits} X'$$

is induced by a unique homothety $\Psi : X \to X'$.

Proof. We denote by G (resp. G') the identity component of the isometry group of X (resp. X'). By lemma 8.3.7 the homomorphisms $\partial_\infty : G \to Homeo(\partial_\infty X)$ and $\partial'_\infty : G' \to Homeo(\partial_\infty X')$ are topological embeddings, where $Homeo(\partial_\infty X)$ and $Homeo(\partial_\infty X')$ are given the compact-open topology. According to [Mos, p.123, cor. 16.2], conjugation by ψ carries $\partial_\infty G$ to $\partial'_\infty G'$. Hence ψ induces a continuous isomorphism $G \to G'$. Such an isomorphism carries (maximal) compact subgroups to (maximal) compact subgroups and it is a classical fact that the induced map $\hat{\Psi} : X \to X'$ of the symmetric spaces is a homothety. ψ and the induced isometry $\partial_{Tits}\hat{\Psi}$ at infinity are G-equivariant with respect to the actions of G on $\partial_{Tits} X$ and $\partial_{Tits} X'$ and we conclude that $\partial_{Tits}\hat{\Psi} = \psi$. □

8.3.3 $(1, A)$-quasi-isometries between Euclidean buildings

Here we prove

Theorem 8.3.9 *Let X, X' be thick Euclidean buildings with Moufang Tits boundary, and assume that the canonical product decomposition of X has no 1-dimensional factors*[12]*. Then for every A there is a C so that for every $(1, A)$ quasi-isometry $\Phi : X \longrightarrow X'$ there is an isometry $\Phi_0 : X \longrightarrow X'$ with $d(\Phi, \Phi_0) < C$.*

The proof of theorem 8.3.9 combines corollary 7.1.5 and material from sections 3.12 and 4.10. We first sketch the argument in the case that X and X' are irreducible, of rank at least 2, and have cocompact affine Weyl groups.

Let (B, Δ_{mod}) be a spherical building. Attached to each root (i.e. half-apartment) in B is a *root group* $U_a \subseteq Aut(B, \Delta_{mod})$ (see 3.12). Remarkably, when B is irreducible and has rank at least 2, the U_a's – and consequently the group $G \subseteq Aut(B, \Delta_{mod})$ generated by them – act canonically and isometrically on any Euclidean building with Tits boundary B (see 4.10). Now let $(B, \Delta_{mod}) = (\partial_{Tits} X, \Delta_{mod})$. If $\Phi : X \longrightarrow X'$ is an (L, A) quasi-isometry, then by 2.3.9 we

[12]The statement is false for $(1, A)$ quasi-isometries between trees.

get an induced isometry $\partial_{Tits}\Phi : \partial_{Tits}X \longrightarrow \partial_{Tits}X'$, so the group $G \subseteq Aut(B, \Delta_{mod})$ acts on $\partial_{Tits}X$, $\partial_{Tits}X'$, and hence on X and X'. By comparing images of apartments (and using the quasi-isometry Φ), ones sees that a subgroup $K \subseteq G$ has bounded orbits in X iff it has bounded orbits in X'. Because B is Moufang (3.12) the maximal bounded subgroups $M \subset G$ pick out "spots" $v_M \in X$ and $\bar{v}_M \in X'$ (proposition 4.10.6), and the resulting 1-1 correspondence between the spots of X and the spots of X' determines a homothety $\Phi_0 : X \longrightarrow X'$ with $\partial_{Tits}\Phi_0 = \partial_{Tits}\Phi$.

Proof of theorem 8.3.9. Step 1: Reduction to the irreducible case.

Lemma 8.3.10 *Every $(1, A)$ quasi-isometry $\phi : \mathbb{E}^r \longrightarrow \mathbb{E}^r$ lies within uniform distance of a homothety.*

For every distance function $d : \mathbb{E}^r \longrightarrow \mathbb{E}^r$ the function $d \circ \phi$ lies within uniform distance of a distance function. By taking limits we see that for every Busemann function $b : \mathbb{E}^r \longrightarrow \mathbb{E}^r$, $b \circ \phi$ is uniformly close to a Busemann function. But the Busemann functions are affine functions, so ϕ is uniformly close to an affine map ϕ_0. Obviously ϕ_0 is an isometry. ☐

By corollary 7.1.5 , there is a constant $D(A, X, X')$ so that the image of every apartment $A \subset X$ is D Hausdorff close to an apartment $A' \subset X'$. Composing $\Phi\big|_A$ with the projection onto A' we get a map which is uniformly close to an isometry $\Psi_A : A \longrightarrow A'$. Hence if $F \subset A$ is a flat, then $\Phi(F) \subset X'$ is uniformly Hausdorff close to the flat $\Psi_A(F) \subset A'$. Therefore we may repeat the reasoning of 8.2 to see that if $X = \prod X_i$, $X' = \prod X'_j$ are the decompositions of X and X' into thick irreducible factors, then after reindexing the factors X'_j there are $(1, \bar{A})$ quasi-isometries $\Phi_i : X_i \longrightarrow X'_i$ so that Φ is uniformly close to $\prod \Phi_i$ ($\bar{A}$ depends only on the quasi-isometry constant of Φ and X, X'). Hence we are reduced to the irreducible case.

Step 2: X and X' are irreducible. The affine Weyl groups W_{aff}, W'_{aff} of X, X' are either finite or cocompact, since their Tits boundaries are irreducible. If W_{aff} is finite then it has a fixed point, so all apartments intersect in a point $p \in X$ and X is a metric cone over $\partial_{Tits}X$. If $\alpha \in \Delta_{mod}$ is a regular point, then $\theta^{-1}(\alpha) \subset \partial_{Tits}X$ is clearly discrete in the cone topology. On the other hand, if W_{aff} is cocompact then $\theta^{-1}(\alpha) \subset \partial_{Tits}X$ is nondiscrete since any regular geodesic ray $\overline{p\xi} \subset A$ can branch off at many singular walls. Since Φ induces a homeomorphism of geometric boundaries $\partial_\infty\Phi : \partial_\infty X \longrightarrow \partial_\infty X'$ by 2.3.9, and this induces an isomorphism of spherical buildings $\partial_{Tits}\Phi : \partial_{Tits}X \longrightarrow \partial_{Tits}X'$, either X and X' are both metric cones, or they both have cocompact affine Weyl groups. If they are both cones, we may produce an isometry $\Phi_0 : X \longrightarrow X'$ by taking the cone over $\partial_{Tits}\Phi : \partial_{Tits}X \longrightarrow \partial_{Tits}X'$. This induces the same bijection of apartments as Φ, and lies at uniform distance from Φ by lemma 8.3.10.

Step 3: X and X' are thick, irreducible, and have cocompact affine Weyl group. Letting $G \subset Aut(\partial_{Tits}X) \overset{\partial_{Tits}\Phi_*}{\simeq} Aut(\partial_{Tits}X')$ be the group generated by the root groups of $\partial_{Tits}X$, we get actions of G on $\partial_{Tits}X$, $\partial_{Tits}X'$, and by 3.12.2 actions on X and X' by automorphisms as well.

Lemma 8.3.11 *A subgroup $B \subset G$ has bounded orbits in X iff it has bounded orbits in X'.*

Proof. We show that if K has a bounded orbit $K(p) = \{gp \big| g \in K\} \subset X$ then K has a bounded orbit in X'.

Let $p \in X$ be a vertex, let $\mathcal{F}_p$ be the collection of apartments passing through p, and let $\mathcal{F}_{K(p)} = \cup_{g \in K}\mathcal{F}_{gp}$. $\mathcal{F}_{K(p)}$ is a K-invariant collection of apartments in X, and when $R > Diam(K(p))$

we have $p \in \cap_{A\in\mathcal{F}_{K(p)}} N_R(A)$. Let $\Phi(\mathcal{F}_p)$ and $\Phi(\mathcal{F}_{K(p)})$ denote the corresponding collections of apartments in X'. Then $\Phi(\mathcal{F}_{K(p)})$ is K-invariant, and $\Phi(p) \in \cap_{A'\in\Phi(\mathcal{F}_{K(p)})} N_{R+C_1}(A')$, where C_1 is a constant such that for every apartment $A \subset X$, the Hausdorff distance $d_H(\Phi(A), A') < C_1$. By proposition 8.1.2, $\cap_{A'\in\Phi(\mathcal{F}_p)} N_{R+C_1}(A')$ is bounded. Thus $\cap_{A'\in\Phi(\mathcal{F}_{K(p)})} N_{R+C_1}(A')$ is a nonempty K-invariant bounded set. □

Proof of theorem 8.3.9 continued. By proposition 4.10.6 we now have a bijection

$$Spot(\Phi) : Spot(X) \to Spot(X')$$

between spots in X and X' via their correpondence with maximal bounded subgroups in G. Moreover by item 2 of proposition 4.10.6 for every apartment $A \subset X$, we have $Spot(\Phi)(Spot(A)) = Spot(A')$ where $A' \subset X'$ is the unique apartment with $\partial_{Tits} A' = \partial_{Tits}\Phi(\partial_{Tits} A)$. Since by item 3 of proposition 4.10.6 $Spot(\Phi)\big|_{Spot(A)} : Spot(A) \to Spot(A')$ is a homeomorphism with respect to half-apartment topologies we see that X is discrete iff X' is discrete.
Case 1: Both X and X' are non-discrete, i.e. their affine Weyl groups have a dense orbit. In this case $Spot(A) = A$, $Spot(A') = A'$, and $Spot(\Phi)\big|_A : A \to A'$ is a homeomorphism since the half-apartment topology is the metric topology. By item 3 of proposition 4.10.6 $Spot(\Phi)\big|_A$ maps singular half-apartments $H \subset A$ with $\partial_{Tits} H = a$ to singular half-apartments $Spot(\Phi)(H) \subset A'$ with $\partial_{Tits}(Spot(\Phi)(H)) = \partial_{Tits}\Phi(a)$. By considering infinite intersections of singular half-apartments with Tits boundary $a \subset \partial_{Tits} A$, it follows that $Spot(\Phi)$ carries all half-spaces $H \subset A$ with $\partial_{Tits} H = a$ to half-spaces $Spot(\Phi)(H)$ with $\partial_{Tits}(Spot(\Phi)(H)) = \partial_{Tits}\Phi(a)$. By considering intersections of half-spaces $H_\pm$ with opposite Tits boundaries, we see that $Spot(\Phi)$ carries hyperplanes whose boundary is a wall $m \subset \partial_{Tits} A$ to hyperplanes in A' with boundary $\partial_{Tits}\Phi(m) \subset \partial_{Tits} A'$. By section 6.4.4 it follows that $\Phi_0 \overset{def}{=} Spot(\Phi) : X \to X'$ is a homothety and $\partial_{Tits}\Phi_0 = \partial_{Tits}\Phi$.
Case 2: X and X' are both discrete. In this case A and A' are crystallographic Euclidean Coxeter complexes; $Spot(A)$ and $Spot(A')$ coincide with the 0-skeleta of A and A'. Again by item 3 of proposition 4.10.6, if $S \subset A$ is either a singular subspace or singular half-apartment, then there is a unique singular subspace or singular half-apartment $S' \subset A'$ so that $Spot(\Phi)(S \cap Spot(A)) = S' \cap Spot(A')$. $k+1$ spots $s_0, \ldots, s_k \in Spot(A)$ are the vertices of a k-simplex in the simplicial complex iff they don't lie in a singular subspace of dimension $< k$ and the intersection of all singular half-apartments containing $\{s_0, \ldots, s_k\}$ contains the $k+1$ spots s_i. Hence $Spot(\Phi)\big|_{Spot(A)} : Spot(A) \to Spot(A')$ is a simplicial isomorphism and hence is induced by a unique homothety $A \to A'$. It follows that $Spot(\Phi) : Spot(X) \to Spot(X')$ is the restriction of a unique homothety $\Phi_0 : X \to X'$ with $\partial_{Tits}\Phi_0 = \partial_{Tits}\Phi$.

Since vertices are uniform in X, we may apply proposition 8.1.2 to conclude that in both cases $d(\Phi_0, \Phi) < D'(L, C, X, X')$, forcing Φ_0 to be an isometry.

□

9 A abridged version of the argument

In this appendix we offer an introduction to the proof of theorem 1.1.2 via the special case when $X = X' = \mathbb{H}^2 \times \mathbb{H}^2$.

Step 1: The structure of asymptotic cones ω-lim$(\lambda_i(\mathbb{H}^2 \times \mathbb{H}^2), z_i)$. Readers unfamiliar with asymptotic cones should read section 2.4. By 2.4.4, any asymptotic cone $\omega\text{-lim}(\lambda_i \mathbb{H}^2, x_i)$ is a $CAT(\kappa)$ space for every κ, so it is a metric tree; since there are large equilateral triangles centered at any point in $\mathbb{H}^2$, the metric tree branches everywhere. The ultralimit operation commutes with taking products, so one concludes that $\omega\text{-lim}(\lambda_i(\mathbb{H}^2 \times \mathbb{H}^2), z_i) \simeq \omega\text{-lim}(\lambda_i \mathbb{H}^2, x_i) \times \omega\text{-lim}(\lambda_i \mathbb{H}^2, y_i)$ where $z_i = x_i \times y_i$ and $\times$ denotes the Euclidean product of metric spaces. So any asymptotic cone of $\mathbb{H}^2 \times \mathbb{H}^2$ is a product of metric trees which branch everywhere.

Step 2: Planes in a product of metric trees are "locally finite". For $i = 1, 2$ let T_i be a metric tree. For simplicity we assume that geodesic segments and rays are extendible to complete geodesics. Since the convex hull of two geodesics in a metric tree is contained in the union of at most 3 geodesics, the convex hull of two 2-flats $\gamma_i \times \delta_i \subset T_1 \times T_2$ is contained in at most nine 2-flats. Section 6 may now be read up to the paragraph after lemma 6.2.1, replacing the word "apartment" with "2-flat", and corollary 4.6.8 with the observation above. Hence every topologically embedded plane in $T_1 \times T_2$ is locally contained in a finite number of 2-flats.

Step 3: Homeomorphisms of products of nondegenerate trees preserve the product structure. We now make the additional assumption that our metric trees T_i branch everywhere: for every $x \in T_i$, $T_i \setminus x$ has at least 3 components. Let $P \subset T_1 \times T_2$ be a topologically embedded plane, and let $z = x \times y \in P$. We know that there are finite trees $\bar{T}_i \subset T_i$ with $z \in \bar{T}_1 \times \bar{T}_2 \subset T_1 \times T_2$ so that $B_z(r) \cap P \subset B_z(r) \cap (\bar{T}_1 \times \bar{T}_2)$. Shrinking r if necessary, we may assume that $\bar{T}_1$ and $\bar{T}_2$ are cones ($x \in \bar{T}_1$ and $y \in \bar{T}_2$ are the only vertices). Elementary topological arguments using local homology groups show that $B_z(r) \cap P$ coincides with $B_z(r) \cap (\cup Q_i)$, where each $Q_i \subset \bar{T}_1 \times \bar{T}_2$ is a quarter plane with vertex at z, i.e. a set of the form $\gamma \times \delta \subset \bar{T}_1 \times \bar{T}_2$ where $\gamma \subset T_1$ (resp. $\delta \subset T_2$) is a geodesic leaving x (resp. y).

Say that two sets $S_1, S_2 \subset T_1 \times T_2$ have the same germ at z if $S_1 \cap U = S_2 \cap U$ for some neighborhood U of z. We see from the above that for every $z \in P$, P has the same germ at z as a finite union of quarter planes. Moreover, since the intersection of two quarter planes Q_1, Q_2 with vertex at z either has the same germ as Q_i, the same germ as a horizontal or vertical segment, or the same germ as $\{z\}$, it follows that a set $S \subset T_1 \times T_2$ has the germ of a quarter plane with vertex at z iff it has the same germ as a two-dimensional intersection of topologically embedded planes, and is *minimal* among such. Hence we have a topological characterization of 2-flats and vertical/horizontal geodesics: a closed, topologically embedded plane $P \subset T_1 \times T_2$ is a 2-flat if for every $z \in P$, P has the same germ at z a the union of four quarter planes with vertex at z; a closed connected subset $S \subset T_1 \times T_2$ is a vertical or horizontal geodesic if for every $z \in S$, S has the same germ at z as the boundary of two adjacent quarter planes with vertex at z. From this one may easily recover the product structure on $T_1 \times T_2$ using only the topology of $T_1 \times T_2$. Hence a homeomorphism $\phi : T_1 \times T_2 \longrightarrow T_1 \times T_2$ preserves the product structure (although it may swap the factors, of course).

Step 4: Quasi-isometries of $\mathbb{H}^2 \times \mathbb{H}^2$ preserve the product structure. Let $\Phi : \mathbb{H}^2 \times \mathbb{H}^2 \longrightarrow \mathbb{H}^2 \times \mathbb{H}^2$ be a quasi-isometry. If $z, z' \in \mathbb{H}^2 \times \mathbb{H}^2$, let $\theta(z, z')$ be the angle between the segment $\overline{zz'}$ and the horizontal direction.

Sublemma 9.0.12 *There is a function $f : [0, \infty) \longrightarrow \mathbb{R}$ with $\lim_{r \to \infty} f(r) = 0$ so that if z, z' are horizontal, then $|\theta(\Phi(z), \Phi(z')) - \frac{\pi}{4}| > \frac{\pi}{4} - f(r)$.*

Proof. If not, we could find a sequence $z_i, z_i' \in \mathbb{H}^2 \times \mathbb{H}^2$ of horizontal pairs so that $\frac{1}{\lambda_i} = d(z_i, z_i') = \infty$

and $\limsup_{i\to\infty} |\theta(\Phi(z_i), \Phi(z_i')) - \frac{\pi}{4}| < \frac{\pi}{4}$. Then $z_\omega, z_\omega' \in \omega\text{-lim}(\lambda_i(\mathbb{H}^2 \times \mathbb{H}^2), z_\omega)$ is a horizontal pair with $\theta(\Phi_\omega(z_\omega), \Phi_\omega(z_\omega')) \neq 0, \frac{\pi}{2}$. This contradicts step 3. □

Since any two horizontal pairs z_1, z_1' and z_2, z_2' may be joined with a continuous family z_t, z_t' of horizontal pairs with $\min d(z_t, z_t') \geq \min(d(z_1, z_1'), d(z_2, z_2'))$, we see that for horizontal pairs z, z', the limit $\lim_{d(z,z')\to\infty} \theta(\Phi(z), \Phi(z'))$ exists and is either 0 or $\frac{\pi}{2}$. We assume without losing generality that the former holds.

Hence as $y \in \mathbb{H}^2$ varies, the compositions $\mathbb{H}^2 \longrightarrow \mathbb{H}^2 \times \{y\} \xrightarrow{\Phi} \mathbb{H}^2 \times \mathbb{H}^2 \xrightarrow{p_1} \mathbb{H}^2$ are quasi-isometries with quasi-isometry constant independent of y, and they lie at finite distance from one another. It follows that they lie at uniform distance from one another, and so Φ preserves the fibers of p_1 up to bounded Hausdorff error. Repeating this argument for p_2 we see that Φ is within uniform distance of a product $\Phi_1 \times \Phi_2$ of quasi-isometries.

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