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RIGOROUS FULL VECTORIAL ANALYSIS OF ELECTROMAGNETIC WAVE PROPAGATION IN 1D INHOMOGENEOUS MEDIA

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Abstract—We propose a new approach to solve the problem of the propagation of electromagnetic waves in unidimensional media with an arbitrary variation of their dielectric permittivity. This method is deduced from the Maxwell equations with a minimum of approximations and allows a full vectorial description of both the electric and magnetic fields through the direct calculation of their Cartesian coordinates. The problem is then equivalent to the solution of a pair of uncoupled ordinary differential equations. We use a very intuitive, highly accurate, pseudospectral technique to solve these equations. This pseudospectral method is based in a combination of Fourier and polynomial expansions of the solution providing very good precision and excellent stability with a relatively low computational effort. We present a simple model of a photonic crystal as an example of application of this technique to real electromagnetic problems.

1. INTRODUCTION

The study of media with an arbitrary variation of their dielectric permittivity, such as photonic crystals and waveguides, is the subject of intense research in recent years. Many theoretical and numerical advances have been published in the last decade [1-17, 21]. From a practical point of view, the full vectorial analysis of this problem is very important in research areas such as Optical Communications, radio propagation in the Ionosphere, scattering in stratified and artificial materials, etc. [2, 17–19, 21–24].

A complete study of media with a periodic variation of their dielectric permittivity can be done using the formalism of the Floquet-Bloch theory, which is formally equivalent to the expansion of the solution in Fourier series [22]. Unfortunately, in some cases, this method is slowly convergent and it is difficult to adapt it to more general functional dependences of the dielectric permittivity with the spatial coordinates. In such cases, finite differences or finite element approaches are widely used. Powerful techniques exist to solve the scattering problem, such as the Wave Approach for multilayered media [11, 13 23]. In such theories the medium under study is described as a convenient multilayer material with appropriate boundary conditions between adjacent layers.

However, in spite of the good performance of present theories and numerical techniques, it is difficult to find detailed full vectorial studies of electromagnetic wave propagation in arbitrarily inhomogeneous materials, even in relatively simple unidimensional cases. The simulation of unidimensional structures is still very important because many practical systems have symmetries with such property. Besides, 1D structures provide a quick and deep physical insight of the problem and they provide excellent tests of the accuracy of new solution techniques in higher dimensions. In this paper we propose a new method of analysis of this problem directly derived from the Maxwell equations with a minimum of approximations and hypothesis. This method is similar to the full vectorial hybrid mode analysis in waveguides [24] and allows a complete and accurate description of every Cartesian component of both the electric and magnetic fields.

In our approach the problem is reduced to a pair of ordinary differential equations for the z dependent part of the E_z and H_z components of the electromagnetic field. These equations are then solved by an efficient collocation technique based in a suitable combination of Fourier and polynomial series. The combination of both approximations provide improved convergence in this case with good stability for initial value and boundary value problems. Collocation or pseudospectral methods are conceptually simple, but quick, efficient and memory saving, which is very important in real simulations, because physical memory is generally a more limiting factor than CPU speed [20, 25, 26]. This pseudospectral technique has a wide applicability and is specially suited to the analysis of complex and non-periodic dielectric profiles.

We consider a lineal, isotropic, unidimensional medium with rectangular symmetry and an arbitrary dielectric permittivity $\varepsilon(z)$. Mathematically, the problem of propagation of the electromagnetic

field in this medium is described by the Maxwell equations,

$$\nabla \cdot \overline{D} = \rho \tag{1}$$

$$\nabla \cdot \overline{B} = 0 \tag{2}$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \tag{3}$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}. \tag{4}$$

In macroscopic media, the following constitutive relations are used,

$$\overline{D} = \varepsilon \overline{E} \tag{5}$$

$$\overline{J} = \sigma \overline{E} \tag{6}$$

$$\overline{B} = \mu \overline{H} \tag{7}$$

where

$$\varepsilon = \varepsilon_0 \varepsilon_r \tag{8}$$

$$\mu = \mu_0 \mu_r \tag{9}$$

We shall make the following assumptions to simplify the problem.

- The material in which the electromagnetic waves are propagating is a perfect dielectric, so that $\rho = 0$ and $\overline{J} = 0$.
- The medium is not magnetic, $\mu = \mu_0$.
- The medium is isotropic.
- The medium is inhomogeneous only in the direction of the z axis, $\varepsilon(z) = \varepsilon_0 \varepsilon_r(z)$. We shall allow for any arbitrary functional dependence of the dielectric permittivity with the z coordinate.
- The medium can be finite, semi-infinite or infinite and has a rectangular symmetry, so that Cartesian coordinates can be used.

With such hypothesis, the Maxwell equations, (1), (2), (3) and (4), can be rewritten as

$$\varepsilon_0 \left[\varepsilon_r(z) \frac{\partial E_x}{\partial x} + \varepsilon_r(z) \frac{\partial E_y}{\partial y} + \frac{\partial \left(\varepsilon_r(z) E_z \right)}{\partial z} \right] = 0 \tag{10}$$

$$\mu_0 \left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} \right) = 0 \tag{11}$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu_0 \frac{\partial H_x}{\partial t} \tag{12}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu_0 \frac{\partial H_y}{\partial t} \tag{13}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu_0 \frac{\partial H_z}{\partial t} \tag{14}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \varepsilon_0 \varepsilon_r(z) \frac{\partial E_x}{\partial t} \tag{15}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \varepsilon_0 \varepsilon_r(z) \frac{\partial E_y}{\partial t} \tag{16}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \varepsilon_0 \varepsilon_r(z) \frac{\partial E_z}{\partial t}. \tag{17}$$

These equations form a set of eight partial differential equations whose unknowns are the six electric and magnetic field components, so we have some freedom to solve the system. Among all the possibilities, we shall choose the six equations which lead to the most simple and straightforward solution of the problem.

2. METHOD OF CALCULATION

The equations from (10) to (17) are separable in Cartesian coordinates. If ε_r is a constant, we know that the general solutions are plane waves,

$$\overline{E} = \overline{E}_0 \exp(i\overline{k} \cdot \overline{r} - i\omega t). \tag{18}$$

Using this result we shall separate the Maxwell equations using the following *antsatz*,

$$E_x = E_{x0} \exp[i(k_x x + k_y y)] E_1(z) e^{-i\omega t} \tag{19}$$

$$E_y = E_{y0} \exp[i(k_x x + k_y y)] E_2(z) e^{-i\omega t} \tag{20}$$

$$E_z = E_{z0} \exp[i(k_x x + k_y y)] E_3(z) e^{-i\omega t} \tag{21}$$

$$H_x = H_{x0} \exp[i(k_x x + k_y y)] H_1(z) e^{-i\omega t} \tag{22}$$

$$H_y = H_{y0} \exp[i(k_x x + k_y y)] H_2(z) e^{-i\omega t} \tag{23}$$

$$H_z = H_{z0} \exp[i(k_x x + k_y y)] H_3(z) e^{-i\omega t}. \tag{24}$$

We shall solve the system formed by the equations (10), (11), (13), (14), (16) and (17). These equations have derivatives with respect to the x coordinate. Substituting equations from (19) to (24) in the previous system we obtain a set of separated ordinary differential equations. However, the new differential equations are coupled in the $E_1(z), E_2(z), E_3(z), H_1(z), H_2(z)$ and $H_3(z)$ factors. Symmetry considerations suggest to try the decoupling of these equations through

the isolation of $E_3(z)$ and $H_3(z)$, calculating the other factors from them.

The $E_1(z)$, $H_1(z)$, $E_2(z)$ and $H_2(z)$ factors are calculated algebraically solving the system formed by the equations (10), (11), (14) and (17), which do not have derivatives of E_z or H_z with respect to x. The expressions for these factors with respect to $E_3(z)$ and $H_3(z)$ are

$$E_1(z) = \frac{iE_{0z}k_xE_3(z)\frac{d\varepsilon_r(z)}{dz} + iE_{0z}k_x\varepsilon_r(z)\frac{dE_3(z)}{dz} - H_{0z}k_y\mu_0\omega\varepsilon_r(z)H_3(z)}{E_{0x}(k_x^2+k_y^2)\varepsilon_r(z)} \tag{25}$$

$$H_1(z) = \frac{iH_{0z}k_x\frac{dH_3(z)}{dz} + E_{0z}\varepsilon_0\varepsilon_r(z)k_y\omega E_3(z)}{H_{0x}(k_x^2+k_y^2)} \tag{26}$$

$$E_2(z) = \frac{iE_{0z}k_yE_3(z)\frac{d\varepsilon_r(z)}{dz} + iE_{0z}k_y\varepsilon_r(z)\frac{dE_3(z)}{dz} - H_{0z}k_x\mu_0\omega\varepsilon_r(z)H_3(z)}{E_{0y}(k_x^2+k_y^2)\varepsilon_r(z)} \tag{27}$$

$$H_2(z) = \frac{iH_{0z}k_y\frac{dH_3(z)}{dz} - E_{0z}\varepsilon_0\varepsilon_r(z)k_x\omega E_3(z)}{H_{0y}(k_x^2+k_y^2)} \tag{28}$$

If we substitute the equations from (25) to (28) into the equations (13) and (16), which have the derivatives $\frac{\partial E_z}{\partial x}$ and $\frac{\partial H_z}{\partial x}$, we obtain a single equation for $E_3(z)$,

$$\frac{d^2E_3(z)}{dz^2} + a_1(z)\frac{dE_3(z)}{dz} + a_2(z)E_3(z) = 0 \tag{29}$$

where

$$a_1(z) = \frac{1}{\varepsilon_r(z)}\frac{d\varepsilon_r(z)}{dz}$$

$$a_2(z) = \frac{1}{\varepsilon_r(z)}\frac{d^2\varepsilon_r(z)}{dz^2} - \frac{1}{\varepsilon_r^2(z)}\left(\frac{d\varepsilon_r(z)}{dz}\right)^2 + \varepsilon_0\varepsilon_r(z)\mu_0\omega^2 - (k_x^2+k_y^2)$$

and other single equation for $H_3(z)$,

$$\frac{d^2H_3(z)}{dz^2} + c(z)H_3(z) = 0 \tag{30}$$

where

$$c(z) = \varepsilon_0\varepsilon_r(z)\mu_0\omega^2 - (k_x^2+k_y^2).$$

Now, we can write the system formed by the equations (29) and (30) as an eigenvalue problem for and effective wavenumber k making the

change, $k_x^2 + k_y^2 = k^2 - k_z^2$, where k_z is a fixed constant and k is the eigenvalue we want to calculate. In this form, the equations (29) and (30) can be rewritten as

$$\frac{d^2E_3(z)}{dz^2} + A_1(z)\frac{dE_3(z)}{dz} + A_2(z)E_3(z) = k^2E_3(z) \tag{31}$$

$$A_1(z) = \frac{1}{\varepsilon_r(z)}\frac{d\varepsilon_r(z)}{dz}$$

$$A_2(z) = \frac{1}{\varepsilon_r(z)}\frac{d^2\varepsilon_r(z)}{dz^2} - \frac{1}{\varepsilon_r^2(z)}\left(\frac{d\varepsilon_r(z)}{dz}\right)^2 + \varepsilon_0\varepsilon_r(z)\mu_0\omega^2 + k_z^2$$

$$\frac{d^2H_3(z)}{dz^2} + C(z)H_3(z) = k^2H_3(z) \tag{32}$$

$$C(z) = \varepsilon_0\varepsilon_r(z)\mu_0\omega^2 + k_z^2$$

which are the desired equations for electromagnetic wave propagation in arbitrarily inhomogeneous 1D media.

3. SOLUTION OF THE PROPAGATION EQUATIONS

Equations (31) and (32) form a system of uncoupled ordinary differential equations which can be solved by several numerical techniques [25]. We shall study qualitatively these equations before using more powerful numerical approaches.

In this full vectorial description, there are two different equations for the electric and magnetic field components. This implies that, *a priori*, the eigenvalues for the wavenumbers in the equations (31) and (32) can be different. Actually, we can define only one eigenvalue for the whole system, because of equations from (25) to (28). Then, it would be more correct to write the equations (31) and (32) as

$$\frac{d^2E_3(z)}{dz^2} + A_1(z)\frac{dE_3(z)}{dz} + A_2(z)E_3(z) = k_E^2E_3(z) \tag{33}$$

$$\frac{d^2H_3(z)}{dz^2} + C(z)H_3(z) = k_H^2H_3(z). \tag{34}$$

The differential equation for $E_3(z)$ is notably more complex than the analogous equation for $H_3(z)$. If the dielectric permittivity is periodic, $\varepsilon_r(z) = \varepsilon_r(z+a)$, which is often the case, we could calculate the solutions using the Floquet-Bloch theory [21, 22]. However, the convergence of this method would be slow in many cases and the calculation of the other factors could be costly and unstable from a

numerical point of view. In addition, in order to apply the Floquet-Bloch technique to the equation (33) we have to transform it into normal form,

$$\frac{d^2\mathcal{E}_3(z)}{dz^2} + \mathcal{A}(z)\mathcal{E}_3(z) = 0. \tag{35}$$

A suitable transformation for this is defined by

$$\mathcal{A}(z) = \left(A_2(z) - k_{\mathcal{E}}^2\right) - \frac{1}{4}A_1^2(z) - \frac{1}{2}\frac{dA_1(z)}{dz} \tag{36}$$

where the original solution has the form,

$$E_3(z) = \mathcal{E}_3(z)\exp\left(-\frac{1}{2}\int^z A_1(\zeta)d\zeta\right). \tag{37}$$

However, if the functional dependence of ε_r is not simple, $A_1(z)$ would not be easily integrable and the accuracy of the solution would be poor.

Another alternative approach to study the form of the solutions of the equation (35) would be to apply a Prüfer transformation [25],

$$\frac{d\mathcal{E}_3(z)}{dz} = R(z)\cos\theta(z) \tag{38}$$

$$\mathcal{E}_3(z) = R(z)\sin\theta(z) \tag{39}$$

which produces the following pair of first order ODE,

$$\frac{d\theta}{dz} = \mathcal{A}(z)\sin^2\theta + \cos^2\theta \tag{40}$$

$$\frac{dR}{dz} = \left[1 - \mathcal{A}(z)\right]R\sin 2\theta. \tag{41}$$

The study of the phase, θ, in equation (40) would give a detailed description of the qualitative behavior of the solutions.

The propagation equations can be solved by accurate numerical methods such as adaptative Runge-Kutta schemes. There are very good and well tested mathematical libraries which can be used to calculate both the eigenvalues and eigenfunctions. However, if the dielectric permittivity has a complex behavior, the solutions can oscillate quickly forming complicated patterns as can be observed in Figure 3. Then a highly precise and stable algorithm is needed. Due to its easily adjustable accuracy and stable converging properties, we have found very convenient to use a simple pseudospectral technique to solve the equations (33) and (34).

Our pseudospectral method is very easy to program and it can produce highly accurate analytical solutions of the problem with a

relatively low computational effort. The accuracy can be controlled selecting the order of the approximating series, but, in practice, low orders are enough for most purposes. Besides, our method shows good numerical stability for finite, semi-infinite and infinite intervals, even for high orders. It can be easily adapted to any kind of problems, such as initial value, boundary value and eigenvalue problems. Also, it is possible to obtain a quick analytical estimation of the solution, which is very useful to study the qualitative behavior of the electromagnetic waves in the material. The idea of the pseudospectral methods is to produce an optimal global approximation of the solution by means of interpolating functions, instead of the local polynomial approximation of the finite difference techniques [26]. In fact, it can be proved that many Runge-Kutta schemes are equivalent to suitable polynomial collocation algorithms.

The main difference among different pseudospectral approaches is the election of the basis functions. Generally these functions are orthogonal, such as the Chebyshev polynomials for finite intervals and the sinc functions for infinite intervals. This forces to carefully choose the collocation points to avoid undesirable effects such as the Runge phenomenon [26].

We have found empirically, after intensive testing, that a simple cosine series combined with an even polynomial expansion provide optimal accuracy and stability in this case. Another advantage of our pseudospectral method is that it uses simple equispaced collocation points without adverse effects, whose stability is only limited by the machine precision. More sophisticated series such as complete Fourier (sines and cosines), Chebyshev, Laguerre or sinc expansions produced similar or worse results for the same problem. In general, Chebyshev series provide optimal approximations for finite, Laguerre for semi-infinite and sinc for infinite intervals, with a suitable choosing of the collocation points [20, 26]. Our pseudospectral method is competitive in every case and can use evenly distributed points. High orders of approximation can be used, but we have found that $N \sim 3$ is enough for most purposes, if a reasonable integration interval is chosen.

The best empirical series we have found has this simple form,

$$u(z) \approx a_0 + \sum_{n=1}^{M} a_n \cos\left[\frac{n}{d}(z - z_0) + \varphi\right] + \sum_{n=1}^{N} b_n (z - z_0)^{2n} \qquad (42)$$

where M and N are the orders of the approximating series, z_0 is the initial integration point, $d = \frac{\lambda}{\pi}$, where λ is the incident wavelength, and φ is a constant phase chosen to avoid undesired cancellations of the cosine argument. A convenient value of the phase is $\varphi = \sqrt{2}$, although

the concrete value does not affect the result appreciably. The solution of the propagation equations are obtained substituting the series (42) into the equations (33) and (34). The necessary collocation points are equispaced over the interval of integration with a integration step $h = \frac{|z_f - z_0|}{\mathcal{O}}$, where z_f is the end point of the integration interval and $\mathcal{O} = M+N$ is the order of the series. Numerical stability of the solution and the suppression of the Runge phenomenon are observed until roundoff errors due to the machine finite precision become important. The value of d in (42) can be adjusted for non-periodic solutions to improve even more the quick convergence of our pseudospectral method.

4. NUMERICAL EXAMPLES

4.1. The Bessel Equation

The numerical examples presented here have been solved with our pseudospectral method implemented in the free MAXIMA [27] computer algebra system.

The first example will be the solution of the Bessel equation of order 0 to test the accuracy of the calculation scheme in an oscillating, but not periodic, function. This equation can be written as

$$x^2 y'' + xy' + x^2 y = 0 \tag{43}$$

We will solve first the initial value problem $y(0) = 1, y'(0) = 0$ with Laguerre polynomials up to 6th order and 3+3 for sinc and cos + poly, equation (42), expansions in the interval $[0, 5]$ with an integration step of 0.5 (equispaced). The results can be seen in Tables from 1 to 3.

Table 1. Comparison of exact values of the $J_0(x)$ and the results of the Laguerre pseudospectral method of 6th order for the equation (43) with initial values.

x	$J_0(x)$	$y_{\text{Laguerre}}(x)$
0	1	1
1	0.76519768655797	0.765197*72875231*
2	0.22389077914124	0.223890*80535392*
3	-0.26005195490193	-0.260051*97303427*
4	-0.39714980986385	-0.397149*86111586*
5	-0.17759677131434	-0.177596*83725827*

Table 2. Comparison of exact values of the $J_0(x)$ and the results of the sinc pseudospectral method of 6th order for the equation (43) with initial values.

x	$J_0(x)$	$y_{\text{sinc}}(x)$
0	1	0.99999999999*719*
1	0.76519768655797	0.765197*69321807*
2	0.22389077914124	0.2238907*7349499*
3	-0.26005195490193	-0.260051*97903735*
4	-0.39714980986385	-0.397149*84910345*
5	-0.17759677131434	-0.177596*73631765*

Table 3. Comparison of exact values of the $J_0(x)$ and the results of our pseudospectral method of $3 + 3$ order for the equation (43) with initial values.

x	$J_0(x)$	$y_{\cos+\text{poly}}(x)$
0	1	0.999999999999*74*
1	0.76519768655797	0.765197*69688696*
2	0.22389077914124	0.2238907*4022318*
3	-0.26005195490193	-0.260051*9464582*
4	-0.39714980986385	-0.3971498*101612*
5	-0.17759677131434	-0.177596*72509972*

In Table 4 we solve the same equation with boundary conditions $y(0) = 1, y(5) = J_0(5)$ over the whole interval. The order of the series (42) is $10 + 10$.

We can see that for the initial value problem the sinc method is the most accurate, very closely followed by cos + poly. However, the sinc method turns unstable for higher orders and is not suitable for periodic functions. In Table 4 the high accuracy and stability of our method for a boundary value problem is shown. Another advantage of pseudospectral methods over standard Runge-Kutta algorithms is that the pseudospectral codes can integrate many differential equations, such as (43), from $x = 0$ without $\frac{1}{0}$ errors.

Table 4. Comparison of exact values of the $J_0(x)$ and the results of our pseudospectral method of order $10+10$ for the equation (43) with boundary values.

x	$J_0(x)$	$y_{\cos+\text{poly}}(x)$
0	1	1
1	0.76519768655797	0.765197*76843422*
2	0.22389077914124	0.223890*82692518*
3	-0.26005195490193	-0.260051*96055422*
4	-0.39714980986385	-0.397149*84288502*
5	-0.17759677131434	-0.177596*76970203*

4.2. A Simple Model of a Photonic Crystal

We can describe a very simple photonic crystal as a medium with a periodic variation of its relative permittivity such as

$$\varepsilon_r(z) = 1 + \sin^2(z). \tag{44}$$

Using our pseudospectral technique to solve the equation (34), the approximate eigenvalues for $k_z = \frac{2\pi}{\lambda}\cos(0)$, i.e., for an initial wave propagating in the z direction, are presented in the Table 5.

Table 5. Approximate eigenvalues for a 1D photonic crystal with $\varepsilon_r = 1 + \sin^2(z)$ and $k_z = \frac{2\pi}{\lambda}$.

λ	k_H
0.1	89
1	8.9
2	4.5
3	3.0
4	2.3
5	1.9
6	1.6
7	1.4
8	1.2
9	1.1
10	1.0

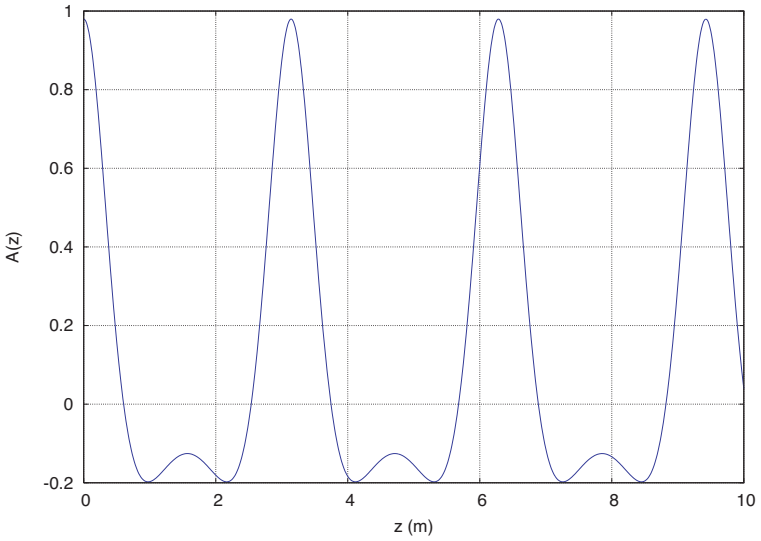

Figure 1. Plot of $\mathcal{A}(z)$ of equation (35) for $\varepsilon_r(z) = 1 + \sin^2(z),\ \lambda = 10\,\text{m},\ k_z = k_y = \frac{2\pi}{\lambda},\ k_x = \sqrt{k^2 - k_y^2 - k_z^2},\ k = 0.9.$

In Figures 3 and 4 we show an example of the calculation of the field components for an electromagnetic wave propagating in a semi-infinite medium with the following initial data: $\varepsilon_r(z) = 1+\sin^2(z), \lambda = 10$ m, $k_z = k_y = \frac{2\pi}{\lambda}, k_x = \sqrt{k^2 - k_y^2 - k_z^2}, k = 0.9$, $E_{x0} = E_{y0} = E_{z0} = 1$ V/m, $H_{x0} = H_{y0} = H_{z0} = 1$ A/m, $E_3(0) = H_3(0) = 1$ and $\frac{dE_3(z)}{dz} = \frac{dH_3(z)}{dz} = 0$.

Figures 1 and 2 show the z dependent terms of equations (33) and (34) in normal form. We can see the more complex pattern of the (33) term, which generates the rich oscillating behavior of E_z in Figure 3. For the specified initial values, the most contributing component is E_x, which is 400 times more intense than the E_z component, but exhibits a much smoother wave form. From the magnetic field components, only H_z, Figure 4, is significative. All the three magnetic components have a relatively simple periodic behavior in agreement with the results of the Bloch-Floquet theory. Our numerical results indicate a good ratio between computational effort and accuracy. In [14] Taylor series up to 100 terms are used to approximate the solution. We obtain similar results with orders less than 10 with the pseudospectral series used in this paper. The very powerful theories based on Green's functions expansions are computationally expensive

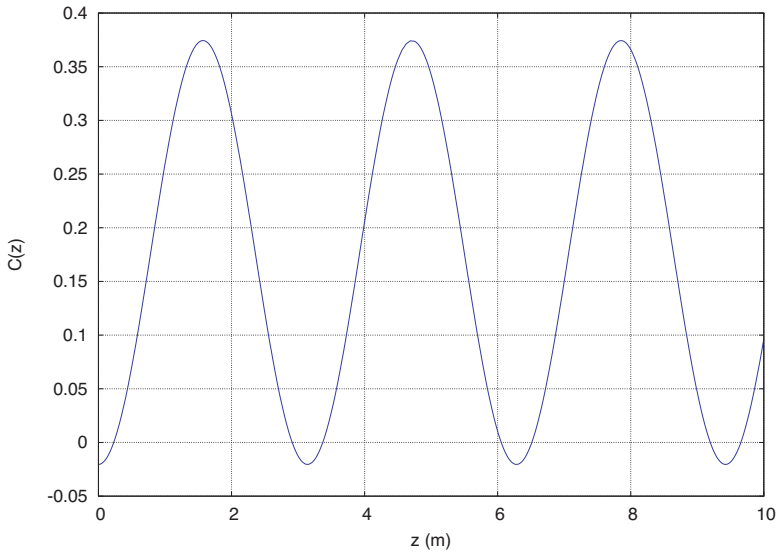

Figure 2. Plot of $C(z)$ of equation (34) for $\varepsilon_r(z) = 1 + \sin^2(z),\ \lambda = 10\,\mathrm{m},\ k_z = k_y = \frac{2\pi}{\lambda},\ k_x = \sqrt{k^2 - k_y^2 - k_z^2},\ k = 0.9$.

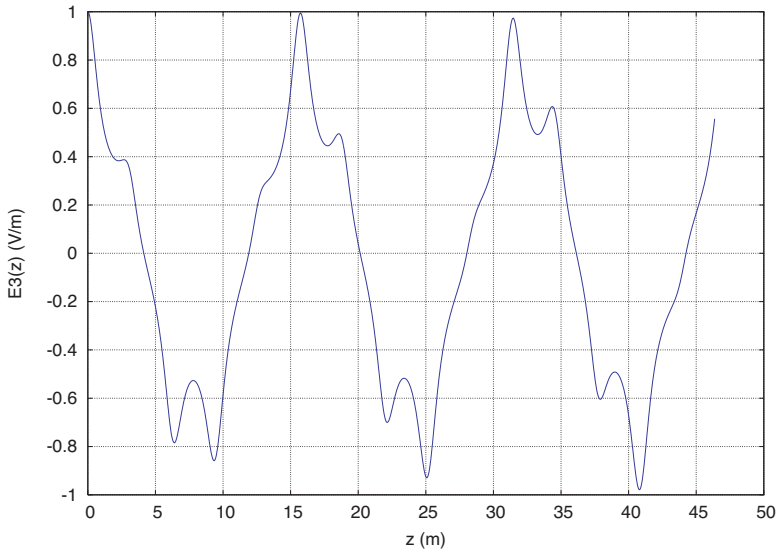

Figure 3. Plot of $E_3(z)$ of equation (33) for $\varepsilon_r(z) = 1 + \sin^2(z),\ \lambda = 10\,\mathrm{m},\ k_z = k_y = \frac{2\pi}{\lambda},\ k_x = \sqrt{k^2 - k_y^2 - k_z^2},\ k = 0.9$.

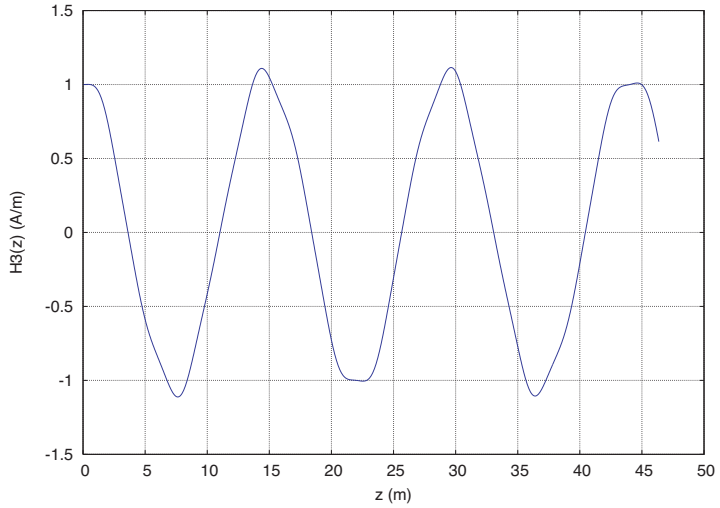

Figure 4. Plot of $H_3(z)$ of equation (34) for $\varepsilon_r(z) = 1 + \sin^2(z),\ \lambda = 10\,\mathrm{m},\ k_z = k_y = \frac{2\pi}{\lambda},\ k_x = \sqrt{k^2 - k_y^2 - k_z^2},\ k = 0.9$.

and have problems with undesirable properties of the integrals, like singularities, oscillating behavior and slow convergence, so that very sophisticated mathematical techniques are required to evaluate or approximate them. Our differential formulation is directly deduced from the Maxwell equations in exact form and it is continuous and fully vectorial. Besides, very accurate approximations to the solutions are possible, at least for the important case of unidimensional media.

5. CONCLUSIONS

A new full vectorial and very general method to study the propagation of electromagnetic waves in inhomogeneous media has been presented. It reduces the problem to the solution of two uncoupled ordinary differential equations for the z-dependent factors of the separated Maxwell equations in Cartesian coordinates. These equations permit the simulation of materials with a continuous variation of their dielectric permittivity without approximating them by discontinuous layered media, reducing so the computational effort of the models. We have developed an effective pseudospectral non-orthogonal collocation technique to solve these equations. It provides very good accuracy and excellent stability with low computational resources and it can be easily adapted to a large class of problems. The obtained analytical

approximations are very useful to perform quick qualitative analysis of the propagation of electromagnetic waves in complex media. Further research work is been done to extend this method to the study of scattering problems in anisotropic, magnetic, conductive and higher dimensional materials in several coordinate systems.

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