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RIGIDITY OF SINGULAR SCHUBERT VARIETIES IN $Gr(m, n)$

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Abstract

Let $\mathbf{a} = (p_1^{q_1}, \dots, p_r^{q_r})$ be a partition and $\mathbf{a}' = (p_1'^{q_1'}, \dots, p_r'^{q_r'})$ be its conjugate. We will prove that if $q_i, q_i' \geq 2$ for all $1 \leq i \leq r$, then any irreducible subvariety X of $Gr(m, n)$ whose homology class is an integral multiple of the Schubert class $[\sigma_{\mathbf{a}}]$ of type $\mathbf{a}$ is a Schubert variety of type $\mathbf{a}$.

1. Introduction

Let $Gr(m, n)$ be the Grassmannian of m-planes in $\mathbb{C}^n$. For a partition $\mathbf{a} = (a_1, \dots, a_m)$, a Schubert variety $\sigma_{\mathbf{a}}$ of type $\mathbf{a}$ is defined by the set of all m-planes E such that $\dim (E \cap \mathbb{C}^{n-m+i-a_i}) \geq i$ for all $1 \leq i \leq m$ for a flag $\{\mathbb{C}^1 \subset \dots \subset \mathbb{C}^n\}$. Then they form a basis for the homology space $H_*(Gr(m, n), \mathbb{Z})$.

For $\mathbf{a} = (p^q)^* = ((n-m)^{m-q}, (n-m-p)^q)$, the Schubert variety $\sigma_{\mathbf{a}}$ of type $\mathbf{a}$ is smooth and every smooth Schubert varieties in $Gr(m, n)$ is of this form. Smooth Schubert varieties in $Gr(m, n)$ are *Schur rigid* with trivial exceptions: for any smooth Schubert variety $\sigma_{\mathbf{a}}$ in $Gr(m, n)$ other than a non-maximal linear space, any irreducible subvariety whose homology class is an integral multiple of the Schubert class $[\sigma_{\mathbf{a}}]$ of type $\mathbf{a}$ is a Schubert variety of type $\mathbf{a}$ ([**W**], [**B**] and [**Ho**]).

In this paper, we will prove the Schur rigidity of singular Schubert varieties of certain types in $Gr(m, n)$.

Theorem. *Let* $\mathbf{a} = (p_1^{q_1}, \dots, p_r^{q_r})$, $p_r \neq 0$ *be a partition and let* $\mathbf{a}' = (p_1'^{q_1'}, \dots, p_s'^{q_r'})$, $p_r' \neq 0$ *be its conjugate. Then, the Schubert variety* $\sigma_{\mathbf{a}}$ *in* $Gr(m, n)$ *is Schur rigid if* $q_i, q_i' \geq 2$ *for all* $1 \leq i \leq r$.

The proof is divided by two parts as in [**W**], [**B**] and [**Ho**]: Schubert rigidity and the equality $\mathcal{B}_{\mathbf{a}} = \mathcal{R}_{\mathbf{a}^*}$. Schubert differential system $\mathcal{B}_{\mathbf{a}}$ is the differential system with a fiber at $x \in Gr(m, n)$ given by the set of all the tangent space of Schubert varieties of type $\mathbf{a}$ passing through x. If any irreducible integral variety of $\mathcal{B}_{\mathbf{a}}$ is a Schubert variety of type $\mathbf{a}$, then we say that the Schubert variety $\sigma_{\mathbf{a}}$ is *Schubert rigid*.

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Putting together the tangent space of all the subvarieties X with $[X] = r[\sigma_{\mathbf{a}}]$, $r \in \mathbb{Z}$ at each point, we get another differential system $\mathcal{R}_{\mathbf{a}^*}$, which we call the Schur differential system ([**B**], [**W**]). By the construction, if any irreducible integral variety of $\mathcal{R}_{\mathbf{a}^*}$ is a Schubert variety of type $\mathbf{a}$, then the Schubert variety $\sigma_{\mathbf{a}}$ is Schur rigid. Furthermore, the Schubert differential system $\mathcal{B}_{\mathbf{a}}$ is always contained in the Schur differential system $\mathcal{R}_{\mathbf{a}^*}$. Thus, the equality $\mathcal{B}_{\mathbf{a}} = \mathcal{R}_{\mathbf{a}^*}$ and the Schubert rigidity is a necessary and sufficient condition for the Schur rigidity ([**B**]). While proving the equality $\mathcal{B}_{\mathbf{a}} = \mathcal{R}_{\mathbf{a}^*}$ is computing integral elements of exterior differential systems, which is an algebraic problem, proving the Schubert rigidity is finding integral varieties of a differential system, which is a local differential geometric problem.

To prove the Schubert rigidity of a singular Schubert variety $\sigma_{\mathbf{a}}$, we use the fact that there is a natural foliation on $\sigma_{\mathbf{a}}$ by smooth Schubert varieties of type, say $\mathbf{b}$. But this does not say that any integral variety of $\mathcal{B}_{\mathbf{a}}$ is foliated by the Schubert varieties of type $\mathbf{b}$. We find a condition that is needed to get such a foliation by applying the theory on the integral varieties of F-structures by Goncharov (Section 3.2).

Once we get such a foliation on the integral variety of $\mathcal{B}_{\mathbf{a}}$, the space of leaves is again a Schubert variety of type, say $\mathbf{c}$, in the parameter space of the Schubert varieties of type $\mathbf{b}$. Then we can apply the induction to get the result. This inductive step forces us to consider the following problem: When $\sigma_{\mathbf{a}}$ is contained in a smooth Schubert variety of type $\mathbf{b}$, then will any integral variety of $\mathcal{B}_{\mathbf{a}}$ be contained in a Schubert variety of type $\mathbf{b}$? We obtain a condition which ensures such an inclusion by using the theory on the integral varieties of F-structures as above (Section 3.3).

To prove the equality $B_{\mathbf{a}} = R_{\mathbf{a}^*} \subset Gr(k, E^* \otimes Q)$ for the fibers of $\mathcal{B}_{\mathbf{a}}$ and $\mathcal{R}_{\mathbf{a}^*}$ at $[E] \in Gr(m, n)$, we use the description $R_{\mathbf{a}} = Gr(k, E^* \otimes Q) \cap \mathbb{P}(\mathbb{S}_{\mathbf{a}}(E^*) \otimes \mathbb{S}_{\mathbf{a}'}(Q))$ as in [**Ho**] and then we compute the complement of the tangent space of $B_{\mathbf{a}}$ in the tangent space of $Gr(k, E^* \otimes Q)$ by hands, while in [**Ho**] the theory of Lie algebra cohomology developed by Kostant used to compute it (Section 4).

One of the applications covered in [**B**] is the one to the holomorphic vector bundles generated by global sections which satisfies vanishing of certain Chern classes. It is related to Schubert varieties in $Gr(m, n)$ and from our result on the rigidity of Schubert varieties in $Gr(m, n)$, one can get the classification of such holomorphic bundles for various vanishing conditions.

Even though Grassmannian $Gr(m, n)$ we considered is a special Hermitian symmetric space, most of the method in this paper can be applied to prove the rigidity of singular Schubert varieties in general Hermitian symmetric space. The only thing to do is to verify the conditions in the above, which needs some works on the representation theory of simple

Lie algebra. We expect that the result in this paper will be generalized to Hermitian symmetric spaces.

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2. Differential systems

2.1. Schubert and Schur differential systems. Let $Gr(m,n)$ be the Grassmannian of m-dimensional subspaces of $V=\mathbb{C}^n$. Let $P(m,n)$ be the set

$$\{\mathbf{a}=(a_1,\dots,a_m)|\; n-m\geq a_1\geq\cdots\geq a_m\geq 0\}$$

of partitions. Fix a flag $\{V_\bullet\}$ of V with $\dim V_i=i$. For $\mathbf{a}\in P(m,n)$, define the *Schubert variety* $\sigma_{\mathbf{a}}(V_\bullet)$ *of type* $\mathbf{a}$ by the set

$$\{E\in Gr(m,n)|\;\dim(E\cap V_{n-m+i-a_i})\geq i\}.$$

Then $\sigma_{\mathbf{a}}(V_\bullet)$ is an irreducible subvariety of $Gr(m,n)$ of codimension $|\mathbf{a}|:=a_1+\cdots+a_m$. By varying the flag $\{V_\bullet\}$, we get a family of Schubert varieties $\sigma_{\mathbf{a}}$ of type $\mathbf{a}$.

For $\mathbf{a}\in P(m,n)$, define its *dual* $\mathbf{a}^*$ by

$$\mathbf{a}^*=(n-m-a_m,\dots,n-m-a_1)$$

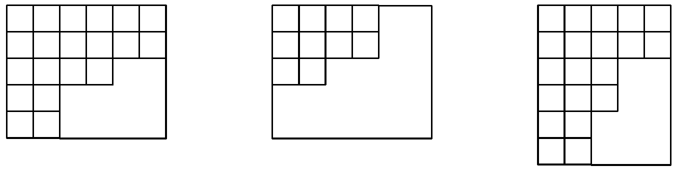
and define its *conjugate* $\mathbf{a}'=(a'_1,\dots,a'_{n-m})$ by

$$a'_i=\text{ the number of }\{j\,|a_j\geq i\}\text{ for }1\leq i\leq n-m.$$

The *Young diagram* $Y_{\mathbf{a}}$ is defined by the set of boxes consisting of a_i boxes in the i-th row, the row of boxes lined up on the left.

Then the Young diagram $Y_{\mathbf{a}^*}$ is obtained by rotating the complement of the Young diagram $Y_{\mathbf{a}}$ by 180 degree and the Young diagram $Y_{\mathbf{a}'}$ is obtained by transposing the Young diagram $Y_{\mathbf{a}}$.

Example. Let $m=5$, $n-m=6$. Young diagrams $Y_{\mathbf{a}},Y_{\mathbf{a}^*},Y_{\mathbf{a}'}$ for $\mathbf{a}=(6,6,4,2,2)$. $\mathbf{a}^*=(4,4,2)$, $\mathbf{a}'=(5,5,3,3,2,2)$ are given by



$Y_{\mathbf{a}}$ $Y_{\mathbf{a}^*}$ $Y_{\mathbf{a}'}$

Define $\mathfrak{n}_{\mathbf{a}}$ to be the vector space of matrices $Z = (z_i^a) \in M_{n-m,m}$ that satisfy $z_i^a = 0$ when $a > n - m - a_i$. This is the tangent space of the Schubert variety $\sigma_{\mathbf{a}}$.

For $\mathbf{a}$ as in the above Example, $\mathfrak{n}_{\mathbf{a}}$ is the spaces of all matrixes of the form

$$\begin{array}{ccccc} 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

We will use the notation $\mathbf{a} = (p_1^{q_1}, \dots, p_r^{q_r})$, $p_r \neq 0$ for the partition with q_1 p_1's, $\dots$, q_r p_r's.

Now, we will define two differential systems $\mathcal{B}_{\mathbf{a}}$ and $\mathcal{R}_{\mathbf{a}^*}$ associated to the Schubert variety $\sigma_{\mathbf{a}}$ following [**W**] and [**B**]. A *differential system* $\mathcal{F}$ on a manifold M is a subvariety of the Grassmannian bundle $Gr(k, TM)$ of tangent k-subspaces of M. So, it assigns a family $\mathcal{F}_x$ of k-subspaces of the tangent space T_xM at each point $x \in M$.

A subvariety X of M is said to be an integral variety of $\mathcal{F}$ if at each smooth point $x \in X$, its tangent space T_xX is contained in the fiber $\mathcal{F}_x$. We say that $\mathcal{F}$ is *integrable* if at each point $x \in M$ and $y \in \mathcal{F}_x$, there is an integral variety passing through x and tangent to the subspace W_y of T_xM corresponding to y.

A typical example of an integrable differential system is obtained from a family of subvarieties of M which covers M, by assigning at each point $x \in M$ the set of all tangent space of the subvarieties in the family passing through x.

Definition. For each $\mathbf{a} \in P(m, n)$, the *Schubert differential system* $\mathcal{B}_{\mathbf{a}}$ of type $\mathbf{a}$ is the differential system with a fiber consisting of the tangent space to the Schubert varieties $\sigma_{\mathbf{a}}$ of type $\mathbf{a}$ passing through a given point. We say that $\sigma_{\mathbf{a}}$ is *Schubert rigid* if Schubert varieties of type $\mathbf{a}$ are the only irreducible integral varieties of $\mathcal{B}_{\mathbf{a}}$.

Let $[E] \in Gr(m, n)$ and $Q = \mathbb{C}^n/E$. Then $T_{[E]}Gr(m, n) = E^* \otimes Q$ and $\wedge^k(E^* \otimes Q) = \oplus_{|\mathbf{a}|=k} \mathbb{S}_{\mathbf{a}}(E^*) \otimes \mathbb{S}_{\mathbf{a}'}(Q)$, where $\mathbb{S}_{\mathbf{a}}$ is the Schur functor of type $\mathbf{a}$. There exists a $SU(n)$-invariant positive (k, k)-form $\phi_{\mathbf{a}}$ which can be written as the sum $(\sqrt{-1})^{k^2}\sum_i \xi_i \wedge \overline{\xi}_i$ at $[E]$, where $\{\xi_i\}$ is an orthonormal basis of $\mathbb{S}_{\mathbf{a}}(E) \otimes \mathbb{S}_{\mathbf{a}'}(Q^*) \subset \wedge^k(T^*_{[E]}Gr(m,n))$. Then we have $\int_{\sigma_{\mathbf{a}^*}} \phi_{\mathbf{b}} = \delta^{\mathbf{b}}_{\mathbf{a}}$ for $\mathbf{a}, \mathbf{b} \in P(m,n)$. (For details, see [**B**].)

Definition. For each $\mathbf{a} \in P(m,n)$, the *Schur differential system* $\mathcal{R}_{\mathbf{a}}$ of type $\mathbf{a}$ is defined by the intersection

$$\cap_{\mathbf{b}\neq\mathbf{a},|\mathbf{b}|=|\mathbf{a}|} Z(\phi_{\mathbf{b}}),$$

where $Z(\phi_{\mathbf{b}})$ is the set of $|b|$-subspace of $T_E(Gr(m,n))$ on which $\phi_{\mathbf{b}}$ vanishes. We say that $\sigma_{\mathbf{a}}$ is *Schur rigid* if Schubert varieties of type $\mathbf{a}^*$ are the only integral varieties of $\mathcal{R}_{\mathbf{a}}$.

Thus $\mathcal{R}_{\mathbf{a}}$ is the differential system on $Gr(m,n)$ such that X is a subvariety of $Gr(m,n)$ with $[X] = r[\sigma_{\mathbf{a}^*}]$ for an integer r if and only if X is an integral variety of $\mathcal{R}_{\mathbf{a}}$([**B**], [**W**]). Clearly, $\mathcal{B}_{\mathbf{a}^*}$ is contained in $\mathcal{R}_{\mathbf{a}}$.

2.2. F-structures, prolongations and integral varieties. In this section, we review the theory on the integral varieties of F-structures ([**G**]).

Definition. Let F be a submanifold of $Gr(k,V)$ with a transitive action of a subgroup of $GL(V)$. A fiber bundle $\mathcal{F} \subset Gr(k,TM)$ on a manifold M of dimension $n = \dim V$ is said to be an *F-structure* if at each point $x \in M$ there is a linear isomorphism $\varphi(x) : V \to T_xM$ such that the induced map $\varphi(x)^k : Gr(k,V) \to Gr(k,T_xM)$ sends F to $\mathcal{F}_x$.

Schubert differential systems are F-structures for various F's. Integrability and the uniqueness can be obtained by studying the cohomology space $H^{k,1}(F)$ and $H^{k,2}(F)$ associated to (W_f, V, T_fF) for $f \in F$([**G**]). When an F-structure is integrable, if $H^{k,1}(F) = 0$, then the family of all integral varieties passing through a fixed point and tangent to a fixed subspace has dimension $\sum_{j\leq k-1} \dim H^{j,1}(F)$. In particular, if $H^{1,1}(F) = 0$, there is only one such integral subvariety. Higher cohomology gives the information on the higher jet of the integral varieties.

In general, Schubert differential system for singular Schubert variety has order ≥ 2, i.e. its integral varieties are determined by higher jets. But smooth Schubert varieties depend only 1-jet and there is a canonical map from the differential system of a singular Schubert variety to that of a certain smooth Schubert variety(Proposition 3.1). So, in this paper, we will consider only the first cohomology $H^{1,1}(F)$, which contains the information on the 2-jets of integral varieties.

Let F be a subvariety of $Gr(k,V)$ with a transitive action of a subgroup of $GL(V)$. Let $\mathcal{F}$ be an F-structure on M with the projection map $\pi : \mathcal{F} \to M$. For $x \in M$ and $y \in \mathcal{F}_x$, let W_y denote the k-subspace of T_xM corresponding to y. For a k-subspace H of $T_y\mathcal{F}$ such that $\pi_* : H \to W_y$ is an isomorphism, define $\partial H : \wedge^2 W_y \to T_xM/W_y$ by

$$\partial H(V_1, V_2) = [\tilde{V}_1, \tilde{V}_2] \mod W_y,$$

$\tilde{V}_i, i = 1, 2$ is a local vector field on M with $\tilde{V}_i(x') \in W_{\psi(x')}$ for a local section ψ of $\mathcal{F}$ with $\psi_*(W_y) = H$. It is well defined([**G**]).

A k-subspace H of $T_y\mathcal{F}$ is said to be a *2-jet of an integral variety* if π_* restricts to an isomorphism $H \subset T_y\mathcal{F} \to W_y \subset T_xM$ and $\partial H = 0$. Such an H is indeed a candidate for the 2-jets of actual integral varieties of $\mathcal{F}$ and the set of such an H is again a subvariety of $Gr(k, V + T_fF)$.

Proposition 2.1. *Let $\mathcal{F}$ be an F-structure on M. Let $x \in M$ and let $y \in \mathcal{F}_x$. If X is an integral variety of $\mathcal{F}$ passing through x and tangent to W_y, then the tangent space $H = T_y\tilde{X} \subset T_y\mathcal{F}$ of the lifting $\tilde{X} := \{(x, [T_xX]) | x \in X, [T_xX] \in Gr(k, T_xM)\}$ of X to $Gr(k, T_xM)$ satisfies*

(1) *$\pi_* : H \to W_y$ is an isomorphism*
(2) *$\partial H = 0$.*

Proof. See Chapter 1 of [**G**]. q.e.d.

The condition that $\pi_* : H \to W_y$ is an isomorphism is equivalent to the condition that H is the graph of a map $p : W_y \to T_y(\mathcal{F}_x) \subset W_y^* \otimes (T_xM/W_y)$. Define $\partial p : \wedge^2 W_y \to T_xM/W_y$ by $\partial p(V_1, V_2) = p(V_1)(V_2) - p(V_2)(V_1)$, considering $T_y(\mathcal{F}_x)$ as a subspace of $W_y^* \otimes (T_xM/W_y)$. Then, $\partial H = 0$ if and only if $\partial p = 0$.

Definition. Let $\mathcal{F}$ be an F-structure on M. The *first prolongation* $\mathcal{F}^{(1)}$ of $\mathcal{F}$ is defined by the union $\cup_{y\in\mathcal{F}}\mathcal{F}_y^{(1)}$ of the set of all 2-jets of integral varieties tangent to W_y for $y \in \mathcal{F}$

Put $H^{1,1}(F) = Ker(\partial : W_f^* \otimes T_fF \to \wedge^2 W_f^* \otimes (V/W_f))$ and put $F^{(1)} = \{H \subset V + T_fF | H \text{ is the graph of a map } p \in H^{1,1}(F)\}$, where $f \in F$. Then, $\mathcal{F}^{(1)}$ is an $F^{(1)}$-structure on $\mathcal{F}$.

If $H^{1,1}(F) = 0$, then $F^{(1)}$ is just a point and the first prolongation defines a distribution on $\mathcal{F}$. In this case, the integrability of this distribution is equivalent to the integrability of the F-structure $\mathcal{F}$. So, there is at most one integral manifold passing through a given point and tangent to a given k-subspace of the tangent space. For the details, see Chapter 1 of [**G**].

3. Schubert rigidity

3.1. Description of the Schubert differential system. For a sequence $(n_1, \ldots, n_r)$ with $n_1 \leq \cdots n_r \leq n$, denote by $F(n_1, \ldots, n_r, n)$ the flag space of all flags $(V_1, \ldots, V_r)$ in $V = \mathbb{C}^n$ with dim $V_i = n_i$ for $1 \leq i \leq r$.

Example. For $\mathbf{a} = (p^q)^*$, the Schubert variety $\sigma_{\mathbf{a}}$ of type $\mathbf{a}$ is the sub-Grassmannian $\{E \in Gr(m,n) | \mathbb{C}^{m-q} \subset E \subset \mathbb{C}^{m+p}\} \simeq Gr(q, p+q)$. The Schubert differential system $\mathcal{B}_{\mathbf{a}}$ is the flag space $F(m-q, m, m+p, n)$ and the parameter space of the family of Schubert varieties of type $\mathbf{a}$ is $F(m-q, m+p, n)$. In this case, there is a double fibration $F(m-q, m+p, n) \leftarrow \mathcal{B}_{\mathbf{a}} \rightarrow Gr(m,n)$.

In general, the Schubert differential systems $\mathcal{B}_{\mathbf{a}}$ is a generalized flag variety because $SL(n)$ acts on $\mathcal{B}_{\mathbf{a}}$ transitively and $\mathcal{B}_{\mathbf{a}}$ is compact. Thus, we have only to find the corresponding subset of simple root system generating the isotropy group which is parabolic.

Let $\mathcal{S} = \{\alpha_1, \dots, \alpha_{n-1}\}$ be the set of simple roots of $G = SL(n)$ and let P be the parabolic subgroup of $SL(n)$ generated by $\mathcal{S}^1 = \mathcal{S} - \{\alpha_m\}$, i.e. $\mathcal{S}^1$ is the set of simple roots of the semisimple part $SL(m) \times SL(n-m)$ of P. Then, $Gr(m,n)$ is expressed as G/P.

Proposition 3.1. *For* $\mathbf{a} = (p_1^{q_1}, \dots, p_r^{q_r}), p_r \neq 0$, *put*

$$\mathcal{S}_{\mathbf{a}} = \mathcal{S}^1 - \{\alpha_{q_1}, \dots, \alpha_{q_1 + \cdots + q_r}\} \cup \{\alpha_{n-p_1} \dots, \alpha_{n-p_r}\}.$$

Let $Q_{\mathbf{a}}$ *be the parabolic subgroup of* $SL(n)$ *generated by* $\mathcal{S}_{\mathbf{a}} \cup \{\alpha_m\}$ *and* $P_{\mathbf{a}}$ *be the parabolic subgroup of* $SL(m) \times SL(n-m)$ *generated by the set* $\mathcal{S}_{\mathbf{a}}$. *Then* $\mathcal{B}_{\mathbf{a}}$ *is the homogeneous manifold* $G/(Q_{\mathbf{a}} \cap P)$ *and* $P_{\mathbf{a}}$ *is the isotropy group of the action of* $SL(m) \times SL(n-m)$ *on the fiber* $B_{\mathbf{a}}$ *of* $\mathcal{B}_{\mathbf{a}} \rightarrow Gr(m,n)$. *If* $\mathcal{S}_{\mathbf{a}} \subset \mathcal{S}_{\mathbf{b}}$, *then there exists a quotient map* $\varphi_{\mathbf{a},\mathbf{b}} : \mathcal{B}_{\mathbf{a}} \rightarrow \mathcal{B}_{\mathbf{b}}$ *which preserves the fibers.*

For example, the following figure describes $\mathfrak{n}_{\mathbf{a}}$, $\mathfrak{m}_{\mathbf{a}}$ and $Q_{\mathbf{a}}$ for $n = 10, m = 4$ and $\mathbf{a} = (6,4,2,2)$, where $\mathfrak{m}_{\mathbf{a}}$ denotes the tangent space of $B_{\mathbf{a}}$. $\mathfrak{n}_{\mathbf{a}}$ is the space of all $n \times n$-matrix with non-zero elements only in $*$ and $\mathfrak{m}_{\mathbf{a}}$ is the space of all $n \times n$-matrix with non-zero elements only in $\bullet$. The Lie algebra of the reductive part of $Q_{\mathbf{a}}$ is the space of all $n \times n$-matrix with non-zero elements only in $\diamond$.

$\diamond$									
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		$*$	$*$	$\bullet$	$\bullet$	$\diamond$	$\diamond$		
		$*$	$*$	$\bullet$	$\bullet$	$\diamond$	$\diamond$		
				$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\diamond$	$\diamond$
				$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\diamond$	$\diamond$

3.2. Foliation by smooth Schubert varieties. In general, a Schubert variety which is Schubert rigid depends on higher jets (See Example 13 in [**B**] and our Theorem). But smooth Schubert varieties in $Gr(m,n)$ depends on only one jet, which will play a central role in proving the Schubert rigidity of singular Schubert varieties.

Proposition 3.2. *For $\mathbf{a} = (p^q)^*$, the Schubert variety $\sigma_{\mathbf{a}}$ of type $\mathbf{a}$ is Schubert rigid except when $(p = 1$ and $q \neq m)$ or $(p \neq n - m$ and $q = 1)$. Furthermore, there is a unique Schubert variety passing through a given point and tangent to a given tangent subspace.*

Proof. See [**W**] or [**Ho**]. q.e.d.

We will say that a Schubert variety $\sigma_{\mathbf{a}}$ is *strongly rigid* if there is a unique Schubert variety passing through a given point and tangent to a given tangent subspace.

To prove the Schubert rigidity of general Schubert varieties, we start with the simplest case and then will use the induction to generalize it.

Example. The case when $\mathbf{a} = (p^q)$ is studied in Example 13 and Remark 33 of [**B**]. Fix a $(n-m-p+q)$-subspace $\Lambda = \mathbb{C}^{n-m-p+q}$ of $\mathbb{C}^n$. Then, $\sigma_{\mathbf{a}}(\Lambda)$ can be expressed as a union of a family of Schubert varieties of type $\mathbf{b} = ((n-m)^q) = ((n-m)^{(m-q)})^*$.

$$\begin{aligned} \sigma_{\mathbf{a}}(\Lambda) &= \{E \in Gr(m,n) | \dim(E \cap \Lambda) \geq q\} \\ &= \bigcup_{\mathbb{C}^q \subset \Lambda} \{E \in Gr(m,n) | \mathbb{C}^q \subset E\} \end{aligned}$$

Note that $Gr(q,n)$ is the parameter space of the Schubert varieties of type $\mathbf{b}$ and $\{\mathbb{C}^q \in Gr(q,n) | \mathbb{C}^q \subset \Lambda\}$ is a Schubert variety of type $\mathbf{c} = ((n-m-p)^q)^*$ in $Gr(q,n)$.

$$\mathbf{a} = \begin{array}{|c|c||c|c|} \hline 0 & 0 & 0 & 0 \\ \hline \mathfrak{m}_{\mathbf{a}} & 0 & 0 & 0 \\ \hline\hline \mathfrak{n}_{\mathbf{a}} & \mathfrak{n}_{\mathbf{a}} & 0 & 0 \\ \hline 0 & \mathfrak{n}_{\mathbf{a}} & \mathfrak{m}_{\mathbf{a}} & 0 \\ \hline \end{array} \quad \mathbf{b} = \begin{array}{|c|c||c|c|} \hline 0 & 0 & 0 & 0 \\ \hline \mathfrak{m}_{\mathbf{b}} & 0 & 0 & 0 \\ \hline\hline 0 & \mathfrak{n}_{\mathbf{b}} & 0 & 0 \\ \hline 0 & \mathfrak{n}_{\mathbf{b}} & 0 & 0 \\ \hline \end{array} \quad \mathbf{c} = \begin{array}{|c||c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline\hline 0 & \mathfrak{m}_{\mathbf{c}} & 0 & 0 \\ \hline \mathfrak{n}_{\mathbf{c}} & 0 & 0 & 0 \\ \hline 0 & \mathfrak{m}_{\mathbf{c}} & 0 & 0 \\ \hline \end{array}$$

From this expression, we get the following desingularisation π_2 of $\sigma_{\mathbf{a}}(\Lambda)$.

$$\begin{array}{l} \mathcal{B}_{\mathbf{b}} = F(q,m,n) \xrightarrow{\pi_1} Gr(q,n) \supset Gr(q,\Lambda) \\ \quad \pi_2 \downarrow \\ \quad Gr(m,n) \supset \sigma_{\mathbf{a}}(\Lambda) \end{array}$$

Here, $\sigma_{\mathbf{a}}(\Lambda)$ is equal to $\pi_2(\pi_1^{-1}(Gr(q,\Lambda)))$ and $\pi_2 : \pi_1^{-1}(Gr(q,\Lambda)) \to \sigma_{\mathbf{a}}(\Lambda)$ is generically one-to-one. The smooth locus of $\sigma_{\mathbf{a}}(\Lambda)$ is foliated by Schubert varieties of type $\mathbf{b}$ in $Gr(m,n)$ and the space of leaves of this foliation is a Schubert variety of type $\mathbf{c}$ in $Gr(q,n)$.

In the same way as above, we get a different desingularisation of $\sigma_{\mathbf{a}}(\Lambda)$ by considering $\sigma_{\mathbf{a}}(\Lambda)$ as a union of a family of another type of Schubert varieties:

$$\sigma_{\mathbf{a}}(\Lambda) = \bigcup_{\Lambda \subset \mathbb{C}^{n-p}} \{E \in Gr(m,n) | E \subset \mathbb{C}^{n-p}\}.$$

Proposition 3.3. *For the partition $\mathbf{a} = (p^q)$, the Schubert variety $\sigma_{\mathbf{a}}$ of type $\mathbf{a}$ is rigid if $p > 1$ and $q > 1$.*

It is proved in Example 13 and Remark 33 of [**B**]. We will prove it again in such a way that can be generalized to the case of the Schubert differential systems of other singular Schubert varieties.

Lemma 3.4. *Let $\mathbf{a}$ and $\mathbf{b}$ be two partitions such that $\mathfrak{n}_{\mathbf{b}}$ is a subspace of $\mathfrak{n}_{\mathbf{a}}$ with $\mathcal{S}_{\mathbf{a}} \subset \mathcal{S}_{\mathbf{b}}$ (and thus, there is a projection $\varphi_{\mathbf{a},\mathbf{b}} : \mathcal{B}_{\mathbf{a}} \to \mathcal{B}_{\mathbf{b}}$ as in Proposition 3.1). Assume that $\mathcal{B}_{\mathbf{b}}$ is strongly rigid. For $p : \mathfrak{n}_{\mathbf{a}} \to \mathfrak{m}_{\mathbf{a}}$, define $\tilde{p}$ by the composition $\mathfrak{n}_{\mathbf{a}} \to \mathfrak{m}_{\mathbf{a}} \twoheadrightarrow \mathfrak{m}_{\mathbf{b}}$.*

If $\tilde{p}(v) = 0$ for all $v \in \mathfrak{n}_{\mathbf{b}}$ and for all $p \in H^{1,1}(B_{\mathbf{a}})$, then the smooth locus of any integral variety of $\mathcal{B}_{\mathbf{a}}$ is foliated by integral varieties of $\mathcal{B}_{\mathbf{b}}$, which are Schubert varieties of type $\mathbf{b}$.

Proof. By the strong rigidity of $\mathcal{B}_{\mathbf{b}}$, its first prolongation $\mathcal{B}_{\mathbf{b}}^{(1)}$ gives a distribution D on $\mathcal{B}_{\mathbf{b}}$ which is integrable because $\mathcal{B}_{\mathbf{b}}$ is integrable. Integral varieties of D are isomorphic to Schubert varieties of type $\mathbf{b}$ via the map $\pi_{\mathbf{b}} : \mathcal{B}_{\mathbf{b}} \to Gr(m,n)$.

Let X be an integral variety of $\mathcal{B}_{\mathbf{a}}$. Let $\tilde{X} \subset \mathcal{B}_{\mathbf{a}}$ be the lifting of X, that is, $\tilde{X} = \{(x, [T_x X]) | x \in X\}$. It suffices to show that $\varphi_{\mathbf{a},\mathbf{b}}(\tilde{X})$ is foliated by the integral varieties of the distribution D induced by $\mathcal{B}_{\mathbf{b}}^{(1)}$, that is, at each point $y \in \tilde{X}$, $(\varphi_{\mathbf{a},\mathbf{b}})_*(T_y \tilde{X})$ contains $D_{\varphi_{\mathbf{a},\mathbf{b}}(y)}$.

$$\begin{array}{ccc} & \mathcal{B}_{\mathbf{b}}^{(1)} & \\ & \downarrow & \\ \tilde{X} \subset \mathcal{B}_{\mathbf{a}} & \xrightarrow{\varphi_{\mathbf{a},\mathbf{b}}} & \mathcal{B}_{\mathbf{b}} \supset \varphi_{\mathbf{a},\mathbf{b}}(\tilde{X}) \\ & \searrow \swarrow & \\ & X \subset Gr(m,n) & \end{array}$$

The map $(\varphi_{\mathbf{a},\mathbf{b}})_*$ is given by the projection $\mathfrak{m} + \mathfrak{m}_{\mathbf{a}} \to \mathfrak{m} + \mathfrak{m}_{\mathbf{b}}$ and $T_y \tilde{X}$ is the graph of a map $p : \mathfrak{n}_{\mathbf{a}} \to \mathfrak{m}_{\mathbf{a}}$ with $\partial p = 0$. So, $(\varphi_{\mathbf{a},\mathbf{b}})_*(T_y \tilde{X})$ is the graph of the map $\mathfrak{n}_{\mathbf{a}} \xrightarrow{p} \mathfrak{m}_{\mathbf{a}} \to \mathfrak{m}_{\mathbf{b}}$ with $\partial p = 0$. On the other hand, $D_{\varphi_{\mathbf{a},\mathbf{b}}(y)}$ is the graph of the zero map $\mathfrak{n}_{\mathbf{b}} \to \mathfrak{m}_{\mathbf{b}}$. By the assumption, $\tilde{p}(v)$ is zero for all $v \in \mathfrak{n}_{\mathbf{b}}$, so $\varphi_{\mathbf{a},\mathbf{b}}(T_y \tilde{X})$ contains $D_{\varphi_{\mathbf{a},\mathbf{b}}(y)}$. q.e.d.

Consider the Schubert differential system $\mathcal{B}_{\mathbf{a}}$ on $Gr(m,n)$ for $\mathbf{a} = (p^q)$ and the Schubert differential system $\mathcal{B}_{\mathbf{c}}$ on $Gr(q,n)$ for $\mathbf{c} = ((n-m-p)^q)^*$.

Lemma 3.5. *For a subvariety $A \subset Gr(q,n)$, define a subvariety X_A of $Gr(m,n)$ by $X_A := \pi_2(\pi_1^{-1}(A))$. If X_A is an integral variety of $\mathcal{B}_{\mathbf{a}}$ on $Gr(m,n)$ and $\dim(\pi_1^{-1}(A))$ is equal to $\dim(X_A)$, then A is an integral variety of $\mathcal{B}_{\mathbf{c}}$ on $Gr(q,n)$.*

$$\begin{array}{lll} \mathcal{B}_{\mathbf{b}} = F(q,m,n) & \xrightarrow{\pi_1} & Gr(q,n) \supset A \\ \pi_2 \downarrow & & \\ Gr(m,n) \supset X_A & & \end{array}$$

Proof. We will follow the arguments in Example 16.6 of [**H**]. Define $\tilde{\pi}_i, i = 1,2$ to be the projection from $Gr(q,n) \times Gr(m,n)$ to the first and second component, respectively. Then $\pi_2(\pi_1^{-1}(A))$ is $\tilde{\pi}_2(\tilde{\pi}_1^{-1}(A) \cap F(q,m,n))$.

Let Γ be a smooth point in A. Then $\tilde{\pi}_1^{-1}(A)$ is smooth at (Γ, E) for all $E \in Gr(m,n)$ with the tangent space

$$T_{(\Gamma,E)}\tilde{\pi}_1^{-1}(A) = \left\{ (\eta,\varphi) | \begin{array}{ll} \eta : \Gamma \to \mathbb{C}^n/\Gamma, & \eta \in T_\Gamma A \\ \varphi : E \to \mathbb{C}^n/E, & \end{array} \right\}.$$

The tangent space of $F(q,m,n)$ at (Γ, E) is

$$T_{(\Gamma,E)}F(q,m,n) = \left\{ (\eta,\varphi) | \begin{array}{ll} \eta : \Gamma \to \mathbb{C}^n/\Gamma, & \\ \varphi : E \to \mathbb{C}^n/E, & \varphi|_\Gamma \equiv \eta \mod E \end{array} \right\}.$$

By dimension counting, we see that the two tangent spaces are transversal so that $\pi_1^{-1}(A) = \tilde{\pi}_1^{-1}(A) \cap F(q,m,n)$ is smooth at all (Γ, E) with $\Gamma \subset E$, and the tangent space of $\pi_1^{-1}(A)$ at (Γ, E) is given by

$$T_{(\Gamma,E)}\pi_1^{-1}(A) = \left\{ (\eta,\varphi) | \begin{array}{ll} \eta : \Gamma \to \mathbb{C}^n/\Gamma, & \eta \in T_\Gamma A \\ \varphi : E \to \mathbb{C}^n/E, & \varphi|_\Gamma \equiv \eta \mod E \end{array} \right\}.$$

For each $E \in X_A$, there are only finitely many $\Gamma \in A$ with $\Gamma \subset E$. If there are more than one $\Gamma \in A$ with $\Gamma \subset E$, then X_A is not smooth at E (Proposition 16.8 of [**H**]). Let E be an element in X_A such that there is only one $\Gamma \in A$ with $\Gamma \subset E$. Then, $\pi_2 : \pi_1^{-1}(A) \to X_A$ is one-to-one over E so X_A is smooth at E with the tangent space

$$T_E(X_A) = \{\varphi : E \to \mathbb{C}^n/E \mid \varphi|_\Gamma \equiv \eta \mod E,\ \eta \in T_\Gamma A, \}.$$

Since $T_E(X_A)$ is of type $\mathbf{a}$ in $Gr(m,n)$, $T_\Gamma A$ is of type $\mathbf{c}$ in $Gr(q,n)$.

q.e.d.

Proof of Proposition 3.3. We will show that for all $p : \mathfrak{n}_{\mathbf{a}} \to \mathfrak{m}_{\mathbf{a}}$ with $p \in H^{1,1}(B_{\mathbf{a}})$, the induced map $\tilde{p} : \mathfrak{n}_{\mathbf{b}} \subset \mathfrak{n}_{\mathbf{a}} \to \mathfrak{m}_{\mathbf{a}} \to \mathfrak{m}_{\mathbf{b}}$ is zero. Then, for an integral variety X of $\mathcal{B}_{\mathbf{a}}$, the space of leaves $A = \pi_1(\varphi_{\mathbf{a},\mathbf{b}}(\tilde{X})) \subset Gr(q,n)$ of the foliation on X given by Lemma 3.4 satisfies the conditions in Lemma 3.5. Thus, $A \subset Gr(q,n)$ is an integral variety of $\mathcal{B}_{\mathbf{c}}(Gr(q,n))$. Since $q > 1$ $\mathcal{B}_{\mathbf{c}}$ is rigid. So, A is the Schubert variety $Gr(q,\Lambda)$ for a

$(n-m-p+q)$-subspace Λ of $\mathbb{C}^n$ and hence, X is the Schubert variety $\sigma_{\mathbf{a}}(\Lambda)$.

Let $E_{i,j}$ be the $n \times n$-matrix with only one non-zero element in the i-th row and j-th column. Then, $[E_{i,j}, E_{k,\ell}] = \delta_{j,k}E_{i\ell} - \delta_{i,\ell}E_{k,j}$ for all $1 \leq i,j,k,\ell \leq n$.

Since $[p(X),Y]-[p(Y),X] \in \mathfrak{n}_{\mathbf{a}}$ for all $X,Y \in \mathfrak{n}_{\mathbf{a}}$ and $[\mathfrak{m}_{\mathbf{a}}/\mathfrak{m}_{\mathbf{b}}, \mathfrak{n}_{\mathbf{b}}] \subset \mathfrak{n}_{\mathbf{b}}$, $[\tilde{p}(X),Y]-[\tilde{p}(Y),X] \in \mathfrak{n}_{\mathbf{a}}$ for all $X,Y \in \mathfrak{n}_{\mathbf{b}}$. Assume that $q+1 \leq i,j \leq m$ and $m+1 \leq r,s \leq n-p$ and $n-p+1 \leq a,b \leq n$. Note that for a fixed a, if $X \in \mathfrak{m}_{\mathbf{b}}$ and $[X, E_{a,i}] = 0$ for all i, then $X = 0$.

0	0	0	0
$\mathfrak{m}_{\mathbf{b}}$	0	0	0
$*$	$E_{r,i}$	0	0
0	$E_{a,i}$	$\bullet$	0

From $[\tilde{p}(E_{a,i}), E_{r,j}] - [\tilde{p}(E_{r,j}), E_{a,i}] \in \mathfrak{n}_{\mathbf{a}}$, we get both $[\tilde{p}(E_{a,i}), E_{r,j}]$ and $[\tilde{p}(E_{r,j}), E_{a,i}]$ are contained in $\mathfrak{n}_{\mathbf{a}}$. Thus, $[\tilde{p}(E_{r,j}), E_{a,i}]$ is zero, which implies that $\tilde{p}(E_{r,j}) = 0$.

Since $[\tilde{p}(E_{a,i}), E_{b,j}] - [\tilde{p}(E_{b,j}), E_{a,i}]$ is contained in $\mathfrak{n}_{\mathbf{a}}$, it should be zero and thus, both $[\tilde{p}(E_{a,i}), E_{b,j}]$ and $[\tilde{p}(E_{b,j}), E_{a,i}]$ should be zero for $a \neq b$. Here, we use the condition that $p > 1$. So, $\tilde{p}(E_{b,j})$ is zero. q.e.d.

3.3. Sub-Grassmannians. To extend the result in Proposition 3.3 to the general case, we consider the following problem: Suppose that a Schubert variety $\sigma_{\mathbf{a}}$ is contained in a proper sub-Grassmannian of $Gr(m,n)$ and $\sigma_{\mathbf{b}}$ is the minimal sub-Grassmannian among them. Then, will any integral variety of $\mathcal{B}_{\mathbf{a}}$ be contained in a sub-Grassmannian $\sigma_{\mathbf{b}}$?

Proposition 3.6. *Let $\mathbf{a} = (p_1^{q_1}, \dots, p_r^{q_r}) \in P(m,n)$ be a partition and let $\mathbf{a}' = ({p'_1}^{q'_1}, \dots, {p'_r}^{q'_r})$ be the conjugate of $\mathbf{a}$. Suppose that $\mathfrak{n}_{\mathbf{a}}$ is contained in a proper rectangle in $\mathfrak{m}$. Let $\mathbf{b}$ be the partition corresponding to the minimal rectangle among them. Then any integral variety of $\mathcal{B}_{\mathbf{a}}$ is contained in a sub-Grassmannian $\sigma_{\mathbf{b}}$ except when $q_1 + \cdots + q_r = m$ and $q_r = 1$ or $q'_1 + \cdots + q'_r = n-m$ and $q'_r = 1$.*

If both $q_1 + \cdots + q_r < m$ and $q'_1 + \cdots + q'_r < n-m$ hold, then there is no proper sub-Grassmnnian containing $\sigma_{\mathbf{a}}$. So, the cases we will consider below are either when $q_1 + \cdots + q_r = m$ and $q_r \geq 2$ or when $q'_1 + \cdots + q'_r = n-m$ and $q'_r \geq 2$.

Lemma 3.7. *Let $\mathbf{a}$ and $\mathbf{b}$ be two partitions such that $\mathfrak{n}_{\mathbf{a}}$ is a subspace of $\mathfrak{n}_{\mathbf{b}}$ with $\mathcal{S}_{\mathbf{a}} \subset \mathcal{S}_{\mathbf{b}}$ and thus, there is a projection $\mathcal{B}_{\mathbf{a}} \to \mathcal{B}_{\mathbf{b}}$ as in Proposition 3.1. Assume that $\mathcal{B}_{\mathbf{b}}$ is strongly rigid. For $p : \mathfrak{n}_{\mathbf{a}} \to \mathfrak{m}_{\mathbf{a}}$, define $\tilde{p}$ by the composite map $\mathfrak{n}_{\mathbf{a}} \to \mathfrak{m}_{\mathbf{a}} \twoheadrightarrow \mathfrak{m}_{\mathbf{b}}$. If $\tilde{p} = 0$ for all $p \in H^{1,1}(B_{\mathbf{a}})$, then any integral subvariety of $\mathcal{B}_{\mathbf{a}}$ is contained in an integral variety of $\mathcal{B}_{\mathbf{b}}$, which is a Schubert variety of type $\mathbf{b}$.*

Proof. The proof is similar to the proof of Lemma 3.4. q.e.d.

Proof of Proposition 3.6. First, we consider the case $\mathbf{a} = (p_1^{q_1}, p_2^{q_2}, p_3^{q_3})$, $p_3 \neq 0$ is a partition with $q_1 + q_2 + q_3 = m$ and $q_3 \geq 2$. As the proof will show, the general case can be obtained in the same way.

Claim. Put $\mathbf{b} = (p_3^m)$.

$$\mathbf{a} = \begin{array}{|c|c|c||c|c|c|}
\hline
0 & 0 & 0 & 0 & 0 & 0 \\ \hline
\bullet & 0 & 0 & 0 & 0 & 0 \\ \hline
\bullet & \bullet & 0 & 0 & 0 & 0 \\ \hline\hline
0 & * & * & 0 & 0 & 0 \\ \hline
0 & 0 & * & \bullet & 0 & 0 \\ \hline
0 & 0 & 0 & \bullet & \bullet & 0 \\ \hline
\end{array}
\qquad
\mathbf{b} = \begin{array}{|c|c|c||c|c|c|}
\hline
0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 \\ \hline\hline
* & * & * & 0 & 0 & 0 \\ \hline
* & * & * & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & \bullet & \bullet & 0 \\ \hline
\end{array}$$

Then any integral variety of $\mathcal{B}_{\mathbf{a}}$ is contained in a Schubert variety of type $\mathbf{b}$.

Proof of the Claim. Let $p : \mathfrak{n}_{\mathbf{a}} \to \mathfrak{m}_{\mathbf{a}}$ be a map with $\partial p = 0$ Then $[p(X), Y] - [p(Y), X] \in \mathfrak{n}_{\mathbf{a}}$ for all $X, Y \in \mathfrak{n}_{\mathbf{a}}$. Put $\tilde{p} : \mathfrak{n}_{\mathbf{a}} \to \mathfrak{m}_{\mathbf{b}}$ to be the composition of p with the projection $\mathfrak{m}_{\mathbf{a}} \to \mathfrak{m}_{\mathbf{b}}$.

Since $[\mathfrak{m}_{\mathbf{a}}/\mathfrak{m}_{\mathbf{b}}, \mathfrak{n}_{\mathbf{a}}] \subset \mathfrak{n}_{\mathbf{b}}$, we have $[\tilde{p}(X), Y] - [\tilde{p}(Y), X] \in \mathfrak{n}_{\mathbf{b}}$ for all $X, Y \in \mathfrak{n}_{\mathbf{a}}$. Assume that $q_1 + 1 \leq k, \ell \leq q_1 + q_2$, $q_1 + q_2 + 1 \leq i, j \leq m$, $m+1 \leq r, s \leq n - p_1$ and $n - p_1 + 1 \leq a, b \leq n - p_2$. Note that for $X \in \mathfrak{m}_{\mathbf{b}}$, if $[X, E_{r,i}] = [X, E_{a,i}] = 0$ for all r and a for a fixed i, then X=0.

$$\begin{array}{|c|c|c||c|c|c|}
\hline
0 & 0 & 0 & 0 & 0 & 0 \\ \hline
\bullet & 0 & 0 & 0 & 0 & 0 \\ \hline
\bullet & \bullet & 0 & 0 & 0 & 0 \\ \hline\hline
0 & E_{r,k} & E_{r,i} & 0 & 0 & 0 \\ \hline
0 & 0 & E_{a,i} & \bullet & 0 & 0 \\ \hline
0 & 0 & 0 & \mathfrak{m}_{\mathbf{b}} & \mathfrak{m}_{\mathbf{b}} & 0 \\ \hline
\end{array}$$

From $[\tilde{p}(E_{r,k}), E_{s,i}] - [\tilde{p}(E_{s,i}), E_{r,k}] \in \mathfrak{n}_{\mathbf{b}}$, we see that both $[\tilde{p}(E_{r,k}), E_{s,i}]$ and $[\tilde{p}(E_{s,i}), E_{r,k}]$ are contained in $\mathfrak{n}_{\mathbf{b}}$. So, $[\tilde{p}(E_{r,k}), E_{s,i}] = 0$. The same equation holds if we replace $E_{s,i}$ by $E_{a,i}$. Thus, $\tilde{p}(E_{r,k}) = 0$.

Put $\mathbf{c} = ((n-m)^{(q_1+q_2)}, p_3^{q_3})$.

$$\mathbf{c} = \begin{array}{|c|c|c||c|c|c|}
\hline
0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 \\ \hline
\bullet & \bullet & 0 & 0 & 0 & 0 \\ \hline\hline
0 & 0 & * & 0 & 0 & 0 \\ \hline
0 & 0 & * & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & \bullet & \bullet & 0 \\ \hline
\end{array}$$

Then $\tilde{p}$ restricts to a map $\mathfrak{n}_{\mathbf{c}} \to \mathfrak{m}_{\mathbf{b}} \subset \mathfrak{m}_{\mathbf{c}}$. Also, we have $[\tilde{p}(X), Y] - [\tilde{p}(Y), X] \in \mathfrak{n}_{\mathbf{c}}$ for all $X, Y \in \mathfrak{n}_{\mathbf{c}}$. Since $q_3 \geq 2$, $\mathcal{B}_{\mathbf{c}}$ is strongly rigid so that $\tilde{p}$ is zero on $\mathfrak{n}_{\mathbf{c}}$. Hence, $\tilde{p}$ is zero. By Lemma 3.7, any integral variety of $\mathcal{B}_{\mathbf{a}}$ is contained in a Schubert variety of type $\mathbf{b}$.

In the same way, we can show that any integral manifold of $\mathcal{B}_{\mathbf{a}}$ is contained in a Schubert variety of type $\hat{\mathbf{b}}$ for $\hat{\mathbf{b}} = ({p'_3}^{n-m})$.

$$\mathbf{a} = \begin{array}{|c|c|c||c|c|c|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \bullet & 0 & 0 & 0 & 0 & 0 \\ \hline \bullet & \bullet & 0 & 0 & 0 & 0 \\ \hline\hline 0 & * & * & 0 & 0 & 0 \\ \hline 0 & 0 & * & \bullet & 0 & 0 \\ \hline 0 & 0 & 0 & \bullet & \bullet & 0 \\ \hline \end{array} \qquad \hat{\mathbf{b}} = \begin{array}{|c|c|c||c|c|c|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \bullet & 0 & 0 & 0 & 0 & 0 \\ \hline \bullet & 0 & 0 & 0 & 0 & 0 \\ \hline\hline 0 & * & * & 0 & 0 & 0 \\ \hline 0 & * & * & 0 & 0 & 0 \\ \hline 0 & * & * & 0 & 0 & 0 \\ \hline \end{array}$$

Here, we use the rigidity of the Schubert differential system $\mathcal{B}_{\hat{\mathbf{c}}}$ with $\hat{\mathbf{c}} = (m^{(q'_1+q'_2)}, {p'_3}^{q'_3}), q'_3 \geq 2$.

$$\hat{\mathbf{c}} = \begin{array}{|c|c|c||c|c|c|}\hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \bullet & 0 & 0 & 0 & 0 & 0 \\ \hline \bullet & 0 & 0 & 0 & 0 & 0 \\ \hline\hline 0 & * & * & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \bullet & 0 & 0 \\ \hline 0 & 0 & 0 & \bullet & 0 & 0 \\ \hline \end{array}$$

q.e.d.

Theorem 3.8. *Let $\mathbf{a} = (p_1^{q_1}, \dots, p_r^{q_r})$ be a partition and let $\mathbf{a}' = ({p'_1}^{q'_1}, \dots, {p'_r}^{q'_r})$ be its conjugate. Then, $\sigma_{\mathbf{a}}$ is Schubert rigid if $q_i, q'_i \geq 2$ for all $i \leq r$.*

Proof. We will use the induction on r. Thanks to Proposition 3.6, we may assume that there is no proper sub-Grassmannian containing $\sigma_{\mathbf{a}}$ so that $q_1 + \cdots q_r < m$ and $q'_1 + \cdots q'_r < n - m$.

By Lemma 3.4 and the same argument as in Proposition 3.3, any integral varieties of $\mathcal{B}_{\mathbf{a}}$ are foliated by the Schubert varieties of type $\mathbf{b} = ((n-m)^q)$, where $q = q_1 + \cdots q_r$. Then the space of leaves A of this foliation will be an integral variety of the Schubert differential system $\mathcal{B}_{\mathbf{c}}$ on $Gr(q,n)$, $\mathbf{c} = ((p_1 + (m-q))^{q_1}, \dots, (p_r + (m-q))^{q_r})$. By Proposition 3.6, any integral varieties of $\mathcal{B}_{\mathbf{c}}$ are contained in a sub-Grassmannian of type $((p_r + (m-q))^q)$ in $Gr(q,n)$. Thus, A is an integral variety of the Schubert differential system $\mathcal{B}_{\mathbf{d}}$, $\mathbf{d} = ((p_1 - p_r)^{q_1}, \dots, (p_{r-1} - p_r)^{q_{r-1}})$ on this sub-Grassmannian, which is rigid by the induction hypothesis.

q.e.d.

4. Schur rigidity

Let $\sigma_{\mathbf{a}}$ be a Schubert variety of type $\mathbf{a}$ in $Gr(m,n)$. If $\sigma_{\mathbf{a}}$ is Schubert rigid and the Schur differential system $\mathcal{R}_{\mathbf{a}^*}$ is equal to the Schubert differential system $\mathcal{B}_{\mathbf{a}}$, then $\sigma_{\mathbf{a}}$ is Schur rigid. In the previous section, we proved the Schubert rigidity of $\sigma_{\mathbf{a}}$ under some assumptions on $\mathbf{a}$ (Theorem 3.8). In this section, we will prove that $\mathcal{B}_{\mathbf{a}}$ is equal to $\mathcal{R}_{\mathbf{a}^*}$ under a weaker assumption (Proposition 4.3) and that this assumption is critical (Remark 2).

4.1. Criterions for the equality $B_{\mathbf{a}} = R_{\mathbf{a}^*}$. Let $\mathbf{a}$ be a partition in $P(m,n)$ and $\mathbf{b}$ be a partition with $|\mathbf{b}| = |\mathbf{a}|$. Since $\phi_{\mathbf{b}}$ is equal to $(\sqrt{-1})^{|\mathbf{b}|^2}\sum_i \xi_i \wedge \overline{\xi}_i$ for an orthonormal basis $\{\xi_i\}$ of $(\mathbb{S}_{\mathbf{b}}(E^*) \otimes \mathbb{S}_{\mathbf{b}'}(Q))^*$ at the origin, the intersection $R_{\mathbf{a}} = \cap_{\mathbf{b}\neq\mathbf{a},|\mathbf{b}|=|\mathbf{a}|} Z(\phi_{\mathbf{b}})$ is equal to $Gr(|\mathbf{a}|, E^* \otimes Q) \cap \mathbb{P}(\mathbb{S}_{\mathbf{a}}(E^*) \otimes \mathbb{S}_{\mathbf{a}'}(Q)) \subset \mathbb{P}(\wedge^{|\mathbf{a}|}(E^* \otimes Q))$ (Proposition 2.8 of [**Ho**]).

Note that we adapt the convention that $\sigma_{\mathbf{a}}$ is a Schubert variety of dimension $|\mathbf{a}^*|$, while X_w is a Schubert variety of dimension $\ell(w)$ in [**Ho**]. Thus, when $\mathbf{a}^* \in P(m,n)$ corresponds to $w \in W^P$, $\mathcal{B}_{\mathbf{a}}$ is equal to $\mathcal{B}_w$ and $\mathcal{R}_{\mathbf{a}^*}$ is equal to $\mathcal{R}_w$.

From the transitive action of P_0 on the fiber $B_{\mathbf{a}}$, we have the decomposition $\mathfrak{p}_0 = \mathfrak{m}_{\mathbf{a}} + \mathfrak{l}_{\mathbf{a}} + \mathfrak{m}^*_{\mathbf{a}}$, where the tangent space of $B_{\mathbf{a}}$ at a point is isomorphic to $\mathfrak{m}_{\mathbf{a}}$. Put $\mathbf{I}_{\mathbf{a}} = \mathbb{S}_{\mathbf{a}^*}(E^*) \otimes \mathbb{S}_{(\mathbf{a}^*)'}(Q)$. By comparing the tangent space $T_{[\mathfrak{n}_{\mathbf{a}}]}B_{\mathbf{a}}$ and the intersection of the tangent spaces $T_{[\mathfrak{n}_{\mathbf{a}}]}Gr(k, E^* \otimes Q)$ and $T_{[\wedge\mathfrak{n}_{\mathbf{a}}]}\mathbb{P}(\mathbf{I}_{\mathbf{a}})$, we get a sufficient condition for the equality $B_{\mathbf{a}} = R_{\mathbf{a}^*}$ as Proposition 3.4 in [**Ho**].

Proposition 4.1. *Assume that for the highest weight vector φ of every irreducible $\mathfrak{l}_{\mathbf{a}}$-representation space in the complement of $\mathfrak{m}_{\mathbf{a}}$ in $\mathfrak{n}^*_{\mathbf{a}} \otimes \mathfrak{m}/\mathfrak{n}_{\mathbf{a}}$, we have*

$$\varphi^k(v_1 \wedge \cdots \wedge v_k) \notin \mathbf{I}_{\mathbf{a}}/\wedge^k \mathfrak{n}_{\mathbf{a}},$$

where $\{v_1, \ldots, v_k\}$ is a basis of $\mathfrak{n}_{\mathbf{a}}$ and $\varphi^k : \wedge^k\mathfrak{n}_{\mathbf{a}} \to \wedge^k\mathfrak{m}/\wedge^k\mathfrak{n}_{\mathbf{a}}$ is defined by

$$\varphi^k(v_1 \wedge \cdots \wedge v_k) = \sum_i v_1 \wedge \cdots \wedge \varphi(v_i) \wedge \cdots \wedge v_k \mod \wedge^k \mathfrak{n}_{\mathbf{a}}.$$

Then $B_{\mathbf{a}}$ is equal to $R_{\mathbf{a}^*}$.

Now, we will compute all the irreducible $\mathfrak{l}_{\mathbf{a}}$-representation spaces in the complement of $\mathfrak{m}_{\mathbf{a}}$ in $\mathfrak{n}^*_{\mathbf{a}} \otimes \mathfrak{m}/\mathfrak{n}_{\mathbf{a}}$. Let E be an m-subspace of $\mathbb{C}^n$ and Q be the quotient $\mathbb{C}^n/E$. Then $\mathfrak{m}$ is equal to $E^* \otimes Q$. Take a partition $\mathbf{a} = (p_1^{q_1}, \ldots, p_r^{q_r}) \in P(m,n)$ and let $\mathbf{a}' = (p_1'^{q_1'}, \ldots, p_r'^{q_r'})$, $p_r' \neq 0$ be its conjugate.

Write $E = \oplus_{i=1}^{r_E} E_i$ and $Q = \oplus_{a=1}^{r_Q} Q_a$ so that $\mathfrak{l}_{\mathbf{a}} = (\oplus_{i=1}^{r_E} sl(E_i)) \oplus (\oplus_{a=1}^{r_Q} sl(Q_a))$. Note that r_E is r if $q_1 + \cdots + q_r = m$ and is $r+1$, otherwise, and r_Q is r if $q'_1 + \cdots q'_r = n - m$ and is $r+1$, otherwise. We indexed E_i and Q_a keeping the order of the basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$ such that $\{e_1, \dots, e_m\}$ is a basis of E. Set $r_i = \dim E_i$ and $s_a = \dim Q_a$ for $1 \leq i \leq r_E$ and $1 \leq a \leq r_Q$. Then $E_i = \langle e_{r_{i-1}+1}, \dots, e_{r_i} \rangle$ and $Q_a = \langle q_{s_{a-1}+1}, \dots, q_{s_a} \rangle$, where $q_p := e_{m+p}$ for $p = 1, \dots, n-m$.

Let Π be the index set of (i,a) such that $\mathfrak{n}_{\mathbf{a}} = \oplus_{(i,a)\in\Pi}(E_i^* \otimes Q_a)$. Then $\mathfrak{m}/\mathfrak{n}_{\mathbf{a}} = \oplus_{(i,a)\notin\Pi}(E_i^* \otimes Q_a)$. As a subspace of $\mathfrak{n}_{\mathbf{a}}^* \otimes \mathfrak{m}/\mathfrak{n}_{\mathbf{a}}$, $\mathfrak{m}_{\mathbf{a}} = (\oplus_{i<j} E_i^* \otimes E_j) \oplus (\oplus_{b<a} Q_b^* \otimes Q_a)$ is equal to

$$(\oplus_{i<j} E_i^* \otimes E_j \otimes \langle Id_{\oplus_{a\in\Pi_{i,j}} Q_a} \rangle_{\mathbb{C}}) \oplus (\oplus_{b<a} \langle Id_{\oplus_{i\in\Pi_{b,a}} E_i} \rangle_{\mathbb{C}} \otimes Q_b^* \otimes Q_a)$$

where $\Pi_{i,j} = \{a : (i,a) \notin \Pi, (j,a) \in \Pi\}$ and $\Pi_{b,a} = \{i : (i,b) \in \Pi, (i,a) \notin \Pi\}$.

We may choose the order of the set of roots of $SL(n)$ in such a way that the maximal root is located in the most left and the lowest box $E_1^* \otimes Q_s$ and the minimal root is located in the most right and the highest box $E_r^* \otimes Q_1$. Then the highest weight vector in $\mathfrak{m}_{\mathbf{a}} \subset \mathfrak{n}_{\mathbf{a}}^* \otimes \mathfrak{m}/\mathfrak{n}_{\mathbf{a}}$ is either

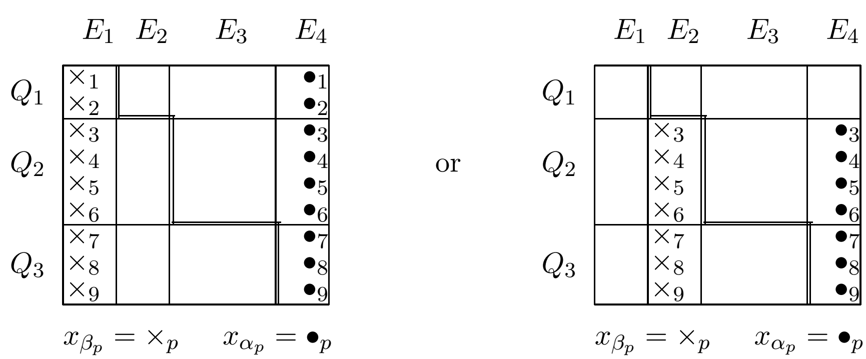
$\sum_{a\in\Pi_{i,j}} \sum_{q_p\in Q_a} x_{\alpha_p}^* \otimes x_{\beta_p}$, where $x_{\alpha_p} = e_{r_j}^* \otimes q_p$ and $x_{\beta_p} = e_{r_{i-1}+1}^* \otimes q_p$ for some $i < j$, or,

$\sum_{i\in\Pi_{b,a}} \sum_{e_p\in E_i} x_{\alpha_p}^* \otimes x_{\beta_p}$, where $x_{\alpha_p} = e_p^* \otimes q_{s_{b-1}+1}$ and $x_{\beta_p} = e_p^* \otimes q_{s_a}$ for some $b < a$,

For example, consider $\mathbf{a} = (9^2, 7^2, 3^4)$ in $P(10, 19)$. Then, we have $E = \oplus_{i=1}^4 E_i$, $Q = \oplus_{a=1}^3 Q_a$ and

$$\Pi = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}.$$

The highest weight vector of $\mathfrak{m}_{\mathbf{a}}$ is like



Proposition 4.2. *Let $E = \oplus_i E_i$ and $Q = \oplus_a Q_a$ be the decomposition associate to $\mathbf{a}$ as in the above and $\{e_1, \dots, e_n\}$ be a basis for $\mathbb{C}^n$ indexed as in the above. Then, the highest weight vector of an irreducible $\mathfrak{l}_{\mathbf{a}}$-representation space in the complement of $\mathfrak{m}_{\mathbf{a}}$ in $\mathfrak{n}_{\mathbf{a}}^* \otimes \mathfrak{m}/\mathfrak{n}_{\mathbf{a}}$ is either*

(1) *a decomposable vector $x_\alpha^* \otimes x_\beta$, where $x_\alpha = e_{r_j}^* \otimes q_{s_{b-1}+1}$ is the lowest weight vector of $E_j^* \otimes Q_b$ and $x_\beta = e_{r_{i-1}+1}^* \otimes q_{s_a}$ is the*

highest weight vector of $E_i^* \otimes Q_a$ *for some* $(j,b) \in \Pi, (i,a) \notin \Pi$, *or,*

(2) $\sum_{q_p \in Q_a} x_{\alpha_p}^* \otimes x_{\beta_p}$, *where* $x_{\alpha_p} = e_{r_j}^* \otimes q_p$ *and* $x_{\beta_p} = e_{r_{i-1}+1}^* \otimes q_p$ *for some* $(j,a) \in \Pi, (i,a) \notin \Pi$ *such that* $(j,a-1) \in \Pi$ *and* $(i,a-1) \notin \Pi$, *or,*

(3) $\sum_{e_p \in E_i} x_{\alpha_p}^* \otimes x_{\beta_p}$, *where* $x_{\alpha_p} = e_p^* \otimes q_{s_{b-1}+1}$ *and* $x_{\beta_p} = e_p^* \otimes q_{s_a}$ *for some* $(i,b) \in \Pi, (i,a) \notin \Pi$ *such that* $(i+1,b) \in \Pi$ *and* $(i+1,a) \notin \Pi$.

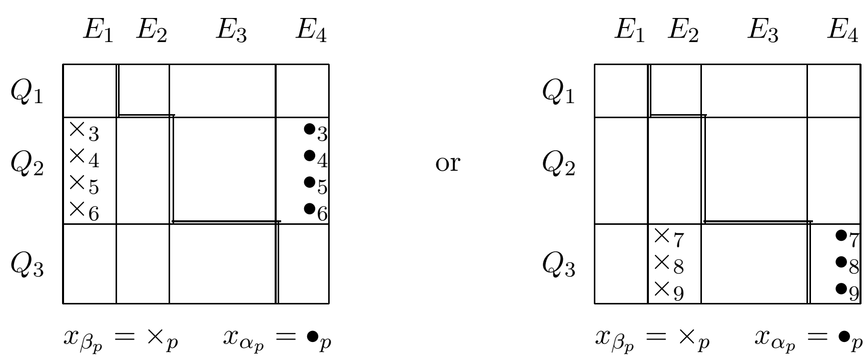
Proof. If $i \neq j$ and $a \neq b$, then $(E_j^* \otimes Q_b)^* \otimes (E_i^* \otimes Q_a)$ is an irreducible $\mathfrak{l}_{\mathbf{a}}$-representation space. The highest weight vector of the irreducible representation spaces $(E_j^* \otimes Q_b)^* \otimes (E_i^* \otimes Q_a)$ is of type (1).

But if $i \neq j$ and $a = b$, then $(E_j^* \otimes Q_a)^* \otimes (E_i^* \otimes Q_a) \simeq E_j \otimes E_i^* \otimes (Q_a^* \otimes Q_a)$ is decomposed as $(E_j \otimes E_i^* \otimes \langle Id_{Q_a} \rangle_{\mathbb{C}}) \oplus (E_j \otimes E_i^* \otimes sl(Q_a))$, each of which are irreducible $sl(E_j) \times sl(E_i) \times sl(Q_a)$-representation spaces. The highest weight vector of the irreducible representation spaces $(E_j \otimes E_i^* \otimes sl(Q_a))$ is of type (1).

The component $E_i^* \otimes E_j$ of $\mathfrak{m}_{\mathbf{a}}$ corresponds to the component $E_i^* \otimes E_j \otimes \langle Id_{\oplus_{a \in \Pi_{i,j}} Q_a} \rangle_{\mathbb{C}}$ in $\mathfrak{n}_{\mathbf{a}}^* \otimes \mathfrak{m}/\mathfrak{n}_{\mathbf{a}}$, so its complement in $\oplus_{a \in \Pi_{i,j}} E_i^* \otimes E_j \otimes \langle Id_{Q_a} \rangle_{\mathbb{C}}$ is $\oplus_{a \in \tilde{\Pi}_{i,j}} E_i^* \otimes E_j \otimes \langle Id_{Q_a} \rangle_{\mathbb{C}}$, where $\tilde{\Pi}_{i,j} = \{a : (i,a) \notin \Pi, (j,a) \in \Pi, (i,a-1) \notin \Pi, (j,a-1) \in \Pi\}$ is obtained from $\Pi_{i,j}$ by excluding the smallest index in $\Pi_{i,j}$. The highest weight of the irreducible representation space in these components is of type (2).

Considering the case when $i = j$ and $a \neq b$, we obtain the highest weight vectors of type (3). q.e.d.

For example, the highest weight vector $\sum_{q_p \in Q_a} x_{\alpha_p}^* \otimes x_{\beta_p}$ of type (2) is



4.2. Proof of the equality $B_{\mathbf{a}} = R_{\mathbf{a}^*}$. We will use the same notations as in the previous section. When $\{v_1, \ldots, v_k\}$ is a basis of $\mathfrak{n}_{\mathbf{a}}$ such that $v_1 = x_\alpha$, we will use the notation $x_\beta \wedge \hat{x}_\alpha \wedge \cdots \wedge v_k$ to denote the k-vector obtained from $v_1 \wedge \cdots \wedge v_k$, which may be considered as a base k-vector, by replacing $v_1 = x_\alpha$ with x_β.

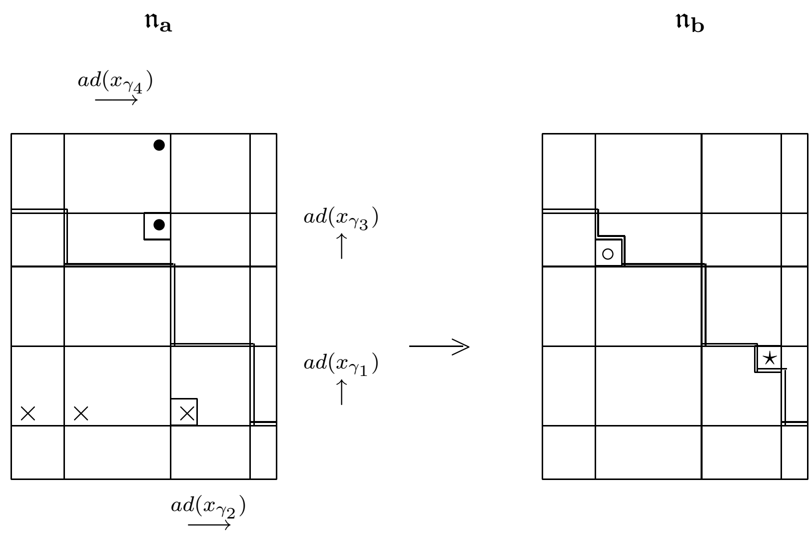
We call the union of all the columns corresponding to e_p's in E_i the E_i*-column*. Similarly, we call the union of all the rows corresponding q_p's in Q_a the Q_a*-row*.

Proposition 4.3. *Let* $\mathbf{a} = (p_1^{q_1}, \dots, p_r^{q_r})$ *be a partition and let* $\mathbf{a}' = ({p'_1}^{q'_1}, \dots, {p'_r}^{q'_r})$ *be its conjugate. Take a decomposition* $E = \oplus_i E_i$ *and* $Q = \oplus_a Q_a$ *associated to* $\mathbf{a}$ *as in the previous section. Let* Π *be the index set of* (i,a) *such that* $\mathfrak{n}_{\mathbf{a}} = \oplus_{(i,a)\in\Pi} E_i^* \otimes Q_a$.

Suppose that for any $(i,a) \notin \Pi$ *with* $(i+1,a) \in \Pi$ *and* $(i,a-1) \in \Pi$, *both* E_i *and* Q_a *are not one dimensional. Then the Schubert differential system* $\mathcal{B}_{\mathbf{a}^*}$ *is equal to the Schur differential system* $\mathcal{R}_{\mathbf{a}}$.

Proof. We will divide the proof into two parts: I. when the highest weight of the irreducible component in the complement of $\mathfrak{m}_{\mathbf{a}}$ in $\mathfrak{n}_{\mathbf{a}}^* \otimes \mathfrak{m}/\mathfrak{n}_{\mathbf{a}}$ is of type (1) and II. when it is of type (2) or (3). (See Proposition 4.2 for the types of the highest weight vectors.)

I. Type (1): Fix $(j,b), \in \Pi, (i,a) \notin \Pi$. Let $\{v_1, \dots, \dots, v_k\}$ be a basis of $\mathfrak{n}_{\mathbf{a}}$ with $v_1 = x_\alpha$ is a lowest weight vector of $E_j^* \otimes Q_b$ and x_β is the highest weight vector of $E_i^* \otimes Q_a$. We will show that $x_\beta \wedge \hat{x}_\alpha \wedge \cdots \wedge v_k$ has a non-zero component in $\mathbf{I_b}$ for a partition $\mathbf{b} \neq \mathbf{a}$ with $|\mathbf{b}| = |\mathbf{a}|$. Then by Proposition 4.1, $B_{\mathbf{a}^*}$ is equal to $R_{\mathbf{a}}$.



$x_\alpha =$ one of $\bullet$'s $\qquad x_{\beta'} = \circ$

$x_\beta =$ one of $\times$'s $\qquad x_{\alpha'} = \star$

Case 1. Assume that $q'_i \geq 2$ for all i. Denote by $x_{\beta'}$ the highest weight in the boxes $E_j^* \otimes Q_c$ for all c with $(j,c) \in \Pi$ and by $x_{\alpha'}$ the lowest weight vector in the boxes $E_k^* \otimes Q_a$ for all k with $(k,a) \notin \Pi$. Then, there is a partition $\mathbf{b}$ such that $\mathfrak{n}_{\mathbf{b}} = \mathfrak{n}_{\mathbf{a}} - \{x_{\beta'}\} \cup \{x_{\alpha'}\}$.

By construction, there is $x_{\gamma_j}, j = 1, \ldots, 4$ in $\mathfrak{p}_0$ such that

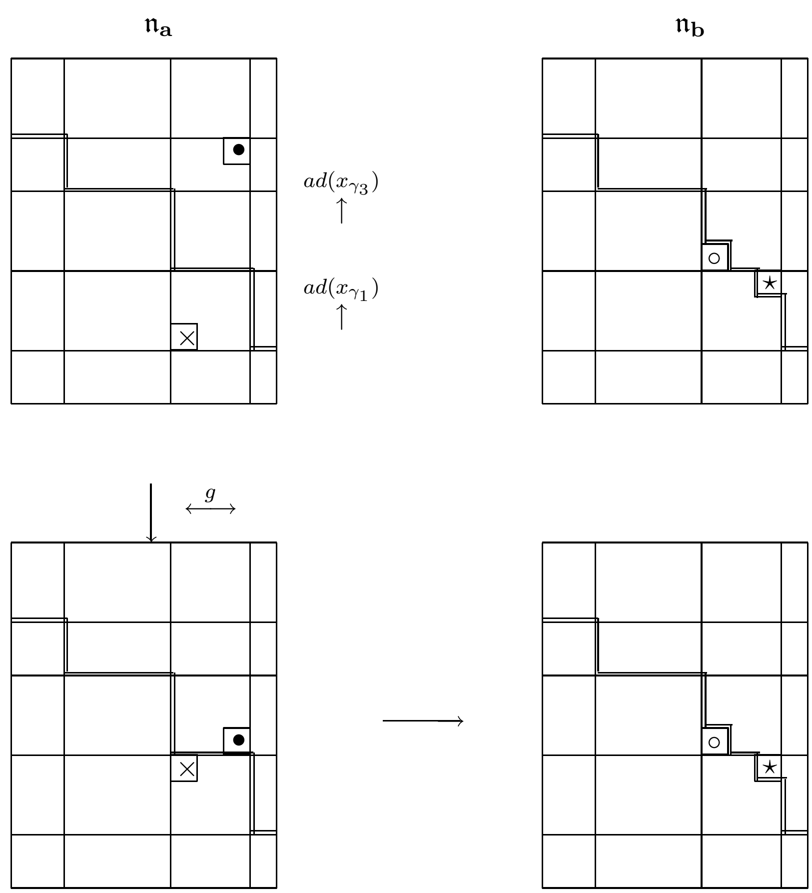
$$ad(x_{\gamma_2})ad(x_{\gamma_1})x_\beta = x_{\alpha'} \quad \text{and} \quad ad(x_{\gamma_4})ad(x_{\gamma_3})x_{\beta'} = x_\alpha.$$

If all x_α, x_β, $x_{\beta'}$ and $x_{\alpha'}$ lie neither in the same E_i-column nor in the same Q_a-raw as in the picture, then all γ_i are distinct. Thus,

$$ad(x_{\gamma_4}) \cdots ad(x_{\gamma_1})(x_\beta \wedge \cdots \wedge \hat{x}_\alpha \wedge \cdots \wedge v_k) = x_{\alpha'} \wedge \cdots \wedge \hat{x}_{\beta'} \wedge \cdots \wedge v_k.$$

But $x_{\alpha'} \wedge \cdots \wedge \hat{x}_{\beta'} \wedge \cdots \wedge v_k$ is the lowest weight vector of $\mathbf{I_b}$. Since $ad(\mathfrak{p}_0)$ preserves $\mathbf{I_b}$, there is a non-zero $\mathbf{I_b}$-component in $x_\beta \wedge \cdots \wedge \hat{x}_\alpha \wedge \cdots \wedge v_k$ and thus, it is not contained in $\mathbf{I_a}$.

If all x_α, x_β, $x_{\beta'}$ and $x_{\alpha'}$ lie either in the same E_i-column or in the same Q_a-row, then either γ_2 is equal to γ_4 or γ_1 is equal to γ_3. We consider the case when $\gamma_2 = \gamma_4$ as in the picture (the proof for the other case is similar to this case).

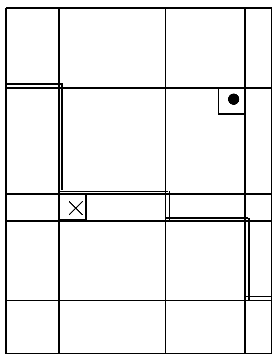


Then the multivector $x_{\alpha'} \wedge \hat{x}_{\beta'} \wedge \cdots \wedge v_k \in \mathbf{I_b}$ can be obtained from the multivector $x_\beta \wedge \hat{x}_\alpha \wedge \cdots \wedge v_k$ by applying $g \circ ad(x_{\gamma_3})ad(x_{\gamma_1})$, where

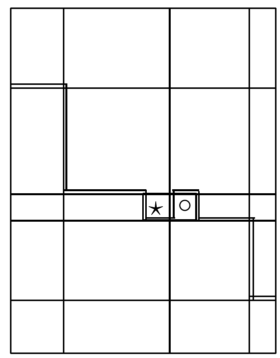
$g \in SL(m)\times SL(n-m)$ which exchanges the most left column $e^*_{s_{i-1}+1}\otimes q_p\,, 1 \le p \le n-m$ with the most right column $e^*_{s_i}\otimes q_p\,, 1 \le p \le n-m$ in the E_i-column. This shows that $x_\beta \wedge \hat{x}_\alpha \wedge \cdots \wedge v_k$ is not contained in $\mathbf{I_a}$.

Case 2. If some $q'_a = 1$, then there may be no partition $\mathbf{b}$ with such property as in Case 1. This is the case when dim Q_a is one and $a = b+1$ and $E^*_{j+1}\otimes Q_a \notin \mathfrak{n}_\mathbf{a}$. Then $x_{\alpha'}$ is left to $x_{\beta'}$ and they are adjacent so we cannot find such a partition $\mathbf{b}$.

$\mathfrak{n}_\mathbf{a}$

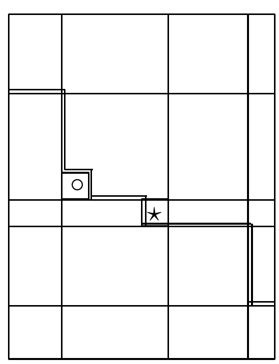


$\mathfrak{n}_\mathbf{b}$



But in this case, consider $\mathbf{b}'$ such that $\mathfrak{n}_{\mathbf{b}'} = \mathfrak{n}_\mathbf{a} - \{x_{\beta''}\}\cup\{x_{\alpha'}\}$ where $x_{\beta''}$ is the highest weight vector in the boxes $E^*_k \otimes Q_b$ for all k with $(k,b) \in \Pi$ and $x_{\alpha'}$ is the lowest weight vector in the boxes $E^*_i \otimes Q_c$ for all c with $(i,c) \notin \Pi$.

$\mathfrak{n}_{\mathbf{b}'}$



If $i \neq j+1$, then there is such a partition $\mathbf{b}'$. If $i = j+1$, then, by the assumption, dim E_i is not equal to one and thus, we can find such a partition $\mathbf{b}'$. Then by the same argument as in the case 1, we can prove that $x_\beta \wedge \hat{x}_\alpha \wedge \cdots \wedge v_k$ is not contained in $\mathbf{I_a}$.

II. Type (2) or (3): Let $\sum_{q_p \in Q_a} x^*_{\alpha_p} \otimes x_{\beta_p}$ be the highest weight vector of type (2) of an irreducible representation space in the complement of $\mathfrak{m}_\mathbf{a}$ in $\mathfrak{n}^*_\mathbf{a} \otimes \mathfrak{m}/\mathfrak{n}_\mathbf{a}$, where $x_{\alpha_p} = e^*_{r_j} \otimes q_p$ and $x_{\beta_p} = e^*_{r_{i-1}+1} \otimes q_p$ for some $(j,a) \in \Pi, (i,a) \notin \Pi$ such that $(j, a-1) \in \Pi$ and $(i, a-1) \notin \Pi$. We will show that $\sum_{q_p \in Q_a} x_{\beta_p} \wedge \hat{x}_{\alpha_p} \wedge \cdots \wedge v_k$ has a non-zero component in $\mathbf{I_b} + \mathbf{I_c}$ if $\dim Q_a \geq 2$, and has a non-zero component in $\mathbf{I_b}$, otherwise, for some partition $\mathbf{b}$ and $\mathbf{c}$.

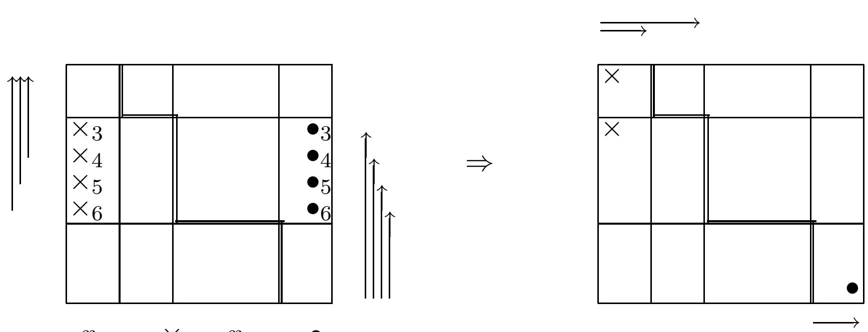
Applying the adjoint actions successively to

$$\sum_{q_p \in Q_a} x_{\beta_p} \wedge \hat{x}_{\alpha_p} \wedge \cdots \wedge v_k \;=\; x_{\beta_{s_{a-1}+1}} \wedge \hat{x}_{\alpha_{s_{a-1}+1}} \wedge \cdots \wedge v_k + \sum_{p=s_{a-1}+2}^{p=s_a} x_{\beta_p} \wedge \hat{x}_{\alpha_p} \wedge \cdots \wedge v_k,$$

we can get

$$x_{\beta_{s_{a-1}+1}} \wedge \hat{x}_{\alpha_{s_c}} \wedge \cdots \wedge v_k \;+\; x_{\beta_{s_{a-2}+1}} \wedge \hat{x}_{\alpha_{s_c}} \wedge \cdots \wedge v_k,$$

where $x_{\beta_{s_{a-2}+1}} = e^*_{r_{i-1}+1} \otimes q_{s_{a-2}+1}$ and $x_{\alpha_{s_c}} = e^*_{r_j} \otimes q_{\alpha_{s_c}}$ and c is the largest index c such that $(j,c) \in \Pi$.



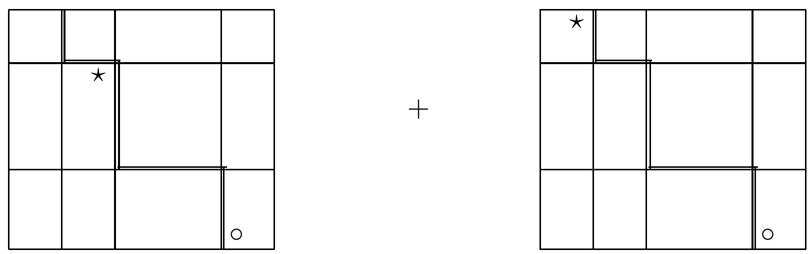
$x_{\beta_p} = \times_p \quad x_{\alpha_p} = \bullet_p$

Applying the adjoint actions again, we can get

$$y_{\beta_{s_{a-1}+1}} \wedge \hat{y}_{\alpha_{s_c}} \wedge \cdots \wedge v_k \;+\; y_{\beta_{s_{a-2}+1}} \wedge \hat{y}_{\alpha_{s_c}} \wedge \cdots \wedge v_k,$$

where $y_{\beta_{s_{a-1}+1}} = e^*_{r_h} \otimes q_{s_{a-1}+1}$ and $y_{\beta_{s_{a-2}+1}} = e^*_{r_i} \otimes q_{s_{a-2}+1}$ and $y_{\alpha_{s_c}} = e^*_{r_{j-1}+1} \otimes q_{\alpha_{s_c}}$ and h is the largest index h such that $(h,a) \notin \Pi$. This is the sum of the lowest weight vector of $\mathbf{I_b}$ and that of $\mathbf{I_c}$ for some partition $\mathbf{b}$ and $\mathbf{c}$ such that all $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are distinct.

$\mathfrak{n}_{\mathbf{b}}$ $\mathfrak{n}_{\mathbf{c}}$



q.e.d.

By Theorem 3.8 and Proposition 4.3, we get

Theorem 4.4. *Let* $\mathbf{a} = (p_1^{q_1}, \ldots, p_r^{q_r}), p_r \neq 0$ *be a partition and let* $\mathbf{a}' = ({p'_1}^{q'_1}, \ldots, {p'_r}^{q'_r})$, $p'_r \neq 0$ *be its conjugate. Then* $\sigma_{\mathbf{a}}$ *is Schur rigid if* $q_i, q'_i \geq 2$ *for all* $i \leq r$.

Remark. One of the problems in algebraic geometry is the smoothability of a singular Schubert variety X_w of G/P. We say X_w is *smoothable* if there is a smooth subvariety X of G/P with $[X] = [X_w]$ in $H_*(G/P, \mathbb{Z})$([**B**]). Assume that $\mathbf{a} = (p^q)$ and that $p = 1$ or $q = 1$, but both are not 1. Then $\sigma_{\mathbf{a}}$ is a singular Schubert variety and the non-smoothability of $X_{\mathbf{a}}$ is proved in [**B**]: if X is a subvariety of $Gr(m,n)$ with $[X] = [\sigma_{\mathbf{a}}]$, then X is a Schubert variety of type $\mathbf{a}$. By Theorem 4.4, for a partition $\mathbf{a} = (p_1^{q_1}, \ldots, p_r^{q_r})$ with its conjugate $\mathbf{a}' = ({p'_1}^{q'_1}, \ldots, {p'_r}^{q'_r})$, if $q_i, q'_i \geq 2$, for all i, then the singular Schubert varieties $\sigma_{\mathbf{a}}$ of type $\mathbf{a}$ is not smoothable, neither.

Remark. With the same notations as in Proposition 4.3, if both E_i and Q_a are one dimensional for some $(i,a) \notin \Pi$ with $E_i^* \otimes Q_a$ adjacent to $\mathfrak{n}_{\mathbf{a}}$, then $B_{\mathbf{a}}$ is a proper subvariety of $R_{\mathbf{a}^*}$.

Consider the highest weight vector $x_\alpha^* \otimes x_\beta$ of an irreducible component in the complement of $\mathfrak{m}_{\mathbf{a}}$ in $\mathfrak{n}_{\mathbf{a}}^* \otimes \mathfrak{m}/\mathfrak{n}_{\mathbf{a}}$ such that $x_\alpha \in \mathfrak{n}_{\mathbf{a}}$ and $x_\beta \in E_i^* \otimes Q_a \subset \mathfrak{m}/\mathfrak{n}_{\mathbf{a}}$. Then one can check that $x_\beta \wedge \hat{x}_\alpha \wedge \cdots \wedge v_k$ is contained in $\mathbf{I}_{\mathbf{a}}$. Thus, this gives a non-trivial element in $T_{\mathbf{a}} := T_{[\mathfrak{n}_{\mathbf{a}}]} Gr(k, \mathfrak{m}) \cap T_{\wedge^k \mathfrak{n}_{\mathbf{a}}} \mathbb{P}(\mathbf{I}_{\mathbf{a}}) \subset \mathfrak{n}_{\mathbf{a}}^* \otimes \mathfrak{m}/\mathfrak{n}_{\mathbf{a}}$. Note that as an element of $T_{[\wedge^k \mathfrak{n}_{\mathbf{a}}]} \mathbb{P}(\mathbf{I}_{\mathbf{a}})$, this tangent vector gives the map

$$x_\alpha \wedge v_2 \wedge \cdots \wedge v_k \longmapsto x_\beta \wedge v_2 \wedge \cdots \wedge v_k \in \mathbf{I}_{\mathbf{a}} / \wedge^k \mathfrak{n}_{\mathbf{a}},$$

where $\{v_1 = x_\alpha, v_2, \ldots, v_k\}$ be a basis of $\mathfrak{n}_{\mathbf{a}}$. Then, $c(t) = (x_\alpha + t x_\beta) \wedge v_2 \wedge \cdots \wedge v_k$, $t \in \mathbb{C}$, is a curve in $R_{\mathbf{a}} = Gr(k, \mathfrak{m}) \cap \mathbb{P}(\mathbf{I}_{\mathbf{a}})$ whose tangent vector is $x_\alpha^* \otimes x_\beta$. But this tangent vector is not contained in $\mathfrak{m}_{\mathbf{a}}$ and thus, $B_{\mathbf{a}}$ is a proper subvariety of $R_{\mathbf{a}^*}$. This is a generalization of the counterexamples considered in [**W**] or Example 9 of [**B**].

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