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Rigorous steps towards holography in asymptotically flat spacetimes

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Abstract. Scalar QFT on the boundary $\Im^+$ at null infinity of a general asymptotically flat $4D$ spacetime is constructed using the algebraic approach based on Weyl algebra associated to a BMS-invariant symplectic form. The constructed theory turns out to be invariant under a suitable strongly-continuous unitary representation of the BMS group with manifest meaning when the fields are interpreted as suitable extensions to $\Im^+$ of massless minimally coupled fields propagating in the bulk. The group theoretical analysis of the found unitary BMS representation proves that such a field on $\Im^+$ coincides with the natural wave function constructed out of the unitary BMS irreducible representation induced from the little group Δ, the semidirect product between $SO(2)$ and the two-dimensional translations group. This wave function is massless with respect to the notion of mass for BMS representation theory. The presented result proposes a natural criterion to solve the long standing problem of the topology of BMS group. Indeed the found natural correspondence of quantum field theories holds only if the BMS group is equipped with the nuclear topology rejecting instead the Hilbert one. Eventually some theorems towards a holographic description on $\Im^+$ of QFT in the bulk are established at level of C^* algebras of fields for strongly asymptotically predictable spacetimes. It is proved that preservation of a certain symplectic form implies the existence of an injective $*$-homomorphism from the Weyl algebra of fields of the bulk into that associated with the boundary $\Im^+$. Those results are, in particular, applied to $4D$ Minkowski spacetime where a nice interplay between Poincaré invariance in the bulk and BMS invariance on the boundary at null infinity is established at level of QFT. It arises that, in this case, the $*$-homomorphism admits unitary implementation and Minkowski vacuum is mapped into the BMS invariant vacuum on $\Im^+$.

Contents

1 Introduction.

1.1. *Holography in asymptotically flat spacetimes.* One of the key obstacles in the current, apparently never-ending, quest to combine in a unique framework general relativity and quantum mechanics consists in a deep-rooted lack of comprehension of the role and the number of quantum degrees of freedom of gravity. Within this respect, a new insight has been gained from the work of G. 't Hooft who suggested to address this problem from a completely new perspective which is now referred to as *the holographic principle* [1]. This principle states, from the most general point of view, that *physical information in spacetime is fully encoded on the boundary of the region under consideration*. 't Hooft paper represented a cornerstone for innumerable research papers which led to an extension of celebrated Bekenstein-Hawking results about black hole entropy to a wider class of spacetime regions (see in particular the *covariant entropy conjecture* in [2]). Furthermore a broader version of the holographic principle arisen from the above-cited developments according to which *any quantum field theory* - gravity included - living on a D-dimensional spacetime can be fully described by means of a second theory living on a suitable submanifold, with codimension 1, which is not necessary (part of the) boundary of the former. However the holographic principle lacks any general prescription on how to concretely construct a holographic counterpart of a given quantum field theory. In high energy physics in the past years the attempt to fill this gap succeeded achieving some remarkable results. The most notable is the so-called AdS/CFT correspondence [3] or Maldacena conjecture, the key remark being the existence of the equivalence between the bulk and the boundary partition function once asymptotically AdS boundary conditions have been imposed on the physical fields. Without entering into details (see [4] for a recent review), it suffice to say that in the low energy limit a supergravity theory living on a $AdS_D \times X^{10-D}$ manifold is (dual to) a $SU(N)$ conformal super Yang-Mills field theory living on the boundary at spatial infinity of AdS_D. Other remarkable versions of holographic principle for AdS-like spacetime are due to Rehren [5, 6] who proved rigorously several holographic results for local quantum fields in a AdS background, establishing a correspondence between bulk and boundary observables without employing string machinery.

It is rather natural to address the question whether similar holographic correspondences hold whenever a different class of spacetimes is considered. In this paper we will deal with the specific case of asymptotically flat spacetimes and we consider fields interacting, in the bulk, only with the gravitational field. The quest to construct a holographic correspondence in this scenario started only recently and a few different approaches have been proposed [7, 8, 9]. In particular, in [7], in order to implement the holographic principle in a *four-dimensional* asymptotically flat spacetimes (M, g), it has been proposed to construct a bulk to boundary correspondence between a theory living on M and a quantum field theory living at future (or past) null infinity $\Im^+$ of M. A key point is that the theory on $\Im^+$ is further assumed to be invariant under the action of the asymptotic symmetry group of this class of spacetimes: the so called Bondi-Metzner-Sachs (BMS) group. The analysis performed along the lines of Wigner approach to Poincaré invariant free quantum field theory has led to construct the full spectrum, the equations of motion and the Hamiltonians for free quantum field theory enjoying BMS invariance [7, 10]. A first and apparently surprising conclusion which has been drawn from these papers is that, in a BMS invariant field theory, there is a natural plethora of different kinds of admissible BMS-invariant fields. As a consequence the one-to-one correspondence between the bulk and boundary particle spectrum, proper of the Maldacena conjecture, does not hold in this context or needs further information to be constructed. Nevertheless such a conclusion should not be seen as a setback, since it represents the symptom of a key feature proper only of asymptotically flat spacetimes. This is the universality of the boundary data, i.e. as explained in more detail in the next section, the structure at future and past null infinity of any asymptotically flat spacetime is the same. Thus, from a holographic perspective, a BMS-invariant field theory on $\Im^+$ should encode the information from all possible asymptotically flat bulk manifolds. Consequently, it is not surprising if there is such a huge number of admissible BMS-invariant free fields. The main question now consists on finding a procedure allowing one to single out information on a specific bulk from the boundary theory.

The aim of this paper is develop part of this programme using the theory of unitary representations of BMS group as well as tools proper of algebraic local quantum field theory. In particular, using the approach introduced in [15, 16, 17] and fully developed in [17], we define quantum field theory on the null surface $\Im^+$ using the algebraic framework based on a suitable representation of Weyl C^* algebra of fields. Then we investigate the interplay of that theory and quantum field theory of a free scalar field in the bulk finding several interesting results. There is a GNS (Fock space) representation of the field theory on $\Im^+$, based on a certain algebraic quasifree state, which admits an irreducible strongly-continuous unitary representation of the BMS group which leaves invariant the vacuum state. The algebra of fields transforms covariantly with respect to that unitary representation. In other words the fields on $\Im^+$ and the above-mentioned unitary action of BMS group have manifest geometrical meaning when the fields on $\Im^+$ are interpreted as suitable extensions of massless minimally coupled fields propagating in the bulk. Furthermore, the group theoretical analysis of the BMS representation proves that the bulk massless field "restricted" on $\Im^+$ coincides with the natural wave function constructed out of the unitary BMS irreducible representation induced from the little group Δ: the semidirect product between $SO(2)$ and the two dimensional translations. This wave-function is massless with respect to a known notion of mass in BMS representation theory. In this context the found extent provides the solution of a long-standing problem concerning the natural topology of BMS group. In fact, the found unitary representation of GNS group takes place only if the BMS group is equipped with the nuclear topology. In this sense the widely considered Hilbert topology must be rejected.

Eventually some theorems towards a holographic description on $\Im^+$ of QFT in the bulk are established at level of Weyl C^* algebras of fields for spacetimes which are both asymptotically flat and strongly asymptotically predictable. It is shown that, if a symplectic form is preserved passing from the bulk to the boundary, the algebra of fields in the bulk can be identified with a subalgebra for the field observables on $\Im^+$ by means of an injective $*$-homomorphism. Moreover, the BMS invariant state of quantum field theory on $\Im^+$ induces a corresponding reference state in the bulk. It could be used to give a definition of particle based only upon asymptotic symmetries, no matter if the bulk admits any isometry group (see also [12]). Those results are, in particular, applied to $4D$ Minkowski spacetime where a nice interplay between Poincaré invariance in the bulk and BMS invariance on the boundary $\Im^+$ is established at level of quantum field theories. Among other results it arises that the above-mentioned injective $*$-homomorphism has unitary implementation such that the Minkowski vacuum is mapped into the BMS invariant vacuum on $\Im^+$.

The outline of the paper is the following.

In *section 2* we review the notion of asymptotically flat space-time and of the Bondi-Metzner-Sachs group. Starting from these premises a field living at null infinity $\Im^+$ is defined as a suitable limit of a bulk scalar field and the set of fields on $\Im^+$ is endowed with a symplectic structure. Eventually the quantum field theory for an uncharged scalar field living on $\Im^+$ is built up within Weyl algebra approach and a preferred Fock representation is selected which also admits a suitable unitary representation of the BMS group.

In *section 3* the theory of unitary and irreducible representation for the BMS group is discussed and quantum field theory on $\Im^+$ is defined along the lines of Wigner analysis for the Poincaré invariant counterpart. Furthermore it is shown that, at least for scalar fields, the approaches discussed in this and in the previous sections are essentially equivalent provided one adopts a nuclear topology on the BMS group.

In *section 4* the issue of an holographic correspondence is discussed for strongly asymptotically predictable spacetimes. We show that preservation of a certain symplectic form implies existence of a injective $*$-homomorphism from the Weyl algebras of the fields in the bulk into that on $\Im^+$. It is done devoting a particular attention to the specific scenario when the bulk is four-dimensional Minkowski spacetime. It arises that, in this case the $*$-homomorphisms admits unitary implementation and Minkowski vacuum is mapped into the BMS invariant vacuum on $\Im^+$ and the standard unitary representation of Poincaré group in the bulk is transformed in a suitable unitary representation of a subgroup of BMS group on $\Im^+$ and the correspondence has a clear geometric interpretation.

In *section 5* we present our conclusion with some comments about possible future developments and investigations. The *appendix* contains the proof of most of the statement within the paper.

1.2. *Basic definitions and notations.* In this paper *smooth* means C^∞ and we adopt the signature $(-,+,+,+)$ for the Lorentzian metric. The proper orthocronous Lorentz group will be denoted by $SO(3,1)\!\uparrow$, while $ISO(3,1)$ indicates the proper orthocronous Poincaré group $SO(3,1)\!\uparrow\!\ltimes T^4$. (Apart from the well-known case of Poincaré group, the symbol $\ltimes$ will be reserved for a semidirect product of a pair of groups and the explicit form of the composition rule will be explicitly defined case by case.)
In a manifold equipped with Lorentzian metric $\Box := \nabla_a \nabla^a$ denotes d'Alembert operator referred to Levi-Civita connection ∇_a, $\pounds_\xi$ denotes the Lie derivative with respect to the vector field f and f^* the push-forward associated with the diffeomorphism f acting on tensor fields of any fixed order. $C^\infty(M;N)$ and $C^\infty_c(M;N)$ respectively indicates the class of smooth functions and compactly supported smooth functions $f : M \to N$. We omit N in the notation if $N = \mathbb{R}$. $\lim_{\Im^+} f$ indicates a function on $\Im^+$ which is the smooth extension to $\Im^+$ of the function f defined in M.
A *spacetime* is a four-dimensional smooth (Hausdorff second countable) manifold M equipped with a Lorentzian metric g assumed to be everywhere smooth (with the possible exception of the point i^0 when M is $\tilde{M}$), finally M is supposed to be time orientable and time oriented. A *vacuum spacetime* is a spacetime satisfying vacuum Einstein equations. We shall make use of several properties of *globally hyperbolic spacetimes* (Chapter 8 in [18]) and we adopt the classic definition of *asymptotically flat at null and spatial infinity* vacuum spacetime given in Sec. 11.1 in [18] and due to Ashtekar. Henceforth "asymptotically flat" means "asymptotically flat at null

and spatial infinity". Finally we shall employ the definition of *strongly asymptotically predictable spacetime* stated in Sec. 12.1 of [18].

2 Scalar QFT on $\Im^+$.

2.1. *Asymptotic flatness, asymptotic Killing symmetries, BMS group and all that.* Consider an asymptotically flat vacuum spacetime (M, g), let $(\tilde{M}, \tilde{g})$ be an associated unphysical spacetime as in Sec. 11.1 in [18] containing the future null infinity $\Im^+$ of M as an embedded submanifold and the point i^0 corresponding to the spatial infinity of M. By construction $\tilde{g}\restriction_M = \Omega^2\restriction_M g$ where $\Omega \in C^\infty(\tilde{M} \setminus \{i^0\})$ and C^2 in i^0 (see [18] for details) is strictly positive on M and vanishes on the null hypersurface $\Im^+$ with $d\Omega\restriction_{\Im^+} \neq 0$ point-wisely. The metric structures of $\Im^+$ are affected by a *gauge freedom* due the possibility of changing the metric $\tilde{g}$ in a neighborhood of $\Im^+$ with a factor ω smooth and strictly positive. It corresponds to the freedom involved in transformations $\Omega \to \omega\Omega$ in a neighborhood of $\Im^+$. The topology of $\Im^+$ (which is that of $\mathbb{R} \times \mathbb{S}^2$) as well as the differentiable structure are not affected by the gauge freedom. Let us stress some features of this extent. Fixing Ω, $\Im^+$ turns out to be the union of complete future-oriented null geodesics of the metric $\tilde{g}$ which are the integral lines of the field $n^a := \tilde{g}^{ab}\tilde{\nabla}_b\Omega$. This property is, in fact, invariant under gauge transformation, but the field n depends on the gauge. For a fixed asymptotically flat vacuum spacetime (M, g), the manifold $\Im^+$ together with its degenerate metric $\tilde{h}$ induced by $\tilde{g}$ and the field n on $\Im^+$ form a triple which, under gauge transformations $\Omega \to \omega\Omega$, transforms as

$$\Im^+ \to \Im^+ , \quad \tilde{h} \to \omega^2 \tilde{h} , \quad n \to \omega^{-1} n . \tag{1}$$

If C denotes the class containing all of the triples $(\Im^+, \tilde{h}, n)$ transforming as in (1) for a fixed asymptotically flat vacuum spacetime (M, g), there is no general physical principle which allows one to select a preferred element in C. Conversely, C is *universal* for all asymptotically flat vacuum spacetimes in the following sense. If C_1 and C_2 are the classes of triples associated respectively to (M_1, g_2) and (M_2, g_2) there is a diffeomorphism $\gamma : \Im_1^+ \to \Im_2^+$ such that for suitable $(\Im_1^+, \tilde{h}_1, n_1) \in C_1$ and $(\Im_2^+, \tilde{h}_2, n_2) \in C_2$,

$$\gamma(\Im_1^+) = \Im_2^+ , \quad \gamma^*\tilde{h}_1 = \tilde{h}_2 , \quad \gamma^* n_1 = n_2 .$$

The proof of this statement relies on the following nontrivial result [18]. For whichever asymptotically flat vacuum spacetime (M, g) (either (M_1, g_1) and (M_2, g_2) in particular) and whichever initial choice for Ω_0, varying the latter with a judicious choice of the gauge ω, one can always fix $\Omega := \omega\Omega_0$ in order that the metric $\tilde{g}$ associated with Ω satisfies

$$\tilde{g}\restriction_{\Im^+} = -2du\, d\Omega + d\Sigma_{\mathbb{S}^2}(x_1, x_2) . \tag{2}$$

This formula uses the fact that in a neighborhood of $\Im^+$, (u, Ω, x_1, x_2) define a meaningful coordinate system. $d\Sigma_{\mathbb{S}^2}(x_1, x_2)$ is the standard metric on a unit 2-sphere (referred to arbitrarily fixed coordinates x_1, x_2) and $u \in \mathbb{R}$ is nothing but an affine parameter along the null geodesics forming $\Im^+$ itself with $n = \partial/\partial u$. In these coordinates $\Im^+$ is just the set of the points with $u \in \mathbb{R}$, $(x_1, x_2) \in \mathbb{S}^2$ and, no-matter the initial spacetime (M, g) (either (M_1, g_1) and (M_2, g_2) in particular), one has finally the triple $(\Im^+, \tilde{h}_B, n_B) := (\mathbb{R} \times \mathbb{S}^2, d\Sigma_{\mathbb{S}^2}, \partial/\partial u)$.

Remark 2.1. It is worthwhile stressing that, although the requirement to have $(\Im^+, \tilde{h}, n) = (\Im^+, \tilde{h}_B, n_B)$ fixes ω completely [18], there are infinite many, isometrically inequivalent, ways to define the standard metric of a unit 2-sphere on the transverse section of $\Im^+$. This is related to the fact that there are infinite many inequivalent embeddings of a manifold with the topology of $\mathbb{S}^2$ in $\mathbb{R}^3$ endowed with the standard metric. This arbitrariness in fixing the metric on $\mathbb{S}^2$ will be made weaker by introducing BMS group and assuming that two metrics connected by an element of this group are equivalent (see also remark 4.1).

Bondi-Metzner-Sachs (BMS) group, G_{BMS} [19, 20, 21, 22], is the group of diffeomorphisms of $\gamma : \Im^+ \to \Im^+$ which preserve the universal structure of $\Im^+$, i.e. $(\gamma(\Im^+), \gamma^*\tilde{h}, \gamma^* n)$ differs from $(\Im^+, \tilde{h}, n)$ at most by a gauge transformation (1). The following proposition holds [18].

Proposition 2.1. *The one-parameter group of diffeomorphisms generated by a smooth vector field ξ' on $\Im^+$ is a subgroup of G_{BMS} if and only if the following holds. ξ' can be extended smoothly to a field ξ (generally not unique) defined in M in some neighborhood of $\Im^+$ such that $\Omega^2 \pounds_\xi g$ has a smooth extension to $\Im^+$ and $\Omega^2 \pounds_\xi g \to 0$ approaching $\Im^+$.*

The requirement $\Omega^2 \pounds_\xi g \to 0$ approaching $\Im^+$ is the best approximation of the Killing requirement $\pounds_\xi g = 0$ for a generic asymptotically flat spacetime which does *not* admits proper Killing symmetries. In this sense BMS group describes *asymptotic null Killing symmetries* valid for all asymptotically flat vacuum spacetimes.

Remark 2.2.
(1) Notice that BMS group is *smaller* than the group of gauge transformations in equations (1) because not all those transformations can be induced by diffeomorphisms of $\Im^+$. On the other hand the restriction of the gauge group to those transformations induced by diffeomorphisms permits to view BMS group as a group of asymptotic Killing symmetries.
Henceforth, whenever it is not explicitly stated otherwise, we consider as admissible realizations of the unphysical metric on $\Im^+$ only those metrics $\tilde{h}$ which can be reached through transformations of BMS group – i.e. through asymptotic symmetries – from a metric whose associated triple is $(\Im^+, \tilde{h}_B, n_B)$.
(2) Therefore $\tilde{h}$ in general may not coincide with the initial metric induced by $\tilde{g}$ on $\Im^+$ but a further, strictly positive on $\Im^+$, factor ω defined in a neighborhood of $\Im^+$ may take place. Notice that $\omega\Omega$ could have singular behaviour at i^0 and i^+ since they do not belong to $\Im^+$ (see footnote on p.279 in [18]). In this sense freedom allowed by rescaling with factors ω is larger that freedom involved in re-defining the unphysical metric $\tilde{g}$ on the whole unphysical spacetime $\tilde{M}$.

To give an explicit representation of G_{BMS} we need a suitable coordinate frame on $\Im^+$. Having fixed the triple $(\Im^+, \tilde{h}_B, n_B)$ one is still free to select an arbitrary coordinate frame on the sphere and, using the parameter u of integral curves of n_B to complete the coordinate system, one is free to fix the origin of u depending on $\zeta, \overline{\zeta}$ generally. Taking advantage of stereographic projection one may adopt complex coordinates $(\zeta, \overline{\zeta})$ on the (Riemann) sphere, $\zeta = e^{i\phi}\cot(\vartheta/2)$, ϕ, ϑ being usual spherical coordinates.
Coordinates $(u, \zeta, \overline{\zeta})$ *on* $\Im^+$ *define a* **Bondi frame** *when* $(\zeta, \overline{\zeta}) \in \mathbb{C}\times\mathbb{C}$ *are complex stereographic coordinates on* $\mathbb{S}^2$*,* $u \in \mathbb{R}$ *(with the origin fixed arbitrarily) is the parameter of the integral curves of* n *and* $(\Im^+, \tilde{h}, n) = (\Im^+, \tilde{h}_B, n_B)$*.*
In this frame the set G_{BMS} is nothing but $SO(3,1)\!\uparrow\!\times C^\infty(\mathbb{S}^2)$, and $(\Lambda, f) \in SO(3,1)\!\uparrow \times C^\infty(\mathbb{S}^2)$ acts on $\Im^+$ as [23]

$$u \rightarrow u' := K_\Lambda(\zeta,\overline{\zeta})(u + f(\zeta,\overline{\zeta}))\,, \tag{3}$$

$$\zeta \rightarrow \zeta' := \Lambda\zeta := \frac{a_\Lambda\zeta + b_\Lambda}{c_\Lambda\zeta + d_\Lambda}\,, \qquad \overline{\zeta} \rightarrow \overline{\zeta}' := \Lambda\overline{\zeta} := \frac{\overline{a_\Lambda}\overline{\zeta} + \overline{b_\Lambda}}{\overline{c_\Lambda}\overline{\zeta} + \overline{d_\Lambda}}\,. \tag{4}$$

$$K_\Lambda(\zeta,\overline{\zeta}) := \frac{(1+\zeta\overline{\zeta})}{(a_\Lambda\zeta + b_\Lambda)(\overline{a_\Lambda}\overline{\zeta} + \overline{b_\Lambda}) + (c_\Lambda\zeta + d_\Lambda)(\overline{c_\Lambda}\overline{\zeta} + \overline{d_\Lambda})} \quad \text{and} \quad \left[\begin{array}{cc} a_\Lambda & b_\Lambda \\ c_\Lambda & d_\Lambda \end{array}\right] = \Pi^{-1}(\Lambda)\,. \tag{5}$$

Π is the well-known surjective covering homomorphism $SL(2,\mathbb{C}) \rightarrow SO(3,1)\!\uparrow$. Thus the matrix of coefficients $a_\Lambda, b_\Lambda, c_\Lambda, d_\Lambda$ is an arbitrary element of $SL(2,\mathbb{C})$ determined by Λ up to an overall sign. However K_Λ and the right hand sides of (4) are manifestly independent from any choice of such a sign. It is clear from (4) and (5) that, *in a fixed Bondi frame*, G_{BMS} can be viewed as the semidirect product of $SO(3,1)\!\uparrow$ and the Abelian additive group $C^\infty(\mathbb{S}^2)$. The elements of this subgroup are called **supertranslations**. In particular, if $\odot$ denotes the product in G_{BMS}, $\circ$ the composition of functions, $\cdot$ the pointwise product of scalar functions and Λ acts on $(\zeta, \overline{\zeta})$ as said in the right-hand sides of (4):

$$K_{\Lambda'}(\Lambda(\zeta,\overline{\zeta}))K_\Lambda(\zeta,\overline{\zeta}) = K_{\Lambda'\Lambda}(\zeta,\overline{\zeta})\,. \tag{6}$$

$$(\Lambda', f') \odot (\Lambda, f) = \left(\Lambda'\Lambda,\ f + (K_{\Lambda^{-1}}\circ\Lambda)\cdot(f'\circ\Lambda)\right)\,. \tag{7}$$

Remark 2.3. *We underline that in the literature the factor* K_Λ *does not always have the same definition. In particular, in [24, 25, 26, 27, 28]*

$$K_\Lambda(\zeta,\overline{\zeta}) := \frac{(a_\Lambda\zeta + b_\Lambda)(\overline{a_\Lambda}\overline{\zeta} + \overline{b_\Lambda}) + (c_\Lambda\zeta + d_\Lambda)(\overline{c_\Lambda}\overline{\zeta} + \overline{d_\Lambda})}{(1+\zeta\overline{\zeta})},$$

but in this paper we stick to the definition (5) as in [23, 29] adapting accordingly the calculations and results from the above mentioned references.

The following proposition arises from the definition of Bondi frame and the equations above.

Proposition 2.2. *Let $(u, \zeta, \overline{\zeta})$ be a Bondi frame on $\Im^+$. The following holds.*
(a) *A global coordinate frame $(u', \zeta', \overline{\zeta}')$ on $\Im^+$ is a Bondi frame if and only if*

$$u = u' + g(\zeta', \overline{\zeta}'), \tag{8}$$

$$\zeta = \frac{a_R \zeta' + b_R}{c_R \zeta' + d_R}, \quad \overline{\zeta} = \frac{\overline{a_R}\overline{\zeta}' + \overline{b_R}}{\overline{c_R}\overline{\zeta}' + \overline{d_R}}, \tag{9}$$

for $g \in C^\infty(\mathbb{S}^2)$ and $R \in SO(3)$ referring to the canonical inclusion $SO(3) \subset SO(3,1)\!\uparrow$ (i.e. the canonical inclusion $SU(2) \subset SL(2,\mathbb{C})$ for matrices of coefficients $(a_\Lambda, b_\Lambda, c_\Lambda, d_\Lambda)$ in (5).)
(b) *The functions K_Λ are smooth on the Riemann sphere $\mathbb{S}^2$. Furthermore $K_\Lambda(\zeta, \overline{\zeta}) = 1$ for all $(\zeta, \overline{\zeta})$ if and only if $\Lambda \in SO(3)$.*
(c) *Let $(u', \zeta', \overline{\zeta}')$ be another Bondi frame as in (a). If $\gamma \in G_{BMS}$ is represented by (Λ, f) in $(u, \zeta, \overline{\zeta})$, the same γ is represented by (Λ', f') in $(u', \zeta', \overline{\zeta}')$ with*

$$(\Lambda', f') = (R, g)^{-1} \odot (\Lambda, f) \odot (R, g). \tag{10}$$

2.2. *Space of fields with BMS representations.* Let us consider QFT on $\Im^+$ developed in the way presented in [15, 16, 17] where QFT on null hypersurfaces was investigated in the case of Killing horizons. $\Im^+$ is not a Killing horizon but the theory can be re-adapted to this case with simple adaptations. The procedure we go to introduce is similar to that sketched in [30] for graviton field.
First of all we fix a relation between scalar fields ϕ in (M, g) and scalar fields ψ on $\Im^+$. The idea is to consider the fields ψ as re-arranged smooth restrictions to $\Im^+$ of fields ϕ. Simple restrictions make no sense because $\Im^+$ does not belong to M. We aspect that a good definition of fields ψ is a suitable smooth limit to $\Im^+$ of products $\Omega^\alpha \phi$ for some fixed real exponent α. A strong suggestion for the value of α is given by the following proposition. (Below $\Box$ is d'Alembert operator referred to $\tilde{g}$ and R and $\tilde{R}$ are the scalar curvatures on M and $\tilde{M}$ respectively.)
We recall the reader that an asymptotically flat spacetime (M, g) is said to be *strongly asymptotically predictable* if in the unphysical associated spacetime there is an open set $\tilde{V} \subset \tilde{M}$ with $\overline{M \cap J^-(\Im^+)} \subset \tilde{V}$ (the closure being referred to $\tilde{M}$) such that $(\tilde{V}, \tilde{g})$ is globally hyperbolic. (Minkowski spacetime and several spacetimes containing black holes are so [18]).

Proposition 2.3. *Assume that (M, g) is asymptotically flat with associated unphysical spacetime $(\tilde{M}, \tilde{g})$ with $\tilde{g}\restriction_M = \Omega^2 g$. Suppose that (M, g) is strongly asymptotically predictable referring to the open set $\tilde{V} \subset \tilde{M}$. If $\phi : M \cap \tilde{V} \to \mathbb{C}$ has compactly supported Cauchy data on some Cauchy surface of $M \cap \tilde{V}$ and satisfies massless conformal Klein-Gordon equation,*

$$\Box\phi - \frac{1}{6} R\phi = 0, \tag{11}$$

(a) *the field* $\tilde{\phi} := \Omega^{-1}\phi$ *can be extended uniquely into a smooth solution in* $(\tilde{V}, \tilde{g})$ *of*

$$\Box\tilde{\phi} - \frac{1}{6}\tilde{R}\tilde{\phi} = 0\,; \tag{12}$$

(b) *for every smooth positive factor* ω *defined in a neighborhood of* $\Im^+$ *used to rescale* $\Omega \to \omega\Omega$ *in such a neighborhood,* $(\omega\Omega)^{-1}\phi$ *extends to a smooth field* ψ *on* $\Im^+$ *uniquely.*

We have assumed the possibility of having $R \neq 0$ in M because, as noticed in [18], all we said in section 2.1 holds true dropping the hypotheses for the spacetime (M, g) to be a vacuum Einstein solution, but requiring that the stress energy tensor T is such that $\Omega^{-2}T$ is smooth on $\Im^+$ and $\Im^-$ and has appropriate limiting behaviour at i^0. A simple and well-known example of the application of the theorem is given by the pair $(M, \tilde{M})$ made of Minkowski spacetime and Einstein static universe [18].

Proof. It is known [18] that, in any open subset of M and under the only hypothesis $\tilde{g} = \Omega^2 g$, (11) is valid for ϕ if and only if (12) is valid for $\Omega^{-1}\phi$. To go on, first of all we recall the reader that, in globally hyperbolic spacetimes, the constraint of compactly supported Cauchy data does not depend on the used Cauchy surface. Therefore ϕ in the thesis has compactly supported Cauchy data on every smooth Cauchy surface S of $M \cap \tilde{V}$. Now fix S as follows. As $(\tilde{V}, g)$ is globally hyperbolic, so are $(M \cap \tilde{V}, \tilde{g})$ and $(M \cap \tilde{V}, g)$ and, in particular, a Cauchy surface $\tilde{S}$ of $(\tilde{V}, g)$ which intersect i^0 give rises, by restriction, to a smooth Cauchy surfaces S for both $(M \cap \tilde{V}, g)$ and $(M \cap \tilde{V}, \tilde{g})$ (see Sec. 12.1 of [18]). Compactly supported Cauchy data C of ϕ on S determine analogous compactly support Cauchy data $\tilde{C}$ on $\tilde{S}$ for (12) through the oblivious extension of the relation $\tilde{\phi} := \Omega^{-1}\phi$ to Cauchy data. Let $\tilde{\Phi}$ be the unique solution of (12) in the whole globally hyperbolic spacetime $(\tilde{V}, \tilde{g})$ associated with Cauchy data $\tilde{C}$. By uniqueness theorem of the solution of Klein-Gordon equation for compact Cauchy data in globally hyperbolic spacetime $\tilde{\Phi}$ must be an extension of $\tilde{\phi}$ to $(\tilde{V}, \tilde{g})$ since both fields have the same Cauchy data on $\tilde{S}$. The proof concludes by noticing that $\Im^+ \subset \tilde{V}$ (and thus $\psi := \tilde{\Phi}\restriction_{\Im^+}$ is, in fact, a smooth extension to $\Im^+$ of $\tilde{\phi}$) because $\overline{M} \cap \overline{J^-(\Im^+)} = \overline{M \cap J^-(\Im^+)} \subset \tilde{V}$ and $\Im^+ \subset \partial M$ by definition of $\Im^+$ and $\Im^+ \subset \overline{J^-(\Im^+)}$ by definition of J^-.

The case with $\omega \neq 1$ is now a trivial consequence of what proved above replacing Ω with $\omega\Omega$ in the considered neighborhood of $\Im^+$ where $\omega > 0$. $\Box$

We go to define a field theory on $\Im^+$ – thought as a pure differentiable manifold – based on smooth scalar fields ψ and assuming G_{BMS} as the natural symmetry group. The latter assumption is in order to try to give some physical interpretation of the theory, since physical information is invariant under G_{BMS} as said above. In particular, we have to handle the extent of a metrical structure on $\Im^+$ which is not invariant under BMS group. The field theory should be viewed, more appropriately, as QFT on the class of all the triples $(\Im^+, \tilde{h}, n)$ connected with $(\Im^+, \tilde{h}_B, n_B)$ by the transformations of G_{BMS}. In this way one takes asymptotic Killing symmetries into account. Therefore we need a representation $G_{BMS} \ni \gamma \mapsto A_\gamma$ in terms of transformations $A_\gamma : C^\infty(\Im^+; \mathbb{C}) \to C^\infty(\Im^+; \mathbb{C})$. The naive idea is to define such an action as the push-forward on scalar fields of diffeomorphisms $\gamma \in G_{BMS}$, i.e. $A_\gamma := \gamma^*$. However this is not a very satisfactory idea, if one wants to maintain the possibility to interpret some of the fields ψ as extensions to $\Im^+$ of fields $(\omega\Omega)^\alpha\phi$ defined in the bulk. Proposition 2.1 shows that there are one-parameter (local) groups of diffeomorphisms $\{\gamma_t\}$ in the physical spacetime (in general not preserving (11)) which induce one-parameter subgroups of G_{BMS}, γ'_t. A natural requirement on the wanted representation $A^{(\alpha)}$ is that, for a scalar field ϕ on M such that $(\omega\Omega)^\alpha\phi$ admits a smooth extension ψ to $\Im^+$

$$A^{(\alpha)}_{\gamma'_t}\psi = \lim_{\Im^+}(\omega\Omega)^\alpha \gamma^*_t(\phi) \tag{13}$$

for every (local) one-parameter group of diffeomorphisms $\{\gamma_t\}$ generated by any vector field ξ as in Proposition 2.1, for every value t of the associated (local) one-parameter group of diffeomorphisms. We have the following result whose proof is in the Appendix.

Proposition 2.4. *Assume that (M, g) is asymptotically flat with associated unphysical spacetime $(\tilde{M}, \tilde{g})$ (with $\tilde{g}\restriction_M = \Omega^2 g$). Fix $\omega > 0$ in a neighborhood of $\Im^+$ such that $\omega\tilde{g}$ is associated with the triple $(\Im^+, \tilde{h}_B, n_B)$. Consider, for a fixed $\alpha \in \mathbb{R}$, a representation $G_{BMS} \ni \gamma \mapsto A^{(\alpha)}_\gamma$ in terms of transformations $A_\gamma : C^\infty(\Im^+;\mathbb{C}) \to C^\infty(\Im^+;\mathbb{C})$ such that $t \mapsto A^{(\alpha)}_{\gamma_t}\psi_0$ is smooth for every fixed ψ_0 and every fixed one-parameter group of diffeomorphisms $\{\gamma_t\}$ subgroup of G_{BMS}. Finally assume that (13) holds for any ψ obtained as smooth extension to $\Im^+$ of $(\omega\Omega)^\alpha\phi$, $\phi \in C^\infty(M;\mathbb{C})$. Then, in any Bondi frame*

$$\left(A^{(\alpha)}_{(\Lambda,f)}\psi\right)(u', \zeta', \overline{\zeta}') := K_\Lambda(\zeta, \overline{\zeta})^{-\alpha}\psi(u, \zeta, \overline{\zeta})\,. \tag{14}$$

for any $(\Lambda, f) \in G_{BMS}$ and referring to (3),(4), (5).

From (6), equation (14) defines, in fact, a representation of G_{BMS} when assumed valid on all the fields $\psi \in C^\infty(\Im^+, \mathbb{C})$ or some BMS-invariant subspace of $C^\infty(\Im^+)$ as $C^\infty_c(\Im^+;\mathbb{C})$ or similar. *From now on we assume that the action of G_{BMS} on scalar fields $\psi \in C^\infty(\Im^+;\mathbb{C})$ is given from a representation $A^{(\alpha)} : G_{BMS} \ni \psi \mapsto A^{(\alpha)}_\gamma$ defined in (14) with α fixed.*

Transformations (14) are well-known and used in the literature [29]. We stress that our interpretation of $A^{(\alpha)}_{(\Lambda,f)}$ is *active* here, in particular the fields ψ are scalar fields and thus they transform as usual scalar fields under change of coordinates related or not by a BMS transformation (passive transformations). Using Proposition 2.2, (5) in particular, the reader can easily prove the following result.

Proposition 2.5. *Consider two Bondi frames B and B' on $\Im^+$. Take $\gamma \in G_{BMS}$ and represent it as (Λ, f) and (Λ', f') in B and B' respectively (so that (10) holds).*
Acting on a scalar fields ψ, $A^{(\alpha)}_{(\Lambda,f)}$ and $A^{(\alpha)}_{(\Lambda',f')}$ produce the same transformed scalar field.

The proposition says that the representation defined in Proposition 2.4 *does not depend* on the particular Bondi frame used to represent $\Im^+$, but it depends only on the diffeomorphisms $\gamma \in G_{BMS}$ individuated by the pairs (Λ, f) in the Bondi frame used to make explicit the representation. In this way we are given a *unique representation* $G_{BMS} \ni \gamma \mapsto A^{(\alpha)}_\gamma$ not depending on the used Bondi frame which can be represented as in (14) when a Bondi frame is selected.

2.3. *BMS-Invariant Symplectic form.* As a second step we introduce the **space of (real) wavefunctions on** $\Im^+$, $\mathcal{S}(\Im^+)$. In a fixed Bondi frame $\mathcal{S}(\Im^+)$ is the *real linear space* of the smooth functions $\psi : \Im^+ \to \mathbb{R}$ such that ψ itself and all of its derivatives in any variable vanish as $|u| \to +\infty$, uniformly in $\zeta, \overline{\zeta}$, faster than any functions $|u|^{-k}$ for every natural k. It is simply proved that actually $\mathcal{S}(\Im^+)$ does not depend on the used Bondi frame (use Proposition 2.2 and the fact that functions f are continuous and thus bounded on the compact $\mathbb{S}^2$). Obviously $C^\infty_c(\Im^+) \subset \mathcal{S}(\Im^+)$ and it is simply proved that $\mathcal{S}(\Im^+)$ is invariant under the representation $A^{(1)}$ of G_{BMS} defined in the previous section.
One has the following result that shows that $\mathcal{S}(\Im^+)$ can be equipped with a symplectic form invariant under the action of BMS group. That symplectic form was also studied in [22] and [17].

Theorem 2.1. *Consider the representations $A^{(\alpha)}$ on $C^\infty(\Im^+;\mathbb{C})$ of G_{BMS} introduced above and the map: $\sigma : \mathcal{S}(\Im^+) \times \mathcal{S}(\Im^+) \to \mathbb{R}$*

$$\sigma(\psi_1, \psi_2) := \int_{\mathbb{R}\times\mathbb{S}^2} \left(\psi_2 \frac{\partial \psi_1}{\partial u} - \psi_1 \frac{\partial \psi_2}{\partial u}\right) du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}) \,, \tag{15}$$

$(u, \zeta, \overline{\zeta})$ being a Bondi frame on $\Im^+$ and $\epsilon_{\mathbb{S}^2}$ being the standard volume form of the unit 2-sphere

$$\epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta}) := \frac{2d\zeta \wedge d\overline{\zeta}}{i(1+\zeta\overline{\zeta})^2} \,. \tag{16}$$

The following holds.
(a) *σ is a nondegenerate symplectic form (that is, it is linear, antisymmetric and $\sigma(\psi_1, \psi_2) = 0$ for all $\psi_1 \in \mathcal{S}(\Im^+)$ entails $\psi_2 = 0$) which does not depend on the used Bondi frame.*
(b) *$\mathcal{S}(\Im^+)$ is invariant under every representation $A^{(\alpha)}$, whereas σ is invariant under $A^{(1)}$.*

Proof. (a) can be proved by direct inspection using Proposition 2.2 to check on the independence from the used Bondi frame and taking advantage of the fact that $\epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta})$ is invariant under three dimensional rotations. Invariance of $\mathcal{S}(\Im^+)$ under $A^{(\alpha)}$ can be established immediately using the fact that the functions f in (3) and the functions K_Λ in (5) and (14) are bounded. Let us prove the non trivial part of item (b). One has

$$\sigma(\psi'_1, \psi'_2) = \int_{\mathbb{R}\times\mathbb{S}^2} \left(\psi'_2 \frac{\partial \psi'_1}{\partial u'} - \psi'_1 \frac{\partial \psi'_2}{\partial u'}\right) du' \wedge \epsilon_{\mathbb{S}^2}(\zeta', \overline{\zeta}') \,.$$

Now we can use (14) together with the known relation

$$\epsilon_{\mathbb{S}^2}(\zeta', \overline{\zeta}') = K_\Lambda(\zeta, \overline{\zeta})^2 \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta})$$

obtaining

$$\sigma(\psi'_1, \psi'_2) = \int_{\mathbb{R}\times\mathbb{S}^2} \left(\psi_2 \frac{\partial \psi_1}{\partial u} - \psi_1 \frac{\partial \psi_2}{\partial u} \right) du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta})$$

which is the thesis. □

Remark 2.4. From now on the restriction to the invariant space $\mathcal{S}(\Im^+)$ of $A^{(1)}_\gamma$ is indicated by A_γ, similarly A denotes the representation $G_{BMS} \ni \gamma \mapsto A_\gamma$.

2.4. *Weyl algebraic quantization and Fock representation.* As third and last step we define QFT on $\Im^+$ for uncharged scalar fields in Weyl approach giving also a preferred Fock space representation.

The formulation of real scalar QFT on the degenerate manifold $\Im^+$ we present here is an almost straightforward adaptation of the theory presented in [17] (see section 4.2 for the corresponding in general curved spacetime [31]). As $\mathcal{S}(\Im^+)$ is a real vector space equipped with a nondegenerate symplectic form σ, there exists a complex C^*-algebra (theorem 5.2.8 in [32]) generated by elements, $W(\psi)$ with $\psi \in \mathcal{S}(\Im^+)$ satisfying, for all $\psi, \psi' \in \mathcal{S}(\Im^+)$,

$$\text{(W1)} \qquad W(-\psi) = W(\psi)^*, \qquad \text{(W2)} \qquad W(\psi)W(\psi') = e^{i\sigma(\psi,\psi')/2}W(\psi + \psi') \, .$$

That C^*-algebra, indicated by $\mathcal{W}(\Im^+)$, is unique up to (isometric) *-isomorphisms (theorem 5.2.8 in [32]). As consequences of (W1) and (W2), $\mathcal{W}(\mathbb{M})$ admits unit $I = W(0)$, each $W(\psi)$ is unitary and, from the nondegenerateness of σ, $W(\psi) = W(\psi_1)$ if and only if $\psi = \psi_1$. $\mathcal{W}(\Im^+)$ is called **Weyl algebra associated with** $\mathcal{S}(\Im^+)$ and σ whereas the $W(\psi)$ are called **(abstract) Weyl operators**. The formal interpretation of elements $W(\psi)$ is $W(\psi) \equiv e^{i\Psi(\psi)}$ where $\Psi(\psi)$ are *symplectically smeared field operators* as we shall see shortly. The definition of σ entails straightforward implementation of *locality principle*:

$$[W(\psi_1), W(\psi_2)] = 0 \quad \text{if} \quad (supp\psi_1) \cap (supp\psi_2) = \emptyset \, . \tag{17}$$

Differently from QFT in curved spacetime, but similarly to [17], here we do not impose any equation of motion. On the other hand the space of wavefunctions, differently from the extent in the case of degenerate manifolds studied in [17], gives rise to direct implementation of locality. No "causal propagator" has to be introduced in this case.

A Fock representation of $\mathcal{W}(\mathbb{M})$ based on a BMS-invariant vacuum state can be introduced as follows. ¿From a physical point of view, the procedure resembles quantization with respect to Killing time in a static spacetime. Fix a Bondi frame $(u, \zeta, \overline{\zeta})$ on $\Im^+$. Any $\psi \in \mathcal{S}(\Im^+)$ can be written as a Fourier integral in the parameter u and one may extract the **positive-frequency part** (with respect to u):

$$\psi_+(u, \zeta, \overline{\zeta}) := \int_{\mathbb{R}^+} \frac{dE}{\sqrt{4\pi E}} e^{-iEu} \widetilde{\psi_+}(E, \zeta, \overline{\zeta}) \, . \tag{18}$$

where $\mathbb{R}^+ := [0, +\infty)$ and

$$\widetilde{\psi_+}(E,\zeta,\overline{\zeta}) := \sqrt{E}\int_{\mathbb{R}} \frac{du}{\sqrt{4\pi}} e^{+iEu}\psi(u,\zeta,\overline{\zeta}) \quad \text{for } E \in \mathbb{R}^+ . \tag{19}$$

Obviously it also holds $\psi = \psi_+ + \overline{\psi_+}$. It could seem that the definition of positive frequency part depend on the used Bondi frame and the coordinate u in particular; actually, by direct inspection based on Proposition 2.2, one finds that:

Proposition 2.6. *Positive-frequency parts do not depend to the Bondi frame and define scalar fields. In other words if $\psi \in \mathcal{S}(\Im^+)$ has positive frequency parts ψ_+ and ψ'_+ respectively in Bondi frames $(u,\zeta,\overline{\zeta})$ and $(u',\zeta',\overline{\zeta}')$, it holds*

$$\psi_+(u,\zeta,\overline{\zeta}) = \psi'_+(u'(u,\zeta,\overline{\zeta}),\zeta'(\zeta,\overline{\zeta}),\overline{\zeta}'(\zeta,\overline{\zeta}))\,, \quad \textit{for all } u\in\mathbb{R},\ (\zeta,\overline{\zeta})\in\mathbb{C}\times\mathbb{C}. \tag{20}$$

We are able to give a definition of one-particle Hilbert space and show that it is isomorphic to a suitable space L^2. Let us denote by $\mathcal{S}(\Im^+)^{\mathbb{C}}_+$ the space made of the complex finite linear combinations of positive-frequency parts of the elements of $\mathcal{S}(\Im^+)$. The proof of the following result is in the appendix.

Theorem 2.2. *With the given definition of $\mathcal{S}(\Im^+)$, σ and $\mathcal{S}(\Im^+)^{\mathbb{C}}_+$, the following holds.*
(a) *The right-hand side of the definition of σ (15) is well-behaved if evaluated on functions in $\mathcal{S}(\Im^+)^{\mathbb{C}}_+$ and it is independent from the used Bondi frame.*
(b) *Using (a) and extending the definition of σ (15) to $\mathcal{S}(\Im^+)^{\mathbb{C}}_+$, consider the complex numbers*

$$\langle \psi_{1+},\psi_{2+}\rangle := -i\sigma(\overline{\psi_{1+}},\psi_{2+})\,, \quad \textit{for every pair } \psi_1,\psi_2 \in \mathcal{S}(\Im^+). \tag{21}$$

There is only one Hermitean scalar product $\langle\cdot,\cdot\rangle$ on $\mathcal{S}(\Im^+)^{\mathbb{C}}_+$ which fulfils (21). $\langle\cdot,\cdot\rangle$ is independent from the used Bondi frame, whereas, referring $\widetilde{\psi_+}$ to a given Bondi frame $(u,\zeta,\overline{\zeta})$,

$$\langle \psi_{1+},\psi_{2+}\rangle = \int_{\mathbb{R}^+\times\mathbb{S}^2} \overline{\widetilde{\psi_{1+}}(E,\zeta,\overline{\zeta})}\,\widetilde{\psi_{2+}}(E,\zeta,\overline{\zeta})\,dE\otimes\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})\,, \quad \textit{for every pair } \psi_1,\psi_2\in\mathcal{S}(\Im^+). \tag{22}$$

(c) *Let $\mathcal{H}$ be the Hilbert completion of $\mathcal{S}(\Im^+)^{\mathbb{C}}_+$ with respect to $\langle\cdot,\cdot\rangle$. The unique complex linear and continuous extension of the map $\psi_+ \mapsto \widetilde{\psi_+}$ (for $\psi \in \mathcal{S}(\Im^+)$) with domain given by the whole $\mathcal{H}$ is a unitary isomorphism onto $L^2(\mathbb{R}^+\times\mathbb{S}^2, dE\otimes\epsilon_{\mathbb{S}^2})$.*

In the following $\mathcal{H}$ will be called **one-particle space**. Quantum field theory on $\Im^+$ relies on the bosonic (i.e. symmetric) Fock space $\mathfrak{F}_+(\mathcal{H})$ built upon the vacuum state Υ (we assume $||\Upsilon|| = 1$ explicitly). The **field operator symplectically smeared with** $\psi \in \mathcal{S}(\Im^+)$ is now defined as [31]

$$\sigma(\psi,\Psi) := ia(\overline{\psi_+}) - ia^\dagger(\psi_+)\,, \tag{23}$$

where the operators $a^\dagger(\psi_+)$ and $a(\overline{\psi_+})$ ($\mathbb{C}$-linear in $\overline{\psi_+}$) respectively create and annihilate the state $\psi_+ \in \mathcal{H}$. The common invariant domain of all the involved operators is the dense linear manifold $F(\mathcal{H})$ spanned by the vectors with finite number of particles. $\Psi(\psi)$ is essentially self-adjoint on $F(\mathcal{H})$ (it is symmetric and $F(\mathcal{H})$ is dense and made of analytic vectors) and satisfies bosonic commutation relations (CCR):

$$[\sigma(\psi,\Psi),\sigma(\psi',\Psi)] = -i\sigma(\psi,\psi')I \, .$$

Since there is no possibility of misunderstandings because we will not introduce other, non symplectic, smearing procedures for field operators defined on $\Im^+$, from now on we use the simpler notation

$$\Psi(\psi) := \sigma(\psi,\Psi)\,, \tag{24}$$

however the reader should bear in his mind that symplectic smearing is understood. Finally the unitary operators

$$\widehat{W}(\psi) := e^{i\overline{\Psi(\psi)}} \tag{25}$$

enjoy properties (W1), (W2) so that they define a unitary representation $\widehat{\mathcal{W}}(\Im^+)$ of $\mathcal{W}(\Im^+)$ which is also irreducible. A proof of these properties is contained in propositions 5.2.3 and 5.2.4 in [32] where the used field operator is $\Phi(f)$ with $f \in \mathfrak{h} := \mathcal{H}$ and it holds $\Psi(\psi) = \sqrt{2}\Phi(i\psi_+)$. If $\Pi : \mathcal{W}(\Im^+) \to \widehat{\mathcal{W}}(\Im^+)$ denotes the unique (σ being nondegenerate) C^*-algebra isomorphism between those two Weyl representations, $(\mathfrak{F}_+(\mathcal{H}), \Pi, \Upsilon)$ coincides, up to unitary transformations, with the GNS triple associated with the algebraic pure state λ on $\mathcal{W}(\Im^+)$ uniquely defined by the requirement (see the appendix)

$$\lambda(W(\psi)) := e^{-\langle\psi_+,\psi_+\rangle/2}\,. \tag{26}$$

2.5. *Unitary BMS invariance.* Let us show that $\mathfrak{F}(\mathcal{H})$ admits a unitary representation of G_{BMS} which is covariant with respect to an analogous representation of the group given in terms of $*$-automorphism of $\widehat{\mathcal{W}}(\Im^+)$. Moreover we show that the vacuum state Υ (or equivalently, the associated algebraic state λ on $\mathcal{W}(\Im^+)$) is invariant under the representation.
Consider the representation A of G_{BMS} in terms of transformations of fields in $\mathcal{S}(\Im^+)$ used in sections 2.2 and 2.3. As a consequence of the invariance of σ under the action of A_γ, by (4) in theorem 5.2.8 of [32] one has the following straightforward result concerning the C^*-algebra $\mathcal{W}(\Im^+)$ constructed with σ.

Proposition 2.7. *With the given definitions of A (remark 2.4) and $\mathcal{W}(\Im^+)$ there is a unique representation of G_{BMS}, indicated by $\alpha : G_{BMS} \ni \gamma \mapsto \alpha_\gamma$, and made of $*$-automorphisms of $\mathcal{W}(\Im^+)$, satisfying*

$$\alpha_\gamma(W(\psi)) = W\left(A_\gamma\psi\right)\,. \tag{27}$$

Let us come to the main result given in the following theorem.

Theorem 2.3. *Consider the representation of $\mathcal{W}(\Im^+)$ built upon Υ in the Fock space $\mathfrak{F}_+(\mathcal{H})$ equipped with the representation of G_{BMS}, α, given above. The following holds.*
(a) *There is unique a unitary representation $U : G_{BMS} \ni \gamma \mapsto U_\gamma$ such that both the requirements below are fulfilled.*
(i) It is covariant with respect to the representation α , i.e.

$$U_\gamma \widehat{W}(\psi) U_\gamma^\dagger = \alpha_\gamma(\widehat{W}(\psi)) \,, \quad \textit{for all } \gamma \in G_{BMS} \textit{ and } \psi \in \mathcal{S}(\Im^+). \tag{28}$$

(ii) The vacuum vector Υ is invariant under U: $U\Upsilon = \Upsilon$.
(b) *Any projective unitary representation[1] $V : G_{BMS} \ni \gamma \mapsto V_\gamma$ on $\mathfrak{F}_+(\mathcal{H})$ which is covariant with respect to α can be made properly unitary, since it must satisfy,*

$$e^{ig(\gamma)} V_\gamma = U_\gamma \,, \quad \textit{with } e^{-ig(\gamma)} = \langle \Upsilon, V_\gamma \Upsilon \rangle, \textit{ for every } \gamma \in G_{BMS}. \tag{29}$$

(c) *The subspaces of $\mathfrak{F}_+(\mathcal{H})$ with fixed number of particles are invariant under U and U itself is constructed canonically by tensorialization of $U\restriction_{\mathcal{H}}$. The latter satisfies, for every $\gamma \in G_{BMS}$ and the positive frequency part of any $\psi \in \mathcal{S}(\Im^+)$*

$$U_\gamma \psi_+ = A^{(1)}_\gamma(\psi_+) \,. \tag{30}$$

Equivalently, in a fixed Bondi frame, where $G_{BMS} \ni \gamma \equiv (\Lambda, f) \in SO(3,1)\!\uparrow\!\ltimes C^\infty(\mathbb{S}^2)$,

$$\left(U_{(\Lambda,f)}\varphi\right)(E,\zeta,\overline{\zeta}) = \frac{e^{iEK_\Lambda(\Lambda^{-1}(\zeta,\overline{\zeta}))f(\Lambda^{-1}(\zeta,\overline{\zeta}))}}{\sqrt{K_\Lambda(\Lambda^{-1}(\zeta,\overline{\zeta}))}} \varphi\left(EK_\Lambda\left(\Lambda^{-1}(\zeta,\overline{\zeta})\right), \Lambda^{-1}(\zeta,\overline{\zeta})\right) \,, \tag{31}$$

is valid for every $\varphi \in L^2(\mathbb{R}^+ \times \mathbb{S}^2; dE \otimes \epsilon_{\mathbb{S}^2})$, $\varphi = \widetilde{\psi_+}$ in particular.

Proof. (a) and (c). The uniqueness property is straightforward consequence of (c), using the further hypothesis (ii) $V\Upsilon = \Upsilon$ in (29), obtaining $e^{-ig(\gamma)} = \langle \Upsilon, V_\gamma \Upsilon \rangle = 1$. Let us pass to prove the existence of U. Consider the positive frequency part ψ_+ of $\psi \in \mathcal{S}(\Im^+)$. From the proof of Theorem 2.3 (in the appendix) we have that $\psi_+ \in C^\infty(\Im^+; \mathbb{C})$ so that $A^{(1)}_\gamma \psi_+$ is well-defined. Furthermore ψ_+ with its derivatives decay as $|u| \to +\infty$ fast enough and uniformly in $\zeta, \overline{\zeta}$, so that it makes sense to apply σ to a pair of functions ψ_+. Moreover the proof of the invariance of σ under the representation $A^{(1)}$ given in Theorem 2.1 can be re-formulated – by changing the relevant domains simply – when working on functions ψ_+ instead of functions in $\mathcal{S}(\Im^+)$. Collecting all together, since $\langle \psi_{1+}, \psi_{2+} \rangle := -i\sigma(\overline{\psi_{1+}}, \psi_{2+})$, it turns out that the map $\psi_+ \mapsto A^{(1)}_\gamma \psi_+$ preserves the values of the scalar product in $\mathcal{H}$ provided any function $A^{(1)}_\gamma \psi_+$ is the positive frequency part of some $\psi' \in \mathcal{S}(\Im^+)$ when $\psi \in \mathcal{S}(\Im^+)$. Now, by direct inspection using (18), (19) as well as (14) and (5), and taking the positivity of K_Λ into account, one finds, in facts, that

[1] See also [33, 34] for an earlier discussion on this issue.

$A^{(1)}_\gamma(\psi_+) = \left(A^{(1)}_\gamma(\psi)\right)_+$. The map $L_\gamma : \psi_+ \mapsto A^{(1)}_\gamma \psi_+$ preserve the scalar product and thus it can be extended by $\mathbb{C}$-linearity and continuity to an isometric transformation S_γ from $\mathcal{H} = \overline{\mathcal{S}(\Im^+)^{\mathbb{C}}_+}$ to $\mathcal{H}$. That transformation is unitary it being surjective because $S_{\gamma^{-1}}$ is its inverse. $\gamma \mapsto S_\gamma$ gives rise, in fact, to a unitary representation of G_{BMS} on $\mathcal{H}$. Let us define the unitary representation $G_{BMS} \ni \gamma \mapsto U_\gamma$ on the whole space $\mathfrak{F}_+(\Im^+)$ by assuming $U_\gamma \Upsilon := \Upsilon$ and using the standard tensorialization of S_γ on every subspaces with finite number of particles. To conclude the proof of (a) and (c) it is now sufficient to establish the validity of (28). (Notice that, with the given definition of U, in proving the validity of the identity $A^{(1)}_\gamma(\psi_+) = \left(A^{(1)}_\gamma(\psi)\right)_+$ one proves, in fact, also (30) and (31)). To prove (28) it is sufficient to note that, in general, whenever the unitary map $V : \mathfrak{F}_+(\mathcal{H}) \to \mathfrak{F}_+(\mathcal{H})$ satisfy $V\Upsilon = \Upsilon$ and it is the standard tensorialization of some unitary map $V_1 : \mathcal{H} \to \mathcal{H}$ then, for any $\phi \in \mathcal{H}$, $Va^\dagger(\phi)V^\dagger = a^\dagger(V_1\phi)$ and $Va(\overline{\phi})V^\dagger = a(\overline{V_1\phi})$. Since $\Psi(\psi) = -ia^\dagger(\psi_+) + ia(\overline{\psi_+})$ one has $U_\gamma\Psi(\psi)U^\dagger_\gamma = U_\gamma\Psi(\psi)U^\dagger_\gamma\Psi(A_\gamma\psi)$. Exponentiating this identity (using the fact that the vectors with finite number of particles are analytic vectors for $\Psi(\psi)$ [32]) (28) arises.

(b) By hypotheses $U_\gamma\widehat{W}(\psi)U^\dagger_\gamma = \alpha_\gamma(\widehat{W}(\psi)) = V_\gamma\widehat{W}(\psi)V^\dagger_\gamma$ so that $[V^\dagger_\gamma U_\gamma, \widehat{W}(\psi)] = 0$. On the other hand the representation of Weyl algebra $\widehat{\mathcal{W}}(\Im^+)$ is irreducible as said above and thus, by Schur's lemma, $V^\dagger_\gamma U_\gamma = \alpha(\gamma)I$. Since $(V^\dagger_\gamma U_\gamma)^{-1} = (V^\dagger_\gamma U_\gamma)^\dagger = \overline{\alpha(\gamma)}I$, it must be $|\alpha(\gamma)|^2 = 1$ and so $e^{ig(\gamma)}V_\gamma = U_\gamma$. Finally $e^{ig(\gamma)}V_\gamma = U_\gamma$ and (ii) imply $e^{-ig(\gamma)} = \langle\Upsilon, V_\gamma\Upsilon\rangle$. $\Box$

2.6. *Topology on G_{BMS} in view of the analysis of irreducible unitary representations and strongly continuity.* Up to now we have assumed no topology on G_{BMS}. As the group is infinite dimensional and made of diffeomorphisms, a very natural topology is that induced by a suitable class of seminorms [35] yielding the so-called *nuclear topology* (see below), though other choices have been made in the literature. We spend some words on this interesting issue. Since its original definition in [23, 36], the BMS group has been recognized as a semidirect product of two groups $G_{BMS} = H \ltimes N$ as it can be directly inferred from (7). The group H stands for the proper orthocronous Lorentz group, whereas the abelian group, the space of supertranslations N, is a suitable set of sufficiently regular real functions on the two sphere equipped with the abelian group structure induced by pointwise sum of functions. Up to now we have chosen $N = C^\infty(\mathbb{S}^2)$, but there are other possibilities connected with the question about the topology to associate to N in order to have the most physically sensible characterization for the Bondi-Metzner-Sachs group. In Penrose construction [19], where the BMS group arises as the group of exact conformal motions (preserving null angles) of the boundaries $\Im^\pm$ of conformally compactified asymptotically simple spacetimes, a specific degree of smoothness on the elements of N was never imposed. Nonetheless, historically the first stringent request has been proposed by Sachs in [23], i.e. each $\alpha \in N$ must be at least twice differentiable. This choice has been abandoned by McCarthy in his study of the BMS theory of representations [24], where he widened the possible supertranslations to the set of real-valued square-integrable functions $N = L^2(\mathbb{S}^2; \epsilon_{\mathbb{S}^2})_\mathbb{R}$ equipped with *Hilbert topology*. The underlying reasons for this proposal are two, the former concerning the great simplification of the treatment of induced representations in this framework[2], the latter related to the conjecture that square integrable supertranslations are more suited to describe *bounded gravitational systems* [26]. It is imperative to notice that, though such assertions may seem at a first glance reasonable (barring a problem with the interpretation of the elements of the group in terms of diffeomorphisms), they have never been really justified besides purely heuristic arguments. As a matter of fact, a natural choice for N and a corresponding topology is, accordingly to the discussion in section 2.2, $N = C^\infty(\mathbb{S}^2)$ equipped with the *nuclear topology*, first proposed in [27]. We recall the reader that the nuclear topology on $C^\infty(\mathbb{S}^2)$ is the unique topology such that $C^\infty(\mathbb{S}^2) \supset \{f_n\}_{n\in\mathbb{N}}$ turns out to converge to $f \in C^\infty(\mathbb{S}^2)$ iff, for every local chart on $\mathbb{S}^2$, $\phi : U \ni p \mapsto (x(p), y(p))$ and in any compact $K \subset U$:

$$\sup_K \left| \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} f_n \circ \phi^{-1} \right| \to \sup_K \left| \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} f \circ \phi^{-1} \right| , \quad \text{as } n \to +\infty,$$

for every choice of $\alpha, \beta = 0, 1, 2 \cdots$. As is well-known, this topology can be induced by a suitable class of seminorms. Although it has been pointed out that this choice for N and its topology should describe more accurately unbounded gravitating sources [26], we will nonetheless find this framework more natural than the Hilbert topology and thus we adopt the nuclear topology on $N = C^\infty(\mathbb{S}^2)$ and equip G_{BMS} with the consequent topology product. In particular we shall show in proposition 3.2 that, with our choice, it is possible to identify a field on $\Im^+$, which transforms with respect to G_{BMS} as said in (31), with an intrinsic BMS field as introduced in the next section. After that proposition we shall remark that the result cannot be achieved using Hilbert topology.
To conclude this section we state a theorem about strongly continuity of the representation of G_{BMS}, $U : G_{BMS} \ni g \mapsto U_g$, defined in theorem 2.3 on $\mathfrak{F}_+(\mathcal{H})$. The relevance of strongly continuity for a unitary representation, is that, through Stone's theorem, it implies the existence of self-adjoint generators of the representation it-self. The proof of the theorem is in the Appendix.

Theorem 2.4. *Consider the unitary representation of the topological group G_{BMS} defined in theorem 2.3, $U : G_{BMS} \ni g \mapsto U_g$, on $\mathfrak{F}_+(\mathcal{H})$ is strongly continuous.*

Remark 2.5. From the proof of the theorem it results that the topology on the Abelian factor of G_{BMS}, N, plays no relevant role in the proof of strong continuity. It can be replaced by whichever topology, taking the product topology on the whole group.

3 BMS theory of representations in nuclear topology.

3.1. *General goals of the section.* In the previous discussions and in particular in section 2.2, we have developed a scalar QFT on $\Im^+$ whose kinematical data are fields ψ which are suitable

[2] Originally it was also thought that, at a level of representation theory, the results were not affected by the choice of the topology of N though this claim was successively falsified.

smooth extensions/restrictions to $\Im^+$ of fields ϕ living in (M,g). Nonetheless a second candidate way to construct a consistent QFT at null infinity consists of considering as kinematical data, the set of wave functions invariant under a unitary irreducible representation of the G_{BMS} group [7]. The support of such functions is not *a priori* the underlying spacetime - $\Im^+$ in our scenario - but it is a suitable manifold modelled on a subgroup of G_{BMS}. For this reason we shall also refer to such fields as **intrinsic $\mathbf{G_{BMS}}$ fields**.
The rationale underlying this section is to demonstrate that, at least for scalar fields, both approaches are fully equivalent. In particular we shall establish that (31) is the transformation proper of an intrinsic scalar G_{BMS} field.

3.2. *The group $\widetilde{G_{BMS}}$ and some associated spaces.* To achieve our task, in the forthcoming discussion on representations of BMS group we shall study the unitary representations of the topological group $\widetilde{G_{BMS}} = SL(2,\mathbb{C}) \ltimes C^\infty(\mathbb{S}^2)$ where the product of the group is given by suitable re-interpretations of (6) and (7) and the topology is the product of the usual topology on $SL(2,\mathbb{C})$ and that nuclear on $C^\infty(\mathbb{S}^2)$ introduced in section 2.6. In a fixed Bondi frame, the composition of two elements $g = (A,\alpha), g' = (A',\alpha') \in \widetilde{G_{BMS}}$ is defined by

$$(A',\alpha') \odot (A,\alpha) \quad = \quad \big(A'A,\ \alpha + (K_{A^{-1}} \circ A)\cdot(\alpha' \circ A)\big)\ , \tag{32}$$

$$A(\zeta,\overline{\zeta}) := \left(\frac{a\zeta+b}{c\zeta+d}, \frac{\overline{a}\overline{\zeta}+\overline{b}}{\overline{c}\overline{\zeta}+\overline{d}}\right)\ , \tag{33}$$

$$K_A(\zeta,\overline{\zeta}) := \frac{(1+\zeta\overline{\zeta})}{(a\zeta+b)(\overline{a}\overline{\zeta}+\overline{b})+(c\zeta+d)(\overline{c}\overline{\zeta}+\overline{d})} \quad \text{and} \quad A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}\ . \tag{34}$$

In a sense, noticing that $SL(2,\mathbb{C})$ is the universal covering of $SO(3,1)\uparrow$, $\widetilde{G_{BMS}}$ could be considered as the universal covering of G_{BMS}. A discussion on this point would be necessary if one tries to interpret the term "universal covering" literally since both G_{BMS} and $\widetilde{G_{BMS}}$ are *infinite* dimensional topological groups. However we limit ourselves to say that, according to [24, 33], replacing in the structure of G_{BMS} the orthocronous proper Lorentz group $SO(3,1)\uparrow$[3] with its universal covering $SL(2,\mathbb{C})$, it introduces only further unitary irreducible representations, induced by the $\mathbb{Z}_2$ subgroup of $SL(2,\mathbb{C})$, beyond the unitary irreducible representations of G_{BMS}. These represent nothing but the symptom that $SL(2,\mathbb{C})$ "covers twice" $SO(3,1)\uparrow$ and they will be not considered in this paper: we shall pick out only representations of $\widetilde{G_{BMS}}$ which are as well representations of G_{BMS}.

The next step consists in the following further definition [28, 37]:

Definition 3.1. *If $n \in \mathbb{Z}$ is fixed, we call $D_{(n,n)}$ the space of real functions f of two complex variables ζ_1,ζ_2 and their conjugate ones $\overline{\zeta}_1,\overline{\zeta}_2$ such that:*

[3] The orthocronous proper Lorentz group is called homogeneous Lorentz group in [24, 33].

- *f is of class C^∞ in its arguments except at most the origin $(0,0,0,0)$;*
- *for any $\sigma \in \mathbb{C}$, $f(\sigma\zeta_1, \bar\sigma\overline{\zeta}_1, \sigma\zeta_2, \bar\sigma\overline{\zeta}_2) = \sigma^{(n-1)}\bar\sigma^{(n-1)} f(\zeta_1, \overline{\zeta}_1, \zeta_2, \overline{\zeta}_2)$ for all $\zeta_1, \zeta_2, \overline{\zeta}_1, \overline{\zeta}_2$.*

Moreover $D_{(n,n)}$ is assumed to be endowed with the topology of uniform convergence on all compact sets not containing the origin for the functions and all their derivatives separately.

The relevance of the definition above arises from the following proposition which, first of all, allows one to identify $C^\infty(\mathbb{S}^2)$ with the space $D_{(2,2)}$ and the subsequent space D_2 introduced below. These spaces will be used later. The relevance of the second statement will be clarified shortly after proposition 3.2. The action of $\Lambda \in SL(2,\mathbb{C})$ on an element α of $C^\infty(\mathbb{S}^2)$, considered in the equation (36) below, is nothing but that arising by the product in $\widetilde{G_{BMS}}$:

$$(I,\alpha) \mapsto g(I,\alpha)g^{-1} =: (I, \Lambda\alpha), \quad g = (\Lambda, \alpha') \in \widetilde{G_{BMS}}\,, \tag{35}$$

I being the unit element of $SL(2,\mathbb{C})$. The dependence on α' is immaterial as the notation suggests.

Proposition 3.1. *There is a one-to-one map $\mathcal{T} : C^\infty(\mathbb{S}^2) \ni \alpha \mapsto f \in D_{(2,2)}$. In this way, the action of $\Lambda \in SL(2,\mathbb{C})$ on an element α of $C^\infty(\mathbb{S}^2)$*

$$(\Lambda\alpha)(\zeta,\overline{\zeta}) = K_\Lambda(\Lambda^{-1}(\zeta,\overline{\zeta}))\alpha(\Lambda^{-1}(\zeta,\overline{\zeta})) \tag{36}$$

is equivalent to the action (defined in [37]) of the same Λ on f

$$f \circ \Lambda^{-1} := f(a\zeta_1 + c\zeta_2, a\overline{\zeta}_1 + c\overline{\zeta}_2, b\zeta_1 + d\zeta_2, b\overline{\zeta}_1 + d\overline{\zeta}_2), \quad \forall \Lambda = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \in SL(2,\mathbb{C})\,. \tag{37}$$

Finally $\mathcal{T}$ is a homeomorphism so that the topology of $D_{(2,2)}$ coincides with that on $C^\infty(\mathbb{S}^2)$.

The proof of this result may be found in the appendix of [39] though we review some of the details which will be important in the forthcoming discussion. The sketch of the argument is the following: the homogeneity condition for the functions $f \in D_{(n,n)}$ allows us to associate to each of such f a pair of C^∞ functions $\xi, \hat\xi$ such that

$$f(\zeta_1, \overline{\zeta}_1, \zeta_2, \overline{\zeta}_2) = |\zeta_1|^{2(n-1)} f\left(\frac{\zeta_2}{\zeta_1}, \frac{\overline{\zeta}_2}{\overline{\zeta}_1}\right) = |\zeta_1|^{2(n-1)}\xi(\zeta,\overline{\zeta}),$$
$$f(\zeta_1, \overline{\zeta}_1, \zeta_2, \overline{\zeta}_2) = |\zeta_2|^{2(n-1)} f\left(\frac{\zeta_1}{\zeta_2}, \frac{\overline{\zeta}_1}{\overline{\zeta}_2}\right) = |\zeta_2|^{2(n-1)}\hat\xi(\zeta,\overline{\zeta}),$$

where $\zeta = \frac{\zeta_1}{\zeta_2}$ and

$$\hat\xi(\zeta,\overline{\zeta}) = |\zeta|^{2(n-1)}\xi(\zeta^{-1}, \overline{\zeta}^{-1}) \tag{38}$$

whenever $(\zeta_1, \zeta_2) \neq (0,0)$. If we call D_n the set of the functions $\hat{\xi}$, the above discussion can be recast as the existence of a bijection between $D_{(n,n)}$ and D_n which thus inherits the same topology as $D_{(n,n)}$ (or *viceversa*). Furthermore (37) becomes, with obvious notation,

$$(\xi \circ \Lambda^{-1})(\zeta, \overline{\zeta}) = |a + c\zeta|^{2(n-1)} \xi \left(\frac{d + b\zeta}{a + c\zeta}, \frac{\bar{d} + \bar{b}\overline{\zeta}}{\bar{a} + \bar{c}\overline{\zeta}} \right), \quad a + c\zeta \neq 0,$$
$$(\hat{\xi} \circ \Lambda^{-1})(\zeta, \overline{\zeta}) = |d + b\zeta|^{2(n-1)} \xi \left(\frac{a + c\zeta}{d + b\zeta}, \frac{\bar{a} + \bar{c}\overline{\zeta}}{\bar{d} + \bar{b}\overline{\zeta}} \right), \quad d + b\zeta \neq 0.$$

If we specialize to $n = 2$, it is now possible to show (see [28, 39]) that the above equations correspond to the canonical realization of the $\widetilde{G_{BMS}}$ group as $SL(2,\mathbb{C}) \ltimes C^\infty(\mathbb{S}^2)$ if we associate the supertranslation $\alpha \in C^\infty(\mathbb{S}^2)$ with $\hat{\xi}$ as:

$$\hat{\xi}(\zeta, \overline{\zeta}) = (1 + |\zeta|^2)\alpha(\zeta, \overline{\zeta}). \tag{39}$$

Within this framework and for every $\Lambda \in SL(2,\mathbb{C})$ and $\alpha \in C^\infty(\mathbb{S}^2)$ (36) turns out to be equivalent to (37) as one can check by direct inspection.

Remark 3.1. Identifying the topological vector space of supertranslations $C^\infty(\mathbb{S}^2)$ with D_2 and equivalently with $D_{(2,2)}$, the $\widetilde{G_{BMS}}$ group turns out to be locally homeomorphic to a nuclear space[4] and thus it is a **nuclear Lie group** as defined by Gelfand and Vilenkin in [38]. In other words, there exists a neighborhood of the unit element of $\widetilde{G_{BMS}}$ which is homeomorphic to a neighborhood of zero in a (separable Hilbert) nuclear space.

If N is the real topological vector space of supertranslation $C^\infty(\mathbb{S}^2)$, N^* indicates its topological dual vector space, whose elements are called **(real) distributions** on N.

Remark 3.2. Since N can be topologically identified as $D_{(2,2)}$, N^* is fully equivalent to the set of continuous linear functionals $D_{(-2,-2)}$ which is obtained setting $n = -2$ in definition 3.1 with the prescription that all the equations should be interpreted in a distributional sense [28, 37]. Consequently each $\phi \in D_{(-2,-2)}$ is a real distribution in two complex variables bijectively determined by a pair $\phi, \hat{\phi} \in D_{-2}$ of real distributions such that $\hat{\phi} = |z|^{-6}\phi$, as in (39). The counterpart of (39) for N^* is the following: to each functional $\phi \in D_{(-2,-2)}$ corresponds the distribution $\beta \in N^*$

$$\beta = (1 + |\zeta|^2)^3 \phi. \tag{40}$$

Furthermore, if $L^2(\mathbb{S}^2, \epsilon_{\mathbb{S}^2})$ is the Hilbert completion of N with respect to the scalar product associated with $\epsilon_{\mathbb{S}^2}$, $N \subset L^2(\mathbb{S}^2, \epsilon_{\mathbb{S}^2}) \subset N^*$ is a *rigged Hilbert space*.

[4] We recall the reader that, given a separable Hilbert space $\mathcal{H}$, $\mathcal{E} \subset \mathcal{H}$ is called a nuclear space if it is the projective limit of a decreasing sequence of Hilbert spaces $\mathcal{H}_k$ such that the canonical imbedding of $\mathcal{H}_k$ in $\mathcal{H}_{k'}$ $(k > k')$ is an Hilbert-Schmidt operator.

3.3. *Main ingredients to study unitary representations of $\widetilde{G_{BMS}}$.* The starting point to study unitary representations of BMS group consists in the detailed analysis of McCarthy [24, 25, 28]. The theory of unitary and irreducible representations for $\widetilde{G_{BMS}}$ with *nuclear topology* has been developed in [28] by means either of Mackey theory of induced representation [41, 42] applied to an infinite dimensional semidirect product [40] either of Gelfand-Vilenkin work on nuclear groups [38, 37]. In the following we briefly discuss some key points. Here we introduce the main mathematical tools in order to construct the intrinsic wave functions. We refer to [7] for a detailed analysis in the Hilbert topology scenario.

Definition 3.2. *If A is an Abelian topological group, a* **character** *(of A) is a continuous group homomorphism $\chi : A \to U(1)$, the latter being equipped with the natural topology induced by $\mathbb{C}$. The set of characters A' is an abelian group called the* **dual character group** *if equipped with the group product*

$$(\chi_1\chi_2)(\alpha) := \chi_1(\alpha)\chi_2(\alpha). \quad \textit{for all } \alpha \in A\,.$$

A central tool concerns an explicit representation of the characters in terms of distributions [28]. The proof of the following relevant proposition is in the appendix.

Proposition 3.2. *Viewing $N := C^\infty(\mathbb{S}^2)$ as an additive continuous group, for every $\chi \in N'$ there is a distribution $\beta \in N^*$ such that*

$$\chi(\alpha) = \exp[i\,(\alpha,\beta)]\,, \quad \textit{for every } \alpha \in N$$

where (α, β) has to be interpreted as the evaluation of the β-distribution on the test function α.

Remark 3.3. With characters one can decompose any unitary representation of $N = C^\infty(\mathbb{S}^2)$. Indeed, a positive finitely normalizable measure μ_{N^*} on N^* exists, which is quasi invariant under group translations (i.e. for any measurable $X \subset N^*$, $\mu_{N^*}(X) = 0$ iff $\mu_{N^*}(N + X) = 0$), and a family of Hilbert spaces $\{\mathcal{H}_\beta\}_{\beta\in N^*}$ such that, for any unitary representation of N, $U : \mathcal{H} \to \mathcal{H}$, $\mathcal{H}$ being any Hilbert space, the following direct-integral decomposition holds (*c.f.* chapter I and chapter IV – theorem 5 and subsequent discussion – in [38]):

$$\mathcal{H} = \int_{N^*}^{\oplus} \mathcal{H}_\beta \, d\mu_{N^*}(\beta).$$

Moreover the spaces $\mathcal{H}_\beta$ are invariant under U and, for every $\alpha \in N$ and $\psi_\beta \in \mathcal{H}_\beta$, one has $U \restriction_{\mathcal{H}_\beta} \psi_\beta = e^{i(\alpha,\beta)}\psi_\beta$. Here (α, β) denotes action of the distribution β on the test function α.

For any $\Lambda \in SL(2,\mathbb{C})$ a natural action $\chi \mapsto \Lambda\chi$ on N' induced by (36) is [24, 28]:

$$(\Lambda\chi)(\alpha) = \chi(\Lambda^{-1}\alpha) \tag{41}$$

whereas an action $\beta \mapsto \Lambda\beta$ on N^* is intrinsically defined from the identity

$$(\Lambda\beta, \alpha) = (\beta, \Lambda^{-1}\alpha). \tag{42}$$

If we associate to the distribution β the pair $(\phi, \hat{\phi})$ as discussed in remark 3.2, the latter $SL(2,\mathbb{C})$ action translates as, if $\Lambda = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \in SL(2,\mathbb{C})$,

$$(\Lambda\phi)(\zeta, \overline{\zeta}) = |a + c\zeta|^{-6} \phi \left(\frac{b + d\zeta}{a + c\zeta}, \frac{\bar{d} + \bar{b}\zeta}{\bar{a} + \bar{c}\overline{\zeta}} \right), \quad \text{with } a + cz \neq 0 \tag{43}$$

$$\left(\Lambda\hat{\phi}\right)(\zeta, \overline{\zeta}) = |d + b\zeta|^{-6} \hat{\phi} \left(\frac{c + a\zeta}{d + b\zeta}, \frac{\bar{a} + \bar{c}\overline{\zeta}}{\bar{d} + \bar{b}\overline{\zeta}} \right), \quad \text{with } d + bz \neq 0\,. \tag{44}$$

Definition 3.3. *Consider a group $G = B \ltimes A$ where A is a topological abelian group and B is any group and a suitable group operation is defined in order to make G the semi direct product of B and A.*
For any $\chi \in A'$, the **orbit** *of χ (with respect to G) is the subset of A'*

$$G\chi := \{g\chi \mid g \in G\}\,, \tag{45}$$

the **isotropy group** *of χ (with respect to G) is the subgroup of G*

$$H_\chi := \{g \in G \mid g\chi = \chi\}\,, \tag{46}$$

and the **little group** *of χ (with respect to G) is the subgroup of H_χ*

$$L_\chi := \{g = (L, 0) \in G \mid g\chi = \chi\}\,. \tag{47}$$

Referring to $\widetilde{G_{BMS}} = SL(2,\mathbb{C}) \ltimes C^\infty(\mathbb{S}^2)$, to (41) and to (42), L_χ can equivalently be seen as the subgroup of $SL(2,\mathbb{C})$ whose elements L satisfy

$$L\bar{\beta} = \bar{\beta}, \tag{48}$$

$\bar{\beta} \in N^*$ being associated to χ according to proposition 3.2.

Remark 3.4. A direct inspection of (41) shows also that the $\widetilde{G_{BMS}}$ action on a character is completely independent form supertranslations. Thus the most general isotropy group has the form

$$H_\chi = L_\chi \ltimes C^\infty(\mathbb{S}^2)\,.$$

We now discuss a last key remark concerning the *mass* of a BMS field. First of all, define a base of *real spherical harmonics* $\{S_{lk}\}_{l=0,1,\cdots,k=1,2,\cdots,2l+1}$, in the real vector space $C^\infty(\mathbb{S}^2)$ as follows:

$$S_{l\,k} := Y_{l0} \qquad \text{if } k = 2l + 1, \tag{49}$$

$$S_{l\,k} := \frac{Y_{l\,-k} - Y_{l\,k}}{\sqrt{2}} \qquad \text{if } 1 < k \leq l, \tag{50}$$

$$S_{l\,k} := i\frac{Y_{l\,-k} + Y_{l\,k}}{\sqrt{2}} \qquad \text{if } l < k \leq 2l, \tag{51}$$

where Y_{lm} are the usual (complex) spherical harmonics with $m \in \mathbb{Z}$ such that $-l \leq m \leq l$. Now, let us consider a generic supertranslation $\alpha \in C^\infty(\mathbb{S}^2)$ and let us decompose (in the sense of $L^2(\mathbb{S}^2, \epsilon^2_{\mathbb{S}})$) it in real spherical harmonics

$$\alpha(\zeta, \bar{\zeta}) = \sum_{l=0}^{1} \sum_{k=1}^{2l+1} a_{lm} S_{lm}(\zeta, \bar{\zeta}) + \sum_{l=2}^{\infty} \sum_{k=1}^{2l+1} a_{lk} S_{lk}(\zeta, \bar{\zeta}), \quad \bar{\alpha}_{lk} \in \mathbb{R} \,. \tag{52}$$

The former double sum defines the *translational component* of α and the latter the *pure supertranslational component* of α. This relation allows one to split $C^\infty(\mathbb{S}^2)$ into an orthogonal direct sum $T^4 \oplus \Sigma$ where T^4 is a four-dimensional real space invariant under $SL(2,\mathbb{C})$ viewed as the subgroup of $\widetilde{G_{BMS}}$ made of elements $(A, 0)$. More precisely (see also proposition 4.2 below):

Proposition 3.3. *The subset $SL(2,\mathbb{C}) \ltimes T^4 \subset \widetilde{G_{BMS}}$ made of the elements (Λ, α) with $\alpha \in T^4$ is a subgroup of $\widetilde{G_{BMS}}$ itself which is invariant under $SL(2,\mathbb{C})$, i.e., if $g \in SL(2,\mathbb{C}) \ltimes T^4$,*

$$(A, 0) \odot g \in SL(2,\mathbb{C}) \ltimes T^4 \,, \quad \textit{for all } A \in SL(2,\mathbb{C}).$$

Remark 3.5. Defining the analogous subset $SL(2,\mathbb{C}) \ltimes \Sigma$, one finds that Σ is *not* $SL(2,\mathbb{C})$ invariant. More precisely breaking of invariance happens when $A \notin SU(2)$.

The decomposition (52) explicitly associates to each $\alpha \in C^\infty(\mathbb{S}^2)$ the 4-vector

$$a_\mu \equiv -\frac{1}{2}\sqrt{\frac{3}{\pi}} \left(\frac{a_{01}}{\sqrt{3}}, a_{11}, a_{12}, a_{13} \right) \tag{53}$$

One has the following very useful proposition which can be proved by direct inspection and which will be used in several key points in the following.

Proposition 3.4. *If $\alpha_a \in T^4$, where a_μ is made of the first four components of α_a as in (53), transforming α_a under the action of $A \in SL(2,\mathbb{C})$ as in (36) is equivalent to transforming the 4-vector a^μ under the action of the Lorentz transformation associated with A itself. In other words:*

$$K_A(\zeta, \overline{\zeta})^{-1} \alpha_a \left(A(\zeta, \overline{\zeta}) \right) = \alpha_{\Pi(A)^{-1}a}(\zeta, \overline{\zeta}) \,, \quad \textit{for all } A \in SL(2,\mathbb{C}), \tag{54}$$

$\Pi : SL(2,\mathbb{C}) \to SO(3,1)\uparrow$ being the canonical covering projection.

According to the discussion in [28], (53) can be translated to the dual space N^* where we shall define the annihilator of T^4 as

$$\left(T^4\right)^0 = \left\{ \beta \in N^* \;\mid\; (\alpha, \beta) = 0, \; \forall \alpha \in T^4 \hookrightarrow C^\infty(\mathbb{S}^2) \right\} \,. \tag{55}$$

$\hookrightarrow$ recalls the reader that T^4 above is seen as a subspace of $C^\infty(\mathbb{S}^2)$ and not as the four-dimensional translation group of vectors a^μ acting in Minkowski space.
¿From now on $(T^4)^* \subset N^*$ denotes the subspace generated by the subset of N^*

$$\{S^*_{lk} \mid -l \leq m \leq l\ ,\ l = 0, 1\}\ ,$$

where each S^*_{lk} is completely defined by the requirements $(S_{l'm'}|S^*_{lm}) := \delta_{ll'}\delta_{mm'}$ together with the expansion (52) and taking into account that each map $N \ni \alpha \mapsto a_{lm}$ is a distribution of N^*. It is simply proved that $(T^4)^*$ and $N^*/(T^4)^0$ are canonically isomorphic and the isomorphism (first introduced in [28]) is invariant under $SL(2,\mathbb{C})$ transformation. As a consequence there is a linear projection of N^* onto $(T^4)^*$ (which is, in fact, the usual projection onto the quotient space composed with the cited isomorphism)

$$\pi : N^* \to (T^4)^* \sim \frac{N^*}{(T^4)^0}\ . \tag{56}$$

That projection enjoys the following remarkable properties [28, 37] which gives the first step in order to introduce the notion of mass for BMS representations:

Proposition 3.5. *Let $\beta \in N^*$ and let $\phi \in D_{(-2,-2)}$ and $\hat{\phi} = |\zeta|^{-6}\phi$ be the distributions associated with β as in remark 3.2. The function*

$$\widehat{\pi(\beta)}(\zeta', \overline{\zeta}') = \frac{i}{2(1+|\zeta'|^2)} \int\limits_{|\zeta|<1} [(\zeta - \zeta')(\overline{\zeta} - \overline{\zeta}')\phi(\zeta, \overline{\zeta}) + (1 - \zeta\zeta')(1 - \overline{\zeta\zeta}')\hat{\phi}(\zeta, \overline{\zeta})] d\zeta d\overline{\zeta}, \tag{57}$$

is well defined for $\zeta, \overline{\zeta} \in \mathbb{C}$ and, in fact, it belongs to T^4. Moreover, as the notation suggests, $\widehat{\pi(\beta)}$ depends on $\pi(\beta)$ and not on the whole distribution β. That is $\widehat{\pi(\beta)} = \widehat{\pi(\beta')}$ if $\pi(\beta) = \pi(\beta')$ for whatever $\beta, \beta' \in N^*$.

The following final proposition [24, 25, 28] is, partially, a straightforward consequence of proposition 3.4. It produces the preannounced notion of mass similar to that used in the theory of Poincaré representations.

Proposition 3.6. *The space $(T^4)^*$ is invariant under the $SL(2,\mathbb{C})$-action on N^* and, according to (39), the supertranslation associated to $\widehat{\pi(\beta)}$ may be expanded in spherical harmonics thus extracting as in (53) a 4-vector $\widehat{\pi(\beta)}_\mu$. Moreover if one defines the real bilinear form on $N^* \ni \beta_1, \beta_2$ as*

$$B\,(\beta_1, \beta_2) := \eta^{\mu\nu}\widehat{\pi(\beta_1)}_\mu \widehat{\pi(\beta_2)}_\nu, \tag{58}$$

with $\eta := diag(-1, 1, 1, 1)$ and it turns out that B is $SL(2,\mathbb{C})$-invariant.

$-B(\beta,\beta) = m^2$ is the equation for the squared-*mass* m^2 of an intrinsic BMS field. It is the analog of the invariant mass of a field in Wigner's approach to define Poincaré-invariant particles. Consequently we we shall refer to N^* as the **supermomentum space** and its elements as the **supermomenta**.

3.4. *Construction of unitary irreducible representations of* $\widetilde{G_{BMS}}$. Consider a group $G = B \ltimes A$ where A is a (possibly infinite dimensional) topological abelian group and B a locally compact topological group and a suitable group operation is defined in order to make G the semi direct product of B and A, which is a topological group with respect to the product of topologies. Using the definitions and propositions given above, the procedure to build up unitary irreducible representations of $\widetilde{G_{BMS}}$ goes on as follows, starting from representations of the little groups of characters. The next proposition has a trivial straightforward proof.

Proposition 3.7. *Take a character $\chi \in A'$ and a closed subgroup of B, K. If $K \ni L \mapsto \sigma_L$ is a unitary representation of K acting on a, non necessarily finite-dimensional, target Hilbert space V, an associated unitary representation $K \ltimes A \ni g \mapsto \chi\sigma_g$ of $K \ltimes A$ acting on V is constructed as follows.*

$$\chi\sigma_{(\Lambda,\alpha)}\,\vec{\psi} := \chi(\alpha)\,\sigma_\Lambda(\vec{\psi})\,, \quad \textit{for all } \vec{\psi} \in V. \tag{59}$$

Furthermore let us define the following equivalence relation in $G \times V$ equipped with the product topology:

$$(g, v) \sim_K (g', v') \quad \text{iff there is } g_K \in K \text{ such that} \quad (g', v') = (gg_K^{-1}, \chi\sigma(g_K)v). \tag{60}$$

The quotient space equipped with its natural topology, will be denoted by

$$G \times_K V := \frac{G \times V}{\sim_K}\,.$$

From now on, concerning the equivalence classes associated with the equivalence relation defined above, we use the notation $[g, v]$ instead of the more appropriate but more complicated $[(g, v)]$.

Remark 3.6. A natural projection map exists

$$\tau : G \times_K V \longrightarrow \frac{G}{K}\,,$$

which associates $[g, v] \in G \times_K V$ with gK. Furthermore the inverse image $\tau^{-1}(p)$ with $p = gK \in G/K$ for some $g \in G$, has the form $[g, v]$ where $v \in V$ is uniquely determined by p. Thus it exists a natural bijection from $\pi^{-1}(p)$ into V such that, automatically, the former acquires the structure of a Hilbert space and this structure does not depend upon the choice of $g \in G$

with $p = gK$. As a matter of fact, if $p = gK = g_1K$ with $g \neq g_1$, then the following diagram commutes:

$$\begin{array}{ccc} \tau^{-1}(p) & \xrightarrow{[g,v]\mapsto v} & V \\ \Big\downarrow{\scriptstyle id.} & & \Big\downarrow{\scriptstyle \sigma^{(k)}(g_1^{-1}g)} \\ \tau^{-1}(p) & \xrightarrow{[g_1,v]\mapsto v} & V \end{array}$$

Consequently since the representation $\sigma^{(k)}(g_1^{-1}g)$ as in (60) is unitary, the above statement naturally follows [41].

According to the above remark we can introduce the following definition

Definition 3.4. *A triple (X, τ, Y) is called* **Hilbert bundle** *if X and Y are topological spaces, π is a continuous surjection of X on Y and $\tau^{-1}(p)$ (the fiber) has an Hilbert space structure for each $p \in Y$ (see chapter 7 in [43]).*

In the following a Hilbert bundle (X, τ, Y) will be also denoted

$$\tau : X \to Y \ .$$

Definition 3.5. *Let (X_1, τ_1, Y_1) and (X_2, τ_2, Y_2) be two Hilbert bundles. A* **Hilbert-bundle isomorphism** *is a pair of homeomorphisms $\lambda_1 : X_1 \to X_2$, $\lambda_2 : Y_1 \to Y_2$, such that*

- $\tau_2\lambda_1 = \lambda_2\tau_1$,
- *λ_1 isometrically maps the fiber $\tau_1^{-1}(p)$ into $\tau_2^{-1}(\lambda_2 p)$ for each $p \in Y_1$.*

Definition 3.6. *Let G be a topological group and (X, τ, Y) an Hilbert bundle. Then (X, τ, Y) is called a* **G-Hilbert bundle** *if there are two continuous actions of G onto X, Y such that the pair $\lambda_{1,g} : X \to X$, with $x \mapsto gx$ and $\lambda_{2,g} : Y \to Y$, with $y \mapsto gy$, is an Hilbert bundle automorphism for each $g \in G$. Accordingly an isomorphism between two different G-Hilbert bundles is an isomorphism between the two Hilbert bundles which commutes with the G-action (see chapter 9 in [43]).*

Proposition 3.8. *According to definitions 3.4 and 3.6, take a representation (59) $\chi\sigma$ associated with a character $\chi \in N'$ and a representation σ of L_χ on the finite dimensional Hilbert space $\mathcal{H}$.*
A $\widetilde{G_{BMS}}$-Hilbert bundle can be built up as follows,

$$\tau^\sigma_\chi : \widetilde{G_{BMS}} \times_{H_\chi} \mathcal{H} \longrightarrow \frac{\widetilde{G_{BMS}}}{H_\chi}, \tag{61}$$

where:
(a) $\widetilde{G_{BMS}} \times_{H_\chi} \mathcal{H}$ *consists of the equivalence classes* $[g, \vec{\psi}]$ *associated with the equivalence relation* $\sim_{H_\chi}$ *in* $\left(\widetilde{G_{BMS}} \times \mathcal{H}\right) \times \left(\widetilde{G_{BMS}} \times \mathcal{H}\right)$

$$(g', \vec{\psi}') \sim_{H_\chi} (g, \vec{\psi}) \,, \quad \text{if and only if } (g', \vec{\psi}') = (gk^{-1}, \chi\sigma(k)\vec{\psi}) \text{ for some } k \in H_\chi$$

(b) *the group actions, respectively on* $\widetilde{G_{BMS}} \times_{H_\chi} \mathcal{H}$ *and* $\frac{\widetilde{G_{BMS}}}{H_\chi}$*, are defined as*

$$g'[g, \vec{\psi}] = [g' \odot g, \vec{\psi}], \quad g'(g \odot H_\chi) = (g \odot g') \odot H_\chi.$$

Eventually if considering two $\widetilde{G_{BMS}}$ *representations* $\chi\sigma$ *on the finite dimensional Hilbert space* $\mathcal{H}$ *and* $\chi\tau$ *on the finite dimensional Hilbert space* $\mathcal{H}'$*, which are unitary equivalent by* $U : \mathcal{H} \to \mathcal{H}'$*, then the Hilbert bundles* $\tau^\sigma_\chi : \widetilde{G_{BMS}} \times_{H_\chi} \mathcal{H} \to \frac{\widetilde{G_{BMS}}}{H_\chi}$ *and* $\tau^\eta_\chi : \widetilde{G_{BMS}} \times_{H_\chi} \mathcal{H}' \to \frac{\widetilde{G_{BMS}}}{H_\chi}$ *are* $\widetilde{G_{BMS}}$*-isomorphic under the map* $[g, \vec{\psi}] \mapsto [g, U\vec{\psi}]$*.*

In order to fully control the theory of $\widetilde{G_{BMS}}$ unitary representations, we also need some measure theoretical notions which will allow us to impose integrability conditions on the set of $\widetilde{G_{BMS}}$ wave functions.
Consider a generic topological space X. Two Borel measures μ, ν on X are said to be lying in the same **measure class** if they assume the value zero for the same Borel sets in X so that μ admits Radon-Nikodym derivative with respect to ν and *viceversa*.

In particular, when we deal with locally compact groups such as $SL(2,\mathbb{C})$, the following theorem holds (see [44] for the demonstration and also section 4 in [43]) and it is of a great importance for our later applications.

Theorem 3.1. *For any closed subgroup* K *of a locally compact group* G*, there is a unique non vanishing measure class* M *on* $\frac{G}{K}$ *such that if* $\mu \in M$*,* $\mu_g \in M$ *for every* $g \in G$*, where* $\mu_g(E) = \mu(g^{-1}E)$ *for every Borel set* $E \subset \frac{G}{K}$*.* M *is called* **invariant measure class** *of* $\frac{G}{K}$*.*

Furthermore, according to [41], consider the Borel-measurable sections of a $\widetilde{G_{BMS}}$-Hilbert bundle (61), i.e. Borel measurable functions $\vec{\psi} : \frac{\widetilde{G_{BMS}}}{H_\chi} \to \widetilde{G_{BMS}} \times_{H_\chi} \mathcal{H}$ such that $\tau^\sigma_\chi \circ \vec{\psi} = id_{\frac{\widetilde{G_{BMS}}}{H_\chi}}$. Since the orbit $\mathcal{O}_\chi = \frac{SL(2,\mathbb{C}) \ltimes C^\infty(\mathbb{S}^2)}{L_\chi \ltimes C^\infty(\mathbb{S}^2)}$ is isomorphic to $\frac{SL(2,\mathbb{C})}{L_\chi}$, we can exploit theorem 3.1 introducing for any orbit $\mathcal{O}_\chi$ and for a $\mu \in M$ the following Hilbert space:

$$\mathcal{H}_\mu = \left\{ \vec{\psi} : \mathcal{O}_\chi \to \mathcal{H} \;\middle|\; \int\limits_{\mathcal{O}_\chi} d\mu(p) \langle \vec{\psi}(p), \vec{\psi}(p) \rangle < \infty \right\} . \tag{62}$$

Above, $\langle \, , \, \rangle$ refers to the $\widetilde{G_{BMS}}$-invariant Hermitean inner product of the fiber $\left(\tau^\sigma_\chi\right)^{-1}(p)$ where p is an element on the orbit $\mathcal{O}_\chi$.

Each element[5] $\vec{\psi}$ in $\mathcal{H}_\mu$, usually called an **"induced wave function"**[6] (or BMS intrinsic free field), inherits a natural $\widetilde{G_{BMS}}$ action as:

$$(g\vec{\psi})(p) = \sqrt{\frac{d\mu(gp)}{d\mu(p)}} g(\vec{\psi}(g^{-1}(p))), \quad \forall g \in \widetilde{G_{BMS}} \tag{63}$$

where $\sqrt{\frac{d\mu(gp)}{d\mu(p)}}$ is the Radon-Nikodym derivative. It is worth stressing that, by construction, the scalar product in $\mathcal{H}_\mu$ is invariant under the above action of $\widetilde{G_{BMS}}$.

Let us fix a little group H_χ and consider the set of all possible $\widetilde{G_{BMS}}$-Hilbert bundles (61) $\zeta^\sigma = (\frac{\widetilde{G_{BMS}}}{H_\chi}, \tau^\sigma_\chi, \widetilde{G_{BMS}} \times_{H_\chi} \mathcal{H})$. We are entitled to directly apply Mackey's theorem (see chapter 16 of [42] and [40, 46]) which grants us that:

Proposition 3.9. *(63) is a unitary strongly continuous $\widetilde{G_{BMS}}$ representation $T_\mu(\zeta^\sigma)$* induced *from σ.*

Remark 3.7. For a fixed little group H_χ, if we consider two invariant measures $\mu, \nu \in M$, then the map which associates to each $\vec{\psi} \in \mathcal{H}_\mu$ the element $\sqrt{\frac{d\mu}{d\nu}}\vec{\psi} \in \mathcal{H}_\nu$ defines an isometry between $\mathcal{H}_\mu$ and $\mathcal{H}_\nu$ and, at the same time, an equivalence between $T_\mu(\zeta^\sigma)$ and $T_\nu(\zeta^\sigma)$. Since, according to theorem 3.1, we have chosen the unique invariant measure class μ of the base space $\frac{\widetilde{G_{BMS}}}{H_\chi}$ on each $\widetilde{G_{BMS}}$-Hilbert bundle, we are entitled to drop the μ-dependence in the induced representation $T_\mu(\zeta^\sigma) \equiv T(\zeta^\sigma)$.

Apparently the last discussion grants us that $T(\zeta^\sigma)$ depends only upon a selected representation of the little group H_χ, but it is rather intuitive that the existence of Hilbert bundle isomorphisms could imply that, a priori different representations of H_χ on different $\widetilde{G_{BMS}}$-Hilbert bundles, could actually induce equivalent full $\widetilde{G_{BMS}}$ representations. In detail, the last assertion can be justified if we notice that (61) depends only on the orbit $\widetilde{G_{BMS}}\chi$ and not on the specific choice of χ. Let us thus choose two different bundles, namely

$$\tau^\sigma_\chi : \widetilde{G_{BMS}} \times_{H_\chi} \mathcal{H} \longrightarrow \frac{\widetilde{G_{BMS}}}{H_\chi}, \quad \tau^{\sigma_1}_{\chi_1} : \widetilde{G_{BMS}} \times_{H_{\chi_1}} \mathcal{H} \longrightarrow \frac{\widetilde{G_{BMS}}}{H_{\chi_1}}$$

such that $\widetilde{G_{BMS}}\chi = \widetilde{G_{BMS}}\chi_1$ for $\chi_1 \neq \chi$. As a consequence, an element $g_1 \in \widetilde{G_{BMS}}$ exists such that $\chi_1 = g_1\chi$ and, according to definition 3.3, $L_{\chi_1} = g_1 L_\chi g_1^{-1}$. This identity translates

[5] We adopt the symbol ψ either for the intrinsic $\widetilde{G_{BMS}}$ field either for the bulk field suitably restricted on $\Im^+$ since they will ultimately be the same object, at least for a scalar $\widetilde{G_{BMS}}$ representation.

[6] For an interested reader, we underline that we adopt the most common name for the wave functions constructed from induced representations. Nonetheless, in the literature, it exists a zoology of different names the most notables being *canonical wave function* (as in [45]) or *Mackey wave function* (as in [42]).

at a level of representation as $\sigma_1(h) = \sigma(g_1^{-1} h g_1)$ for each $h \in H_\chi$. Furthermore, according to definition 3.6, there is an isomorphism

$$(\lambda_1, \lambda_2) : \left(\widetilde{G_{BMS}} \times_{H_\chi} \mathcal{H}, \tau_\chi^\sigma, \frac{\widetilde{G_{BMS}}}{H_\chi} \right) \longrightarrow \left(\widetilde{G_{BMS}} \times_{H_{\chi_1}} \mathcal{H}, \tau_{\chi_1}^\sigma, \frac{\widetilde{G_{BMS}}}{H_{\chi_1}} \right),$$

induced by the maps

$$\lambda_1 : [g, \vec{\psi}] \mapsto [g_1 \odot g \odot g_1^{-1}, \vec{\psi}] \text{ and } \lambda_2 : \widetilde{G_{BMS}}\chi \ni p \mapsto g_1 p \in \widetilde{G_{BMS}}\chi_1 ,$$

where p stands for a generic point on the orbit. Thus, the irreducible representations, induced either from σ i.e. $T(\zeta^\sigma)$ either from σ_1 i.e. $T(\zeta^{\sigma_1})$, are $\widetilde{G_{BMS}}$-equivalent by construction and, consequently, they will be considered as the same. A summary of this discussion lies in the following remark:

Remark 3.8. The σ-dependence of $T(\zeta^\sigma)$ is determined up to $\widetilde{G_{BMS}}$-equivalence.

Remark 3.9. According to the previous discussion and, in particular, according to remarks 3.7 and 3.8, a generic $\widetilde{G_{BMS}}$ (unitary) representation depends only upon the choice of the character χ and of the unitary representation σ of H_χ. Consequently it will be indicated as $T(\zeta_\chi^\sigma)$ making explicit the dependence on χ.

The explicit action of a generic $T(\zeta_\chi^\sigma)$ should be defined on the induced wave function as in (63). However it is more convenient to recast (63) as[7]:

$$\vec{\psi}(gh) = T(h^{-1})\vec{\psi}(g), \ \forall g \in SL(2,\mathbb{C}), \ h \in L_\chi \tag{64}$$

where we write $T(h^{-1})$ instead of $[T(\zeta_\chi^\sigma)](h^{-1})\vec{\psi}(g)$ to stress, that for a fixed $\widetilde{G_{BMS}}$-Hilbert bundle and for a fixed representation σ of L_χ, the dependence of the induced representation T on such data is superfluous. The $\widetilde{G_{BMS}}$ action explicitly reads, for $(\Lambda, \alpha) \in \widetilde{G_{BMS}}$,

$$(\Lambda\vec{\psi})(g) = \vec{\psi}(\Lambda^{-1} g), \tag{65}$$

$$(\alpha\vec{\psi})(g) = \chi\left(g^{-1}\alpha\right)\vec{\psi}(g), \tag{66}$$

which is a unitary representation induced from $T(\zeta_\chi^\sigma)$ as in (64) and thus, according to Mackey theorem, it is also irreducible. From an operative point of view, an equivalent definition of an induced wave function can be constructed dropping the condition (65). In this scenario we introduce the set of μ square-integrable maps of $\mathcal{H}_\mu$ (see theorem 3.1)

$$\vec{\psi} : \mathcal{O}_\chi = \frac{SL(2,\mathbb{C})}{L_\chi} \to \mathcal{H} \,. \tag{67}$$

[7]In the literature such as [24, 25], the argument (ζ_χ^σ) is considered a priori fixed and thus it is not even introduced.

However the absence of (65) requires the introduction of an additional datum, namely an almost everywhere continuous section ω of the bundle $\tau : SL(2,\mathbb{C}) \to \mathcal{O}_\chi$ which satisfies $\omega(p)\chi = p, \ \forall p \in \mathcal{O}_\chi$. Thus we can define, as an **induced wave function**, a map (67) which transforms under $(\Lambda, \alpha) \in \widetilde{G_{BMS}}$ as

$$(\Lambda\vec{\psi}_\omega)(p) = \sqrt{\frac{d\mu(\Lambda p)}{d\mu(p)}}[T(\zeta^\sigma_\chi)]\left(\omega(p)^{-1}\Lambda\omega(\Lambda^{-1}p)\right)\vec{\psi}_\omega(\Lambda^{-1}(p)), \quad \Lambda \in SL(2,\mathbb{C})\,, \ \ p \in \mathcal{O}_\chi \tag{68}$$

$$(\alpha\vec{\psi}_\omega)(p) = p(\alpha)\vec{\psi}_\omega(p), \quad \alpha \in C^\infty(\mathbb{S}^2)\,. \tag{69}$$

Above, $p(\alpha)$ denotes the action of the character $p \in SL(2,\mathbb{C})\chi$ on α and the subscript ω reflects the strict dependence of the induced wave function upon the choice of the section itself.

3.5. *The scalar induced wave function.* The long explicit construction all the $\widetilde{G_{BMS}}$ irreducible unitary representations has been completed and extensively discussed in the Hilbert topology in [24, 25] and in the nuclear topology in [28], thus it will not be reviewed here. It is anyway interesting for our purposes to stress some of the non trivial points in McCarthy analysis; in particular, whereas in the Hilbert topology all the unitary representation for the BMS group can be constructed as induced representations from compact little group, in the nuclear topology the scenario is far more complicated and it can be summarized in the following proposition [28]:

Proposition 3.10. *The following facts hold for representations of* $\widetilde{G_{BMS}}$*.*
(a) *If an unitary representation of* $\widetilde{G_{BMS}}$ *is irreducible then it must arise either from a transitive* $SL(2,\mathbb{C})$ *action on* N^* *or from a cylinder measure with respect to which the* $SL(2,\mathbb{C})$ *action is strictly ergodic.*
(b) *All the unitary induced* $\widetilde{G_{BMS}}$ *representations are irreducible.*

The statement (a) is the reason why the current classification of unitary irreducible representations of $\widetilde{G_{BMS}}$ group is not complete. As a matter of fact the construction of representations arising from strictly ergodic measure is rather challenging and, up to now, it has not been solved nor addressed in detail.
Nonetheless, for our purposes we are mainly interested in induced representations and the statement (b) has been fully exploited in [28] where, starting from the analysis in [47], a plethora of $\widetilde{G_{BMS}}$ possible little groups has been identified. These can be classified in two different families, the connected subgroups of $SL(2,\mathbb{C})$ and the non connected compact subgroups of $SU(2)$. We shall now concentrate on[8] $SU(2)$, $SU(1,1)$, Γ (the universal covering of $SO(2)$ made of all the matrices $diag(e^{\frac{it}{2}}, e^{-\frac{it}{2}})$ with $t \in \mathbb{R}$) and on $\Delta = \Gamma \ltimes T^2$ (the double covering of the two dimensional Euclidean group). The analysis for the $SU(2)$ scenario has been already developed in [7, 11] in the Hilbert topology, where the wave functions of the intrinsic BMS free fields, their kinematical and dynamical configurations have been throughout discussed. On the opposite, we

[8] These compact little groups are also present in the Hilbert topology scenario.

shall now focus attention on the Δ case – proper only of the nuclear topology – which will turn out to be in direct correspondence with scalar fields on $\Im^+$ induced from the bulk.

Δ orbit classification. This little group is the set of matrices

$$\Lambda_{t,v} = \begin{bmatrix} e^{\frac{it}{2}}, & v \\ 0 & e^{-\frac{it}{2}} \end{bmatrix},$$

with $t \in \mathbb{R}$ and $v \in \mathbb{C}$. Thus, according to (48), a fixed point $\bar{\beta}$ (which thus admits Δ as little group) satisfies:

$$(\Delta\bar{\beta}) = \bar{\beta}.$$

In order to solve this distributional equation the rationale is to switch from $\bar{\beta} \in N^*$ to the associated pair $(\bar{\phi}, \hat{\bar{\phi}}) \in D_{-2}$ as in remark 3.2 and to use (43) and (44), i.e.

$$\Delta\bar{\phi} = \bar{\phi}, \quad \Delta\hat{\bar{\phi}} = \hat{\bar{\phi}}\,.$$

As discussed in [28], the general solution to these equations is:

$$\bar{\phi} = S, \tag{70}$$

$$\hat{\bar{\phi}} = S|\zeta|^{-6} + A\delta^{2,2} + C\delta, \tag{71}$$

where $S, A, C \in \mathbb{R}$ are constants and $\delta^{p,q}$ is the p-th derivative on the variable ζ and q-th derivative on the variable $\overline{\zeta}$ of $\delta = \delta(\zeta)\delta(\overline{\zeta})$.

Proposition 3.11. *The mass (58) associated to any orbit of the Δ_χ little group is* 0.

Proof. The demonstration follows from proposition 3.6 and (57) directly according to which

$$\widehat{\pi(\bar{\beta})}(\zeta', \overline{\zeta}') = \frac{C}{1 + |\zeta'|^2} \not\equiv 0;$$

thus (53) grants us that $\widehat{\pi(\bar{\beta})}_\mu = C(1, 0, 0, 1)$; consequently we conclude from (58) that $m^2 = \eta^{\mu\nu}\widehat{\pi(\bar{\beta})}_\mu\widehat{\pi(\bar{\beta})}_\nu = 0$. □

Only a three dimensional orbit $\frac{SL(2,\mathbb{C})}{\Delta} = \mathbb{R} \times \mathbb{S}^2$ with a vanishing mass can be associated to the Δ_χ little group. Furthermore, from (71) we can infer that, besides the constant C which plays the role of the energy, the orbit is fully determined only if we fix the values A, S which from now on are set to 0.

We can now explicitly construct the representation and we can choose an $SL(2, \mathbb{C})$-invariant measure on the orbit $\frac{SL(2,\mathbb{C})}{\Delta_\chi}$ which represent the key data to construct the induced wave function as in proposition 3.3 and 3.4. Leaving the detailed analysis for the other non connected little groups to [7, 25], we concentrate on:

Δ induced wave functions. Unitary and irreducible representations of Δ are of two types [28, 42]. A representation $D^{\lambda,p,q}$ of the first type is individuated by a triple $p, q \in \mathbb{R}\setminus\{0\}$ $\lambda \in \frac{\mathbb{Z}}{2}$ and it is defined by:

$$D^{\lambda,p,q}\left(\begin{bmatrix} e^{\frac{it}{2}}, & \upsilon \\ 0 & e^{-\frac{it}{2}} \end{bmatrix}\right) = e^{i\lambda t}e^{i(pb+qc)}, \quad \upsilon = b + ic\,.$$

This acts on an infinite dimensional complex target Hilbert space by multiplication and it induces an infinite dimensional $\widetilde{G_{BMS}}$ representation. A representation D^s of the second type is individuated by a number $s \in \mathbb{Z}/2$ and it is defined by:

$$D^s\begin{bmatrix} e^{\frac{it}{2}}, & \upsilon \\ 0 & e^{-\frac{it}{2}} \end{bmatrix} = e^{ist}, \tag{72}$$

This acts on a one-dimensional complex target Hilbert space by multiplication.

Remark 3.10. Although the above representations are well known even for a Poincaré invariant theory, the second being faithful the first being unfaithful, in a $\widetilde{G_{BMS}}$ scenario they are both faithful. More generally, it has been shown in [28] that an induced representation built upon an orbit $\mathcal{O}_\chi$ is faithful iff $\widehat{\pi(\bar{\phi})} \neq 0$, $\bar{\phi}$ being the supermomentum associated to $\mathcal{O}_\chi$ solving (48).

It is possible to reinforce the result presented in theorem 3.1: we can exploit either theorem 1 and corollary 1 in chapter 4, section 3 of [42] either the unimodularity[9] of $SL(2,\mathbb{C})$ and Δ to claim that the $SL(2,\mathbb{C})$-invariant measure class M contains a measure μ which is $SL(2,\mathbb{C})$-invariant. Referring to this specific measure μ we can construct the Hilbert space of induced wave functions $\psi : \mathcal{O}_\chi \to \mathbb{C}$

$$\mathcal{H}_\mu = \left\{ \psi : \mathcal{O}_\chi \to \mathbb{C} \;\middle|\; \int_{\mathcal{O}_\chi} d\mu(p)\bar{\psi}(p)\psi(p) < +\infty \right\}. \tag{73}$$

Now we can use remark 3.7 and the formula (68) and (69) to construct the explicit expression of the induced wave function (63).

Remark 3.11. The Δ little group is a rather special case since no global continuous section ω of the bundle $\tau : SL(2,\mathbb{C}) \to \frac{SL(2,\mathbb{C})}{\Delta}$ exists such for the $SU(2)$ or the Γ little group. There are different choices commonly used and, they being far from the aim of this paper, we refer to

[9] A locally compact group G is called *unimodular* if its right-invariant and left-invariant Haar measure coincide (*c.f.* page 69 in [42])

[48] for a complete discussion.

According to (68) and (69), an induced wave function transforms, for any $g = (\Lambda, \alpha) \in \widetilde{G_{BMS}}$ and under the Δ representation (72), as

$$(g\psi)(p) = \sqrt{\frac{d\mu(\Lambda p)}{d\mu(p)}} p(\alpha) D^s(\omega(p)^{-1}\Lambda\omega(\Lambda^{-1}p))\psi_\omega(\Lambda^{-1}p) = \\ p(\alpha)e^{ist}\psi_\omega(\Lambda^{-1}p), \tag{74}$$

where, with the above-said choice of μ, the Radon-Nikodym measure is 1 and e^{ist} is the action of the one dimensional Δ representation associated to $D^s(\omega(p)^{-1}\Lambda\omega(\Lambda^{-1}p))$. Eventually we may write the *induced scalar* Δ $\widetilde{G_{BMS}}$ *wave function* (i.e. $s = 0$ in (74)) as

$$\psi : \mathcal{O}_\chi \longrightarrow \mathbb{C}, \tag{75}$$

$$(g\psi_\omega)(\Lambda p) = p(\alpha)\psi_\omega(p)\,, \quad \forall\, g \in \widetilde{G_{BMS}} \tag{76}$$

3.6. *The covariant scalar wave function and its bulk interpretation.* Although the induced wave function transforms under a unitary irreducible $\widetilde{G_{BMS}}$ representation, thus containing all relevant physical information, from a physical perspective, it is rather common to start from a different wave function. That is the **covariant wave function(al)** (or **covariant free field**) which, in a BMS setting, is [7, 49]:

$$\vec{\Psi} : N^* \longrightarrow \mathcal{H}' \tag{77}$$

where $\mathcal{H}'$ is a suitable *finite dimensional* target Hilbert space either real or complex and N^* is the space of distributions over $\mathbb{S}^2$. Under the action of $g = (\Lambda, \alpha) \in \widetilde{G_{BMS}}$, $\vec{\Psi}$ in (77) transforms as

$$\left[U^\lambda(g)\vec{\Psi}\right][\beta] = \chi_\beta(\Lambda\alpha)D^\lambda(\Lambda)\vec{\Psi}[\Lambda^{-1}\beta], \tag{78}$$

where $D^\lambda(\Lambda)$ is a unitary, *but not necessary irreducible*, representation of $SL(2,\mathbb{C})$ labeled by the superscript λ. χ_β is the character associated with β as in definition 3.2 and it acts according to remark 3.3.

Remark 3.12. At first glance (77) and (63) are *a priori* unrelated, the main striking difference consisting in the existence of a different induced wave function for each isotropy subgroup H_χ, whereas the covariant wave equation is unique up to the choice of a unitary $SL(2,\mathbb{C})$ representation. Nonetheless it is possible to show that both kinds of wave functions are ultimately equivalent if suitable constraints are imposed on the covariant wave function[10] [42, 45]. In the

[10] We shall also refer to the construction of the covariant wave equation and of the associated equations of motion as the **Wigner's program** in relation to Wigner seminal paper [50] where he dealt with the Poincaré case.

BMS scenario [7], upon selecting a specific covariant wave function and a representation U^λ as in (78), the restriction to the induced wave function associated to a fixed little group H_χ operatively corresponds to:

1. restrict the support of (78) from N^* to the orbit $\frac{SL(2,\mathbb{C})}{L_\chi} \hookrightarrow N^*$ ($\hookrightarrow$ denoting an embedding);
2. act on $\vec{\Psi}$ by the linear transformation $U^\lambda\left[\omega^{-1}(\beta)\right]$ where β is a point on the orbit and where ω is the section chosen in (68);
3. select in (78) the irreducible unitary representation σ of (63) contained in D^λ.

We discuss now in details the last step in the above construction which is rather counterintuitive. Let us start either from a generic unitary, but fixed, representation D^λ of $SL(2,\mathbb{C})$ either from a generic, but fixed, irreducible unitary representation σ^j of a fixed little group L_χ.
Let us now consider the restriction of D^λ to L_χ which decomposes as the finite sum $D^\lambda\restriction_{L_\chi}= \bigoplus\limits_{j'} C_{\lambda,j'}\sigma^{j'}$, where $\sigma^{j'}$ is a unitary irreducible representation of L_χ and $C_{\lambda,j'}$ are suitable integers standing for the multiplicity of the $\sigma^{j'}$ in D^λ. According to theorem 16.2.1 in [42], the above decomposition translates either to the full $\widetilde{G_{BMS}}$ representation either to the target space of (78) i.e.

$$\mathcal{H}' = \bigoplus_{j'} C_{\lambda,j'}\mathcal{H}^{j'}. \tag{79}$$

Let us now recognize a fixed $\mathcal{H}^{j'}$ as the target space of an induced wave function (63) which transforms under the action of the unitary and irreducible representation $\sigma^{j'}$. The selection of a fixed representation $\sigma^{j'}$ in D^λ is now equivalent to constrain $\mathcal{H}'$ – the target space of (77) – to $\mathcal{H}^{j'}$. This result can be operatively achieved imposing the subsidiary condition on the covariant wave function (with support on $\mathcal{O}_\chi$)

$$\rho\vec{\Psi}[\bar{\beta}] = \vec{\Psi}[\bar{\beta}], \tag{80}$$

where ρ is the projector selecting $\mathcal{H}^j \subset \mathcal{H}'$ and where $\bar{\beta}$ is the supermomentum associated to the fixed point on $\mathcal{O}_\chi$.
If we now remember that the following identity holds:

$$\vec{\Psi}[\beta] = [D^\lambda(\Lambda)\vec{\Psi}][\bar{\beta}], \tag{81}$$

where $\beta = \Lambda^{-1}\bar{\beta} \in \frac{SL(2,\mathbb{C})}{L_\chi}$ and where $\Lambda \in SL(2,\mathbb{C})$, (80) becomes

$$D^\lambda(\Lambda)\rho[D^\lambda(\Lambda)]^{-1}\vec{\Psi}[\beta] = \vec{\Psi}[\beta].$$

If we set[11] $\rho[\beta] = D^\lambda(\Lambda)\rho[D^\lambda(\Lambda)]^{-1}$, (81) becomes the so-called *projection equation*

$$\rho[\beta]\vec{\Psi}[\beta] = \vec{\Psi}[\beta]\,. \tag{82}$$

[11] The β dependence of ρ should not be interpreted literally: it means that Λ in (81) is the unique value such that $\Lambda^{-1}\bar{\beta} = \beta$.

Remark 3.13. According to the discussion of section 1B in chapter 21 of [42], which easily generalize to the $\widetilde{G_{BMS}}$ scenario, (82) is also a covariant matrix operator and, since induced wave functions are in one-to-one correspondence with pairs $\left\{D^{\lambda}(\Lambda), \rho\right\}$, it is customary to claim that (82) represents the most general covariant wave equation[12].

Since our final goal is to show the equivalence between (31) and the $\widetilde{G_{BMS}}$ Δ scalar induced wave function (76), let us restrict our attention to this specific case.

Definition 3.7. *A covariant scalar wave function is a map*

$$\Psi : N^* \longrightarrow \mathbb{C}, \tag{83}$$

which transforms as

$$\left[U^{\lambda}(g)\Psi\right](\beta) = \chi_{\beta}(\Lambda\alpha)\Psi[\Lambda^{-1}\beta], \tag{84}$$

under a scalar $SL(2,\mathbb{C})$ unitary representation U_g, with $g = (\Lambda, \alpha) \in \widetilde{G_{BMS}}$. In (84) $\Lambda^{-1}\beta$ is defined as in (41).

Proposition 3.12. *Referring to the definition above, the constraint to impose to reduce (84) to (76) is*

$$\left[\beta - \frac{SL(2,\mathbb{C})}{\Delta}\bar{\beta}\right]\Psi[\beta] = 0, \;\; \widehat{\pi(\bar{\beta})} \neq 0 \tag{85}$$

where $\beta \in N^*$ and $\bar{\beta}$ is the fixed point of the Δ orbit constructed out of (70) and (71).

The proof of this proposition is a straightforward consequence of the analysis in [7] and of the coincidence between the scalar covariant $SL(2,\mathbb{C})$ representation and the scalar representation induced from the Δ little group (i.e. (82) is identically satisfied).
Furthermore the mass equation which usually appears in the Hilbert topology [7], i.e.

$$\left[\eta^{\mu\nu}\widehat{\pi(\beta)}_{\mu}\widehat{\pi(\beta)}_{\nu}\right]\Psi[\beta] = 0,$$

is automatically satisfied by (85) since the little group Δ is associated only to a vanishing mass whenever $\widehat{\pi(\bar{\beta})} \neq 0$.

To conclude, we want now to establish the main result of this section, namely that a covariant massless scalar field which satisfies (85) is equivalent to (31). The meaning of "equivalent" will be clarified in the following discussion. Let us remember that $N^* \sim D_{(-2,-2)} \sim D_{-2}$ as well

[12]Chapter 21 of [42] contains the specific discussion for the Poincaré scenario where it is shown that (82) is simply a compact expression for the usual Dirac, Proca equations, etc... whereas, for the $\widetilde{G_{BMS}}$ counterpart in the Hilbert topology, we refer to [7].

as $N \sim D_{(2,2)} \sim D_2$. Thus an element $\beta \in N^*$ is bijectively related with the pair $\hat{\phi}, \phi \in D_{-2}$ introduced in proposition 3.1. Furthermore let us recall that the fixed point of the Δ orbit is $\bar{\hat{\phi}} = S|z|^{-6} + K\delta^{2,2} + C\delta$ with $C \neq 0$; if we select the specific values $S = K = 0$, then $\bar{\hat{\phi}} = C\delta$ and, according to proposition 3.1 and to remark 3.2, the associated supermomentum is

$$\bar{\beta} = \frac{\bar{\hat{\phi}}}{(1+ \mid \zeta \mid^2)^3},$$

We need now the following Lemma:

Lemma 3.1. *The supermomentum $\bar{\beta}$ lies in $(T^4)^*$.*

Proof: consider the isomorphism discussed about (56) first introduced in [28]

$$\frac{N^*}{(T^4)^0} \sim (T^4)^*, \tag{86}$$

where both sides are preserved under $SL(2,\mathbb{C})$ transformation and the isomorphism commute with the action of that group.

It is straightforward that $\bar{\beta} \notin (T^4)^0$ since if we consider any supertranslation $f \in T^4 \hookrightarrow C^\infty(\mathbb{S}^2)$ such that $f(0) \neq 0$, then $(f, \bar{\beta}) = Cf(0) \neq 0$. As a consequence we are free to choose $\bar{\beta}$ as the representative of a conjugacy class in $\frac{N^*}{(T^4)^0}$ and, according to (86), $\bar{\beta}$ also lies in $(T^4)^*$. $\square$

Furthermore, since the orbit $\mathcal{O}_{\bar{\beta}}$ is generated as $\frac{SL(2,\mathbb{C})}{\Delta}\bar{\beta}$, (86) also grants us that $\mathcal{O}_{\bar{\beta}} \in (T^4)^*$ i.e., according to proposition (3.5), $\pi(\beta) = \beta$ for any $\beta \in \mathcal{O}_{\bar{\beta}}$. This last remark entitles to substitute in (84) β with $\pi(\beta)$.

$$\Psi : (T^4)^* \longrightarrow \mathbb{C},$$
$$[U(g)\Psi](\pi(\beta)) = \chi_{\pi(\beta)}(\Lambda\alpha)\Psi\left[\Lambda^{-1}\pi(\beta)\right], \quad \forall\, g \in \widetilde{G_{BMS}}$$

which still satisfies the orbit constraint $\left[\pi(\beta) - \frac{SL(2,\mathbb{C})}{\Delta}\pi(\bar{\beta})\right]\Psi[\pi(\beta)] = 0$.

The next step consists in bearing in mind that $(T^4)^* \sim T^4$ [28], i.e., according to proposition (3.5) and (57), any element $\pi(\beta) \in (T^4)^*$ is in one to one correspondence with the element $\widehat{\pi(\beta)} \in T^4$.

Furthermore, according to (53) and to proposition 3.11, we can identify each $\widehat{\pi(\beta)} \in \mathcal{O}_{\widehat{\pi(\bar{\beta})}}$ with a four-vector p_μ which satisfies the mass relation $\eta^{\mu\nu}p_\mu p_\nu = 0$. Thus we can write $p_\mu \equiv (E, E\mathbf{n}(\zeta', \overline{\zeta}'))$ where $\mathbf{n}(\zeta', \overline{\zeta}')$ is a three dimensional spatial versor spanning a two-sphere of unit radius whose coordinates are $\zeta', \overline{\zeta}'$.

At this stage the reader should bear in mind that we are ultimately dealing with a Gelfand triplet i.e. $N \subset L^2(\mathbb{S}^2) \subset N^*$; thus, according to these last remarks we are entitled to switch from the covariant wave function living on $(T^4)^* \subset N^*$ to a second one living on $T^4 \subset N$ which reads[13]:

$$\Psi : \mathcal{O}_{\widehat{\pi(\bar\beta)}} \hookrightarrow T^4 \mapsto \mathbb{C},$$
$$(U(g)\Psi)[\widehat{\pi(\beta)}] = \chi(\Lambda\alpha)\Psi\left[\Lambda^{-1}\widehat{\pi(\beta)}\right] . \quad \forall\, g \in \widetilde{G_{BMS}}$$

Since $\widehat{\pi(\beta)}$ now lies in $C^\infty(\mathbb{S}^2)$ the net effect of an $SL(2,\mathbb{C})$ action is according to (36) and to (42)

$$\left(\Lambda^{-1}\widehat{\pi(\beta)}\right)(\zeta',\overline{\zeta}') = K_\Lambda(\Lambda^{-1}\zeta',\Lambda^{-1}\overline{\zeta}')\widehat{\pi(\beta)}(\Lambda^{-1}\zeta',\Lambda^{-1}\overline{\zeta}').$$

In terms of 4-vectors, $\Lambda^{-1}\widehat{\pi(\beta)}$ corresponds to the 4-vector whose components are

$$p_0 = K_\Lambda(\Lambda^{-1}\zeta',\Lambda^{-1}\overline{\zeta}')E\,, \quad \mathbf{p} = K_\Lambda(\Lambda^{-1}\zeta',\Lambda^{-1}\overline{\zeta}')E\mathbf{n}(\Lambda^{-1}\zeta',\Lambda^{-1}\overline{\zeta}')\,.$$

The character can be directly evaluated as $\chi(\Lambda\alpha) = e^{iEK_\Lambda^{-1}(\zeta',\overline{\zeta}')\alpha(\Lambda^{-1}\zeta',\Lambda^{-1}\overline{\zeta}')}$. Substituting these results in the scalar covariant wave equation and taking into account that each $(\widehat{\pi(\beta)})(\zeta',\overline{\zeta}')$ is uniquely determined by its associated four vector p_μ which, in turn, is determined by the coordinates $(E,\zeta,\overline{\zeta})$, we can eventually recast (84) in terms of a field $\varphi(E,\zeta,\overline{\zeta}) := \Psi[\widehat{\pi(\beta)}]$ as:

$$[U(g)\varphi](E,\zeta',\overline{\zeta}') = \frac{e^{iEK_\Lambda(\Lambda^{-1}\zeta',\Lambda^{-1}\overline{\zeta}')\alpha(\Lambda^{-1}\zeta',\Lambda^{-1}\overline{\zeta}')}}{\sqrt{K_\Lambda(\Lambda^{-1}(\zeta,\overline{\zeta}))}}\varphi(K_\Lambda^{-1}(\zeta',\overline{\zeta}')E,\Lambda^{-1}\zeta',\Lambda^{-1}\overline{\zeta}')\,.$$

The square root is due to the fact that we passed to the measure $dE \otimes \epsilon_{\mathbb{S}^2}$ from the invariant one $d\mathbf{p}/E(\mathbf{p})$. We have found nothing but the unitary representation of G_{BMS} given in (31). Therefore this fact shows also that the representation of $\widetilde{G_{BMS}}$ obtained by (85) is a unitary representation of G_{BMS} as well.

We have eventually proved that:

Theorem 3.2. *A field on $\Im^+$ satisfying (31) is nothing but a $\widetilde{G_{BMS}}$-covariant massless scalar field which satisfies (85). Furthermore, the representation of $\widetilde{G_{BMS}}$ obtained by (85) is a unitary representation of G_{BMS} as well.*

As a last remark we wish to clarify why the above theorem holds only when a suitable nuclear topology is imposed on the set of supertranslations. If we choose $N = L^2(\mathbb{S}^2,\epsilon_{\mathbb{S}^2})$ (where the field of the Hilbert space is $\mathbb{R}$), it is still possible to construct a massless scalar wave function induced from the Γ little group living on an orbit whose fixed point has a vanishing pure supertranslational component. Nonetheless, in this framework, according to the Riesz-Fisher

[13] Alternatively it is possible to interpret the covariant wave function on T^4 as the one on $(T^4)^*$ where the argument $\beta \in (T^4)^*$ has been evaluated with a fixed test function as in (57).

theorem, a character $\chi(\alpha)$ can be always associated with an element $\beta \in L^2(\mathbb{S}^2, \epsilon_{\mathbb{S}^2})$ [24] such that

$$\chi(\alpha) = e^{i\int\limits_{\mathbb{S}^2} \epsilon_{\mathbb{S}^2}\alpha\beta}, \quad \forall\alpha \in L^2(\mathbb{S}^2, \epsilon_{\mathbb{S}^2}).$$

This formula represents the key obstruction to obtain (31) in an Hilbert topology framework since, whenever we consider a scalar covariant wave function $\Psi : N^* = L^2(\mathbb{S}^2, \epsilon_{\mathbb{S}^2}) \longrightarrow \mathbb{C}$ with a support restricted on $T^4 \subset N^*$, we are requiring that $\beta \in N^*$ can be written as $\beta(\zeta, \overline{\zeta}) = \sum\limits_{k=0}^{1}\sum\limits_{l=1}^{2k+1} \beta_{lk} S_{lk}(\zeta, \overline{\zeta})$ where S_{lk} are the real spherical harmonics. Accordingly a character will always be written as:

$$\chi(\alpha) = e^{i\left(\sum\limits_{k=0}^{1}\sum\limits_{l=1}^{2k+1} \beta_{lk}\alpha_{lk}\right)},$$

which cannot produce a phase as that in (31) $e^{iEK_\Lambda(\Lambda^{-1}\zeta', \Lambda^{-1}\overline{\zeta}')\alpha(\Lambda^{-1}\zeta', \Lambda^{-1}\overline{\zeta}')}$ whenever α includes components in the space of pure supertranslations. Thus, it is this expression which represents the symptom that a correspondence between (31) and an intrinsic BMS field could be achieved only if a distributional support for the covariant wave function is considered.

4 A few holographic issues.

4.1. *General goals of the section.* We want to start to investigate the issue of holographic correspondence between QFT formulated in the *bulk* for fields ϕ satisfying Klein-Gordon equation (11) as in Proposition 2.3, and QFT formulated on the *boundary* $\Im^+$ as showed in the previous section. In this sense the bulk is the globally hyperbolic subregion near the null infinity $\Im^+$ of an asymptotically flat and strongly asymptotically predictable spacetime. We know from Proposition 2.3 that, at level of classical fields, there is a correspondence between solutions of field equations $(\Box - \frac{1}{6}R)\phi = 0$ and associated fields ψ defined on $\Im^+$. We want to investigate whether or not such a correspondence can be implemented at level of algebras of observables associated with the relevant fields. If the correspondence can be implemented in terms of an injective $*$-homomorphism, the algebra of the bulk can be realized as a (sub)algebra of the observables of the boundary. *In this sense it would realize a sort of holographic machinery which encodes complete information of QFT defined in the bulk in QFT living in the boundary.*
To this end we have to recall some features of linear QFT in globally hyperbolic spacetime [31].

4.2. *Linear QFT in the bulk.* *Let us assume that the spacetime* (M, g) *is globally hyperbolic*, $K := \Box + P$, P being any smooth real valued function on M, denotes a Klein-Gordon-like operator in that spacetime and $\mathcal{S}_K(M)$ indicates the real space of solutions ϕ of $K\phi = 0$ with compactly supported Cauchy data on a (and thus every) Cauchy surface of (M, g).
A natural nondegenerate symplectic form on $\mathcal{S}_K(M)$ can be defined as

$$\sigma_M(\phi_1, \phi_2) := \int_S (\phi_2 \nabla_N \phi_1 - \phi_1 \nabla_N \phi_2)\; d\mu_g^{(S)}\,, \tag{87}$$

the choice of the Cauchy surface S being immaterial because the right-hand side does not depend on such a choice. N is the unit future directed normal vector to S and $d\mu_g^{(S)}$ the measure associated with the metric induced on S by g. Nondegenerateness implies that there is a unique C^* algebra generated by **(abstract) Weyl operators** $W(\phi)$, with $\phi \in \mathcal{S}_K(M)$, such that

$$(Wb1) \quad W_M(-\phi) = W_M(\phi)^* \,, \quad (Wb2) \quad W_M(\phi_1)W_M(\phi_2) = e^{i\sigma(\phi_1,\phi_2)/2}\, W_M(\phi_1+\phi_2)\,.$$

That C^* algebra is **Weyl algebra,** $\mathcal{W}_K(M)$**, associated with** K **in the spacetime** (M, g). The formal interpretation of elements $W(\phi)$ is $e^{i\sigma_M(\phi,\Phi)}$, $\sigma_M(\phi, \Phi)$ being the usual field operator symplectically smeared with smooth field equations with compactly supported Cauchy data. There is an equivalent construction of $\mathcal{W}_K(M)$ which allows a straightforward representation of locality based on the linear, real, formally anti self-adjoint operator $E_K : C_c^\infty(M) \to C^\infty(M)$ called **causal propagator** of K. It is defined as the difference of advanced and retarded fundamental solutions of $Kf = 0$ which are known to exist globally provided the spacetime is globally hyperbolic. Let us focus attention on remarkable features of E_K we go to list.
(i) $E_K f \in \mathcal{S}_K(M)$ for $f \in C_c^\infty(M)$. (ii) E_K is surjective onto $\mathcal{S}_K(M)$. (iii) $supp(E_K f) \subset J(suppf)$. (iv) $E_K f = 0$ if and only if $f = Kg$ for some $g \in C_c^\infty(M)$.
As consequence of those properties the identity holds [31]

$$\int_M \phi f \, d\mu_g = \sigma_M(E_K f, \phi) \quad \text{and thus} \quad \int_M f E_K g \, d\mu_g = \sigma_M(E_K f, E_K g)\,. \tag{88}$$

where $d\mu_g$ is the volume form of M induced by the metric g. To go on, it is convenient to define

$$V_M(f) := W_M(E_K f)\,, \quad \text{for every } \; f \in C_c^\infty(M)\,. \tag{89}$$

Taking the former of (88) into account, the formal interpretation of elements $V_M(f)$ is $e^{i\Phi(f)}$, $\Phi(f) = \int_M \Phi f \, d\mu_g$ being the usual field operator smeared with smooth compactly supported functions. The interpretation given above makes sense in terms of operators whenever a regular state is fixed, by applying GNS theorem. It turns out, for (iv), that

$$V_M(f) = V_M(g)\,, \quad \text{if and only if } \; f - g = Kh \; \text{ for some } \; h \in C_c^\infty(M). \tag{90}$$

This is nothing but the constraint due to field equation $K_K\Phi = 0$ given in a distributional-like fashion, using the fact that K is formally self-adjoint and $KE_K = 0$ by definition of E_K. By construction, generators $V_M(f)$ generate the same C^*-algebra, $\mathcal{W}(M)$, as $W_M(\phi)$. The improvement is due to the fact that, now , property (iii) together with (Wb2) and the latter in (88) entail

$$[V_M(f), V_M(g)] = 0 \quad \text{whenever the supports of } f \text{ and } g \text{ are causally separated.}$$

This is the natural formulation of locality in spacetime.

4.3. *General holographic tools.* All results and tools introduced above can be used in the globally hyperbolic spacetime $(\tilde{V} \cap M, g)$ whenever (M, g) is asymptotically flat and strongly asymptotically predictable with respect the open set $\tilde{V}$ (see comments before proposition 2.3) equipped with Klein-Gordon operator for a conformally coupled massless scalar field $K := \Box - \frac{1}{6}R$. The main proposition concerning holographic relations between $\mathcal{W}_K(\tilde{V} \cap M)$ and $\mathcal{W}(\Im^+)$ consists of the following proposition. We need a preliminary definition. If (M, g) is an asymptotically flat spacetime, strongly asymptotically predictable with respect to $\tilde{V} \subset \tilde{M}$ and $K := \Box - \frac{1}{6}R$, the **projection map** $\Gamma_{M_{\tilde{V}}} : \mathcal{S}_K(M_{\tilde{V}}) \to \mathcal{S}(\Im^+)$ is the real linear map which associates every $\phi \in \mathcal{S}_K(M_{\tilde{V}})$ with the smooth extension to $\Im^+$ of $(\omega\Omega)^{-1}\phi$ as in Proposition 2.3, where $(\omega\Omega)^2 g$ induces the triple $(\Im^+, \tilde{h}_B, n_B)$ on $\Im^+$.

Proposition 4.1. *Let (M, g) be an asymptotically flat spacetime, strongly asymptotically predictable with respect to $\tilde{V} \subset \tilde{M}$ and let E_K denote the causal propagator in $M_{\tilde{V}} := \tilde{V} \cap M$ of $K := \Box - \frac{1}{6}R$. Assume that both conditions below hold true for the projection map $\Gamma_{M_{\tilde{V}}}$:*
(a) $\Gamma_{M_{\tilde{V}}}(\mathcal{S}_K(M_{\tilde{V}})) \subset \mathcal{S}(\Im^+)$,
(b) *symplectic forms are preserved by $\Gamma_{M_{\tilde{V}}}$, that is, for all $\phi_1, \phi_2 \in \mathcal{S}(M_{\tilde{V}})$,*

$$\sigma_{M_{\tilde{V}}}(\phi_1, \phi_2) = \sigma(\Gamma_{M_{\tilde{V}}}\phi_1, \Gamma_{M_{\tilde{V}}}\phi_2)\,, \tag{91}$$

Then $\mathcal{W}(M_{\tilde{V}})$ can be identified with a sub C^*-algebra of $\mathcal{W}(\Im^+)$ by means of a C^*-algebra isomorphism $\imath$ uniquely determined by the requirement

$$\imath(W_{M_{\tilde{V}}}(\phi)) = W(\Gamma_{M_{\tilde{V}}}\phi)\,, \quad \textit{for all } \phi \in \mathcal{S}_K(M_{\tilde{V}})\,, \tag{92}$$

or, equivalently,

$$\imath(V_{M_{\tilde{V}}}(f)) = W(\Gamma_{M_{\tilde{V}}}E_K f)\,, \quad \textit{for all } f \in C_c^\infty(M_{\tilde{V}})\,. \tag{93}$$

Proof. For (89), the thesis can be proved referring to generators $V_{M_{\tilde{V}}}(\phi)$ only. We start by fixing the relevant sub C^*-algebra of $\mathcal{W}(\Im^+)$ as follows. As a consequence of (a), it makes sense to consider the *-algebra in $\mathcal{W}(\Im^+)$, $\mathcal{A}$, finitely generated by the elements $V_{M_{\tilde{V}}}(\Gamma_{M_{\tilde{V}}}\phi)$ for all $\phi \in \mathcal{S}(M_{\tilde{V}})$. The closure (in $\mathcal{W}(\Im^+)$) of that *-algebra, $\overline{\mathcal{A}}$, is a sub C^*-algebra of $\mathcal{W}(\Im^+)$ by construction. On the other hand, by construction and using the uniqueness of the norm of a C^*-algebra, $\overline{\mathcal{A}}$ must coincide with Weyl algebra associated with the real vector space $\mathcal{S}_0 := \Gamma_{M_{\tilde{V}}}(\mathcal{S}(M_{\tilde{V}}))$ and the nondegenerate symplectic form σ restricted to that space. Whenever the real linear application $\Gamma : \mathcal{S}(M_{\tilde{V}}) \to \mathcal{S}_0$ is bijective, the validity of requirement (b) entails (as an immediate consequence of the main statement of theorem 5.2.8 in [32]) that there is a *-algebra isomorphism $\imath : \mathcal{W}(\mathbb{M}) \to \mathcal{W}(\mathcal{S}_0) \equiv \overline{\mathcal{A}}$ uniquely determined by the requirement $\imath(V_{M_{\tilde{V}}}(\phi)) = W(\Gamma_{M_{\tilde{V}}}\phi))$, which is nothing but (92). As is well known, *-algebra isomorphisms of C^*-algebras are C^*-algebra isomorphisms. Hence the thesis holds true provided the map $\Gamma_{M_{\tilde{V}}} : \mathcal{S}(M_{\tilde{V}}) \to \mathcal{S}_0$ is bijective. $\Gamma_{M_{\tilde{V}}}$ is surjective by construction. Assume that $\phi \in Ker(\Gamma)$ then, by condition (b) and using left-argument linearity of σ one has $\sigma_{M_{\tilde{V}}}(\phi, \psi) = 0$ for all $\psi \in \mathcal{S}(M_{\tilde{V}})$. Thus it must hold $\phi = 0$ because $\sigma_{M_{\tilde{V}}}$ is nondegenerate. It implies that $\Gamma_{M_{\tilde{V}}}$ is also injective concluding the proof. $\Box$

In case the hypotheses of Proposition 4.1 is fulfilled, another relevant consequence will take place. In that case any algebraic state $E : \mathcal{W}(\Im^+) \to \mathbb{C}$ can be pulled back on $\mathcal{S}(M_{\tilde{V}})$ through $\imath$ to the state $\nu_\imath : \mathcal{W}(M_{\tilde{V}}) \to \mathbb{C}$, defined as $\nu_\imath(a) := \nu(\imath(a))$ for all $a \in \mathcal{W}(M_{\tilde{V}})$. In particular it happens for the BMS-invariant state λ (corresponding to Υ in its GNS representation) of section 2.4: the state $\lambda_\imath$ could be used to build up QFT in the bulk. For instance, it may give a notion of particle also if the bulk spacetime does not admit any group of isometries (Poincaré group in particular). From the fact that bulk isometries, barring pathological situations prepared *ad hoc*, give rise to asymptotic symmetries and λ is invariant under all asymptotic symmetries, we expect that $\lambda_\imath$ is invariant under isometries of the bulk. The formal investigation on this fact in the general case will be performed elsewhere. Another relevant point which deserves investigation concerns the short distance behaviour of n-point functions associated with $\lambda_\imath$. In fact it is a well-established result that physically meaningful states must have Hadamard behaviour property (see [31] for a general discussion on this extent). There is no evidence, from our construction, that $\lambda_\imath$ is Hadamard. However all those properties can be studied in the particular and relevant case of Minkowski spacetime. This is the content of next section.

4.4. *Holographic interplay of Minkowski space and* $\Im^+$. Let us consider the case of four dimensional Minkowski space $(\mathbb{M}^4, \eta)$. That spacetime is asymptotically flat. Starting from a fixed Minkowski frame referred to coordinates $(t, \mathbf{x})$, the unphysical spacetime $(\tilde{M}, \tilde{g})$ can be fixed to be Einstein static universe [18] as follows. One passes to spherical coordinates in the rest space of the initial Minkowski frame, obtaining coordinates (t, r, ϑ, ϕ) on $\mathbb{M}^4$, and finally one adopts null coordinates $u := t - r \in \mathbb{R}$, $v := t + r \in \mathbb{R}$ obtaining global coordinates (u, v, ϑ, ϕ) on $\mathbb{M}^4$. Using these initial coordinates, define coordinates $\vartheta = \vartheta$, $\varphi = \phi$, $T = \tan^{-1} v + \tan^{-1} u$ and $R = \tan^{-1} v - \tan^{-1} u$ and assume $\Omega^2\restriction_{\mathbb{M}^4}= 4[(1 + v^2)(1 + u^2)]^{-1}$. With these definitions $\tilde{g} := \Omega^2\eta$ reads

$$\tilde{g} = -dT^2 + dR^2 + \sin^2 R(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) . \tag{94}$$

This metric makes sense in a larger spacetime $(\tilde{M}, \tilde{g})$ obtained by assuming $T \in \mathbb{R}$, $R \in (0, \pi)$ and ϑ, φ varying everywhere on $\mathbb{S}^2$. That is Einstein static spacetime. (The singularities for $R \to 0, \pi$ in $\tilde{M}$ are only apparent they being "origins of spherical coordinates" and the expression of the metric (94) is valid throughout all Einstein spacetime except for the two one-dimensional submanifolds corresponding to values "$R = 0$" and "$R = \pi$". To cover the whole manifold $\tilde{M}$ two charts at least are necessary.)

With that procedure $(\mathbb{M}^4, \eta)$ turns out to be embedded into $(\tilde{M}, \tilde{g})$ as a globally hyperbolic submanifold and $\Im^+$ is completely represented by the set of points with $T + R = \pi$, $R \in (0, \pi)$. Rescaling $\tilde{g}$ on $\Im^+$ by the further regular factor $\omega^2 := (\sin R)^{-2}$ (i.e. $\omega^2 := 1 + u^2$) and changing coordinates in the sector T, R one gets a metric with associated triple $(\Im^+, \tilde{h}_B, n_B)$.

A natural Bondi frame on $\Im^+$, which we say to be **associated with the Minkowski frame** $(t, \mathbf{x})$ in $\mathbb{M}^4$, is finally obtained as $(u, \zeta, \overline{\zeta})$ where u is just the (limit to $\Im^+$ of the) null coordinate u in the reference frame initially fixed in Minkowski spacetime and $\zeta := e^{i\phi} \cot \frac{\vartheta}{2}$, also (ϑ, ϕ) being angular spherical coordinates in the reference frame initially fixed in Minkowski spacetime.

Remark 4.1. The case of Minkowski spacetime permits to clarify better the arbitrariness in fixing the metric on the transverse section of $\Im^+$ to be that standard of the unit 2-sphere (see remark 2.1). The metric of the 2-sphere determined by $\tilde{h}_B$ is nothing but that of the unit 2-spheres $4d\zeta d\overline{\zeta}/(1+\zeta\overline{\zeta})^2$ in the rest frame of initial Minkowski coordinates $(t, \mathbf{x})$. Starting form a different initial Minkowski frame $(t', \mathbf{x}')$ connected with the initial one by means of a Poincaré transformation (Λ, a), one would determine the same asymptotic manifold $\Im^+$ but he would find a different metric $\tilde{h}'_B$ on $\Im^+$ itself, $\tilde{h}_B = 0du' + 4d\zeta' d\overline{\zeta}'/(1+\zeta'\overline{\zeta}')^2$. Notice that the non degenerate part is again the standard metric of the unit 2-sphere but, as $\zeta \neq \zeta'$ and $\overline{\zeta} \neq \overline{\zeta}'$, it is not the standard metric of the unit 2-sphere determined in the former case: Conversely, it is that of the unit 2-spheres in the rest frame of Minkowski coordinates $(t', \mathbf{x}')$. However, a closer scrutiny shows that the triples $(\Im^+, \tilde{h}_B, n_B)$ and $(\Im^+, \tilde{h}'_B, n'_B)$ are connected by a transformation of BMS group (Λ, f_a). Indeed one has the following result whose (simple) proof is left to the reader (see also [51, 52]).

Proposition 4.2. *Let $(t, \mathbf{x}) = (x^0, x^1, x^2, x^3)$ and $(t, \mathbf{x}) = (x'^0, x'^1, x'^2, x'^3)$ be Minkowski frames in $(\mathbb{M}^4, \eta)$ such that $x'^\mu = \Lambda^\mu{}_\nu(x^\nu + a^\nu)$ for some $(\Lambda, a) \in ISO(3,1)$ and let $(u, \zeta, \overline{\zeta})$ and $(u', \zeta', \overline{\zeta}')$ be the respectively associated Bondi frames on $\Im^+$. The following holds.*
(a) *The Bondi frames are connected by means of the BMS transformation*

$$u' := K_\Lambda(\zeta, \overline{\zeta})(u + f_a(\zeta, \overline{\zeta}))\,, \quad (\zeta', \overline{\zeta}') = \Lambda(\zeta, \overline{\zeta})\,,$$

where the action of Λ on $(\zeta, \overline{\zeta})$ is that in (4) and the function f_a belongs to the space T^4 spanned by the first four real spherical harmonics as defined in Section 3.3, that is[14]

$$f_a := a^0 - \frac{a^1(\zeta + \overline{\zeta})}{\zeta\overline{\zeta} + 1} - \frac{a^2(\zeta - \overline{\zeta})}{i(\zeta\overline{\zeta} + 1)} - \frac{a^3(\zeta\overline{\zeta} - 1)}{\zeta\overline{\zeta} + 1}\,. \tag{95}$$

(b) *The set*

$$\mathfrak{R} := \left\{ (\Lambda, f_a) \in G_{BMS} \;\middle|\; f_a = a^0 - \frac{a^1(\zeta + \overline{\zeta})}{\zeta\overline{\zeta} + 1} - \frac{a^2(\zeta - \overline{\zeta})}{i(\zeta\overline{\zeta} + 1)} - \frac{a^3(\zeta\overline{\zeta} - 1)}{\zeta\overline{\zeta} + 1}\,,\;\; a \in \mathbb{R}^4 \right\}$$

is a subgroup of G_{BMS}, the map $ISO(3,1) \ni (\Lambda, a) \mapsto (\Lambda, f_a) \in \mathfrak{R}$ being a continuous-group isomorphism.

The second statement is a straightforward consequence of proposition 3.4.
The quantum version of proposition above will be established in Theorem 4.2 below. These results are due to the fact that Poincaré isometries are also asymptotic symmetries (see also [51]).

[14] In angular spherical coordinates, we recognize in the factors below in front of $-a^1$, $-a^2$ and $-a^3$ the components of the radial versor, respectively, $\sin\vartheta\cos\phi$, $\sin\vartheta\sin\phi$ and $\cos\vartheta$.

Einstein static universe is globally hyperbolic because it is static and T-constant sections are compact (see chapter 6 in [53]). As a consequence $(\mathbb{M}^4, \eta)$ is strongly asymptotically predictable with respect to $\tilde{V} := \tilde{M}$ itself, and so we may define $\mathbb{M}^4_{\tilde{V}} := \mathbb{M}^4$.
The part of standard free QFT in Minkowski spacetime [54, 55] for a massless scalar field ϕ, we are interested in, can be summarized as follows in Weyl quantization referred to Weyl algebra $\mathcal{W}(\mathbb{M}^4)$ with $K := -\Box$. Standard free QFT can be viewed as the GNS realization of $\mathcal{W}(\mathbb{M}^4)$ based on a preferred algebraic state $\lambda_{\mathbb{M}^4}$ invariant under Poincaré group and individuated as we go to describe. Take a Minkowski frame with coordinates $(t, \mathbf{x}) \in \mathbb{R}^4$ and, for every $\phi \in \mathcal{S}_K(\mathbb{M}^4)$, define its positive frequency part, ϕ_+,

$$\phi_+(t,\mathbf{x}) := \int_{\mathbb{R}^3} \frac{d\mathbf{p}\, e^{i(\mathbf{p}\cdot\mathbf{x}-t|\mathbf{p}|)}}{\sqrt{16\pi^3|\mathbf{p}|}} \widetilde{\phi_+}(\mathbf{p}) \,, \quad \widetilde{\phi_+}(\mathbf{p}) := \sqrt{\frac{|\mathbf{p}|}{16\pi^3}} \int_{\mathbb{R}^3} d\mathbf{x} \left(\phi(0,\mathbf{x}) - i\frac{(\partial_t\phi)(0,\mathbf{x})}{|\mathbf{p}|}\right) e^{-i\mathbf{p}\cdot\mathbf{x}} \,. \quad (96)$$

ϕ_+ has no compactly supported Cauchy data and $\phi = \phi_+ + \overline{\phi_+}$. The sesquilinear form

$$\langle \phi_{1+}, \phi_{2+} \rangle_{\mathbb{M}^4} := -i\sigma_{\mathbb{M}^4}(\overline{\phi_{1+}}, \phi_{2+}) \,, \quad \text{for every pair } \phi_1, \phi_2 \in \mathcal{S}_K(\mathbb{M}^4) \quad (97)$$

is well-defined and give rise to a Hermitean scalar product on the space $\mathcal{S}_K(\mathbb{M}^4)^{\mathbb{C}}_+$ of complex linear combinations of positive frequency parts and

$$\langle \phi_{1+}, \phi_{2+} \rangle_{\mathbb{M}^4} = \int_{\mathbb{R}^3} d\mathbf{p}\, \overline{\widetilde{\phi_{1+}}(\mathbf{p})} \widetilde{\phi_{2+}}(\mathbf{p}) \,, \quad \text{for every pair } \phi_1, \phi_2 \in \mathcal{S}_K(\mathbb{M}^4). \quad (98)$$

As a consequence $\mathcal{S}_K(\mathbb{M}^4)^{\mathbb{C}}_+$ is isomorphic to a subspace of $L^2(\mathbb{R}^3, d\mathbf{p})$. Since the former is also dense in the latter[15] by (96), one finds that the **one-Minkowski-particle space** $\mathcal{H}_{\mathbb{M}^4}$, i.e. the Hilbert completion of $\mathcal{S}_K(\mathbb{M}^4)^{\mathbb{C}}_+$, is isomorphic to $L^2(\mathbb{R}^3, d\mathbf{p})$ itself.
The orthocronous proper Poincaré group $ISO(3,1)$ acts naturally on wavefunctions via push-forward: $g^* : \mathcal{S}_K(\mathbb{M}^4) \ni \phi \mapsto \phi \circ g^{-1}$ for every $g \in ISO(3,1)$. The symplectic form $\sigma_{\mathbb{M}^4}$ is invariant under such g^*, g being an isometry. Furthermore, it turns out that there is an irreducible strongly-continuous unitary representation $L^{(1)} : ISO(3,1) \ni g \mapsto L^{(1)}_g$ with $L^{(1)}_g : \mathcal{H}_{\mathbb{M}^4} \to \mathcal{H}_{\mathbb{M}^4}$ such that $(g^*\phi)_+ = L_g \phi_+$ for every $g \in ISO(3,1)$ and every $\phi \in \mathcal{S}_K(\mathbb{M}^4)$. In particular this implies that the decomposition in positive and negative frequency parties as well as the scalar product, do not depend on the particular Minkowski frame used. An irreducible operator representation $\widehat{\mathcal{W}}(\mathbb{M}^4)$ of Weyl algebra $\mathcal{W}(\mathbb{M}^4)$ is constructed on $\mathfrak{F}_+(\mathcal{H}_{\mathbb{M}^4})$ in terms of usual symplectically-smeared field operators and their exponentials

$$\sigma_{\mathbb{M}^4}(\phi, \Phi) := ia(\overline{\phi_+}) - ia^\dagger(\phi_+) \,, \quad \widehat{W}_{\mathbb{M}^4}(\psi) := e^{\overline{i\sigma_{\mathbb{M}^4}(\phi,\Phi)}} \,. \quad (99)$$

[15] As is well known, the map (see (96)) $C^\infty_c(\mathbb{R}^3) \ni f \mapsto \int_{\mathbb{R}^3} d\mathbf{p} f(\mathbf{x}) e^{-i\mathbf{p}\cdot\mathbf{x}}$ has range dense in $L^2(\mathbb{R}^3, d\mathbf{p})$ because Fourier transform is a Hilbert-space isomorphism and $C^\infty_c(\mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3, d\mathbf{p})$, therefore the range is also L^2-dense in the space $B \subset L^2(\mathbb{R}^3, d\mathbf{p})$ of functions which are in $C^\infty_c(\mathbb{R}^3)$ and vanish in a neighborhood of $\mathbf{p} = 0$. Finally B is dense in $L^2(\mathbb{R}^3, d\mathbf{p})$ and it is invariant under multiplication of its elements with either $\sqrt{|\mathbf{p}|}$ and $1/\sqrt{|\mathbf{p}|}$. Thus, by the latter equation in (96), we find that, up to Hilbert-space isomorphisms, $\overline{\mathcal{S}_K(\mathbb{M}^4)^{\mathbb{C}}_+} = L^2(\mathbb{R}^3, d\mathbf{p})$.

The vacuum state $\Upsilon_{\mathbb{M}^4}$ of $\mathfrak{F}_+(\mathcal{H}_{\mathbb{M}^4})$ is, by definition, invariant under the unitary representation L of $ISO(3,1)$ obtained by tensorialization of $L^{(1)}$ and the following covariance relations hold

$$L_g \widehat{W}_{\mathbb{M}^4}(\phi) L_g^\dagger = \widehat{W}_{\mathbb{M}^4}(g^*\phi) \,, \quad \text{for every } \phi \in \mathcal{S}_K(\mathbb{M}^4) \text{ and } g \in ISO(3,1) \,. \tag{100}$$

If $\Pi_{\mathbb{M}^4} : \mathcal{W}(\mathbb{M}^4) \to \widehat{\mathcal{W}}(\mathbb{M}^4)$ denotes the unique ($\sigma_{\mathbb{M}^4}$ being nondegenerate) C^*-algebra isomorphism between those two Weyl representations, $(\mathfrak{F}_+(\mathcal{H}_{\mathbb{M}^4}), \Pi_{\mathbb{M}^4}, \Upsilon_{\mathbb{M}^4})$ coincides, up to unitary transformations, with the GNS triple associated with the algebraic pure state $\lambda_{\mathbb{M}^4}$ on $\mathcal{W}(\mathbb{M}^4)$ uniquely defined by the requirement (see the appendix)

$$\lambda_{\mathbb{M}^4}(W_{\mathbb{M}^4}(\phi)) := e^{-\langle \phi_+, \phi_+ \rangle_{\mathbb{M}^4}/2} \,. \tag{101}$$

We can now state and prove the main results of this section.

Theorem 4.1. *Consider free QFT for a real scalar field ϕ propagating in four-dimensional Minkowski spacetime $(\mathbb{M}^4, \eta)$ and QFT for a real scalar field on $\Im^+$. Let $\mathcal{W}(\mathbb{M}^4)$ be the Weyl algebra associated with the space $\mathcal{S}_K(\mathbb{M}^4)$ and the symplectic form $\sigma_{\mathbb{M}^4}$ as defined in section 4.2 (specialized to the case $M_{\tilde{V}} := \mathbb{M}^4$ and $K := -\Box$). The following holds.*
(a) $\Gamma_{\mathbb{M}^4}(\mathcal{S}_K(\mathbb{M}^4)) \subset \mathcal{S}(\Im^+)$ *because $\Gamma_{\mathbb{M}^4}\phi$ has compact support for $\phi \in \mathcal{S}_K(\mathbb{M}^4)$, moreover $\Gamma_{\mathbb{M}^4}$ preserves symplectic forms. As a consequence $\mathcal{W}(\mathbb{M}^4)$ can be identified with a sub C^*-algebra of $\mathcal{W}(\Im^+)$ by means of a C^*-algebra isomorphism $\imath_{\mathbb{M}^4}$ uniquely determined by the requirement*

$$\imath_{\mathbb{M}^4}(W_{\mathbb{M}^4}(\phi)) = W(\Gamma_{\mathbb{M}^4}\phi) \,, \quad \textit{for all } \phi \in \mathcal{S}_K(\mathbb{M}^4) \,. \tag{102}$$

(b) *Consider Minkowski vacuum $\lambda_{\mathbb{M}^4}$ on $\mathcal{W}(\mathbb{M}^4)$ and the BMS-invariant vacuum λ on $\mathcal{W}(\Im^+)$ and focus on the respectively associated GNS realizations $(\mathfrak{F}(\mathcal{H}_{\mathbb{M}^4}), \Pi_{\mathbb{M}^4}, \Upsilon_{\mathbb{M}^4})$ and $(\mathfrak{F}(\mathcal{H}), \Pi, \Upsilon)$. The C^*-algebra isomorphism $\imath_{\mathbb{M}^4}$ corresponds to a unitary (i.e. isometric surjective) operator $\mathcal{U} : \mathfrak{F}(\mathcal{H}_{\mathbb{M}^4}) \to \mathfrak{F}(\mathcal{H})$ such that*
(i) $\mathcal{U} : \Upsilon_{\mathbb{M}^4} \mapsto \Upsilon$,
(ii) $\mathcal{U}\widehat{W}_{\mathbb{M}^4}(\phi)\mathcal{U}^{-1} = \widehat{W}(\Gamma_{\mathbb{M}^4}\phi)$.
Therefore the algebraic state induced by λ on $\mathcal{W}(\mathbb{M}^4)$ through $\imath_{\mathbb{M}^4}$ is Minkowski vacuum $\lambda_{\mathbb{M}^4}$.

Proof. (a) Fix a Minkowski reference frame $(t, \mathbf{x})$ in $\mathbb{M}^4$, pass to spherical coordinates in the rest frame obtaining coordinates $(t, r, \zeta, \overline{\zeta})$, next pass to null coordinates in the sector t, r and, finally, construct coordinates $(u, \zeta, \overline{\zeta})$ on $\Im^+$ referred to a Bondi frame as described at the beginning of this section. In Minkowski spacetime solutions of $K\phi = 0$ propagate along null geodesics [56]. In other words, if $\phi = Ef$, the support of ϕ is included in the union of null geodesics originated from every point $q \in supp f$. On the the hand the map $u = Z(q, \zeta, \overline{\zeta})$ that associates the unique null geodesics starting from the point $q \in \mathbb{M}^4$ and direction $(\zeta, \overline{\zeta})$ with the coordinate u where the geodesics reaches $\Im^+$ (the remaining coordinates being $(\zeta, \overline{\zeta})$) is well defined and smooth [57, 58]. If $\phi \in \mathcal{S}_K(\mathbb{M}^4)$, $\phi = Ef$ where f is smooth with compact support, as a consequence $supp\, \Gamma_{\mathbb{M}^4}\phi \subset \{Z(q, \zeta, \overline{\zeta}) \mid q \in supp f \,, (\zeta, \overline{\zeta}) \in \mathbb{S}^2\} \times \mathbb{S}^2$ is compact because Z is continuous and defined on a compact set. Since $\Gamma_{\mathbb{M}^4}\phi$ is smooth by definition, we have proved that $\Gamma_{\mathbb{M}^4}(\mathcal{S}_K(\mathbb{M}^4)) \subset \mathcal{S}(\Im^+)$. Now we pass to prove that $\Gamma_{\mathbb{M}^4}$ preserves the symplectic forms. To this end we notice that, if $\phi, \phi' \in \mathcal{S}_{\mathbb{M}^4}$ then $\sigma_{\mathbb{M}^4}(\phi, \phi') = i2Re\langle \phi_+, \phi'_+ \rangle_{\mathbb{M}^4}$ and the analog holds for wavefunctions $\psi, \psi' \in \mathcal{S}(\Im^+)$ referring to the corresponding symplectic form σ and scalar product $\langle \cdot, \cdot \rangle$ as in Theorem 2.2. (The proof is immediate, taking into account the fact that positive frequency parts satisfy $\sigma_{\mathbb{M}^4}(\phi_+, \phi'_+) = 0$ and the analog for the other case.) As a consequence, to show that $\sigma_{\mathbb{M}^4}(\phi, \phi') = \sigma(\Gamma_{\mathbb{M}^4}\phi, \Gamma_{\mathbb{M}^4}\phi')$, it is *completely equivalent* to show that

$$\langle \phi_+, \phi'_+ \rangle_{\mathbb{M}^4} = \langle (\Gamma_{\mathbb{M}^4}\phi)_+, (\Gamma_{\mathbb{M}^4}\phi')_+ \rangle \;, \quad \text{for every pair of wavefunctions } \phi, \phi' \in \mathcal{S}_{\mathbb{M}^4}. \tag{103}$$

Notice that the positive frequency parts in the left-hand side are referred to Minkowski time t in $\mathbb{M}^4$, whereas those in the right-hand side are referred to coordinate u in $\Im^+$. Proof of (103) is a consequence of the following lemma whose proof is quite technical and presented in the Appendix.

Lemma 4.1. *In the hypotheses of theorem 4.1, fix a Minkowski reference frame $(t, \mathbf{x})$ in $\mathbb{M}^4$, and consider the associated Bondi frame $(u, \zeta, \overline{\zeta})$ on $\Im^+$. If $(E, \zeta, \overline{\zeta})$ are the spherical coordinates of $\mathbf{p}$ in the rest frame (where $E := |\mathbf{p}|$ in particular), it holds*

$$\widetilde{(\Gamma_{\mathbb{M}^4}\phi)}_+(E, \zeta, \overline{\zeta}) = -iE\widetilde{\phi_+}(\mathbf{p}(E, \zeta, \overline{\zeta})) \;, \quad \textit{for all } \phi \in \mathcal{S}_{\mathbb{M}^4}, \tag{104}$$

the function in the left-hand side being that of definition (19) with $\mathcal{S}(\Im^+) \ni \psi = \Gamma_{\mathbb{M}^4}\phi$.

From (104) one proves (103). Indeed, starting from (98), passing in spherical coordinates in the integral in the right-hand side and taking (22) into account, one gets (103)

$$\langle \phi_+, \phi'_+ \rangle_{\mathbb{M}^4} = \int_{\mathbb{R}^+ \times \mathbb{S}^2} dE E^2 \, \epsilon_{\mathbb{S}^2} \, \overline{\widetilde{\phi_+}(\mathbf{p}(E, \zeta, \overline{\zeta}))} \widetilde{\phi'_+}(\mathbf{p}(E, \zeta, \overline{\zeta}))$$

$$= \int_{\mathbb{R}^+ \times \mathbb{S}^2} dE \epsilon_{\mathbb{S}^2} \, \overline{-iE\widetilde{\phi_+}(\mathbf{p}(E, \zeta, \overline{\zeta}))} \, (-i)E\widetilde{\phi'_+}(\mathbf{p}(E, \zeta, \overline{\zeta})) = \langle (\Gamma_{\mathbb{M}^4}\phi)_+, (\Gamma_{\mathbb{M}^4}\phi')_+ \rangle \;.$$

(b) Referring to lemma 4.1, start from the $\mathbb{C}$-linear isometric map $h_0 : \mathcal{S}_K(\mathbb{M}^4)^{\mathbb{C}}_+ \to \mathcal{S}(\Im^+)^{\mathbb{C}}_+$ which associates the function $\widetilde{\phi_+}(\mathbf{p})$ with the function $-iE\widetilde{\phi_+}(\mathbf{p}(E, \zeta, \overline{\zeta})) = \widetilde{(\Gamma_{\mathbb{M}^4}\phi)}_+(E, \zeta, \overline{\zeta})$. The domain and the range of that map are dense in $\mathcal{H}_{\mathbb{M}^4}$ and $\mathcal{H}$ respectively: In the first case it has been proved previously, the proof for the latter case is immediate using the density property in the former and the measures in the relevant L^2 spaces corresponding to the two Hilbert spaces. As a consequence, h_0 extends to a unitary map $h : \mathcal{H}_{\mathbb{M}^4} \to \mathcal{H}$. In turn, this second map extends to a unitary map $\mathcal{U} : \mathfrak{F}(\mathcal{H}_{\mathbb{M}^4}) \to \mathfrak{F}(\mathcal{H})$ by tensorialization and assuming that (i) $\mathcal{U}\Upsilon_{\mathbb{M}^4} = \Upsilon$. By construction it also holds $\mathcal{U}\sigma_{\mathbb{M}^4}(\phi, \Phi)\mathcal{U}^{-1} = \Psi(\Gamma_{\mathbb{M}^4}\phi)$ working in the dense space of analytic vectors containing a finite number of particles. Passing to exponentials one finds (ii). $\Box$

The second theorem concerns the interplay of orthocronous proper Poincaré group $ISO(3,1)$ and G_{BMS}. We knows that in the bulk there is a strongly-continuous unitary irreducible representation $ISO(3,1) \ni g \mapsto L_g$ satisfying (100). Referring to the Minkowski frame $(t, \mathbf{x})$ used to build up the metric on $\Im^+$ and all that, if $g = (\Lambda, T)$ with $\Lambda \in SO(3,1)\Uparrow$ and $a \in \mathbb{R}^4$, the action of L_g on a positive frequency part $\widetilde{\phi}_+$ reads

$$\left(L_{(\Lambda,a)}\widetilde{\phi}_+\right)(\mathbf{p}) = \sqrt{\frac{E_{\Lambda^{-1}}}{E}} e^{-i(p|\Lambda a)}\widetilde{\phi}_+(\mathbf{p}_{\Lambda^{-1}}) \,, \tag{105}$$

where $p := (E, \mathbf{p})$, $(E_\Lambda, \mathbf{p}_\Lambda) := \Lambda p$, whereas $(a|b)$ denotes the standard product of 4-vectors a and b. The question is: *what is the meaning of the representation $ISO(3,1) \ni g \mapsto \mathcal{U}L_g\mathcal{U}^{-1}$ acting on quantum states for QFT defined in $\Im^+$?*
The following theorem gives an answer to the question which is the quantum version of Proposition 4.2.

Theorem 4.2. *With the same hypotheses as in Theorem 4.1, represent G_{BMS} as the semidirect product of $ISO(3,1)$ and $C^\infty(\mathbb{S}^2)$ in the Bondi frame on $\Im^+$ associated with the Minkowski frame $(t, \mathbf{x})$. Consider the natural unitary representation of $ISO(3,1)$ in QFT in $(\mathbb{M}^4, \eta)$ given in (105). The representation on $\mathfrak{F}(\mathcal{H})$, induced on QFT on $\Im^+$ by means of $\mathcal{U}$, is $ISO(3,1) \ni g \mapsto \mathcal{U}L_g\mathcal{U}^{-1}$ and it coincides with the restriction of the representation of G_{BMS}, U, defined in Theorem 2.3, to the subgroup isomorphic to $ISO(3,1)$ (see proposition 4.2)*

$$\left\{(\Lambda, f_a) \in G_{BMS} \,\middle|\, f_a = a^0 - \frac{a^1(\zeta+\overline{\zeta})}{\zeta\overline{\zeta}+1} - \frac{a^2(\zeta-\overline{\zeta})}{i(\zeta\overline{\zeta}+1)} - \frac{a^3(\zeta\overline{\zeta}-1)}{\zeta\overline{\zeta}+1}\,,\ (\Lambda, a) \in ISO(3,1)\right\}$$

Proof. By lemma 4.1 one has

$$\left(\mathcal{U}\widetilde{\phi_+}\right)(\mathbf{p}, \zeta, \overline{\zeta}) = -iE\widetilde{\phi_+}(\mathbf{p}(E, \zeta, \overline{\zeta}))\,. \tag{106}$$

Representing the right-hand side of (105) in complex spherical coordinates $\zeta, \overline{\zeta}$ and applying $\mathcal{U}$ on the final result making use of (106), a straightforward, but tedious, computation based on (54) proves that, for every $\widetilde{\psi_+} \in \mathcal{U}(\mathcal{H}_{\mathbb{M}^4})$, $\mathcal{U}L_{(\Lambda,a)}\mathcal{U}^{-1}\widetilde{\psi_+} = U_{(\Lambda,f_a)}\widetilde{\psi_+}$ holds true whenever (Λ, a) is any pure translation, any pure rotation and any boost along z. Hence the decomposition theorem of Lorentz group and the structure of the group product in $ISO(3,1)$ and in G_{BMS} imply that the identity holds for every element $(\Lambda, a) \in ISO(3,1)$. Since $\mathcal{U}(\mathcal{H}_{\mathbb{M}^4}) \subset \mathcal{H}$ is dense in $\mathcal{H}$ (and $\mathcal{U}$ preserve one-particle spaces), we have obtained that $\mathcal{U}L_{(\Lambda,a)}\restriction_{\mathcal{H}} \mathcal{U}^{-1} = U_{(\Lambda,f_a)}\restriction_{\mathcal{H}}$. Finally, since $\mathcal{U} : \mathfrak{F}(\mathcal{H}_{\mathbb{M}^4}) \to \mathfrak{F}(\mathcal{H})$, $L_{(\Lambda,a)} : \mathfrak{F}(\mathcal{H}_{\mathbb{M}^4}) \to \mathfrak{F}(\mathcal{H}_{\mathbb{M}^4})$ and $U_{(\Lambda,f_a)} : \mathfrak{F}(\mathcal{H}) \to \mathfrak{F}(\mathcal{H})$ are all obtained by tensorialization procedure, it must hold $\mathcal{U}L_{(\Lambda,a)}\mathcal{U}^{-1} = U_{(\Lambda,f_a)}$. $\Box$

5 Conclusions

The main purpose underlying this paper has been to show that, at least in the scalar case, it is possible to start from a scalar free field ϕ living in the bulk of an asymptotically flat four-dimensional spacetime M and to relate it by means of a suitable extension/restriction procedure with a second field ψ living on $\Im^+$, the boundary of M at future null infinity. Under suitable hypotheses (preservation of a symplectic form), this relation preserves information at level of quantum field theories when passing from the bulk to the boundary thus implementing the holographic principle. This has been proved at level of Weyl C^* algebras associated with the fields. Within this framework, ψ is interpreted as a kinematical datum of a quantum field theory intrinsically defined on $\Im^+$ and invariant under the action of the BMS group as discussed in section 1. We have shown that such physical intuitive idea can be made rigorously precise identifying ψ with an intrinsic BMS field constructed out of the induced unitary irreducible representations. Such result has been achieved by means of a technology similar to celebrated Wigner's one used to classify and construct explicitly all possible Poincaré-invariant wavefunctions. Universality of such an approach and the techniques handled in section 2 and 3 suggest that our results, achieved for massless fields, may be extended far beyond the case of vanishing "spin". Furthermore it would be interesting to investigate the interplay of these results with the *asymptotic quantization* procedure proposed by Ashtekar [30] where the main variable is played by BMS-invariant *gauge* massless fields living on $\Im^\pm$. To this end it is worth noticing that, in [30] and in most of the paper concerning applications of the BMS group, the peculiar role played by the unitary BMS irreducible representation induced from the subgroup Γ (instead of our Δ), suggests that it has been always implicitly assigned an Hilbert topology to the set of supertranslations N. This is in apparent contrast with our results and the issue deserves future investigation. This is because the results presented in section 3 indicates that, in order to "relate" a bulk field with the boundary BMS-invariant counterpart, it is necessary to adopt a nuclear topology on N. The relevance of this result does not only lie in the realm of a rigorous mathematical analysis of the BMS group, but it mainly affects the physical kinematical configuration of the field theory living on $\Im^+$ since, as discussed in [28] and partly in section 3, in the "nuclear" scenario, it arises a plethora of possible free fields (or equivalently little groups) which are *not* present in the Hilbert topology.

A complete survey of the bulk to boundary relation for free fields should also comprise the rather elusive case of massive fields. Within this specific framework, the extension/restriction procedure proposed in section 2 fails mainly due to the presence of an intrinsic scale length represented by the mass. Nonetheless we believe that an "holographic investigation" along the lines proposed in [11] is still possible and it is currently under investigation.

Other key results of this paper appear in section 4 where the holographic interplay between a bulk theory living on a strongly asymptotically predictable spacetime and the BMS boundary theory has been discussed within the framework of C^* algebras of field-observables and their isomorphisms. In particular, in the specific scenario of Minkowski spacetime, a key achievement consists on establishing an unitary correspondence between the bulk vacuum and the BMS counterpart on $\Im^+$, though the uniqueness of the latter has not been proved and it should be analyzed in detail. The uniqueness problem of a BMS-invariant quasifree (algebraic) state λ on $\Im^+$ has relevance in the issue of the notion of particle in the absence of Poincaré group. If the BMS-invariant quasifree state is uniquely determined, it could be used to give a definition of particle for spacetime which does not admit a group of isometries but are asymptotically flat and the algebra of the field in the bulk can be identified with a subalgebra of the fields on $\Im^+$ by means of an injective $*$-homomorphism $\imath$ as in proposition 4.1 In this case, λ induces a quasifree state $\lambda_\imath$ for the algebra of fields in the bulk with an associated definition of particle. We have established in theorems 4.1 and 4.2 that such a notion of particle, whenever available, must agree with the usual one in four dimensional Minkowski spacetime since $\lambda_\imath$, in that case, is just Minkowski vacuum. Another relevant point which deserves investigation concerns the short distance behaviour of n-point functions associated with $\lambda_\imath$. There is no evidence, from our construction, that $\lambda_\imath$ is Hadamard also if it happens in Minkowski spacetime trivially.

To conclude, we wish also to pinpoint that, within this paper, we have completely discarded the role of interactions. Nonetheless, in order to construct a full holographic bulk to $\Im^\pm$ correspondence it is imperative to understand how to couple the boundary free field either with self/external interactions (barring gravitational field) either with gauge degrees of freedom. A complete and concrete solution of this challenging issue would possibly rule out whether it is really possible or not to define a full asymptotically flat/BMS correspondence and, thus, we believe it is worth to be deeply analyzed.

A Appendix

A.1. *GNS reconstruction.* The interplay of the Fock representation presented in section 2.4 and GNS theorem [55, 59] is simply sketched. (The same extent holds for QFT in Minkowski spacetime presented in section 4.4 if replacing $\mathcal{W}(\Im^+)$ with $\mathcal{W}(\mathbb{M}^4)$, $W(\psi)$ with $W_{\mathbb{M}^4}(\phi)$, Π with $\Pi_{\mathbb{M}^4}$, $\Psi(\psi)$ with $\sigma_{\mathbb{M}^4}(\phi,\Phi)$ and λ with $\lambda_{\mathbb{M}^4}$.) Using notation introduced in section 2.4, if $\Pi : \mathcal{W}(\Im^+) \to \widehat{\mathcal{W}}(\Im^+)$ denotes the unique (σ being nondegenerate) C^*-algebra isomorphism between those two Weyl representations, it turns out that $(\mathfrak{F}_+(\mathcal{H}), \Pi, \Upsilon)$ is the GNS triple associated with a particular pure algebraic state λ (*quasifree* [59] and invariant under the automorphism group associated with G_{BMS}) on $\mathcal{W}(\Im^+)$ we go to introduce. Define

$$\lambda(W(\psi)) := e^{-\langle\psi_+,\psi_+\rangle/2}$$

then extend λ to the *-algebra finitely generated by all the elements $W(\psi)$ with $\psi \in \mathcal{S}(\Im^+)$, by linearity and using (W1), (W2). It is simply proved that, $\lambda(\mathbb{I}) = 1$ and $\lambda(a^*a) \geq 0$ for every element a of that *-algebra so that λ is a state. As the map $\mathbb{R} \ni t \mapsto \lambda(W(t\psi))$ is continuous, known theorems [60] imply that λ extends uniquely to a state λ on the complete Weyl algebra $\mathcal{W}(\Im^+)$. On the other hand, by direct computation, one finds that $\lambda(W(\psi)) = \left\langle \Upsilon, \widehat{W}(\psi)\Upsilon \right\rangle$. Since a state on a C^* algebra is continuous, this relation can be extended to the whole algebras by linearity and continuity and using (W1), (W2) so that a general GNS relation is verified:

$$\lambda(a)\,\langle \Upsilon, \Pi(a)\Upsilon\rangle \quad \text{for all } a \in \mathcal{W}(\Im^+)\,. \tag{107}$$

To conclude, it is sufficient to show that Υ is cyclic with respect to Π. Let us show it. If $\widehat{\mathcal{F}}(\Im^+)$ denotes the *-algebra generated by field operators $\Psi(\psi)$, $\psi \in \mathcal{S}(\Im^+)$, defined on $F(\mathcal{H})$, $\widehat{\mathcal{F}}(\Im^+)\Upsilon$ is dense in the Fock space (see proposition 5.2.3 in [32]). Let $\Phi \in \mathfrak{F}_+(\mathcal{H})$ be a vector orthogonal to both Υ and to all the vectors $\widehat{W}(t_1\psi_1)\cdots\widehat{W}(t_n\psi_n)\Upsilon$ for $n = 1, 2, \ldots$ and $t_i \in \mathbb{R}$ and $\psi_i \in \mathcal{S}(\Im^+)$. Using Stone theorem to differentiate in t_i for $t_i = 0$, starting from $i = n$

and proceeding backwardly up to $i = 1$, one finds that Φ must also be orthogonal to all of the vectors $\Psi(\psi_1)\cdots\Psi(\psi_n)\Upsilon$ and thus vanishes because $\widehat{\mathfrak{F}}(\Im^+)\Upsilon$ is dense. This result means that $\Pi(\mathcal{W}(\Im^+))\Upsilon$ is dense in the Fock space too, i.e. Υ is cyclic with respect to Π. Since Υ satisfies also (107), the uniqueness of the GNS triple proves that the triple $(\mathfrak{F}_+(\mathcal{H}), \Pi, \Upsilon)$ is just (up to unitary transformations) the GNS triple associated with λ. Since the Fock representation is irreducible, λ is pure.

A.2. *Proof of some propositions.*
Proof of Proposition 2.4. In the following we assume that Ω includes the further factor ω. Referring to the expression of BMS group in a Bondi frame, we prove the thesis for any element (Λ, f) of BMS group of the form either $(\Lambda(t), 0)$ or (I, tf) where $f \in C^\infty(\mathbb{S}^2)$ and $t \to \Lambda(t)$ is a one-parameter subgroup of $SO(3,1)\uparrow$. Notice that the subgroups $t \mapsto (\Lambda(t), 0)$ and $t \mapsto (I, tf)$ are also one-parameter group of diffeomorphisms of $\Im^+$ generated by a smooth vector fields ξ' on $\Im^+$ as in Proposition 2.1 as one may check by direct inspection. ¿From decomposition theorem of Lorentz group, it is simply proved that every element of G_{BMS} is a finite product of those elements $(\Lambda(t), 0)$ and (I, tf). Hence, using the property (6), the thesis turns out to be valid for a generic element of BMS group.
Assume that (A, f) is an element of the one-parameter group of $\Im^+$-diffeomorphisms $\{\gamma'_t\}$ generated by ξ' and let ξ be a smooth extension of ξ to M (i.e. $\tilde{M}$) as in Proposition 2.1 generating $\{\gamma_t\}$. In coordinates $(\Omega, u, \zeta, \overline{\zeta})$ about $\Im^+$, (13) can be written down

$$(A_{\gamma'_t}\psi)(\Omega_t, u_t, \zeta_t, \overline{\zeta}_t) = \lim_{\Omega_t \to 0} \Omega_t^\alpha \phi_t(\Omega_t, u_t, \zeta_t, \overline{\zeta}_t), \tag{108}$$

where $\gamma_t : (\Omega, u, \zeta, \overline{\zeta}) \to (\Omega_t, u_t, \zeta_t, \overline{\zeta}_t)$ and $\phi_t := \gamma_t^* \phi$ so that

$$\phi_t(\Omega_t, u_t, \zeta_t, \overline{\zeta}_t) = \phi(\Omega, u, \zeta, \overline{\zeta}) .$$

(108) can be re-written

$$(A_{\gamma'_t}\psi)(u_t, \zeta_t, \overline{\zeta}_t) = \lim_{\Omega_t \to 0} \frac{\Omega_t^\alpha}{\Omega^\alpha} \, \Omega^\alpha \phi(\Omega, u, \zeta, \overline{\zeta}) ,$$

that is, since on $\Im^+$ γ_t coincides with γ'_t which preserves $\Im^+$ itself,

$$(A_{\gamma'_t}\psi)(u_t, \zeta_t, \overline{\zeta}_t) = \left(\lim_{\Omega \to 0} \frac{\Omega_t}{\Omega}\right)^\alpha \psi(u, \zeta, \overline{\zeta}) . \tag{109}$$

Using Hôpital rule

$$(A_{\gamma'_t}\psi)(u_t, \zeta_t, \overline{\zeta}_t) = \left(\frac{\partial \Omega_t}{\partial \Omega}|_{\Omega=0}\right)^\alpha \psi(u, \zeta, \overline{\zeta}) . \tag{110}$$

Our task is computing the derivative in the right-hand side of (110). By definition of ξ one finds

$$\frac{d}{dt}\left(\frac{\partial \Omega_t}{\partial \Omega}|_{(\Omega=0,u,\zeta,\overline{\zeta})}\right) = \frac{\partial \xi^\Omega(\gamma_t(\Omega, u, \zeta, \overline{\zeta}))}{\partial \Omega}|_{\Omega=0} . \tag{111}$$

Now making explicit the condition that $(\Omega^2 \pounds_\xi g)_{\alpha\beta}$ extends smoothly to a vanishing field approaching $\Im^+$ (Proposition 2.1) in the considered coordinates, one easily finds for components $\alpha = \Omega, \beta = u$:

$$\frac{\partial \xi^\Omega(\Omega, u, \zeta, \overline{\zeta})}{\partial \Omega}|_{\Omega=0} = -\frac{\partial \xi'^u(u, \zeta, \overline{\zeta})}{\partial u} ,$$

where we also used $\xi' = \xi$ on $\Im^+$. Finally, from (111)

$$\frac{d}{dt} \ln \left| \frac{\partial \Omega_t}{\partial \Omega}|_{(\Omega=0,u,\zeta,\overline{\zeta})} \right| = -\frac{\partial \xi'^u(u_t, \zeta_t, \overline{\zeta}_t)}{\partial u_t}|_{(u_t,\zeta_t,\overline{\zeta}_t)=\gamma'_t(u,\zeta,\overline{\zeta})} . \tag{112}$$

Let us solve this equation in the relevant cases. By direct inspection one finds that the right hand side vanishes when the one-parameter subgroup $\{\gamma'_t\}$ generated by ξ' has the form $t \mapsto (I, tf)$ and so $\frac{\partial \Omega_t}{\partial \Omega}|_{(\Omega=0,u,\zeta,\overline{\zeta})}$ = constant in this case. Since $\frac{\partial \Omega_0}{\partial \Omega}|_{(\Omega=0,u,\zeta,\overline{\zeta})} = 1$, (110) produces

$$(A_{\gamma'_t}\psi)(u_t, \zeta_t, \overline{\zeta}_t) = \psi(u, \zeta, \overline{\zeta}) ,$$

which is just the thesis in the considered case. Let us consider the other case with γ'_t having the form $t \mapsto (\Lambda_t, 0)$. In this case one gets

$$\xi^u(u_t, \zeta_t, \overline{\zeta}_t) = \frac{u_t}{K_{\Lambda_t}(\Lambda_t^{-1}(\zeta_t, \overline{\zeta}_t))} \left(\frac{dK_{\Lambda_t}(\zeta, \overline{\zeta})}{dt} \right)|_{(\zeta,\overline{\zeta})=\Lambda_t^{-1}(\zeta_t,\overline{\zeta}_t)} u \frac{d \ln |K_{\Lambda_t}(\zeta, \overline{\zeta})|}{dt}|_{(\zeta,\overline{\zeta})=\Lambda_t^{-1}(\zeta_t,\overline{\zeta}_t)} .$$

From (112), using the fact that $\frac{\partial \Omega_0}{\partial \Omega}|_{(\Omega=0,u,\zeta,\overline{\zeta})} = 1$, one finds at the end via (110):

$$(A_{\gamma'_t}\psi)(u_t, \zeta_t, \overline{\zeta}_t) = K_{\Lambda_t}(\zeta, \overline{\zeta})^{-\alpha} \psi(u, \zeta, \overline{\zeta}) ,$$

which is the thesis in the considered case. $\Box$

Proof Theorem 2.2. (a) and (b). Take $\psi \in \mathcal{S}(\Im^+)$. Using integration by parts in (19) and standard theorem (Lesbegue's dominate convergence) to interchange the symbol of derivative with that of integral, it is simply proved that, if $\psi \in \mathcal{S}(\Im^+)$, $(E, \zeta, \overline{\zeta}) \mapsto \widetilde{\psi_+}(E, \zeta, \overline{\zeta})/\sqrt{E}$ belongs to $C^\infty(\mathbb{R}^+ \times \mathbb{S}^2; \mathbb{C})$ and, as $E \to +\infty$, it decays, uniformly in $\zeta, \overline{\zeta}$ with all derivatives in any variable, faster than any negative power of E. Using the same procedure in (18), one finds straightforwardly that $\zeta, \overline{\zeta}$ uniform estimates hold for ψ_+:

$$\left| \frac{\partial^k}{\partial u^k} \frac{\partial^c}{\partial \zeta^c} \frac{\partial^d}{\partial \overline{\zeta}^d} \psi_+(u, \zeta, \overline{\zeta}) \right| \leq \frac{C_{k,c,d}}{1 + |u|^{k+1}} \tag{113}$$

for nonnegative constants $C_{k,c,d}$ depending on $k, c, d = 0, 1, 2, \ldots$. Therefore it make sense to apply σ defined in (15) to a pair of positive frequency parts ψ_{1+}, ψ_{2+} when $\psi_1, \psi_2 \in \mathcal{S}(\Im^+)$. The independence from the used Bondi frame can be proved by direct inspection using (20), Proposition 2.2 to check on the independence from the used Bondi frame and taking advantage of the fact that $\epsilon_{\mathbb{S}^2}(\zeta, \overline{\zeta})$ is invariant under three dimensional rotations.

Let us prove the item (b). In the following we use the notation $\psi'(u,\zeta,\overline{\zeta}) := (A_{(\Lambda,f)}\psi)(u,\zeta,\overline{\zeta})$. Finally (22) is a straightforward application of Fubini-Tonelli theorem in the explicit expression for $\Omega(\overline{\psi_{1+}},\psi_{2+})$, the hypotheses being fulfilled due to the decaying estimates said above, using (18) (take into account that actually the apparent singularity due to the factor $E^{-1/2}$ does not exist because of (19) where the integral produces a smooth function in E). The remaining part of (b) is an immediate consequence of (22). Let us prove (c). First of all notice that the map $\psi_+ \mapsto \widetilde{\psi_+}$ for $\psi \in \mathcal{S}(\Im^+)$ is well-defined because the map $\widetilde{\psi_+} \mapsto \psi_+$ is injective. The proof follows straightforwardly from injectivity of Fourier transformation in Schwartz space referring to Fourier transform involved in (19) and using the fact that ψ is real. By (21) the complex linear extension of $\psi_+ \mapsto \widetilde{\psi_+}$ is bounded and thus, it being defined in a dense subspace, it admits a unique bounded extension from the completion of $\mathcal{S}(\Im^+)^{\mathbb{C}}_+$ to a closed subspace of $L^2(\mathbb{R}^+\times\mathbb{S}^2, dE\otimes\epsilon_{\mathbb{S}^2})$. To prove the thesis it is sufficient to show that the subspace includes $C_c^\infty((0,+\infty)\times\mathbb{S}^2;\mathbb{C})$ because the latter is dense in $L^2(\mathbb{R}^+\times\mathbb{S}^2, dE\otimes\epsilon_{\mathbb{S}^2})$. To this end, take $\phi\in C_c^\infty((0,+\infty)\times\mathbb{S}^2;\mathbb{C})$ and define ψ as:

$$\psi(u,\zeta,\overline{\zeta}) := \int_{\mathbb{R}^+}\frac{dE}{\sqrt{4\pi E}}e^{-iEu}\phi(E,\zeta,\overline{\zeta}) + \int_{\mathbb{R}^+}\frac{dE}{\sqrt{4\pi E}}\overline{e^{-iEu}\phi(E,\zeta,\overline{\zeta})}\,.$$

Notice that the singularity of $E^{-1/2}$ at $E=0$ is harmless since the support of ϕ does not include that point and thus the whole integrand is smooth and compactly supported. Finally, by direct inspection, one finds that $\psi\in\mathcal{S}(\Im^+)$ and $\widetilde{\psi_+}=\phi$. This concludes the proof. $\Box$

Proof of Theorem 2.4. As is well-known working with group representations, to prove the thesis it is sufficient to show that strong continuity holds for $g\to I$ (the unit element of G_{BMS}). Let us to prove strong continuity as $g\to I$ for the restriction of the representation U to $\mathcal{H}$. To this end we prove, as the first step, the strong continuity of U when it works on one-particle states represented by smooth compactly supported functions $\tilde{\phi}(E,\zeta,\overline{\zeta})$. (In the following, for sake of simplicity, we write $\zeta,\overline{\zeta}$ concerning coordinates on $\mathbb{S}^2$, but actually one needs at least two charts to cover the compact smooth manifold $\mathbb{S}^2$. The use of two charts removes the apparent singularity of the coordinates $\zeta,\overline{\zeta}$ on the point ∞ of the Riemann sphere.) Using the fact that every U_g is unitary, one sees that $||U_g\tilde{\phi}-\tilde{\phi}||\to 0$ as $g\to I$ is equivalent to $(\tilde{\phi},U_g\tilde{\phi})\to(\tilde{\phi},\tilde{\phi})$ as $g\to I$. With an explicit representation (by means of (31)) we have to prove that, as $g\to I$ and for a smooth compactly supported $\tilde{\phi}$,

$$\lim_{(\Lambda,f)\to(I,0)}\int_{\mathbb{R}^+\times\mathbb{S}^2}\sqrt{K_\Lambda(\zeta,\overline{\zeta})}\,e^{iEf(\zeta,\overline{\zeta})}\overline{\widetilde{\psi}\left(\frac{E}{K_\Lambda(\zeta\,\overline{\zeta})},\Lambda(\zeta,\overline{\zeta})\right)}\,\widetilde{\psi}(E,\zeta,\overline{\zeta})\,dE\otimes\epsilon_{\mathbb{S}^2}(\zeta,\overline{\zeta})$$
$$=\int_{\mathbb{R}^+\times\mathbb{S}^2}\overline{\widetilde{\psi}(E,\zeta,\overline{\zeta})}\widetilde{\psi}(E,\zeta,\overline{\zeta})\,dE\otimes\epsilon_{\mathbb{S}^2}(\zeta\,\overline{\zeta})\,. \tag{114}$$

Taking Λ in a relatively compact neighborhood B of the unit element of $SO(3,1)\!\uparrow$, (for any fixed f) the smooth compactly supported map

$$(\Lambda,E,\zeta\,\overline{\zeta})\mapsto\left|e^{iEf(\zeta,\overline{\zeta})}K_\Lambda(\zeta\,\overline{\zeta})\,\overline{\widetilde{\psi}\left(\frac{E}{K_\Lambda(\zeta,\overline{\zeta})},\Lambda(\zeta,\overline{\zeta})\right)}\,\widetilde{\psi}(E,\zeta,\overline{\zeta})\right|$$

is bounded by construction by some constant K not depending on f (which does not give contribution to the considered functions since it is real valued). On the other hand, there is a compact $C \subset \mathbb{R}^+ \times \mathbb{S}^2$ containing all the supports of the maps

$$(E, \zeta\,\overline{\zeta}) \mapsto \left| e^{iEf(\zeta,\overline{\zeta})} K_\Lambda(\zeta\,\overline{\zeta})\, \overline{\widetilde{\psi}\left(\frac{E}{K_\Lambda(\zeta,\overline{\zeta})}, \Lambda(\zeta,\overline{\zeta})\right)}\, \widetilde{\psi}(E,\zeta,\overline{\zeta}) \right|,$$

for all $\Lambda \in B$ and all $f \in C^\infty(\mathbb{S}^2)$. As a consequence all those maps are (Λ, f)-uniformly bounded by a smooth compactly supported function on $\mathbb{R}^+ \times \mathbb{S}^2$ which assumes the value K in C. Thus we can use Lebesgue's dominate convergence theorem in the right-hand of (114) establishing the validity of (114) itself. We have proved strong continuity on smooth compactly supported functions in $\mathcal{H}$. As the space of those functions is dense in $\mathcal{H}$, it implies strong continuity on the whole $\mathcal{H}$. Indeed, if $\phi \in \mathcal{H}$, by triangular inequality

$$||\phi - U_g\phi|| \leq ||\phi - \phi_n|| + ||\phi_n - U_g\phi_n|| + ||U_g(\phi_n - \phi)||2||\phi - \phi_n|| + ||\phi_n - U_g\phi_n||\,.$$

Therefore $\lim_{g\to I} ||\phi - U_g\phi|| \leq 2||\phi-\phi_n|| + \lim_{g\to I} ||\phi_n - U_g\phi_n||$. Taking ϕ_n smooth and compactly supported with $\phi_n \to \phi$ for $n \to +\infty$, one gets $\lim_{g\to I} ||\phi - U_g\phi|| = 0$, i.e. strong continuity holds for $U\restriction_{\mathcal{H}}$.
To conclude the proof we show that the strong continuity in $\mathcal{H}$ implies strong continuity in the whole Fock space. By construction, if $V_g := U\restriction_{\mathcal{H}}$, on the U invariant subspace $\mathcal{H}^N \subset \mathfrak{F}_+(\mathcal{H})$ containing N particles one has $V_g^{(N)} := U\restriction_{\mathcal{H}^N} = V_g \otimes \cdots \otimes V_g$ where the number of factors is N. (Obviously $V_g^{(0)} := I$.) As a consequence $g \mapsto V_g^{(N)}$ is strongly continuous. Now consider a generic element of $\mathfrak{F}_+(\mathcal{H})$ which can be viewed as a sequence $\Phi = \{\Psi_N\}_{N=0,1,\ldots}$ with $\Psi_N \in \mathcal{H}^N$. Let us show that $(\Phi, V_g\Phi) \to ||\Phi||^2$ as $g \to I$. (Using either the fact that V_g is unitary either the group representation structure, this is equivalent to $||V_{g'}\Phi - V_h\Phi||^2 \to 0$ as $g' \to h$). Spaces $\mathcal{H}^N$ are invariant, pairwisely orthogonal and $V_g^{(N)}$ are isometric; as a consequence one has

$$(\Phi, V_g\Phi) = \sum_{N=0}^{+\infty} (\Psi^{(N)}, V_g^{(N)}\Psi^{(N)})\,,$$

where $|(\Psi^{(N)}, V_g^{(N)}\Psi^{(N)})| \leq ||\Psi^{(N)}||||V_g^{(N)}\Psi^{(N)}|| = ||\Psi^{(N)}||^2$ and thus

$$\sum_{N=0}^{+\infty} |(\Psi^{(N)}, V_g^{(N)}\Psi^{(N)})| \leq \sum_{N=0}^{+\infty} ||\Psi^{(N)}||^2 = ||\Phi||^2\,.$$

This g-uniform bound (essentially via Lebesgue dominate convergence theorem) allows one to interchange symbols of summation and limit:

$$\lim_{g\to I}(\Phi, V_g\Phi) = \sum_{N=0}^{+\infty} \lim_{g\to I}(\Psi^{(N)}, V_g^{(N)}\Psi^{(N)}) = \sum_{N=0}^{+\infty}(\Psi^{(N)}, V_I^{(N)}\Psi^{(N)})||\Phi||^2\,,$$

where we have used strong continuity of each representation $V^{(N)}$. This is what we wanted to prove. $\Box$

Proof of Proposition 3.2. It is sufficient to show that each $\chi \in N'$ admits a corresponding function $\beta : N \to \mathbb{R}$ continuous and linear such that $\chi(\alpha) = e^{i\beta(\alpha)}$ for every $\alpha \in N$. (In fact, a continuous linear functional $\beta : N = C^\infty(\mathbb{S}^2) \to \mathbb{R}$ is a distribution by definition and thus one can write (α, β) instead of $\beta(\alpha)$.) Let us prove it. Actually, the following proof holds true in the more general hypothesis on N to be a topological vector space.
Fix $\chi \in N'$. First of all we identify $U(1)$ with $\mathbb{S}^1$ and, in turn, we identify $\mathbb{S}^1$ with $(-\pi, \pi]$ where $\pi \equiv +\pi$. In this picture, for our fixed $\chi \in N'$, there is a continuous map $f : N \to (-\pi, \pi]$ such that $\chi(\alpha) = e^{if(\alpha)}$ for all $\alpha \in N$. From continuity there is an open set $B_0 \subset N$ such that $B_0 = f^{-1}((-\pi, \pi))$. B_0 is a neighborhood of the zero vector of N. Indeed $e^{if(0)} = \chi(0) = 1$ since χ is a homomorphism. We have found that $f(0) = 2k\pi$ for some $k \in \mathbb{Z}$. On the other hand, because $f(0) \in (-\pi, \pi]$ by hypotheses, it must be $f(0) = 0$. In particular $f(0) \in (-\pi, \pi)$ hence $0 \in B_0$ and thus, as we said, B_0 is a open neighborhood of 0. As N is a topological vector space, there is an open balanced (also said star-shaped) neighborhood of 0, $B \subset B_0$.
In general the function f does not satisfy $f(u) + f(v) = f(u + v)$ because $f(u) + f(v)$ may not belong to B_0 also if $f(u), f(v)$ do. Nevertheless we define the map $\beta : N \to \mathbb{R}$ such that:

$$\beta(v) := n_v f\left(\frac{1}{n_v} v\right) , \quad \text{for all } v \in N,\ n_v > 0 \text{ being the first natural with } (1/n_v)v \in B\,. \tag{115}$$

We have the following results.

(a) For every $\alpha \in N$ it holds

$$e^{i\beta(\alpha)} = \chi(\alpha)\,.$$

Indeed, using $\chi(v)^m = \chi(mv)$ valid for every natural $m > 0$ and $e^{if(\alpha/n_\alpha)} = \chi(\alpha/n_\alpha)$, one has $e^{i\beta(\alpha)} = e^{in_\alpha f(\alpha/n_\alpha)} = (\chi(\alpha/n_\alpha))^{n_\alpha} = \chi(n_\alpha(\alpha/n_\alpha)) = \chi(\alpha)$.

(b) If β is continuous it is additive as well, i.e.

$$\beta(u + v) = \beta(u) + \beta(v)\,, \quad \text{for all } u, v \in N\,.$$

Indeed, from $\chi(u)\chi(v) = \chi(u+v)$ and (a), one obtains $e^{i(\beta(u)+\beta(v))} = e^{i\beta(u+v)}$. Fix $u, v \in N$ and let t range in $[0, 1]$. The function $g : t \mapsto \beta(u) + \beta(tv) - \beta(u + tv)$ must be continuous because straightforward composition of continuous functions. On the other hand, since $e^{i(\beta(u)+\beta(tv))} = e^{i\beta(u+tv)}$, g must take values in the non connected and discrete set $2\pi\mathbb{Z}$. Since continuous functions transforms connected sets to connected sets, g must take a constant value in $2\pi\mathbb{Z}$. As $g(0) = 0$, we conclude that $\beta(u) + \beta(tv) - \beta(u + tv) = 0$ for $t \in [0, 1]$, in particular $\beta(u) + \beta(v) = \beta(u + v)$.

(c) If β is continuous it is linear as well.
Indeed from (b) one has $m\beta(v) = \beta(mv)$ for every natural $m > 0$ and $v \in N$. As a consequence, defining $u := nv$, one obtains $\beta(u/n) = (1/n)\beta(u)$ valid for every natural $n > 0$ and $u \in N$. Both these results entail that $r\beta(w) = \beta(rw)$ for every rational $r > 0$ and $w \in N$. By continuity

one finds $r\beta(w) = \beta(rw)$ for every real $r > 0$ and $w \in N$. Finally (b) implies also that $\beta(0u) = 0\beta(u) = 0$ and $\beta(-u) = -\beta(u)$ for every $u \in N$. Putting all together one obtains that $r\beta(w) = \beta(rw)$ for every $r \in \mathbb{R}$ and $w \in N$. Taking additivity into account we have proved that β is linear.
To conclude, the proof it is sufficient to show that β defined in (115) is continuous. Let us demonstrate it proving that β is continuous at each point $\alpha \in N$. The difficult point to handle in the proof is that n_α in (115) is a function of α itself in spite of f being continuous. If $\alpha \in N$, by definition of n_α one has $\alpha/n_\alpha \in B$, but $\alpha/(n_\alpha - 1) \notin B$. If B^c denotes $N \setminus B$, there are now two possibilities concerning the requirement $\alpha/(n_\alpha - 1) \notin B$: (1) $\alpha/(n_\alpha - 1) \in int(B^c)$ or (2) $\alpha/(n_\alpha - 1) \in \partial B$.
Suppose that (1) holds, i.e. $\alpha/(n_\alpha - 1) \in intB^c$, together with $\alpha/n_\alpha \in B$. In this case $\alpha \in n_\alpha B$ as well as $\alpha \in int(n_\alpha - 1)B^c$. These sets are open by construction. As a consequence, there is an open neighborhood V of α such that, if $\alpha' \in V$, $\alpha'/(n_\alpha - 1) \in intB^c$ – so $\alpha'/(n_\alpha - 1) \notin B$ – and furthermore $\alpha'/n_\alpha \in B$. In other words $n_{\alpha'} = n_\alpha$. In this case, there is a constant $C = n_\alpha > 0$ such that, if α' lies in a neighborhood V of α, $\beta(\alpha') = Cf(\alpha'/C)$. Since f is continuous, β is such in V and thus β is continuous at α.
To conclude, suppose that (2) is valid, that is $\alpha/(n_\alpha - 1) \in \partial B$, together with $\alpha/n_\alpha \in B$. Consider a sufficiently small open neighborhood V of such a α. If $\alpha' \in V$ there are two possibilities: $\alpha' \in (n_\alpha - 1)B^c$ or $\alpha' \in (n_\alpha - 1)B$.
If $\alpha' \in (n_\alpha - 1)B^c$ one has $\alpha'/(n_\alpha - 1) \notin B$, but $\alpha'/n_\alpha \in B$ so that $n_{\alpha'} = n_\alpha$ and thus

$$\beta(\alpha') = n_\alpha f(\alpha'/n_\alpha) . \tag{116}$$

Conversely, if $\alpha' \in (n_\alpha - 1)B$, it must hold $\alpha'/(n_\alpha - 1) \in B$ so that n_α is not the first positive natural $n_{\alpha'}$ such that $\alpha'/n_{\alpha'} \in B$. In this case $n_{\alpha'} < n_\alpha$ and thus

$$\beta(\alpha') = n_{\alpha'} f(\alpha'/n_{\alpha'}) , \quad \text{where } n_{\alpha'} < n_\alpha . \tag{117}$$

Let us prove that in this second case, actually,

$$n_{\alpha'} f(\alpha'/n_{\alpha'}) = n_\alpha f(\alpha'/n_\alpha) , \tag{118}$$

holds true anyway so that $\beta(\alpha') = n_\alpha f(\alpha'/n_\alpha)$ and (116) is valid in every case. Defining $\gamma := n_{\alpha'}\alpha'$ (notice that $\gamma \in B$ by hypotheses) and $m = n_\alpha - n_{\alpha'}$ (notice that $0 < m < n_\alpha$ by construction), (118) is equivalent to

$$n_\alpha f(\gamma) - m f(\gamma) = n_\alpha f\left(\gamma - \frac{m}{n_\alpha}\gamma\right) . \tag{119}$$

To prove (119) notice that, from $\chi(\alpha) = e^{if(\alpha)}$ one gets (use the fact that χ is a homomorphism and $\mathbb{N} \ni n_\alpha, m > 0$),

$$n_\alpha f(\gamma) - m f(t\gamma) - n_\alpha f\left(\gamma - \frac{m}{n_\alpha}\gamma\right) \in 2\pi\mathbb{Z} .$$

Finally consider the map, with $\gamma \in B$ fixed,

$$[0,1] \ni t \mapsto h(t) := n_\alpha f(t\gamma) - m f(t\gamma) - n_\alpha f\left(t\gamma - \frac{m}{n_\alpha} t\gamma\right).$$

This map is continuous because f is continuous on B, $t\gamma \in B$ and $t\gamma - \frac{m}{n_\alpha}t\gamma \in B$ for $t \in [0,1]$ since $\gamma \in B$, B is balanced and $0 \leq 1 - m/n_\alpha < 1$. As $2\pi\mathbb{Z}$ is not connected and discrete but $[0,1]$ is connected, it must be $h(t) =$ constant. On the other hand $h(0) = 0$, thus $h(t) = 0$ for $t \in [0,1]$ and (119) must hold true.
We have proved once again that there is a constant $C = n_\alpha > 0$ such that, if α' lies in a neighborhood V of α, $\beta(\alpha') = Cf(\alpha'/C)$. Since f is continuous, β is such in V and thus β is continuous at α. $\Box$

Proof of Lemma 4.1. ¿From the decomposition in the former formula in (96), passing in spherical coordinates one gets, for $\phi \in \mathcal{S}_K(\mathbb{M}^4)$ (remind that $\epsilon_{\mathbb{S}^2} = \sin\vartheta\, d\vartheta \wedge d\varphi$ is the standard volume form of the unit 2-sphere),

$$\phi(t,r,\vartheta',\varphi') = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^+} \frac{dE\, E^2}{\sqrt{2E}} \int_{\mathbb{S}^2} \epsilon_{\mathbb{S}^2}(\vartheta,\phi) e^{iE(r\cos\alpha(\vartheta,\varphi,\vartheta',\varphi')-t)} \widetilde{\phi_+}(E,\vartheta,\varphi) + c.c.$$

where $\alpha = \alpha(\vartheta,\varphi,\vartheta',\varphi')$ is the angle between vectors $\mathbf{x} = (r\sin\vartheta'\cos\varphi', r\sin\vartheta'\sin\varphi', r\cos\vartheta')$ and $\mathbf{p} = (E\sin\vartheta\cos\varphi, r\sin\vartheta\sin\varphi, r\cos\vartheta)$. Passing to null coordinates $u := t - r$, $v := t + r$ and using the function $\widetilde{\phi'_+}(E,\vartheta,\varphi) := \sqrt{E}\widetilde{\phi_+}(E,\vartheta,\varphi)$ which, by the second formula in (96), turns out to be bounded, smooth and ϑ,φ-uniformly rapidly decaying as $E \to +\infty$ by construction (to prove it use the latter in (96) taking into account that Cauchy surfaces are smooth and compactly supported and Fourier transform maps such functions into Schwartz functions), the equation above can be rearranged as

$$\phi(t,r,\vartheta',\varphi') = \frac{1}{4\pi^{3/2}} \int_{\mathbb{R}^+} dE \int_{\mathbb{S}^2} \epsilon_{\mathbb{S}^2}(\vartheta,\varphi) e^{iEv(\cos\alpha-1)/2} e^{-iEu(\cos\alpha+1)/2} E\widetilde{\phi'_+}(E,\vartheta,\varphi) + c.c.$$

By definition of $\Gamma_{\mathbb{M}^4}$ and using the fact that $\omega^2\Omega^2\restriction_{\mathbb{M}^4} = 4(1+v^2)^{-1}$ (see the beginning of section 4.4), one has

$$(\Gamma_{\mathbb{M}^4}\phi)(u,\vartheta',\varphi') = \lim_{v\to+\infty} \frac{\sqrt{1+v^2}}{2}\phi(u,v,\vartheta',\varphi')$$

Since we know that this limit does exist by Proposition 2.3 and the factor in front of ϕ diverges, we conclude that ϕ itself must vanish as $v \to +\infty$. As a consequence, expanding $\sqrt{1+v^2}$ in powers of v^{-1}, we conclude that it must also hold

$$(\Gamma_{\mathbb{M}^4}\phi)(u,\vartheta',\varphi') \lim_{v\to+\infty} \frac{v}{2}\phi(u,v,\vartheta',\varphi').$$

In other words

$$(\Gamma_{\mathbb{M}^4}\phi)(u,\vartheta',\varphi') \lim_{v\to+\infty} \frac{1}{8\pi^{3/2}} \int_{\mathbb{R}^+}\!dE\!\int_{\mathbb{S}^2}\!\epsilon_{\mathbb{S}^2}(\vartheta,\varphi) e^{\frac{iEv(\cos\alpha-1)}{2}} e^{\frac{-iEu(\cos\alpha+1)}{2}} vE\widetilde{\phi'_+}(E,\vartheta,\varphi) + c.c. \quad (120)$$

Notice that the former exponential in the integrand, essentially due to Riemann-Lebesgue's lemma, makes vanishing the integral except for the case $\cos\alpha - 1 = 0$, that is when $(\vartheta,\varphi) = (\vartheta',\varphi')$; on the other hand the factor v blows up in this point giving rise to a Dirac δ. Indeed the limit can be computed using standard Dirac-δ-regularization procedures of distributional calculus obtaining (see below)

$$(\Gamma_{\mathbb{M}^4}\phi)(u,\vartheta,\varphi) = \frac{-i}{\sqrt{4\pi}} \int_{\mathbb{R}^+} dE\, \widetilde{\phi'_+}(E,\vartheta,\varphi)e^{iEu} + c.c. \tag{121}$$

We have found out that

$$(\Gamma_{\mathbb{M}^4}\phi)(u,\vartheta,\varphi) = \int_{\mathbb{R}^+} dE\, \frac{(-i)E\widetilde{\phi_+}(E,\vartheta,\varphi)e^{iEu}}{\sqrt{4\pi E}} + \int_{\mathbb{R}^+} dE\, \overline{\frac{(-i)E\widetilde{\phi_+}(E,\vartheta,\varphi)e^{iEu}}{\sqrt{4\pi E}}}\,.$$

¿From that expression for $(\Gamma_{\mathbb{M}^4}\phi)(u,\vartheta,\varphi)$, applying the definition (19) and standard properties of Fourier transform for L^1 functions, one straightforwardly gets

$$\widetilde{(\Gamma_{\mathbb{M}^4}\phi)}_+(E,\vartheta,\varphi) = (-i)E\widetilde{\phi_+}(E,\vartheta,\varphi)\,,$$

which is the thesis we wanted to prove. To conclude let us prove (121). Without loss of generality we can rotate the used Cartesian frame to have $\mathbf{p}$ with the direction of the positive axis z. In this case (120) reads, if $c := \cos\vartheta$,

$$(\Gamma_{\mathbb{M}^4}\phi)(u,0,\varphi') \lim_{v\to+\infty} \frac{1}{8\pi^{3/2}} \int_{\mathbb{R}^+}\!\!dE\!\int_0^{2\pi}\!\!d\varphi \int_{-1}^{1}\!dc\, e^{\frac{iEv(c-1)}{2}} e^{\frac{-iEu(c+1)}{2}} vE\widetilde{\phi'_+}(E,\vartheta,\varphi) + c.c.$$

$$= \lim_{v\to+\infty} \frac{-2i}{8\pi^{3/2}} \int_{\mathbb{R}^+}\!\!dE\!\int_0^{2\pi}\!\!d\varphi \int_{-1}^{1}\!dc\, \frac{d}{dc}\left(e^{\frac{iEv(c-1)}{2}}\right) e^{\frac{-iEu(c+1)}{2}} \widetilde{\phi'_+}(E,\vartheta,\varphi) + c.c.$$

Integration by parts gives (noticing that the dependence from φ vanishes for $\vartheta = 0,\pi$, i.e. $c = 1,-1$, and, thus, integration in $d\varphi$ trivially produces a factor 2π)

$$(\Gamma_{\mathbb{M}^4}\phi)(u,0,\varphi') \lim_{v\to+\infty} \frac{-i4\pi}{8\pi^{3/2}} \int_{\mathbb{R}^+}\!\!dE e^{-iEu}\widetilde{\phi'_+}(E,0,\varphi) - \lim_{v\to+\infty} \frac{-i4\pi}{8\pi^{3/2}} \int_{\mathbb{R}^+}\!\!dE e^{-iEv}\widetilde{\phi'_+}(E,0,\varphi) + c.c.$$

$$+ \lim_{v\to+\infty} \frac{-2i}{8\pi^{3/2}} \int_{\mathbb{R}^+}\!\!dE\!\int_0^{2\pi}\!\!d\varphi \int_{-1}^{1}\!dc\, e^{\frac{iEv(c-1)}{2}} e^{\frac{-iEu(c+1)}{2}} (-i)Eu\widetilde{\phi'_+}(E,\vartheta,\varphi) + c.c.$$

In other words

$$(\Gamma_{\mathbb{M}^4}\phi)(u,0,\varphi') = \frac{-i}{\sqrt{4\pi}} \int_{\mathbb{R}^+} dE\, \widetilde{\phi'_+}(E,0,\varphi)e^{-iEu} + \lim_{v\to+\infty} \frac{i}{\sqrt{4\pi}} \int_{\mathbb{R}^+}\!\!dE e^{-iEv}\widetilde{\phi'_+}(E,0,\varphi) + c.c.$$

$$+ \lim_{v\to+\infty} \frac{-1}{4\pi^{3/2}} \int_{\mathbb{R}^+}\!\!dE\!\int_0^{2\pi}\!\!d\varphi \int_{-1}^{1}\!dc\, e^{\frac{iEv(c-1)}{2}} e^{\frac{-iEu(c+1)}{2}} Eu\widetilde{\phi'_+}(E,\vartheta,\varphi) + c.c.$$

As the map $E \mapsto \widetilde{\phi'_+}(E,0,\varphi)$ (with φ constant) is smooth and rapidly decaying, Riemann-Lebesgue's lemma implies that the limit in the former line vanishes. Let us focus on the last limit. As the integrand is L^1, we can interchange the order of integration via Fubini-Tonelli theorem obtaining in particular that the considered limit can be re-written (up to an overall constant)

$$\lim_{v\to+\infty}\int_{[0,2\pi]\times[-1,1]} d\varphi dc \left\{\int_{\mathbb{R}^+} dE e^{\frac{iEv(c-1)}{2}} e^{\frac{-iEu(c+1)}{2}} Eu\widetilde{\phi'_+}(E,\vartheta,\varphi)\right\} + c.c. \tag{122}$$

By Riemann-Lebesgue's lemma, the integral in brackets vanishes, as $v \to +\infty$, almost everywhere in (c,φ). On the other hand, since the following c,φ-uniform bound holds

$$\left|\int_{\mathbb{R}^+} dE e^{\frac{iEv(c-1)}{2}} e^{\frac{-iEu(c+1)}{2}} Eu\widetilde{\phi'_+}(E,\vartheta,\varphi)\right| \le \int_{\mathbb{R}^+} dE \left|Eu\widetilde{\phi'_+}(E,\vartheta,\varphi)\right| = M < +\infty,$$

and the domain of integration of the external integral in (122) has measure finite, we can use Lebesgue's dominate theorem getting:

$$\lim_{v\to+\infty}\int_{[0,2\pi]\times[-1,1]} d\varphi dc \left\{\int_{\mathbb{R}^+} dE e^{\frac{iEv(c-1)}{2}} e^{\frac{-iEu(c+1)}{2}} Eu\widetilde{\phi'_+}(E,\vartheta,\varphi)\right\} + c.c.$$

$$= \int_{[0,2\pi]\times[-1,1]} d\varphi dc \lim_{v\to+\infty} \left\{\int_{\mathbb{R}^+} dE e^{\frac{iEv(c-1)}{2}} e^{\frac{-iEu(c+1)}{2}} Eu\widetilde{\phi'_+}(E,\vartheta,\varphi)\right\} + c.c.$$

$$= \int_{[0,2\pi]\times[-1,1]} d\varphi dc \;\; 0 + c.c. = 0\,.$$

We conclude that

$$(\Gamma_{\mathbb{M}^4}\phi)(u,0,\varphi') = \frac{-i}{\sqrt{4\pi}}\int_{\mathbb{R}^+} dE\, \widetilde{\phi'_+}(E,0,\varphi) e^{-iEu} + c.c.$$

Notice that the values φ and φ' are arbitrary only because of the singularity of spherical coordinates for $\vartheta = 0$ (the problem is harmless here because the singular set has measure zero). What is relevant in the expression above is that, barring the problem with coordinates, it says that the versor $\mathbf{n}'$ on $\mathbb{S}^2$ in the argument of the function in the left-hand side, $(\Gamma_{\mathbb{M}^4}\phi)(u,\mathbf{n}')$, coincides with the analog, $\mathbf{n}$, in the argument of the integrated function $\widetilde{\phi'_+}(E,\mathbf{n})$. Rotating back the used Cartesian frame to work with a generic value of ϑ the equation above transforms into:

$$(\Gamma_{\mathbb{M}^4}\phi)(u,\vartheta,\varphi) = \frac{-i}{\sqrt{4\pi}}\int_{\mathbb{R}^+} dE\, \widetilde{\phi'_+}(E,\vartheta,\varphi) e^{-iEu} + c.c.$$

where we have identified the angles φ and φ' as it is due working for $\vartheta \neq 0,\pi$ because $\mathbf{n}' = \mathbf{n}$. This equation is (121). □

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