# RIGOROUS ASYMPTOTIC EXPANSIONS FOR LAGERSTROM'S MODEL EQUATION—A GEOMETRIC APPROACH

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Abstract. The present work is a continuation of the geometric singular perturbation analysis of the Lagerstrom model problem which was commenced in [PS04]. We establish the same framework here, reinterpreting Lagerstrom's equation as a dynamical system which is subsequently analyzed by means of methods from dynamical systems theory as well as of the blow-up technique. We show how rigorous asymptotic expansions for the Lagerstrom problem can be obtained using geometric methods, thereby establishing a connection to the method of matched asymptotic expansions. We explain the structure of these expansions and demonstrate that the occurrence of the well-known logarithmic (switchback) terms therein is caused by a resonance phenomenon.

## 1. Introduction

Singular perturbation problems in general and singularly perturbed differential equations in particular are characterized by the presence of at least two fundamentally different scales. The existence of these independent scales gives one a small parameter and thus permits one to use perturbation methods. The aim of these methods is to obtain uniformly valid approximations. It is, however, the essence of a singular perturbation problem that a straightforward perturbation fails to be uniformly valid. Indeed, it is typical of singular perturbation techniques that one works with approximations which are valid in restricted domains only.

Traditionally, this type of problems has been treated using the method of *matched asymptotic expansions*: one proceeds by constructing two (or more) asymptotic expansions which together cover the entire domain, although neither is uniformly valid there. To obtain a uniformly valid approximation on the entire domain, these individual expansions have to be matched; the essence of *matching* lies in comparing two expansions on a suitable domain of overlap. An excellent account of the fundamental notions and concepts in perturbation theory is given in [LC72].

More recently, an alternative approach to such problems, known as *geometric singular perturbation theory*, has emerged (see [Fen79] or [Jon95] for details and references). This approach, which in general requires certain hyperbolicity assumptions, is based on methods from the theory of dynamical systems, in particular on invariant manifold theory.

A classical singular perturbation problem from fluid dynamics occurs in the asymptotic treatment of viscous flow past a solid at low Reynolds number, see e.g. [vD75]. Though first attempts at clarification date back to [Sto51], it was not until a century later that the conceptual structure of the problem was at last resolved

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in [Kap57, KL57, PP57]. To illustrate the mathematical ideas and techniques used by Kaplun in his asymptotic treatment of low Reynolds number flow, Lagerstrom [Lag66] proposed an analytically rather simple model problem which was subsequently analyzed by himself as well as by numerous other researchers, see [Lag88] and the references therein.
Much of the interest in Lagerstrom's model problem has been directed at the development of matched asymptotic expansions to describe its solutions, see e.g. [KC81, Lag88], or [HTB90]. Our goal is to show how such expansions can be obtained using geometric methods, thereby establishing a connection between the two approaches. As observed already in [Fen79], in the context of layer-type problems, determining outer solutions is equivalent to computing expansions of slow manifolds. However, the well-developed geometric theory is not applicable at points where hyperbolicity is lost. In several instances it has been possible to extend the geometric approach past such points using the blow-up technique. Blow-up is essentially a sophisticated rescaling which allows one to analyze the dynamics near a singularity; details can be found in [DR96]. In particular, blow-up has been employed by [KS01] and [vGKS] to give a detailed geometric analysis of the singularly perturbed planar fold. Apart from deriving asymptotic expansions of slow manifolds continued beyond the fold point, they also explained the structure of these expansions and gave an algorithm for the computation of their coefficients. Our line of attack is very similar to theirs.
A distinctive feature of asymptotic expansions for the singularly perturbed planar fold as well as for Lagerstrom's model is the occurrence of logarithmic terms. The nature of the expansions in these and similar problems is both complicated and unexpected, as the governing equations typically give no immediate indication of the presence of such terms; indeed, this is why they are often so tricky to obtain. Traditionally, logarithmic terms have been accounted for under the notions of *switchback* and *integrated effects*. We show that the occurrence of logarithms in the expansions for Lagerstrom's model equation is caused by a resonance phenomenon. At this point, we conjecture that similar resonance phenomena are responsible for the occurrence of logarithmic terms in many other singular perturbation problems, at least after reinterpretation in a dynamical systems framework.

This article is organized as follows: Section 2 contains some background information on the Lagerstrom model problem as well as on the blow-up transformation used in our analysis; in Section 3, we derive asymptotic expansions for Lagerstrom's model, whereas in Section 4, we briefly indicate how these expansions are related to the classical ones known from the literature. This work should be regarded as a continuation of [PS04].

## 2. Background information

**2.1. Lagerstrom's model equation.** In its simplest form, Lagerstrom's model equation is given by the non-autonomous second-order boundary value problem

$$\ddot{u} + \frac{n-1}{x}\dot{u} + u\dot{u} = 0, \tag{1a}$$

$$u(\varepsilon) = 0, \qquad u(\infty) = 1, \tag{1b}$$

with $n \in \mathbb{N}$, $0 < \varepsilon \leq x \leq \infty$, and the overdot denoting differentiation with respect to $x$. Equivalently, by introducing the rescaling

$$\xi = \frac{x}{\varepsilon}, \tag{2}$$

one can write (1) as

$$u'' + \frac{n-1}{\xi}u' + \varepsilon u u' = 0, \tag{3a}$$

$$u(1) = 0, \qquad u(\infty) = 1, \tag{3b}$$

with $1 \leq \xi \leq \infty$ and the prime denoting differentiation with respect to $\xi$.

Originally, (1) and (3) are the versions of the model which was first introduced in [Lag66] and [KL57] to elucidate certain basic ideas used in the asymptotic treatment of viscous flow past a solid at low Reynolds number. In the following, we will only consider the cases $n = 2$ and $n = 3$, which represent the physically relevant settings of flow in two and three dimensions, respectively. For more background information and further references on Lagerstrom's model equation, we refer to [PS04].

As in [PS04], replacing $\xi \in [1, \infty)$ by $\eta := \xi^{-1} \in (0, 1]$, appending the (trivial) equation $\varepsilon' = 0$, and setting $u' = v$, we find

$$\begin{aligned} u' &= v, \\ v' &= -(n-1)\eta v - \varepsilon u v, \\ \eta' &= -\eta^2, \\ \varepsilon' &= 0 \end{aligned} \tag{4}$$

in extended phase space, with boundary conditions given by

$$u(1) = 0, \qquad \eta(1) = 1, \qquad u(\infty) = 1; \tag{5}$$

obviously, (5) means that $v(\infty) = 0$ for the solution to (4), whereas $v(1)$ still is to be determined.

For $\varepsilon > 0$ fixed, let $\mathcal{V}_\varepsilon$ be defined by

$$\mathcal{V}_\varepsilon := \big\{(0, v, 1) \,\big|\, v \in [\underline{v}, \overline{v}]\big\}, \tag{6}$$

where $0 \leq \underline{v} < \overline{v} < \infty$, and let the point $Q$ be given by $Q := (1, 0, 0)$ (see again [PS04]). Note that $\mathcal{V}_\varepsilon$ and $Q$ correspond to the inner and outer boundary conditions in (3b), respectively. The saturation of $\mathcal{V}_\varepsilon$ under the flow induced by (4) we denote by $\mathcal{M}_\varepsilon$. Correspondingly, the manifolds $\mathcal{V}$ and $\mathcal{M}$ in extended phase space are defined by $\mathcal{V} := \bigcup_{\varepsilon \in [0, \varepsilon_0]} \mathcal{V}_\varepsilon \times \{\varepsilon\}$ and $\mathcal{M} := \bigcup_{\varepsilon \in [0, \varepsilon_0]} \mathcal{M}_\varepsilon \times \{\varepsilon\}$; here, the parameter $\varepsilon$ is not fixed, but is allowed to vary in some interval $[0, \varepsilon_0]$, with $\varepsilon_0 > 0$ small.

The equilibria of (4) are located on the line $\ell_\varepsilon := \big\{(u, 0, 0) \,\big|\, u \in \mathbb{R}\big\}$; obviously, $Q \in \ell_\varepsilon$. For $\varepsilon > 0$, the one-dimensional strongly stable manifold of $Q$, which we call $\mathcal{W}_\varepsilon^{ss}$, can be computed explicitly due to the simple structure of (4) for $\eta = 0$, whence e.g.

$$v(u) = \frac{\varepsilon}{2}\left(1 - u^2\right); \tag{7}$$

here, we have used $v(1) = 0$. The following result can be found in [PS04]:

**Proposition 2.1** ([PS04]). *Let $k \in \mathbb{N}$ be arbitrary, and let $\varepsilon > 0$.*

(1) *There exists an attracting two-dimensional center manifold $\mathcal{W}_\varepsilon^c$ for (4) which is given by $\{v = 0\}$.*

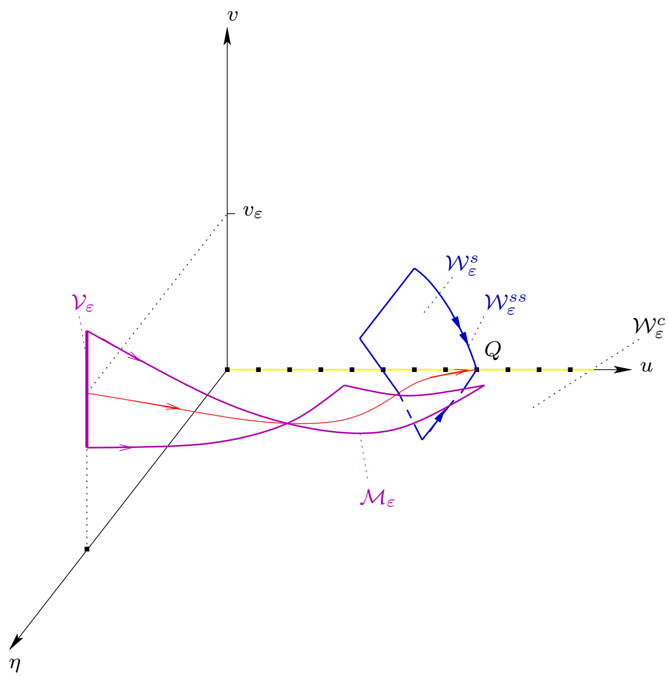


FIGURE 1. Geometry of system (4) for $\varepsilon > 0$ fixed.

(2) *For $|u-1|$, $v$, and $\eta$ sufficiently small, there is a stable invariant $\mathcal{C}^k$-smooth foliation $\mathcal{F}^s_\varepsilon$ with base $\mathcal{W}^c_\varepsilon$ and one-dimensional $\mathcal{C}^k$-smooth fibers.*

Given Proposition 2.1, one can define the stable manifold $\mathcal{W}^s_\varepsilon$ of $Q$ as

$$\mathcal{W}^s_\varepsilon := \bigcup_{P\in\Upsilon} F^s_\varepsilon(P), \tag{8}$$

where $\Upsilon := \big\{(1,0,\eta)\,\big|\,0 \le \eta \ll 1\big\}$, i.e., as a union of fibers $F^s_\varepsilon \in \mathcal{F}^s_\varepsilon$ with base points in the weakly stable orbit $\Upsilon$. The situation is illustrated in Figure 1.
The main result in [PS04] is the following theorem on the existence and (local) uniqueness of solutions to (4) (and, consequently, to (1)):

**Theorem 2.2** ([PS04]). *For $\varepsilon \in (0, \varepsilon_0]$, with $\varepsilon_0 > 0$ sufficiently small and $n = 2, 3$, there exists a locally unique solution to (4),(5).*

The proof is constructive, and is performed by tracking $\mathcal{M}_\varepsilon$ through phase space and by showing that its intersection with $\mathcal{W}^s_\varepsilon$ is transverse under the resulting flow. As we are only interested in small values of $\varepsilon$, we took a perturbational approach: given transversality for $\varepsilon = 0$, we concluded that the intersection remained transverse for $\varepsilon > 0$ small. On a technical level, the tracking was done along singular orbits of (4) connecting $\mathcal{V}_0$ to $Q$. These orbits, which we denoted by $\Gamma$, were used as templates for orbits of the full problem obtained for $\varepsilon > 0$. However, due to the non-hyperbolic character of the problem for $\varepsilon = 0$, we could not deduce the existence of a stable manifold $\mathcal{W}^s_0$ from standard invariant manifold theory. It was shown in [PS04], however, that $\mathcal{W}^{ss}$ and $\mathcal{W}^s$ can still be smoothly defined down to $\varepsilon = 0$ via the blow-up technique.

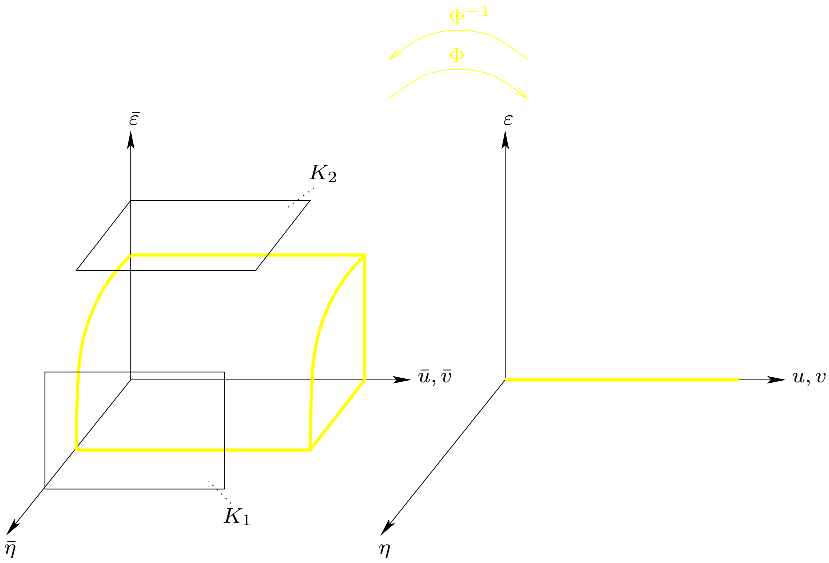


FIGURE 2. The blow-up transformation $\Phi$.

**2.2. The blow-up transformation.** The *(polar) blow-up transformation* $\Phi$ introduced in [PS04] to analyze the dynamics of (4) near $\ell := \{(u,0,0,0) \mid u \in \mathbb{R}^+\}$ is given by

$$\Phi : \begin{cases} \mathbb{R} \times B \to \mathbb{R}^4, \\ (\bar u, \bar v, \bar\eta, \bar\varepsilon, \bar r) \mapsto (\bar u, \bar r\bar v, \bar r\bar\eta, \bar r\bar\varepsilon), \end{cases} \tag{9}$$

where $B := \mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{S}^2$ denotes the two-sphere in $\mathbb{R}^3$, i.e., $\mathbb{S}^2 = \{(\bar v, \bar\eta, \bar\varepsilon) \mid \bar v^2 + \bar\eta^2 + \bar\varepsilon^2 = 1\}$. Note that obviously $\Phi^{-1}(\ell) = \mathbb{R} \times \mathbb{S}^2 \times \{0\}$, which is the blown-up locus obtained by setting $\bar r = 0$. Moreover, for $\bar r \neq 0$, i.e., away from $\Phi^{-1}(\ell)$, $\Phi$ is a $\mathcal{C}^\infty$-diffeomorphism. We will only be interested in $\bar r \in [0, r_0]$, with $r_0 > 0$ small.
The reason for introducing (9) is that degenerate equilibria, such as those in $\ell$, are often amenable to analysis by means of blow-up techniques. The blow-up is a (singular) coordinate transformation whereby the degenerate equilibrium is blown up to some $n$-sphere. Transverse to the sphere and even on the sphere itself one often gains enough hyperbolicity to allow for a complete analysis by standard techniques. For a general discussion of blow-up we refer to [DR91] and to [Dum93], whereas applications to singular perturbation problems can be found in [DS95] and [DR96] as well as in [KS01] and [vGKS].
The vector field on $\mathbb{R} \times B$, which is induced by the vector field corresponding to (4), is most conveniently studied by introducing different charts for the manifold $\mathbb{R} \times B$. In the following, we will only be concerned with two charts $K_1$ and $K_2$ corresponding to $\bar\eta > 0$ and $\bar\varepsilon > 0$ in (9), respectively, see Figure 2. The reason is that these two charts correspond precisely to the inner and outer regions in the classical approach, see [PS04].

The *directional blow-up* in the direction of positive $\eta$ (i.e., in $K_1$) is given by

$$\Phi_1 : \begin{cases} \mathbb{R}^4 \to \mathbb{R}^4, \\ (u_1, v_1, r_1, \varepsilon_1) \mapsto (u_1, r_1 v_1, r_1, r_1 \varepsilon_1), \end{cases} \tag{10}$$

whence

$$u = u_1, \quad v = r_1 v_1, \quad \eta = r_1, \quad \varepsilon = r_1 \varepsilon_1. \tag{11}$$

After transformation to $K_1$ and desingularization, the equations in (4) have the following form:

$$\begin{aligned} u_1' &= v_1, \\ v_1' &= (2-n)v_1 - \varepsilon_1 u_1 v_1, \\ r_1' &= -r_1, \\ \varepsilon_1' &= \varepsilon_1. \end{aligned} \tag{12}$$

The desingularization, which is necessary to obtain a non-trivial flow for $r_1 = 0$, is performed by dividing out the common factor $r_1$ on both sides of the equations; it corresponds to a rescaling of time, leaving the phase portrait unchanged. The equilibria of (12) are easily seen to lie in $\ell_1 := \{(u_1, 0, 0, 0) \,|\, u_1 \in \mathbb{R}^+\}$. A simple computation shows that the corresponding eigenvalues are given by $-1$, 0, and 1 both for $n = 3$ and for $n = 2$; these eigenvalues obviously are in *resonance*. In fact, it is these resonances in $K_1$ which are responsible for the occurrence of logarithmic switchback terms in the Lagerstrom model, as will become clear later on.
Similarly, in chart $K_2$, (9) is given by

$$\Phi_2 : \begin{cases} \mathbb{R}^4 \to \mathbb{R}^4, \\ (u_2, v_2, \eta_2, r_2) \mapsto (u_2, r_2 v_2, r_2 \eta_2, r_2), \end{cases} \tag{13}$$

respectively, by

$$u = u_2, \quad v = r_2 v_2, \quad \eta = r_2 \eta_2, \quad \varepsilon = r_2, \tag{14}$$

which is simply an $\varepsilon$-dependent rescaling of the original variables, since $r_2 = \varepsilon$. Desingularizing once again, we obtain

$$\begin{aligned} u_2' &= v_2, \\ v_2' &= (1-n)\eta_2 v_2 - u_2 v_2, \\ \eta_2' &= -\eta_2^2, \\ r_2' &= 0 \end{aligned} \tag{15}$$

for the blown-up vector field in $K_2$; these equations are simple insofar as $r_2$ occurs only as a parameter. For $r_2 \in [0, r_0]$, the equilibria of (15) are given by $\ell_2 := \{(u_2, 0, 0, r_2) \,|\, u_2 \in \mathbb{R}^+\}$.
The following observation, which is valid in both cases alike, is the starting point of our analysis:

**Proposition 2.3** ([PS04])**.** *Let $k \in \mathbb{N}$ be arbitrary.*

(1) *There exists an attracting three-dimensional center manifold $\mathcal{W}_2^c$ for (15) which is given by $\{v_2 = 0\}$.*
(2) *For $|u_2 - 1|$, $v_2$, $\eta_2$, and $r_2$ sufficiently small, there is a stable invariant $\mathcal{C}^k$-smooth foliation $\mathcal{F}_2^s$ with base $\mathcal{W}_2^c$ and one-dimensional $\mathcal{C}^k$-smooth fibers.*

Let $\mathcal{W}_2^{ss}$ denote the fiber $F_2^s(Q_2) \in \mathcal{F}_2^s$ with base point $Q_2 := (1, 0, 0, r_2)$; note that for any $r_2 = \varepsilon \in [0, \varepsilon_0]$ fixed, $Q_2$ and $\mathcal{W}_2^{ss}$ correspond to the original equilibrium $Q$ and its stable fiber $\mathcal{W}_\varepsilon^{ss}$, respectively.

***Remark* 1.** Just as was the case with $\mathcal{W}_\varepsilon^{ss}$, $\mathcal{W}_2^{ss}$ is also known explicitly: given $v_2(1) = 0$, one obtains from (15) with $\eta_2 = 0$ that

$$v_2(u_2) = \frac{1}{2}\left(1 - u_2^2\right); \tag{16}$$

hence, $\mathcal{W}_2^{ss}$ is independent of both $\varepsilon$ and $n$. □

As in [PS04], let the singular orbit $\gamma_2$ be defined by

$$\gamma_2(\xi_2) := \left\{\left(1, 0, \xi_2^{-1}, 0\right) \,\middle|\, \xi_2 \in (0, \infty)\right\}, \tag{17}$$

and let $\Gamma_2 := \gamma_2 \cup \{Q_2\}$; note that $\gamma_2(\xi_2) \to Q_2$ as $\xi_2 \to \infty$. With Proposition 2.3, we obtain the following

**Proposition 2.4** ([PS04])**.** *The manifold $\mathcal{W}_2^s$ defined by*

$$\mathcal{W}_2^s := \bigcup_{P_2 \in \Gamma_2} F_2^s(P_2) \tag{18}$$

*is an invariant, $\mathcal{C}^k$-smooth manifold, namely the stable manifold of $Q_2$.*

Tracking the manifold $\mathcal{W}_2^s$ along the singular orbit $\overline{\Gamma}$ to the inner boundary in $K_1$ defines a global manifold $\overline{\mathcal{W}}^s$ which determines the solution to (4),(5), as given by Theorem 2.2, see Figure 3.
The relation between charts $K_1$ and $K_2$ on their overlap domain can be described as follows:

**Lemma 2.5** ([PS04])**.** *Let $\kappa_{12}$ denote the change of coordinates from $K_1$ to $K_2$, and let $\kappa_{21} = \kappa_{12}^{-1}$ be its inverse. Then, $\kappa_{12}$ is given by*

$$u_2 = u_1, \quad v_2 = v_1\varepsilon_1^{-1}, \quad \eta_2 = \varepsilon_1^{-1}, \quad r_2 = r_1\varepsilon_1, \tag{19}$$

*and $\kappa_{21}$ is given by*

$$u_1 = u_2, \quad v_1 = v_2\eta_2^{-1}, \quad r_1 = r_2\eta_2, \quad \varepsilon_1 = \eta_2^{-1}. \tag{20}$$

***Remark* 2** (Notation)**.** Let us introduce the following notation: for any object $\square$ in the original setting, let $\overline{\square}$ denote the corresponding object after the blow-up; in charts $K_i$, $i = 1, 2$, the same object will appear as $\square_i$ when necessary. □

Moreover, as in [PS04], we define the *sections* $\Sigma_1^{in}$, $\Sigma_1^{out}$, and $\Sigma_2^{in}$, where

$$\Sigma_1^{in} := \left\{(u_1, v_1, r_1, \varepsilon_1) \,\middle|\, u_1 \geq 0, v_1 \geq 0, \varepsilon_1 \geq 0, r_1 = \rho\right\}, \tag{21a}$$

$$\Sigma_1^{out} := \left\{(u_1, v_1, r_1, \varepsilon_1) \,\middle|\, u_1 \geq 0, v_1 \geq 0, r_1 \geq 0, \varepsilon_1 = \delta\right\}, \tag{21b}$$

$$\Sigma_2^{in} := \left\{(u_2, v_2, \eta_2, r_2) \,\middle|\, u_2 \geq 0, v_2 \geq 0, r_2 \geq 0, \eta_2 = \delta^{-1}\right\}, \tag{21c}$$

with $0 < \rho, \delta \ll 1$ arbitrary, but fixed; obviously, $\kappa_{12}\left(\Sigma_1^{out}\right) = \Sigma_2^{in}$.

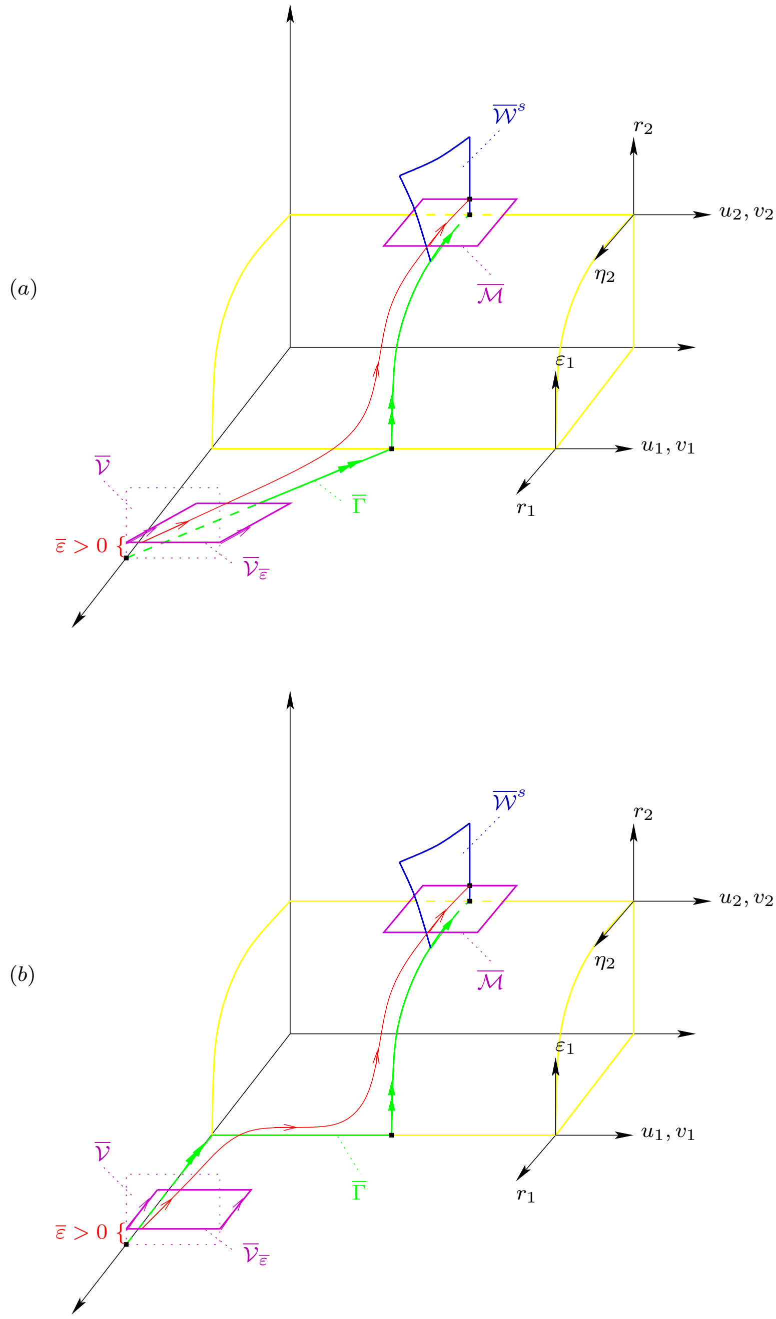


FIGURE 3. Geometry of the blown-up system for $(a)$ $n = 3$ and $(b)$ $n = 2$.

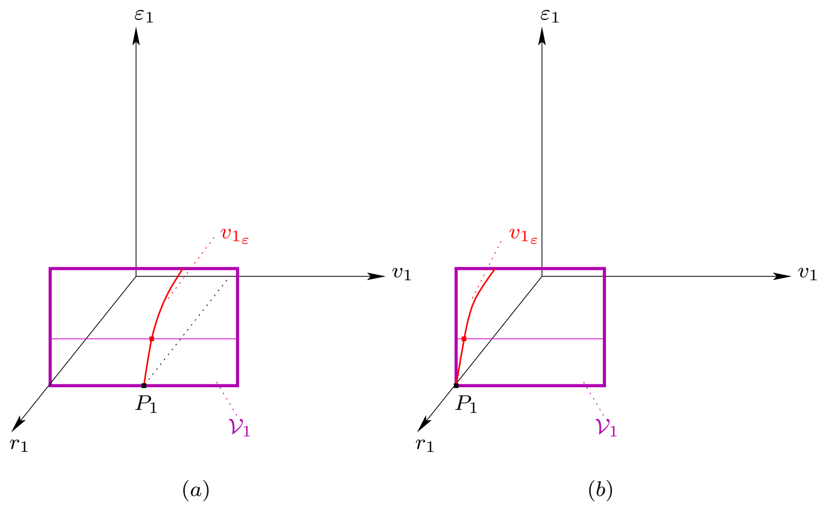


FIGURE 4. $v_{1_\varepsilon}$ for $(a)$ $n = 3$ and $(b)$ $n = 2$.

## 3. Rigorous asymptotic expansions

We now set out to derive asymptotic expansions for the function $v_{1_\varepsilon} := v_1\big|_{\xi=1}$, as defined by the unique solution to (4),(5) given in Theorem 2.2, see Figure 4. It is well-known that, to leading order, the method of matched asymptotic expansions gives

$$v_{1_\varepsilon} = 1 - \varepsilon \ln \varepsilon - (\gamma + 1)\varepsilon + \mathcal{O}(\varepsilon^2) \tag{22}$$

for $n = 3$ and

$$v_{1_\varepsilon} = -\frac{1}{\ln \varepsilon} + \frac{\gamma}{(\ln \varepsilon)^2} + \mathcal{O}\left(\frac{1}{(\ln \varepsilon)^3}\right) \tag{23}$$

for $n = 2$, respectively, see e.g. [Lag88]. Classically, the somewhat surprising occurrence of logarithmic terms in these expansions has been accounted for under the notion of switchback; we will show that these terms are in fact due to a resonance phenomenon. Incidentally, note that $v_{1_\varepsilon}$ equals $\frac{du}{d\xi}\big|_{\xi=1}$, the analogue of the drag on the solid, which is a quantity of considerable interest in the original fluid dynamical problem.

Our approach is rigorous in the sense that the expansions we compute are approximations to a well-defined geometric object, namely, to the invariant manifold $\overline{\mathcal{W}}^s$ introduced in the previous section. Roughly speaking, our strategy is the following: we begin by computing expansions for $\mathcal{W}_2^s$ in $K_2$; these expansions, when translated to $K_1$ and evaluated at the inner boundary there, will provide us with expansions for $v_{1_\varepsilon}$. We again distinguish between $n = 3$ and $n = 2$ here, the case $n = 3$ being considerably simpler.

***Remark* 3.** This strategy, which is slightly different from the strategy applied in [PS04], is somewhat more efficient as far as computing expansions for $v_{1_\varepsilon}$ is concerned. Later on, we will indicate how asymptotic *solution* expansions for (4),(5) can be obtained. $\square$

The complicated structure of the expansions in $K_1$ arises as $\overline{\mathcal{W}}^s$ passes near the line of equilibria $\ell_1$ in $K_1$. As indicated above, the logarithmic terms in (22) and (23), respectively, are due to the resonant eigenvalues $-1$, 0, and 1 which occur in $K_1$, see [PS04]. We now present a simple argument to substantiate our claim: for $n = 3$, consider the equations in $K_1$ given by (12). After introducing the new variable $\tilde{v}_1 = e^{\xi_1} v_1$, one obtains for the first two equations in (12)

$$
\begin{aligned}
u_1' &= e^{-\xi_1}\tilde{v}_1, \\
\tilde{v}_1' &= -\varepsilon_1 u_1 \tilde{v}_1.
\end{aligned} \tag{24}
$$

Integration of (24) yields[1]

$$
\begin{aligned}
u_1(\xi_1) &= u_{1_0} + \int_0^{\xi_1} e^{-\xi'}\tilde{v}_1(\xi')\,d\xi', \\
\tilde{v}_1(\xi_1) &= v_{1_0} - \varepsilon_{1_0}\int_0^{\xi_1} e^{\xi'} u_1(\xi')\tilde{v}_1(\xi')\,d\xi',
\end{aligned} \tag{25}
$$

where we have used $\varepsilon_1 = \varepsilon_{1_0} e^{\xi_1}$ and $u_{1_0}, v_{1_0}$, and $\varepsilon_{1_0}$ are constants. Note that near $\ell_1$, $v_1$ and $\varepsilon_1$ are small, which implies $\tilde{v}_1 = \mathcal{O}(1)$ there. Hence, a *Picard iteration scheme* can be applied to (25), with the starting point given by $\big(u_1^{(0)}, \tilde{v}_1^{(0)}\big) = (u_{1_0}, v_{1_0})$. In fact, one easily sees that (25) defines a contraction operator for $u_1$ and $\tilde{v}_1$ in $\mathcal{L}^\infty\big[0, \ln\frac{\varepsilon_1}{\varepsilon_{1_0}}\big]$, which ensures convergence of the scheme.[2] A straightforward computation gives

$$u_1^{(1)} = u_{1_0} + v_{1_0}\big(1 - e^{-\xi_1}\big), \tag{26a}$$

$$\tilde{v}_1^{(1)} = v_{1_0} + \varepsilon_{1_0} u_{1_0} v_{1_0}\big(1 - e^{\xi_1}\big), \tag{26b}$$

$$u_1^{(2)} = u_1^{(1)} + \varepsilon_{1_0} u_{1_0} v_{1_0}\big(1 - \xi_1 - e^{-\xi_1}\big), \tag{26c}$$

$$
\begin{aligned}
\tilde{v}_1^{(2)} = \tilde{v}_1^{(1)} &+ \varepsilon_{1_0} v_{1_0}^2\big(1 + \xi_1 - e^{\xi_1}\big) + \frac{1}{2}\varepsilon_{1_0} u_{1_0}^2 v_{1_0}\big(1 - 2e^{\xi_1} + e^{2\xi_1}\big) \\
&+ \frac{1}{2}\varepsilon_{1_0} u_{1_0} v_{1_0}^2\big(3 + 2\xi_1 - 4e^{\xi_1} + e^{2\xi_1}\big)
\end{aligned} \tag{26d}
$$

for the first two iterates in (25). As $\xi_1 = \ln\frac{\varepsilon_1}{\varepsilon_{1_0}}$, this then generates a logarithmic term in $\varepsilon_1$ after rewriting (26c) as a function of $\varepsilon_1$. Similarly, the products of powers of $\xi_1$ and $e^{\xi_1}$ which occur for higher iterates in (25) will give rise to products of powers of $\ln\varepsilon_1$ and $\varepsilon_1$ after those iterates have been rewritten in terms of $\varepsilon_1$.

In fact, it is thus possible to obtain successive approximations to the *transition map* $\Pi$ from $\Sigma_1^{in}$ to $\Sigma_1^{out}$ in (12), see [KS01]; the above computation gives the leading-order behavior of $\Pi$. Note that it is precisely the resonant terms in (12) which cannot be eliminated by a normal form transformation and which preclude the existence of a linearizing transformation for (12) proper, see e.g. [CLW94].

[1] Without loss of generality, we set $\xi_{1_0} = 0$ here.

[2] Incidentally, the introduction of $\tilde{v}_1$ in (12) is required for the proof of contractivity of (25).

### 3.1. The case $n = 3$.

**3.1.1.** *Expansions in chart $K_2$.* As the equations in $K_2$ are completely independent of $r_2$, we can simply omit the last equation in (15), which leaves us with the essentially three-dimensional system

$$\begin{aligned} u_2' &= v_2, \\ v_2' &= -2\eta_2 v_2 - u_2 v_2, \\ \eta_2' &= -\eta_2^2. \end{aligned} \tag{27}$$

***Remark* 4.** In contrast to what is usually done in the literature, we do not intend to derive asymptotic expansions for the *solutions* to (4) here, but rather for the *manifold* $\mathcal{W}^s$ as defined in Section 2. As the solutions to (4),(5) clearly do depend on $\varepsilon$, however, any ansatz aimed at obtaining solution expansions would of course have to take into account this dependence on $\varepsilon$. Note that in our approach, $\varepsilon$ enters only in chart $K_1$, see below. □

Given Proposition 2.3, we can make an ansatz for the expansion of $\mathcal{W}_2^s$ of the form

$$v_2(u_2, \eta_2) = \sum_{j=0}^{\infty} C_j(\eta_2)(u_2 - 1)^j, \tag{28}$$

where

$$C_j(\eta_2) := \frac{1}{j!}\frac{\partial^j}{\partial u_2^j} v_2(u_2, \eta_2)\Big|_{u_2=1}, \tag{29}$$

see [vGKS]. Hence,

$$\left| v_2(u_2, \eta_2) - \sum_{j=0}^{N} C_j(\eta_2)(u_2 - 1)^j \right| = \mathcal{O}\big((u_2 - 1)^{N+1}\big) \tag{30}$$

for any $N \in \mathbb{N}$, and the above estimate is uniform for $\eta_2$ bounded.

***Remark* 5.** An equally valid ansatz would be to set

$$u_2(v_2, \eta_2) = \sum_{j=0}^{\infty} D_j(\eta_2) v_2^j, \tag{31}$$

with

$$D_j(\eta_2) := \frac{1}{j!}\frac{\partial^j}{\partial v_2^j} u_2(v_2, \eta_2)\Big|_{v_2=0}. \tag{32}$$

However, the reason for considering (28) and not (31) is that, ultimately, we are interested in deriving an expansion for $v_{1_\varepsilon}$. Given Lemma 2.5, it is therefore the ansatz in (28) we have to use. □

Rewriting (27) with $\eta_2$ as the independent variable and omitting the subscript 2, we obtain

$$\frac{du}{d\eta} = -\frac{v}{\eta^2}, \tag{33a}$$

$$\frac{dv}{d\eta} = \frac{2}{\eta} v + \frac{u-1}{\eta^2} v + \frac{v}{\eta^2}; \tag{33b}$$

inserting (28) into (33b) yields

$$\sum_{j=0}^{\infty}\left[\frac{dC_j}{d\eta}(u-1)^j - C_j j(u-1)^{j-1}\left(\frac{1}{\eta^2}\sum_{k=0}^{\infty}C_k(u-1)^k\right)\right] \\ = \frac{2}{\eta}\sum_{j=0}^{\infty}C_j(u-1)^j + \frac{1}{\eta^2}\sum_{j=0}^{\infty}C_j(u-1)^{j+1} + \frac{1}{\eta^2}\sum_{j=0}^{\infty}C_j(u-1)^j, \tag{34}$$

where we have used (33a). Collecting powers of $u-1$ in (34) gives a recursive sequence of differential equations for the coefficient functions in (28),

$$C_1' - \frac{C_1}{\eta}\left(\frac{C_1}{\eta} + \frac{1}{\eta} + 2\right) = 0, \tag{35a}$$

$$C_j' - \frac{C_j}{\eta}\left(2 + \frac{1}{\eta}\right) - \frac{j+1}{\eta^2}C_1C_j = \frac{1}{\eta^2}C_{j-1} + \frac{1}{\eta^2}\sum_{\substack{k+l=j+1\\ k,l\geq 2}} kC_kC_l, \quad j \geq 2, \tag{35b}$$

with initial conditions

$$C_1(0) = -1, \quad C_2(0) = -\frac{1}{2}, \quad C_j(0) = 0, \ j \geq 3; \tag{36}$$

note that $C_0 \equiv 0$ due to $v(1) = 0$. The conditions in (36) are obtained from Remark 1, as $\mathcal{W}^{ss}$ is given by

$$v(u,0) = -(u-1) - \frac{1}{2}(u-1)^2. \tag{37}$$

We will first explicitly solve these equations for $j = 1$ and afterwards derive the general form of the solution for $j$ arbitrary.

***Remark* 6.** Most of the following computations have been performed with the help of the computer algebra package MAPLE, see e.g. [Cor02]. □

From (35a), it follows that

$$C_1(\eta) = -\frac{\eta^2}{\eta - e^{\eta^{-1}}\tilde{E}_1(\eta^{-1}) - \gamma_1 e^{\eta^{-1}}}, \tag{38}$$

where $\tilde{E}_k$ is in general defined by[3]

$$\tilde{E}_k(z) := \int_1^{\infty} e^{-z\tau}\tau^{-k}\,d\tau, \qquad z \in \mathbb{C}, \ \Re(z) > 0, \ k \in \mathbb{N}, \tag{39}$$

see e.g. [AS64], and $\gamma_1 \in \mathbb{R}$ is some constant that is to be determined. With (36) and de l'Hôspital's rule, we obtain $\lim_{\eta\to 0} C_1(\eta) = -1$ for $\gamma_1 = 0$; hence, $\gamma_1 = 0$. Due to

$$\tilde{E}_2\big(\eta^{-1}\big) = e^{-\eta^{-1}} - \eta^{-1}\tilde{E}_1\big(\eta^{-1}\big), \tag{40}$$

we can write

$$C_1(\eta) = -\frac{\eta e^{-\eta^{-1}}}{\tilde{E}_2(\eta^{-1})}. \tag{41}$$

[3]Here, $\Re(z)$ denotes the real part of $z$.

Since we are interested in (28) for $\eta \to \infty$ (which corresponds to the overlap domain between the two charts $K_1$ and $K_2$), we expand $\tilde{E}_2\left(\eta^{-1}\right)^{-1}$ about $\eta = \infty$ to obtain an indication as to what $C_j$ might look like in general:

$$\tilde{E}_2\left(\eta^{-1}\right)^{-1} = 1 + (1-\gamma)\eta^{-1} + \eta^{-1}\ln\eta + \left(\gamma^2 - 2\gamma + \frac{3}{2}\right)\eta^{-2} + 2\gamma\eta^{-2}\ln\eta + \eta^{-2}(\ln\eta)^2 + \mathcal{O}\left(\eta^{-3}\right), \tag{42}$$

which implies

$$C_1(\eta) = \eta e^{-\eta^{-1}} \sum_{k,l=0}^{\infty} \gamma_{kl}^1 \eta^{-k}(\ln\eta)^l. \tag{43}$$

We will show that $C_j$ can in fact be expanded as in (43) for any $j \in \mathbb{N}$. To that end, note that e.g. for $j = 2$, equation (35b) becomes

$$C_2' - \frac{C_2}{\eta}\left(2 + \frac{1}{\eta}\right) + \frac{3e^{-\eta^{-1}}}{\eta\tilde{E}_2(\eta^{-1})}C_2 = -\frac{e^{-\eta^{-1}}}{\eta\tilde{E}_2(\eta^{-1})}, \tag{44}$$

which has the solution

$$C_2(\eta) = \left(-\int \exp\left[3\int e^{-\eta'^{-1}}\eta'^{-1}\tilde{E}_2\left(\eta'^{-1}\right)^{-1}d\eta'\right]\eta^{-3}\tilde{E}_2\left(\eta^{-1}\right)^{-1}d\eta + \gamma_2\right) \times \eta^2 e^{-\eta^{-1}} \exp\left[-3\int e^{-\eta^{-1}}\eta^{-1}\tilde{E}_2\left(\eta^{-1}\right)^{-1}d\eta\right]; \tag{45}$$

here, we have used (41). Equation (45) obviously cannot be integrated in closed form. Still, one can derive the following result concerning the structure not only of $C_2$, but of any $C_j$ with $j \geq 2$:

**Proposition 3.1.** *For $j \geq 1$, the solution $C_j(\eta)$ to (35),(36) can be written as*

$$C_j(\eta) = \eta e^{-\eta^{-1}} \sum_{k,l=0}^{\infty} \gamma_{kl}^j \eta^{-k}(\ln\eta)^l. \tag{46}$$

*Here, $\gamma_{kl}^j \in \mathbb{R}$ are constants, to be determined from (36).*

*Proof.* The proof is by an induction argument: for $i = 1$, the assertion is obviously valid, see (43); let us assume that it holds for $i = 1, \ldots, j-1$. For the homogeneous solution to (35b), one finds

$$C_j^{hom}(\eta) = \gamma_j \eta^2 e^{-\eta^{-1}} \exp\left[-(j+1)\int e^{-\eta^{-1}}\eta^{-1}\tilde{E}_2\left(\eta^{-1}\right)^{-1}d\eta\right]; \tag{47}$$

the integrand in (47) can be expanded as

$$-\frac{e^{-\eta^{-1}}}{\eta\tilde{E}_2(\eta^{-1})} = -\eta^{-1} + \gamma\eta^{-2} - \eta^{-2}\ln\eta + \mathcal{O}\left(\eta^{-3}\right), \tag{48}$$

whence

$$\exp\left[(j+1)\int\left[-\eta^{-1} + \gamma\eta^{-2} - \eta^{-2}\ln\eta + \mathcal{O}\left(\eta^{-3}\right)\right]d\eta\right] = \mathcal{O}\left(\eta^{-j-1}\right), \quad j \geq 2. \tag{49}$$

For (47), the claim now follows from (42), (49), and the following lemma:

**Lemma 3.2.** *For any* $\alpha$, $\beta \in \mathbb{Z}$,

$$\int z^\alpha (\ln z)^\beta \, dz = \begin{cases} \dfrac{z^{\alpha+1}(\ln z)^\beta}{\alpha+1} - \dfrac{\beta}{\alpha+1}\displaystyle\int z^\alpha (\ln z)^{\beta-1}\, dz, & \alpha \neq -1 \\ \dfrac{(\ln z)^{\beta+1}}{\beta+1}, & \alpha = -1. \end{cases} \tag{50}$$

For the particular solution, note first that by the induction hypothesis, the right-hand side of (35b) can be written as

$$\frac{1}{\eta^2}C_{j-1} + \frac{1}{\eta^2}\sum_{\substack{k+l=j+1\\ k,l\geq 2}} kC_kC_l = e^{-\eta^{-1}} \sum_{m,n=0}^{\infty} \tilde{\gamma}_{mn}^j \eta^{-m}(\ln\eta)^n. \tag{51}$$

A particular solution to

$$C_j' - \frac{C_j}{\eta}\left(2+\frac{1}{\eta}\right) + (j+1)\frac{e^{-\eta^{-1}}}{\eta\tilde{E}_2(\eta^{-1})}C_j = e^{-\eta^{-1}}\eta^{-m}(\ln\eta)^n \tag{52}$$

is given by

$$\begin{aligned} C_j^{part}(\eta) = \int \eta^{-m-2}(\ln\eta)^n \exp\left[(j+1)\int e^{-\eta'^{-1}}\eta'^{-1}\tilde{E}_2\big(\eta'^{-1}\big)^{-1} d\eta'\right] d\eta \\ \times \eta^2 e^{-\eta^{-1}} \exp\left[-(j+1)\int e^{-\eta^{-1}}\eta^{-1}\tilde{E}_2\big(\eta^{-1}\big)^{-1} d\eta\right]; \end{aligned} \tag{53}$$

with (42), (49), and Lemma 3.2, this concludes the proof, as $m, n \geq 0$ and $j \geq 2$. □

We can even obtain a somewhat more precise result on the structure of $C_j$, $j \geq 1$. Let $(k,l)$ denote the *index* of a term $\eta^{-k}(\ln\eta)^l$ in (46); given this notation, we have the following

**Proposition 3.3.** *A term with index* $(k,l)$ *can occur in* (46) *only if* $l \leq k$.

*Proof.* The proof is again by induction: for $i = 1$, the assertion is obvious from (41) and (42). Given the assertion for $i = 1, \ldots, j-1$, it follows immediately from (49) and Lemma 3.2 that it holds for the homogeneous part (47) of $C_j$, as well. To prove the assertion for (53), we proceed as follows: as in [vGKS] we define a map, say, $\iota(m,n)$, which assigns to the index of a term in (51) the set of indices of the terms it generates in (53). By (49) and the proof of Proposition 3.1, one then easily sees that with Lemma 3.2,

$$\iota(m,n) = \begin{cases} \{(m+m', n+n'), (m+m', n+n'-1), \ldots, (m+m', 0)\}, & m \neq j \\ \{(j+m'+1, n+n'+1)\}, & m = j; \end{cases} \tag{54}$$

here, $m', n' \in \mathbb{N}$, with $n' \leq m'$. This completes the proof, as $n \leq m$ by assumption. □

**3.1**.2. *Expansions in chart* $K_1$. Given Proposition 3.1, we are able to derive asymptotic expansions for $\mathcal{W}_2^s$ in $K_2$ for $\eta_2 \to \infty$. To obtain the desired expansion for $v_{1_\varepsilon}$, however, we need to know what these expansions correspond to in $K_1$. First, note that with Lemma 2.5, (28) becomes

$$\varepsilon_1^{-1} v_1(u_1, \varepsilon_1) = \sum_{j=0}^{\infty} A_j(\varepsilon_1)(u_1-1)^j, \tag{55}$$

where

$$A_j(\varepsilon_1) = \frac{e^{-\varepsilon_1}}{\varepsilon_1} \sum_{\substack{k,l=0 \\ l \le k}}^{\infty} \alpha_{kl}^j \varepsilon_1^k (\ln \varepsilon_1)^l; \tag{56}$$

here, $\alpha_{kl}^j = (-1)^l \gamma_{kl}^j$. It remains to show that (56) does indeed make sense for $\varepsilon_1 \to 0$ (which is equivalent to $\eta_2 \to \infty$ in $K_2$). To that end, we assume that a curve of initial conditions in $\Sigma_1^{out}$ of the form

$$(u_1, v_1, \varepsilon_1) = \left(u_1^{out}, v_1^{out}\left(u_1^{out}\right), \delta\right), \qquad v_1^{out}(1) = 0 \tag{57}$$

is given, and we investigate the corresponding invariant manifold consisting of segments of solutions of (12). By variation of constants, integrating backwards from $\Sigma_1^{out}$, this manifold can be represented as follows:

$$u_1\left(\xi_1, u_1^{out}\right) = u_1^{out} - \int_{\xi_1}^{\Xi} v_1\left(\xi', u_1^{out}\right) d\xi', \tag{58a}$$

$$\begin{aligned} v_1\left(\xi_1, u_1^{out}\right) &= \frac{\delta}{\varepsilon} v_1^{out}\left(u_1^{out}\right) e^{-\xi_1} \\ &\quad + e^{-\xi_1} \int_{\xi_1}^{\Xi} e^{\xi'} \varepsilon_1(\xi') u_1\left(\xi', u_1^{out}\right) v_1\left(\xi', u_1^{out}\right) d\xi', \end{aligned} \tag{58b}$$

$$\varepsilon_1(\xi_1) = \varepsilon e^{\xi_1}, \tag{58c}$$

where $\Xi = \ln \frac{\delta}{\varepsilon}$. We have the following result:

**Proposition 3.4.** *Let $v_1^{out}\left(u_1^{out}\right)$ be $\mathcal{C}^k$-smooth for some $k \in \mathbb{N}$. Then, for $j = 0, \ldots, k$, $\frac{\partial^j}{\partial u_1^j} v_1(u_1, \varepsilon_1)$ exists and is continuous for $\varepsilon_1 \in [0, \delta]$ and $|u_1 - 1| \le \beta$, with $\beta > 0$ sufficiently small.*

*Proof.* Changing the integration variable to $\varepsilon_1$ in (58), we obtain

$$u_1(\varepsilon_1) = u_1^{out} - \int_{\varepsilon_1}^{\delta} v_1(u_1(\varepsilon'), \varepsilon') \, \frac{d\varepsilon'}{\varepsilon'}, \tag{59a}$$

$$v_1(u_1, \varepsilon_1) = \frac{\delta}{\varepsilon_1} v_1^{out} + \frac{1}{\varepsilon_1} \int_{\varepsilon_1}^{\delta} \varepsilon' u_1(\varepsilon') v_1(u_1(\varepsilon'), \varepsilon') \, d\varepsilon'; \tag{59b}$$

here, $\varepsilon' = \varepsilon e^{\xi'}$. Then, (59) together with

$$u_1(\varepsilon') \sim u_1^{out} + v_1^{out} \left(1 - \frac{\delta}{\varepsilon'}\right), \qquad v_1(u_1(\varepsilon'), \varepsilon') \sim \frac{\delta}{\varepsilon'} v_1^{out} \tag{60}$$

implies that $v_1$ remains continuous in $(u_1, \varepsilon_1)$ for $\varepsilon_1 \to 0$. Differentiating (59) formally with respect to $u_1$ yields

$$1 = \frac{du_1^{out}}{du_1} - \int_{\varepsilon_1}^{\delta} \frac{\partial v_1(u_1(\varepsilon'), \varepsilon')}{\partial u_1} \, d\varepsilon', \tag{61a}$$

$$\begin{aligned} \frac{\partial v_1(u_1, \varepsilon_1)}{\partial u_1} &= \frac{\delta}{\varepsilon_1} \frac{dv_1^{out}}{du_1} \\ &\quad + \frac{1}{\varepsilon_1} \int_{\varepsilon_1}^{\delta} \varepsilon' \left[ \frac{du_1(\varepsilon')}{du_1} v_1(u_1(\varepsilon'), \varepsilon') + u_1(\varepsilon') \frac{\partial v_1(u_1(\varepsilon'), \varepsilon')}{\partial u_1} \right] d\varepsilon'; \end{aligned} \tag{61b}$$

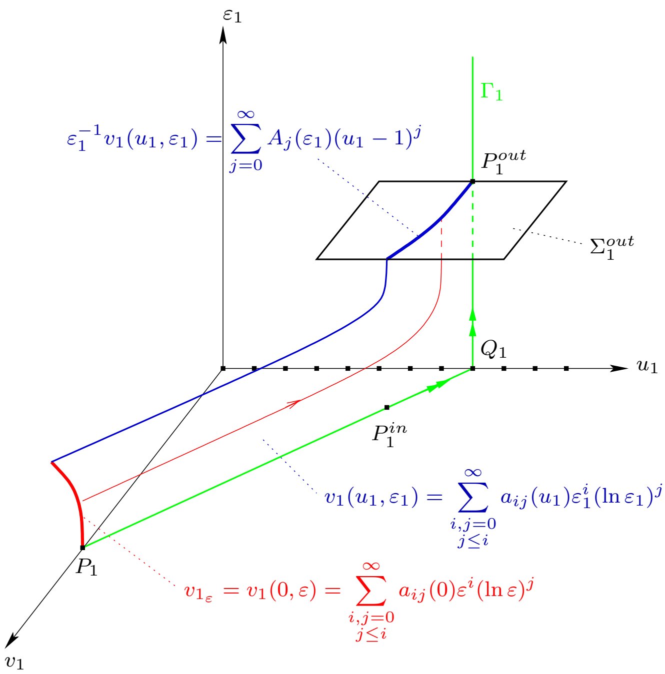


FIGURE 5. Strategy for deriving an expansion for $v_{1_\varepsilon}$ in $K_1$ $(n = 3)$.

as we have $\frac{d\varepsilon'}{d\varepsilon_1} = \frac{\varepsilon'}{\varepsilon_1}$, it follows that

$$\frac{du_1(\varepsilon')}{du_1} = \frac{v_1(u_1(\varepsilon'), \varepsilon')}{v_1(u_1(\varepsilon_1), \varepsilon_1)}, \tag{62}$$

whence

$$\frac{\partial v_1(u_1(\varepsilon'), \varepsilon')}{\partial u_1} = \frac{\partial v_1(u_1(\varepsilon'), \varepsilon')}{\partial u_1(\varepsilon')} \frac{v_1(u_1(\varepsilon'), \varepsilon')}{v_1(u_1(\varepsilon_1), \varepsilon_1)}. \tag{63}$$

This, together with (61a), gives us a formula for $\frac{dv_1^{out}}{du_1} = \frac{dv_1^{out}}{du_1^{out}} \frac{du_1^{out}}{du_1}$,

$$\frac{dv_1^{out}}{du_1} = \frac{dv_1^{out}}{du_1^{out}} \left( 1 + \frac{1}{v_1(u_1, \varepsilon_1)} \int_{\varepsilon_1}^{\delta} \frac{\partial v_1(u_1(\varepsilon'), \varepsilon')}{\partial u_1(\varepsilon')} \frac{v_1(u_1(\varepsilon'), \varepsilon')}{\varepsilon'} \, d\varepsilon' \right); \tag{64}$$

in sum, we obtain

$$\begin{aligned} \frac{\partial v_1(u_1, \varepsilon_1)}{\partial u_1} &= \frac{\delta}{\varepsilon_1} \frac{dv_1^{out}}{du_1^{out}} \left( 1 + \frac{1}{v_1(u_1, \varepsilon_1)} \int_{\varepsilon_1}^{\delta} \frac{\partial v_1(u_1(\varepsilon'), \varepsilon')}{\partial u_1(\varepsilon')} \frac{v_1(u_1(\varepsilon'), \varepsilon')}{\varepsilon'} \, d\varepsilon' \right) \\ &+ \frac{1}{\varepsilon_1 v_1(u_1, \varepsilon_1)} \int_{\varepsilon_1}^{\delta} \varepsilon' \left[ v_1(u_1(\varepsilon'), \varepsilon') + u_1(\varepsilon') \frac{\partial v_1(u_1(\varepsilon'), \varepsilon')}{\partial u_1(\varepsilon')} \right] v_1(u_1(\varepsilon'), \varepsilon') \, d\varepsilon'. \end{aligned} \tag{65}$$

Suppose now that $v_1^{out}(u_1^{out})$ is $\mathcal{C}^1$-smooth; using a standard fixed point argument, one can show that (65) has a unique solution $\frac{\partial v_1(u_1, \varepsilon_1)}{\partial u_1}$ which is continuous in $(u_1, \varepsilon_1)$. This concludes the proof for $k = 1$; the argument for $k \ge 2$ is similar. □

Inspired by (55) and (56), we attempt an expansion for $v_1(u_1, \varepsilon_1)$ as

$$v_1(u_1, \varepsilon_1) = \sum_{\substack{i,j=0 \\ j\le i}}^{\infty} a_{ij}(u_1)\varepsilon_1^i(\ln\varepsilon_1)^j, \tag{66}$$

see Figure 5; the requirement that $j \le i$ in the summation in (66) is a consequence of Proposition 3.3. The coefficient functions $a_{ij}$, $0 \le j \le i$, will be determined uniquely by the demand for smoothness in $u_1$ and by the requirement that, seen as a double expansion, (66) should agree with (55). To that end, let us introduce $u_1$ as the independent variable in (12), whence[4]

$$\begin{aligned} \frac{dv}{du} &= -1 - \varepsilon u, \\ \frac{d\varepsilon}{du} &= \frac{\varepsilon}{v}. \end{aligned} \tag{67}$$

***Remark* 7.** With (31) instead of (28) in $K_2$, we would now have

$$u(v, \varepsilon) = \sum_{\substack{i,j=0 \\ j\le i}}^{\infty} b_{ij}(v)\varepsilon^i(\ln\varepsilon)^j \tag{68}$$

instead of (66). One can easily check that the following considerations would then remain valid, with only a few minor adjustments required. □

Proceeding as in $K_2$ and multiplying the resulting equations with $v$, we obtain

$$\begin{aligned} \sum_{\substack{i,j=0 \\ j\le i}}^{\infty} \left[ a_{ij}'\varepsilon^i(\ln\varepsilon)^j \left( \sum_{\substack{k,l=0 \\ l\le k}}^{\infty} a_{kl}\varepsilon^k(\ln\varepsilon)^l \right) + a_{ij}\varepsilon^i(\ln\varepsilon)^{j-1}(i\ln\varepsilon + j) \right] \\ = (-1-\varepsilon u)\sum_{\substack{i,j=0 \\ j\le i}}^{\infty} a_{ij}\varepsilon^i(\ln\varepsilon)^j, \end{aligned} \tag{69}$$

where $' = \frac{d}{du}$ now. We will not look for the general solution to (69) at this point, but will for the moment only consider the first few terms in (66); hence, for $0 \le j \le i \le 2$,

$$a_{00}'a_{00} = -a_{00}, \tag{70a}$$

$$a_{11}'a_{00} + a_{00}'a_{11} + a_{11} = -a_{11}, \tag{70b}$$

$$a_{10}'a_{00} + a_{00}'a_{10} + a_{10} + a_{11} = -a_{10} - ua_{00}, \tag{70c}$$

$$a_{22}'a_{00} + a_{11}'a_{11} + a_{00}'a_{22} + 2a_{22} = -a_{22}, \tag{70d}$$

$$a_{21}'a_{00} + a_{11}'a_{10} + a_{10}'a_{11} + a_{00}'a_{21} + 2a_{21} + 2a_{22} = -a_{21} - ua_{11}, \tag{70e}$$

$$a_{20}'a_{00} + a_{10}'a_{10} + a_{00}'a_{20} + 2a_{20} + a_{21} = -a_{20} - ua_{10}. \tag{70f}$$

Note that these equations can be solved recursively: (70a) yields either $a_{00} \equiv 0$ or $a_{00}' = -1$; however, for (55) and (66) to agree when seen as double expansions, we have to take the latter, see (41), whence

$$a_{00} = -u + \alpha^{00}. \tag{71}$$

[4] Again, the subscript 1 will be omitted.

Here, $\alpha^{00}$ is a constant which is to be determined from (55); indeed, it follows from (41) and (42) that an expansion for $v$ is given by

$$v(u,\varepsilon) = \left[-1 + \gamma\varepsilon + \varepsilon\ln\varepsilon + \mathcal{O}(\varepsilon^2)\right](u-1) + \mathcal{O}\big((u-1)^2\big). \tag{72}$$

To obtain agreement to lowest order between (55) and (66), we hence have to take $\alpha^{00} = 1$.
Substituting (71) into (70b) and solving the resulting equation

$$-a_{11}'(u-1) + a_{11} = 0, \tag{73}$$

one then has

$$a_{11} = \alpha^{11}(u-1). \tag{74}$$

Similarly, (71) and (74) together with (70c) give

$$-a_{10}'(u-1) + a_{10} = (u-\alpha^{11})(u-1), \tag{75}$$

which has the solution

$$a_{10} = -(u-1)^2 - (1-\alpha^{10})(u-1) - (1-\alpha^{11})(u-1)\ln(u-1); \tag{76}$$

for (76) to be smooth, $\alpha^{11}$ has to be chosen such that the $\ln(u-1)$-terms in (77) vanish, which implies $\alpha^{11} = 1$. The requirement that (55) and (66) should agree then gives $\alpha^{10} = 1+\gamma$, see (72):

$$a_{10} = -(u-1)^2 + \gamma(u-1). \tag{77}$$

For (70d), (70e), and (70f), one obtains by the same procedure

$$a_{22} = (u-1)^2 - (u-1), \tag{78a}$$

$$a_{21} = (2\gamma+1)(u-1)^2 - (2\gamma-1)(u-1), \tag{78b}$$

$$a_{20} = (u-1)^3 + (1+\alpha^{20})(u-1)^2 - (\gamma^2-\gamma+1)(u-1), \tag{78c}$$

where $\alpha^{22} = 1$ and $\alpha^{21} = 2\gamma+1$ have again been chosen such that the $\ln(u-1)$-terms in (78b) and (78c) cancel, and $\alpha^{20}$ has to be determined by comparing the leading-order terms in (55) and (66), see Figure 6.

***Remark* 8.** The above procedure is in fact closely related to the approach one would classically take when matching (55) and (66). As these two expansions have to agree on the overlap domain between the two charts $K_1$ and $K_2$, it is there one would have to define an *intermediate variable*. Note that in [HTB90], say, logarithmic switchback is handled using a modified version of the block matching principle introduced by [vD75]: terms are matched in blocks according to the powers of $\varepsilon$ they contain, with no distinction being made for any additional logarithmic factors. Our approach seems to justify this principle, as the logarithmic factors in (66) are determined simply by the requirement that $a_{ij}$ be smooth. □

One expects, of course, that the above procedure can be carried out to any order in $i$ and $j$ in (69), where, for $i$ fixed, one starts with $j = i$ and then proceeds recursively down to $j = 0$. That this is indeed possible is contained in the following result:

**Proposition 3.5.** *There exist unique smooth functions $a_{ij}(u_1)$ such that* (55) *and* (66)*, seen as double expansions, are the same.*

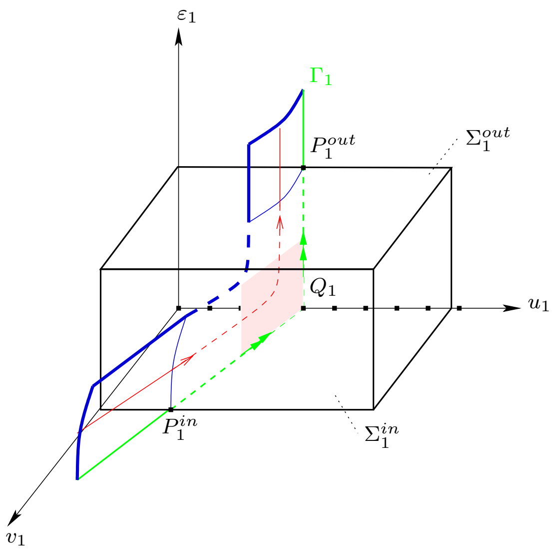


FIGURE 6. Overlap domain (shaded) of expansions (55) and (66).

*Proof.* We proceed as in the computation of the first few coefficients in (66) above: for fixed $i$, we successively solve (69), starting with $j = i$. By induction, we will establish

$$a_{ij} = \sum_{k=1}^{i} \alpha_k^{ij}(u-1)^k, \qquad 1 \le j \le i, \tag{79a}$$

$$a_{i0} = \sum_{k=1}^{i+1} \alpha_k^{i0}(u-1)^k, \qquad j = 0 \tag{79b}$$

for any $i \ge 1$ and $0 \le j \le i$, with constants $\alpha_k^{ij} \in \mathbb{R}$ that have to be chosen appropriately. Indeed, for $i = 1$, the claim follows from (74) and (77) by inspection. Let us assume that (79) is valid for $a_{kl}$, where $k = 1, \dots, i-1$ and $l \le k$. Substituting (71) into (69) and collecting powers of $\varepsilon \ln \varepsilon$, one obtains

$$-a'_{ij}(u-1) + i a_{ij} = -(j+1)a_{i,j+1} - u a_{i-1,j} - \sum_{\substack{k+m=i \\ l+n=j \\ l \le k \le i-1,\ n \le m \le i-1}} a'_{kl} a_{mn}; \tag{80}$$

here, we have used $a'_{00} = -1$. The homogeneous solution to (80) is given by

$$\alpha^{ij}(u-1)^i, \qquad 0 \le j \le i, \tag{81}$$

with some constant $\alpha^{ij}$; to complete the proof, we have to consider the following cases:

- for $j = i$, equation (80) becomes

$$-a'_{ii}(u-1) + i a_{ii} = -\sum_{\substack{k+m=i \\ k,m \ge 1}} a'_{kk} a_{mm}, \tag{82}$$

which has the solution

$$a_{ii} = \alpha^{ii}(u-1)^i + \sum_{k=1}^{i-1} \alpha_k^{ii}(u-1)^k, \tag{83}$$

since, by (79),

$$-\sum_{\substack{k+m=i \\ k,m\geq 1}} a'_{kk}a_{mm} = -\sum_{k=1}^{i-1} \tilde{\alpha}_k^{ii}(u-1)^k, \tag{84}$$

and since the term $-\tilde{\alpha}_k^{ii}(u-1)^k$ gives a particular solution of the form

$$-\frac{\tilde{\alpha}_k^{ii}}{i-k}(u-1)^k, \qquad 1 \leq k \leq i-1; \tag{85}$$

the constant $\alpha^{ii}$ remains to be determined in one of the next steps.

- for $j = i-1$ (which is indeed representative of all further cases), one obtains

$$\begin{aligned} -a'_{i,i-1}(u-1) + ia_{i,i-1} = &-ia_{ii} - ua_{i-1,i-1} \\ &- \sum_{\substack{k+m=i \\ k,m\geq 1}} \left[a'_{kk}a_{m,m-1} + a_{kk}a'_{m,m-1}\right], \end{aligned} \tag{86}$$

where the homogeneous solution is again given by (81). As for the inhomogeneity, note that terms of the form $-i\alpha_k^{ii}(u-1)^k$ in $-ia_{ii}$ generate particular solutions of the form

$$i\alpha^{ii}(u-1)^i \ln(u-1), \qquad k = i, \tag{87a}$$

$$-\frac{i\alpha_k^{ii}}{i-k}(u-1)^k, \qquad 1 \leq k \leq i-1. \tag{87b}$$

Similarly, for the terms $-\alpha_k^{i-1,i-1}u(u-1)^k$ in $-ua_{i-1,i-1}$, one obtains

$$\alpha_{i-1}^{i-1,i-1}(u-1)^{i-1}\big((u-1)\ln(u-1) - 1\big), \qquad k = i-1, \tag{88a}$$

$$-\frac{\alpha_{i-2}^{i-1,i-1}}{(i-k)(i-k-1)}(u-1)^k\big(-1 + (i-k)u\big), \qquad 1 \leq k \leq i-2. \tag{88b}$$

By the induction hypothesis, for the remaining terms, one has

$$-\sum_{\substack{k+m=i \\ k,m\geq 1}} \left[a'_{kk}a_{m,m-1} + a_{kk}a'_{m,m-1}\right] = -\sum_{k=0}^{i} \tilde{\alpha}_k^{i,i-1}(u-1)^k, \tag{89}$$

with constants $\tilde{\alpha}_k^{i,i-1} \in \mathbb{R}$. Just as above, the terms $-\tilde{\alpha}_k^{i,i-1}(u-1)^k$ give rise to terms of the form

$$\tilde{\alpha}_i^{i,i-1}(u-1)^i \ln(u-1), \qquad k = i, \tag{90a}$$

$$-\frac{\tilde{\alpha}_k^{i,i-1}}{i-k}(u-1)^k, \qquad 1 \leq k \leq i-1. \tag{90b}$$

In sum, one thus has

$$a_{i,i-1} = \alpha^{i,i-1}(u-1)^i + \sum_{k=1}^{i-1} \alpha_k^{i,i-1}(u-1)^k + \big(i\alpha^{ii} + \alpha_{i-1}^{i-1,i-1} + \tilde{\alpha}_i^{i,i-1}\big) \ln{(u-1)}(u-1)^i, \tag{91}$$

where $\alpha^{ii}$ is now chosen such that (91) is smooth, i.e., such that the $\ln{(u-1)}$-terms cancel, and $\alpha^{i,i-1}$ is still at our disposal.

- for $1 \leq j \leq i-2$ in general, one repeats the same procedure, i.e., one solves (80) and subsequently fixes $\alpha^{i,j+1}$ appropriately so as to eliminate any $\ln{(u-1)}$-terms in $a_{ij}$, guaranteeing the smoothness of $a_{ij}$.
- in the final step, for $j = 0$, additional terms of $\mathcal{O}\big((u-1)^{i+1}\big)$ are generated as claimed, due to the terms $-\alpha_i^{i-1,0}u(u-1)^i$ and $-\tilde{\alpha}_i^{i0}(u-1)^{i+1}$ in the right-hand side of (80), giving

$$\alpha_i^{i-1,0}\big(u + \ln{(u-1)}\big)(u-1)^i \tag{92}$$

and

$$\tilde{\alpha}_i^{i0}u(u-1)^i \tag{93}$$

in $a_{i0}$, respectively. One is then left with $\alpha^{i1}$ and $\alpha^{i0}$, which one chooses such as to make sure that $a_{i0}$ is smooth and that (55) and (66) agree if both are seen as double expansions, which concludes the proof.

□

We are now ready to formulate the main result of this section, namely, to give an expansion for $v_{1_\varepsilon}$ when $n = 3$:

**Proposition 3.6.** *For $\varepsilon \in (0, \varepsilon_0]$, with $\varepsilon_0 > 0$ sufficiently small, $v_{1_\varepsilon} = v_1(0, \varepsilon)$ can be expanded as*

$$v_{1_\varepsilon} = 1 - \varepsilon \ln \varepsilon - (\gamma + 1)\varepsilon + 2\varepsilon^2 (\ln \varepsilon)^2 + 4\gamma\varepsilon^2 \ln \varepsilon + \mathcal{O}(\varepsilon^3). \tag{94}$$

*Proof.* Substituting (71), (74), (77), (78a), and (78b) into (66), one obtains the following expansion for $v_1$:

$$\begin{aligned} v_1(u_1, \varepsilon_1) = &-(u_1 - 1) + (u_1 - 1)\varepsilon_1 \ln \varepsilon_1 \\ &+ \big[-(u_1-1)^2 + \gamma(u_1-1)\big]\,\varepsilon_1 + \big[(u_1-1)^2 - (u_1-1)\big]\,\varepsilon_1^2 (\ln \varepsilon_1)^2 \\ &+ \big[(2\gamma+1)(u_1-1)^2 - (2\gamma-1)(u_1-1)\big]\,\varepsilon_1^2 \ln \varepsilon_1 + \mathcal{O}\big(\varepsilon_1^3\big). \end{aligned} \tag{95}$$

The assertion now follows with $u_1 = 0$ and $\varepsilon_1 = \varepsilon$ in (95). □

***Remark* 9.** Note that the expansion in (94) is equally valid in the original setting of (4), i.e., $v_\varepsilon = v_{1_\varepsilon}$ holds, as is easily seen by performing the appropriate blow-down transformation, which is trivial here. Analogous results have been obtained in the literature, see e.g. [Lag88]. □

**3.2. The case $n = 2$.** Although the case $n = 2$ is potentially more difficult, we can use the same strategy as before to compute expansions for $v_{1_\varepsilon}$. Not surprisingly, however, the analysis is computationally more involved now, due to the extensive switchback which arises in the matching process for $n = 2$.

**3.2**.1. *Expansions in chart $K_2$.* As the situation in $K_2$ is very similar to that for $n=3$, we will not go into too many details: given the ansatz

$$v_2(u_2,\eta_2)=\sum_{j=0}^{\infty}C_j(\eta_2)(u_2-1)^j, \tag{96}$$

which we substitute into (27) rewritten with $\eta_2$ as the independent variable,

$$\frac{du}{d\eta}=-\frac{v}{\eta^2}, \tag{97a}$$

$$\frac{dv}{d\eta}=\frac{v}{\eta}+\frac{u-1}{\eta^2}v+\frac{v}{\eta^2}, \tag{97b}$$

we obtain a recursive sequence of equations for $C_j$, $j\geq 1$, by comparing powers of $u-1$:

$$C_1'-\frac{C_1}{\eta}\left(\frac{C_1}{\eta}+\frac{1}{\eta}+1\right)=0, \tag{98a}$$

$$C_j'-\frac{C_j}{\eta}\left(1+\frac{1}{\eta}\right)-\frac{j+1}{\eta^2}C_1C_j=\frac{1}{\eta^2}C_{j-1}+\frac{1}{\eta^2}\sum_{\substack{k+l=j+1\\k,l\geq 2}}kC_kC_l,\quad j\geq 2. \tag{98b}$$

The initial conditions are again given by

$$C_1(0)=-1,\quad C_2(0)=-\frac{1}{2},\quad C_j(0)=0,\ j\geq 3; \tag{99}$$

moreover, $C_0\equiv 0$, as was the case for $n=3$. From (98a), we have

$$C_1(\eta)=-\frac{\eta e^{-\eta^{-1}}}{\tilde{E}_1(\eta^{-1})+\gamma_1}; \tag{100}$$

as $\lim_{\eta\to 0}C_1(\eta)=-1$ for $\gamma_1=0$, we conclude that, again, $\gamma_1=0$. The expansion of $\tilde{E}_1\big(\eta^{-1}\big)^{-1}$ about $\eta=\infty$ is given by

$$\begin{aligned}\tilde{E}_1\big(\eta^{-1}\big)^{-1}=(\ln\eta-\gamma)^{-1}-\eta^{-1}(\ln\eta-\gamma)^{-2}+\frac{1}{4}\eta^{-2}(\ln\eta-\gamma)^{-2}\\+\eta^{-2}(\ln\eta-\gamma)^{-3}+\mathcal{O}\big(\eta^{-3}\big),\end{aligned} \tag{101}$$

whence

$$C_1(\eta)=\eta e^{-\eta^{-1}}\sum_{k,l=0}^{\infty}\gamma_{kl}^1\eta^{-k}(\ln\eta-\gamma)^{-l}; \tag{102}$$

here, $\gamma$ is Euler's constant, as before. For $j=2$, (98b) gives

$$C_2'-\frac{C_2}{\eta}\left(1+\frac{1}{\eta}\right)+\frac{3e^{-\eta^{-1}}}{\eta\tilde{E}_1(\eta^{-1})}C_2=-\frac{e^{-\eta^{-1}}}{\eta\tilde{E}_1(\eta^{-1})}, \tag{103}$$

which has the solution

$$C_2(\eta)=\left(-\int\eta^{-2}\tilde{E}_1\big(\eta^{-1}\big)^2d\eta+\gamma_2\right)\eta e^{-\eta^{-1}}\tilde{E}_1\big(\eta^{-1}\big)^{-3}. \tag{104}$$

Although (104) cannot be integrated in closed form, we still have the following result, which is similar to the one obtained for $n=3$:

**Proposition 3.7.** *For $j \geq 1$, the solution $C_j(\eta)$ to* (98),(99) *can be written as*

$$C_j(\eta) = \eta e^{-\eta^{-1}} \sum_{k,l=0}^{\infty} \gamma_{kl}^j \eta^{-k} (\ln \eta - \gamma)^{-l}. \tag{105}$$

*Here, $\gamma_{kl}^j \in \mathbb{R}$ are constants to be determined from* (36).

*Proof.* The proof is very similar to that of Proposition 3.1: for $i = 1$, the assertion holds by (100); let it be valid for $i = 1, \ldots, j-1$. The homogeneous solution to (98b) is given by

$$C_j^{hom}(\eta) = \gamma_j \frac{\eta e^{-\eta^{-1}}}{\tilde{E}_1(\eta^{-1})^{j+1}}, \tag{106}$$

where $\tilde{E}_1\big(\eta^{-1}\big)^{-j-1}$ can be expanded as

$$\begin{aligned} \tilde{E}_1\big(\eta^{-1}\big)^{-j-1} &= (\ln \eta - \gamma)^{-j-1} - (j+1)\eta^{-1}(\ln \eta - \gamma)^{-j-2} \\ &+ \frac{1}{4}(j+1)\eta^{-2}(\ln \eta - \gamma)^{-j-2} + \frac{1}{2}(j+1)(j+2)\eta^{-2}(\ln \eta - \gamma)^{-j-3} + \mathcal{O}\big(\eta^{-3}\big). \end{aligned} \tag{107}$$

Given the induction hypothesis, the right-hand side of (98b) has the following form:

$$\frac{1}{\eta^2} C_{j-1} + \frac{1}{\eta^2} \sum_{\substack{k+l=j+1 \\ k,l \geq 1}} k C_k C_l = e^{-\eta^{-1}} \sum_{m,n=0}^{\infty} \tilde{\gamma}_{mn}^j \eta^{-m} (\ln \eta - \gamma)^{-n}; \tag{108}$$

a particular solution of (98b), corresponding to a term $e^{-\eta^{-1}} \eta^{-m} (\ln \eta - \gamma)^{-n}$, is given by

$$C_j^{part}(\eta) = \int \eta^{-m-1} (\ln \eta - \gamma)^{-n} \tilde{E}_1\big(\eta^{-1}\big)^{j+1} d\eta \cdot \eta e^{-\eta^{-1}} \tilde{E}_1\big(\eta^{-1}\big)^{-j-1}. \tag{109}$$

With the substitution $\eta' = \frac{\eta}{\gamma}$ and (107), the integrand in (109) can be written as

$$\sum_{\substack{m'=m+1 \\ n'=n-j-1}}^{\infty} \tilde{\gamma}_{m'n'}^{mn} \eta'^{-m'} (\ln \eta')^{-n'}; \tag{110}$$

the claim now follows from Lemma 3.2, with $\eta'$ again replaced by $\eta$. □

Note, however, that we can state no analogue to Proposition 3.3 here; in fact, it seems that any index pair $(k, l)$ can occur in (66) now.

**3.2**.2. *Expansions in chart $K_1$.* As was the case for $n = 3$, it now remains to translate (28) to $K_1$ in order to obtain an expansion for $v_{1_\varepsilon}$, see Figure 7. Lemma 2.5 again gives

$$\varepsilon_1^{-1} v_1(u_1, \varepsilon_1) = \sum_{j=0}^{\infty} A_j(\varepsilon_1)(u_1 - 1)^j, \tag{111}$$

see (55), where

$$A_j(\varepsilon_1) = \frac{e^{-\varepsilon_1}}{\varepsilon_1} \sum_{k,l=0}^{\infty} \alpha_{kl}^j \frac{\varepsilon_1^k}{(\ln \varepsilon_1 + \gamma)^l}, \tag{112}$$

with $\alpha_{kl}^j = (-1)^l \gamma_{kl}^j$.

***Remark* 10** (Transcendentally small terms)**.** Terms in powers of $\varepsilon_1$, as well as terms in powers of $\varepsilon_1$ multiplied by powers of $(\ln\varepsilon_1+\gamma)^{-1}$, are smaller than all positive powers of $(\ln\varepsilon_1+\gamma)^{-1}$: they are said to be beyond all orders of $(\ln\varepsilon_1+\gamma)^{-1}$, or to be transcendentally small terms. In the original setting of flow around a circular cylinder, [Ski75] showed how to calculate a few of these terms. He pointed out, however, that they are in fact negligible: it is the only very slight asymmetry in the flow field which indicates the relative insignificance of these inertial terms for low Reynolds numbers, see [Ski75] for a detailed analysis. □

As for $n=3$, using variation of constants, we can again write

$$u_1\big(\xi_1,u_1^{out}\big)=u_1^{out}-\int_{\xi_1}^{\Xi}v_1\big(\xi',u_1^{out}\big)d\xi', \tag{113a}$$

$$v_1\big(\xi_1,u_1^{out}\big)=v_1^{out}\big(u_1^{out}\big)+\int_{\xi_1}^{\Xi}\varepsilon_1(\xi')u_1\big(\xi',u_1^{out}\big)v_1\big(\xi',u_1^{out}\big)d\xi', \tag{113b}$$

$$\varepsilon_1(\xi_1)=\varepsilon e^{\xi_1} \tag{113c}$$

for the manifold consisting of segments of solutions to (12), given the initial curve (57). In analogy to Proposition 3.4 we now have

**Proposition 3.8.** *Let $v_1^{out}\big(u_1^{out}\big)$ be $\mathcal{C}^k$-smooth for some $k\in\mathbb{N}$. Then, for $j=0,\dots,k$, $\frac{\partial^j}{\partial u_1^j}v_1(u_1,\varepsilon_1)$ exists and is continuous for $\varepsilon_1\in[0,\delta]$ and $|u_1-1|\le\beta$, with $\beta>0$ sufficiently small.*

*Proof.* The proof is the same as for $n=3$, with the relevant relations given by

$$u_1(\varepsilon_1)=u_1^{out}-\int_{\varepsilon_1}^{\delta}v_1(u_1(\varepsilon'),\varepsilon')\,\frac{d\varepsilon'}{\varepsilon'}, \tag{114a}$$

$$v_1(u_1,\varepsilon_1)=\frac{\delta}{\varepsilon_1}v_1^{out}+\frac{1}{\varepsilon_1}\int_{\varepsilon_1}^{\delta}u_1(\varepsilon')v_1(u_1(\varepsilon'),\varepsilon')\,d\varepsilon' \tag{114b}$$

and

$$\begin{aligned}\frac{\partial v_1(u_1,\varepsilon_1)}{\partial u_1}&=\frac{dv_1^{out}}{du_1^{out}}\left(1+\frac{1}{v_1(u_1,\varepsilon_1)}\int_{\varepsilon_1}^{\delta}\frac{\partial v_1(u_1(\varepsilon'),\varepsilon')}{\partial u_1(\varepsilon')}\frac{v_1(u_1(\varepsilon'),\varepsilon')}{\varepsilon'}\,d\varepsilon'\right)\\&\quad+\frac{1}{v_1(u_1,\varepsilon_1)}\int_{\varepsilon_1}^{\delta}\left[v_1(u_1(\varepsilon'),\varepsilon')+u_1(\varepsilon')\frac{\partial v_1(u_1(\varepsilon'),\varepsilon')}{\partial u_1(\varepsilon')}\right]v_1(u_1(\varepsilon'),\varepsilon')\,d\varepsilon'\end{aligned} \tag{115}$$

now, respectively. □

To derive an expansion for $v_1(u_1,\varepsilon_1)$ of the form

$$v_1(u_1,\varepsilon_1)=\sum_{i,j=0}^{\infty}a_{ij}(u_1)\frac{\varepsilon_1^i}{(\ln\varepsilon_1+\gamma)^j}, \tag{116}$$

we have to consider

$$\begin{aligned}\frac{dv}{du}&=-\varepsilon u,\\\frac{d\varepsilon}{du}&=\frac{\varepsilon}{v},\end{aligned} \tag{117}$$

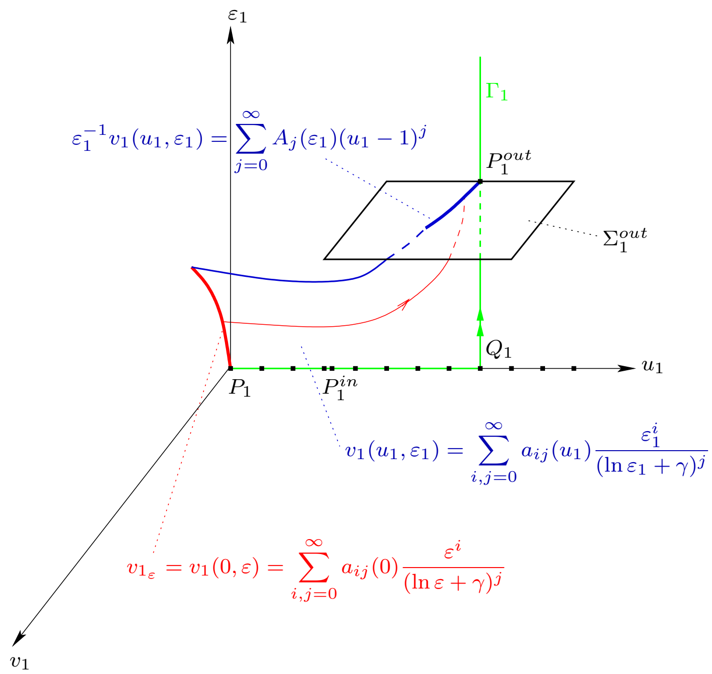


FIGURE 7. Strategy for deriving an expansion for $v_{1_\varepsilon}$ in $K_1$ ($n = 2$).

which yields

(118)
$$\sum_{i,j=0}^{\infty}\left[a'_{ij}\frac{\varepsilon^i}{(\ln\varepsilon+\gamma)^j}\left(\sum_{k,l=0}^{\infty}a_{kl}\frac{\varepsilon^k}{(\ln\varepsilon+\gamma)^l}\right)+a_{ij}\frac{\varepsilon^i}{(\ln\varepsilon+\gamma)^j}\left(i-\frac{j}{\ln\varepsilon+\gamma}\right)\right]$$
$$=-\varepsilon u\sum_{i,j=0}^{\infty}a_{ij}\frac{\varepsilon^i}{(\ln\varepsilon+\gamma)^j}.$$

Collecting powers of $\varepsilon(\ln\varepsilon+\gamma)^{-1}$ in (118), we obtain the following sequence of equations for $a_{ij}$, with $0 \le i+j \le 3$:

$$a'_{00}a_{00} = 0, \tag{119a}$$
$$a'_{01}a_{00} + a'_{00}a_{01} = 0, \tag{119b}$$
$$a'_{10}a_{00} + a'_{00}a_{10} + a_{10} = -ua_{00}, \tag{119c}$$
$$a'_{02}a_{00} + a'_{00}a_{02} + a'_{01}a_{01} - a_{01} = 0, \tag{119d}$$
$$a'_{11}a_{00} + a'_{00}a_{11} + a'_{10}a_{01} + a'_{01}a_{10} + a_{11} = -ua_{01}, \tag{119e}$$
$$a'_{20}a_{00} + a'_{00}a_{20} + a'_{10}a_{10} + 2a_{20} = -ua_{10}, \tag{119f}$$
$$a'_{03}a_{00} + a'_{00}a_{03} + a'_{02}a_{01} + a'_{01}a_{02} - 2a_{02} = 0, \tag{119g}$$
$$a'_{12}a_{00} + a'_{00}a_{12} + a'_{11}a_{01} + a'_{01}a_{11} + a'_{10}a_{02} + a'_{02}a_{10} + a_{12} - a_{11} = -ua_{02}, \tag{119h}$$
$$a'_{21}a_{00} + a'_{00}a_{21} + a'_{20}a_{01} + a'_{01}a_{20} + a'_{11}a_{10} + a'_{10}a_{11} + 2a_{21} = -ua_{11}, \tag{119i}$$
$$a'_{30}a_{00} + a'_{00}a_{30} + a'_{20}a_{10} + a'_{10}a_{20} + 3a_{30} = -ua_{20}. \tag{119j}$$

Here, we have proceeded by diagonalization: since we do not have such precise information on the structure of (105) as we did for $n = 3$, a simple recursion will not work. We thus have to take a different approach, comparing coefficients in (118) for $i + j = p$ constant.

Let us illustrate the procedure by explicitly solving the first few equations in (119): from (119a), we have $a_{00} \equiv 0$ or $a'_{00} = 0$; however, for (111) and (116) to agree, we have to take the former, which substantially simplifies the following analysis. Equation (119b) is vacuous, as, indeed, $0 = 0$, whereas (119c) then immediately yields $a_{10} \equiv 0$. Given $a_{00} \equiv 0$, (119d) implies either $a_{01} \equiv 0$ or $a'_{01} = 1$. Here, the requirement that (111) and (116) agree to lowest order fixes $a'_{01} = 1$, whence

$$a_{01} = u + \alpha^{01} \tag{120}$$

for some $\alpha^{01} \in \mathbb{R}$. In fact, with (100) and (101), one finds

$$v(u,\varepsilon) = \Big[\frac{1}{\ln\varepsilon+\gamma} - \frac{\varepsilon}{\ln\varepsilon+\gamma} + \frac{\varepsilon}{(\ln\varepsilon+\gamma)^2} + \frac{1}{2}\frac{\varepsilon^2}{\ln\varepsilon+\gamma} - \frac{5}{4}\frac{\varepsilon^2}{(\ln\varepsilon+\gamma)^2} + \frac{\varepsilon^2}{(\ln\varepsilon+\gamma)^3} + \mathcal{O}(\varepsilon^3)\Big](u-1) + \mathcal{O}\big((u-1)^2\big), \tag{121}$$

which implies $\alpha^{01} = -1$. Substituting (120) into (119e) then gives

$$a_{11} = -(u-1)^2 - (u-1); \tag{122}$$

moreover, it follows from (119f) that $a_{20} \equiv 0$, as well. Again with (120), equation (119g) becomes

$$a'_{02}(u-1) - a_{02} = 0, \tag{123}$$

which has the solution

$$a_{02} = \alpha^{02}(u-1) \tag{124}$$

for some constant $\alpha^{02}$. With reference to (121), one can fix $\alpha^{02}$, whence $\alpha^{02} = 0$. As for $a_{12}$ and $a_{21}$, one easily obtains from (119h) and (119i) that

$$a_{12} = 2(u-1)^2 + u - 1 \tag{125}$$

and

$$a_{21} = \frac{1}{2}(u-1)^3 + (u-1)^2 + \frac{1}{2}(u-1), \tag{126}$$

respectively, whereas $a_{30} \equiv 0$. As for $n = 3$, we can now prove the following general result:

**Proposition 3.9.** *There exist unique smooth functions $a_{ij}(u_1)$ such that* (111) *and* (116)*, seen as double expansions, are the same.*

*Proof.* The proof differs significantly from the one we gave for $n = 3$, although it is again by induction, now on the sum $i + j$ instead of on $i$ alone, however. We will show

$$a_{ij} = \sum_{k=1}^{i+j} \alpha_k^{ij} (u-1)^k, \qquad i, j \geq 1, \tag{127a}$$

$$a_{0j} = \sum_{k=1}^{j-1} \alpha_k^{0j} (u-1)^k, \qquad j \geq 2, \tag{127b}$$

$$a_{i0} \equiv 0, \qquad i \geq 2; \tag{127c}$$

indeed, for $i + j = 2$, the assertion is obvious from (122) and (124). Let (127) be valid for $i + j = 2, \dots, p-1$; we have to show that it is valid for $i + j = p$, as well. Collecting powers of $\varepsilon(\ln \varepsilon + \gamma)^{-1}$ in (118), we obtain

$$\sum_{\substack{k+m=i \\ l+n=j}} [a'_{kl} a_{mn} + a'_{mn} a_{kl}] + i a_{ij} - (j-1) a_{i,j-1} = -u a_{i-1,j}. \tag{128}$$

Substituting $a_{00} \equiv 0$ and (120) into (128) gives the following equation for $a_{0p}$:

$$a'_{0p}(u-1) - (p-1) a_{0p} = - \sum_{\substack{k+l=p+1 \\ k,l \geq 2}} [a'_{0k} a_{0l} + a'_{0l} a_{0k}]; \tag{129}$$

by the induction hypothesis, the above sum can be written as

$$- \sum_{k=1}^{p-2} \tilde{\alpha}_k^{0p} (u-1)^k. \tag{130}$$

As terms of the form $-\tilde{\alpha}_k^{0p}(u-1)^k$ generate particular solutions of the form

$$- \frac{\tilde{\alpha}_k^{0p}}{k - p + 1} (u-1)^k, \qquad 1 \leq k \leq p-2 \tag{131}$$

in (129) and as the homogeneous solution is given by

$$\alpha^{0p} (u-1)^{p-1}, \tag{132}$$

the claim follows for $a_{0p}$. Note that the constant $\alpha^{0p}$ has to be chosen such that (111) and (116) agree when seen as double expansions; the smoothness of $a_{0p}$ is granted irrespective of the choice of $\alpha^{0p}$. For $a_{ij}$, with $i + j = p$ and $i \geq 1$, (128) yields

$$i a_{ij} = j a_{i,j-1} - u a_{i-1,j} - \sum_{\substack{k+m=i \\ l+n=j}} [a'_{kl} a_{mn} + a'_{mn} a_{kl}]; \tag{133}$$

due to the fact that $a_{00} \equiv 0$, this completely determines $a_{ij}$. Moreover, it follows from the induction hypothesis that $a_{ij}$ is of the desired form and that $a_{p0} \equiv 0$, respectively, which concludes the proof. $\square$

***Remark* 11.** The above proof shows that once the leading-order behavior in (116) is determined, the transcendentally small terms in (116) are given as solutions not of differential, but of algebraic equations. Our analysis thus immediately provides us with these terms, whereas in the classical approach, rather cumbersome computations are required for their determination, as matching is typically done only up to transcendentally small quantities there. $\square$

We can now give an expansion for $v_{1_\varepsilon}$ when $n = 2$:

**Proposition 3.10.** *For $\varepsilon \in (0, \varepsilon_0]$, with $\varepsilon_0 > 0$ sufficiently small, $v_{1_\varepsilon} = v_1(0, \varepsilon)$ can be expanded as*

$$v_{1_\varepsilon} = -\frac{1}{\ln\varepsilon + \gamma} + \mathcal{O}\bigg(\frac{1}{(\ln\varepsilon + \gamma)^2}\bigg). \tag{134}$$

*Proof.* As for $n = 3$, (116) gives

$$\begin{aligned} v_1(u_1, \varepsilon_1) &= (u_1 - 1)\frac{1}{\ln\varepsilon_1 + \gamma} + \left[-(u_1 - 1)^2 - (u_1 - 1)\right] \frac{\varepsilon_1}{\ln\varepsilon_1 + \gamma} \\ &\quad + \left[\frac{1}{2}(u_1 - 1)^3 + (u_1 - 1)^2 + \frac{1}{2}(u_1 - 1)\right] \frac{\varepsilon_1^2}{\ln\varepsilon_1 + \gamma} + \mathcal{O}\bigg(\frac{1}{(\ln\varepsilon_1 + \gamma)^2}\bigg), \end{aligned} \tag{135}$$

whence we obtain the assertion with $u_1 = 0$. $\square$

***Remark* 12.** As $-(\ln\varepsilon + \gamma)^{-1}$ can be expanded as

$$-\frac{1}{\ln\varepsilon + \gamma} = -\frac{1}{\ln\varepsilon}\sum_{j=0}^{\infty}\Big(-\frac{\gamma}{\ln\varepsilon}\Big)^j \tag{136}$$

for $0 < \varepsilon < \varepsilon_0$ sufficiently small, $v_{1_\varepsilon}$ can be written as

$$v_{1_\varepsilon} = -\frac{1}{\ln\varepsilon} + \frac{\gamma}{(\ln\varepsilon)^2} + \mathcal{O}\bigg(\frac{1}{(\ln\varepsilon)^3}\bigg), \tag{137}$$

which agrees with the expansion found e.g. in [HTB90]. In fact, as was pointed out by [LC72], the expansion is more compact if it is *telescoped*, i.e., arranged in powers of $(\ln\varepsilon + \gamma)^{-1}$. However, this arrangement is not helpful numerically, as (134) becomes undefined for $\varepsilon = 0.5614\ldots$, whereas (137) allows for values of $\varepsilon$ up to 1. $\square$

## 4. Asymptotic solution expansions

**4.1. Expansions in chart $K_1$.** Given the expansion for $v_{1_\varepsilon}$ from Proposition 3.6, which we can write as

$$v_{1_\varepsilon} = \sum_{\substack{i,j=0 \\ j \le i}}^{\infty} \beta_{ij}\varepsilon^i(\ln\varepsilon)^j, \tag{138}$$

with constants $\beta_{ij} \in \mathbb{R}$, it makes sense to define expansions

$$u_1(r_1,\varepsilon_1) = \sum_{\substack{i,j=0\\ j\le i}}^{\infty} a_{ij}(r_1)\varepsilon_1^i(\ln\varepsilon_1)^j, \quad v_1(r_1,\varepsilon_1) = \sum_{\substack{i,j=0\\ j\le i}}^{\infty} b_{ij}(r_1)\varepsilon_1^i(\ln\varepsilon_1)^j, \tag{139}$$

where we regard both $u_1$ and $v_1$ as functions of $(r_1, \varepsilon_1)$ now. The ansatz in (139) is closer to the classical approach than what was done in the previous sections; however, as the techniques we apply are very similar to the ones used before, we only sketch the procedure here, leaving out most of the details. Rewriting (12) with $r_1$ as the independent variable now and omitting again the subscript 1, we obtain

$$\begin{aligned} \frac{du}{dr} &= -\frac{v}{r}, \\ \frac{dv}{dr} &= \frac{v}{r} + \frac{\varepsilon uv}{r}, \\ \frac{d\varepsilon}{dr} &= -\frac{\varepsilon}{r}. \end{aligned} \tag{140}$$

With (140), we find

$$\sum_{\substack{i,j=0\\ j\le i}}^{\infty} \Big[a_{ij}'\varepsilon^i(\ln\varepsilon)^j - \frac{a_{ij}}{r}\varepsilon^i(\ln\varepsilon)^{j-1}(i\ln\varepsilon + j)\Big] = -\frac{1}{r}\sum_{\substack{i,j=0\\ j\le i}}^{\infty} b_{ij}\varepsilon^i(\ln\varepsilon)^j, \tag{141a}$$

$$\begin{aligned} \sum_{\substack{i,j=0\\ j\le i}}^{\infty} \Big[b_{ij}'\varepsilon^i(\ln\varepsilon)^j - \frac{b_{ij}}{r}\varepsilon^i(\ln\varepsilon)^{j-1}(i\ln\varepsilon + j)\Big] &= \frac{1}{r}\sum_{\substack{i,j=0\\ j\le i}}^{\infty} b_{ij}\varepsilon^i(\ln\varepsilon)^j \\ &+\frac{1}{r}\left(\sum_{\substack{i,j=0\\ j\le i}}^{\infty} a_{ij}\varepsilon^i(\ln\varepsilon)^j\right)\cdot\left(\sum_{\substack{k,l=0\\ l\le k}}^{\infty} b_{kl}\varepsilon^k(\ln\varepsilon)^l\right); \end{aligned} \tag{141b}$$

comparing powers of $\varepsilon\ln\varepsilon$ yields the following recursive sequence of differential equations,

$$ra_{ij}' - ia_{ij} - (j+1)a_{i,j+1} = -b_{ij}, \tag{142a}$$

$$rb_{ij}' - ib_{ij} - (j+1)b_{i,j+1} = b_{ij} + \sum_{\substack{k+m=i-1\\ l+n=j\\ l\le k,\ n\le m}} a_{kl}b_{mn}, \tag{142b}$$

with initial conditions given by

$$a_{ij}(1) = 0, \quad b_{ij}(1) = \beta_{ij}, \qquad 0\le j\le i. \tag{143}$$

***Remark* 13.** For $v_1$, it follows directly from (138) that the ansatz in (139) is plausible, whereas for $u_1$, it can be justified a posteriori using (142a). □

For the first few coefficients in (139), we thus have

$$ra_{00}' = -b_{00}, \tag{144a}$$
$$rb_{00}' = b_{00}, \tag{144b}$$
$$ra_{11}' - a_{11} = -b_{11}, \tag{144c}$$
$$rb_{11}' - b_{11} = b_{11}, \tag{144d}$$
$$ra_{10}' - a_{10} - a_{11} = -b_{10}, \tag{144e}$$
$$rb_{10}' - b_{10} - b_{11} = b_{10} + a_{00}b_{00}; \tag{144f}$$

with Proposition 3.6, the solution to (144) is easily found to be given by

$$a_{00} = 1 - r, \qquad b_{00} = r, \tag{145a}$$
$$a_{11} = -r(1-r), \qquad b_{11} = -r^2, \tag{145b}$$
$$a_{10} = (1-\gamma)r(1-r) + 2r^2 \ln r, \qquad b_{10} = -r(1+\gamma r) - 2r^2 \ln r, \tag{145c}$$

which allows us to state the following result:

**Proposition 4.1.** *For $n = 3$, the solution to* (1) *can be expanded as*

$$u(\xi, \varepsilon) = 1 - \frac{1}{\xi} + \varepsilon(1 - \gamma - \ln \varepsilon)\left(1 - \frac{1}{\xi}\right) - \varepsilon\left(1 + \frac{1}{\xi}\right)\ln \xi + \mathcal{O}(\varepsilon^2) \tag{146}$$

*(*inner expansion*); here, $\xi$ is as defined in* (2)*.*

*Proof.* The result is immediate from (139) and (145), after application of the appropriate blow-down transformations $r_1 = \xi^{-1}$ and $\varepsilon_1 = \varepsilon\xi$, as

$$\begin{aligned} u_1(r_1, \varepsilon_1) = 1 - r_1 - r_1(1-r_1)\varepsilon_1 \ln \varepsilon_1 + (1-\gamma)r_1(1-r_1)\varepsilon_1 + \\ + 2r_1^2 \ln r_1 \varepsilon_1 + \mathcal{O}(\varepsilon_1^2). \end{aligned} \tag{147}$$

□

Similarly, for $n = 2$, the equations

$$\begin{aligned} \frac{du}{dr} &= -\frac{v}{r}, \\ \frac{dv}{dr} &= \frac{\varepsilon u v}{r}, \\ \frac{d\varepsilon}{dr} &= -\frac{\varepsilon}{r} \end{aligned} \tag{148}$$

in combination with an ansatz of the form

$$u_1(r_1, \varepsilon_1) = \sum_{i,j=0}^{\infty} a_{ij}(r_1)\frac{\varepsilon_1^i}{(\ln \varepsilon_1 + \gamma)^j}, \quad v_1(r_1, \varepsilon_1) = \sum_{i,j=0}^{\infty} b_{ij}(r_1)\frac{\varepsilon_1^i}{(\ln \varepsilon_1 + \gamma)^j} \tag{149}$$

lead to the recursive sequence of equations

$$ra_{ij}' - ia_{ij} + (j-1)a_{i,j-1} = -b_{ij}, \tag{150a}$$
$$rb_{ij}' - ib_{ij} + (j-1)b_{i,j-1} = \sum_{\substack{k+m=i-1 \\ l+n=j}} a_{kl}b_{mn}; \tag{150b}$$

the initial conditions are again given by

$$a_{ij}(1) = 0, \quad b_{ij}(1) = \beta_{ij}, \qquad i, j \geq 0 \tag{151}$$

for some $\beta_{ij} \in \mathbb{R}$. To leading order, one finds

$$a_{01} = \ln r, \qquad b_{01} = -1, \tag{152}$$

whence one obtains the following

**Proposition 4.2.** *For $n = 2$, the inner expansion of the solution to* (1) *is given by*

$$u(\xi, \varepsilon) = -\frac{\ln \xi}{\ln \varepsilon\xi + \gamma} + \mathcal{O}\Big(\frac{1}{(\ln \varepsilon\xi + \gamma)^2}\Big). \tag{153}$$

*Proof.* The proof is analogous to that for $n = 3$. □

***Remark* 14.** By expanding $(\ln \varepsilon\xi + \gamma)^{-1}$ for $0 < \varepsilon < \varepsilon_0$ small, one can write

$$u(\xi, \varepsilon) = -\frac{\ln \xi}{\ln \varepsilon + \gamma} + \mathcal{O}\Big(\frac{1}{(\ln \varepsilon + \gamma)^2}\Big) \tag{154}$$

or

$$u(\xi, \varepsilon) = -\frac{\ln \xi}{\ln \varepsilon} + \frac{\gamma \ln \xi}{(\ln \varepsilon)^2} + \frac{(\ln \xi)^2}{(\ln \varepsilon)^2} + \mathcal{O}\Big(\frac{1}{(\ln \varepsilon)^3}\Big), \tag{155}$$

which are the expansions usually found in the literature, see e.g. [LC72] or [HTB90]. □

**4.2. Expansions in chart $K_2$.** To derive solution expansions in $K_2$, we would have to proceed as above, i.e., we would first determine the general structure of the coefficients in (139) and (149), respectively, which we would then use to rearrange (139) and (149) with respect to a new basis. It is these expansions which would provide us with the proper ansatz for the corresponding expansions in $K_2$. In sum, we expect to obtain the following result which we cite for reference only, see [LC72]:

**Proposition 4.3.** *The outer solution expansion for* (1) *is given by*

$$u(x, \varepsilon) = 1 - \varepsilon E_2(x) + \mathcal{O}(\varepsilon^2) \tag{156}$$

*for $n = 3$ and by*

$$u(x, \varepsilon) = 1 + \frac{1}{\ln \varepsilon + \gamma} E_1(x) + \mathcal{O}\Big(\frac{1}{(\ln \varepsilon + \gamma)^2}\Big) \tag{157}$$

*for $n = 2$, respectively; here, $E_k$ is defined by*

$$E_k(z) := \int_z^\infty e^{-t} t^{-k}\, dt, \qquad z \in \mathbb{C},\ \Re(z) > 0,\ k \in \mathbb{N}. \tag{158}$$

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