Ring vortex solitons in nonlocal nonlinear media

D. Briedis
Department of Mathematics, Australian National University, Canberra ACT 0200, Australia
briedis@maths.anu.edu.au

D.E. Petersen
Laser Physics Centre, Australian National University, Canberra ACT 0200, Australia
dep111@rsphysse.anu.edu.au

D. Edmundson
ANU Supercomputer Facility, Australian National University, Canberra ACT 0200, Australia
darran.edmundson@anu.edu.au

W. Krolikowski
Laser Physics Centre, Australian National University, Canberra ACT 0200, Australia
wzk111@rsphysse.anu.edu.au

O. Bang
Research Center COM, Technical University of Denmark, DK-2800 Kongens Lyngby, Denmark
bang@com.dtu.dk

Abstract: We study the formation and propagation of two-dimensional vortex solitons, i.e. solitons with a phase singularity, in optical materials with a nonlocal focusing nonlinearity. We show that nonlocality stabilizes the dynamics of an otherwise unstable vortex beam. This occurs for either single or higher charge fundamental vortices as well as higher order (multiple ring) vortex solitons. Our results pave the way for experimental observation of stable vortex rings in other nonlocal nonlinear systems including Bose-Einstein condensates with pronounced long-range interparticle interaction.

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1. Introduction

A soliton is a localized wave that propagates without change through a nonlinear medium. Such a localized wave forms when the dispersion or diffraction associated with the finite size of the wave is balanced by the nonlinear change of the properties of the medium induced by the wave itself. Solitons, a topic of great interest in recent years, are universal in nature having been identified in such diverse physical systems as fluids, plasmas, solids, matter waves, and classical field theory [1]. In the field of optics, solitons hold ongoing promise for their potential application in telecommunications and all-optical switching [2]. The soliton concept is also important in the context of Bose-Einstein condensates (BECs) where it represents a stable localized coherent excitation of the condensate. Depending on whether the bosonic interaction is attractive or repulsive, bright or dark BEC solitons can be observed [3, 4].

Solitons exist in a variety of forms, including bright, dark, scalar, vector, and numerous others [5]. Perhaps the most commonly studied are fundamental bright solitons in the form of self-localized single peak structures with transversely constant phase. However, in recent years so-called bright vortex (or spinning) solitons have also attracted a great deal of attention. These are finite-size beams characterized by a phase singularity at the center. The phase of such a beam rotates around the singularity, the number of rotations defining the topological charge of the vortex [6]. As a consequence of this phase singularity, a vortex beam carries angular momentum and its amplitude in the center is identically equal to zero. That is, these beams have the form of a bright ring.

The formation and dynamics of bright vortex solitons has been extensively studied in the context of optics [7, 8, 9, 10, 11] and BECs with an attractive interparticle potential [12]. The early studies of vortex beams in saturable self-focusing nonlinear media demonstrated their intrinsic instability [8]. Extensive numerical and theoretical investigations showed that vortex rings decompose during propagation into a number of filaments that travel off tangentially to conserve the total angular momentum. It has been established that fragmentation of the vortex is caused by the excitation of exponentially growing azimuthal linear modes [8, 13].

In 1997 Quiroga and Michinel [10] demonstrated the stable propagation of vortex rings in the case of competing third-order self-focusing and fifth-order self-defocusing nonlinearities. At low light intensity the focusing dominates while for high intensity the self-defocusing effect prevails. This model supports the stable propagation of vortex rings at sufficiently large power when the vortex profile becomes very flat. Soon other groups confirmed the existence of stable vortex rings in other competing nonlinearity models [11], both in two and three spatial dimensions. From a practical perspective, it should be mentioned that while the vortex beam is stabilized by the competing nonlinearities, this happens in the regime when the higher-order contribution to the nonlinearity dominates — a rather peculiar situation not observed in real physical systems, except for, perhaps certain engineered meta-materials.

A more physical and experimentally feasible situation has been considered by Yang and Pelinovsky [14]. They studied the so-called vortex vector soliton which consists of two mutually incoherent components propagating in a self-focusing medium. One component is a fundamental beam with constant phase while the second is a vortex ring [15]. In contrast to the then common belief that such structures are unstable (confirmed by simulations and experiments), Yang and Pelinovsky showed the existence of a region in parameter space supporting stable vortex vector solitons. From a waveguide perspective, one can look at this object as a vortex ring guided by a waveguide structure (or external potential) which provides stability for the coupled system. Recently a method of stabilizing vortex solitons utilizing a partially coherent light beam has also been proposed [16].

Here we consider coherent vortex ring solitons in nonlinear media with a spatially nonlocal self-focusing nonlinearity. We will show that such models support *stable propagation* of vortex solitons of arbitrary charge for a sufficiently high degree of nonlocality [17].

Spatial nonlocality, an established concept in plasma physics [18, 19, 20], implies that the response of the medium at a particular point is not determined solely by the wave intensity at that point (the case in local media) but also depends on the wave intensity in its vicinity. The nonlocal nature of the nonlinearity often results from an underlying transport process such as atom diffusion [21] or heat transfer [22]. It can also originate from long-range molecular interactions as in nematic liquid crystals with an orientational nonlocal nonlinearity [23, 24]. Spatial nonlocality of the nonlinear response is also naturally present in the description of BECs where it represents the finite range of the bosonic interaction. Often neglected in theoretical models of weakly localized BECs, the nonlocality has to be taken into account in situations when, for example, strong localization occurs during catastrophic self-focusing (collapse) or when the bosonic particles exhibit long-range interactions [25, 26].

Nonlocality has also recently become important in optics [27, 28, 29, 30]. Studies of spatially nonlocal nonlinearities reveal a number of interesting effects. Perhaps most importantly, nonlocality tends to suppress the modulational instability (MI) of plane waves propagating in self-focusing media. (While this is generally the case, it is worth noting that certain types of nonlocality may actually promote MI, even in defocusing media [31, 32].) It is well-known that localized multi-dimensional waves in media with a focusing nonlinearity may exhibit strong self-focusing which can lead to a catastrophic increase (blow-up, or collapse) of the intensity over finite time (or propagation distance). However, as first shown by Turitsyn for a restricted class of models [33] and more recently by our group [29, 34] for general nonlocal models, nonlocality can prevent catastrophic collapse of beams and stabilize multidimensional solitons.

2. Model

We consider an optical beam propagating along the z-axis of a nonlinear self-focusing material with the scalar amplitude of the electric field $E(\mathbf{r},z) = \psi(\mathbf{r},z)\exp(iKz - i\Omega t) + c.c.$ Here $\mathbf{r} = (x,y)$, K is the wavenumber, Ω is the optical frequency, and $\psi(\mathbf{r},z)$ is the slowly varying amplitude. We assume that the refractive index change $N(I)$ induced by the beam with intensity $I(\mathbf{r},z) = |\psi(\mathbf{r},z)|^2$ can be described by the nonlocal model

$$N(I) = \int R(\mathbf{r}' - \mathbf{r}) I(\mathbf{r}',z)\, \mathrm{d}^2\mathbf{r}', \tag{1}$$

where $\int \mathrm{d}^2\mathbf{r} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y$. The response function $R(\mathbf{r})$, assumed to be a real, positive definite, localized, and symmetric function (i.e. $R(\mathbf{r}) = R(r)$, where $r = |\mathbf{r}|$), satisfies the normalization condition $\int R(\mathbf{r})\, \mathrm{d}^2\mathbf{r} = 1$. The width of the response function $R(r)$ determines the degree of nonlocality. For a singular response, $R(r) = \delta(r)$, the refractive index becomes a local function of the light intensity, $N(I) = I(\mathbf{r},z)$, i.e., the refractive index change at a given point is solely determined by the light intensity at that very point. With increasing width of $R(r)$ the light intensity in the vicinity of the point $\mathbf{r}$ also contributes to the index change at that point. The nonlinear response (1) leads to the following *nonlocal* nonlinear Schrödinger (NLS) equation governing the evolution of the optical beam

$$i\partial_z\psi + \left(\partial_x^2 + \partial_y^2\right)\psi + N(I)\psi = 0. \tag{2}$$

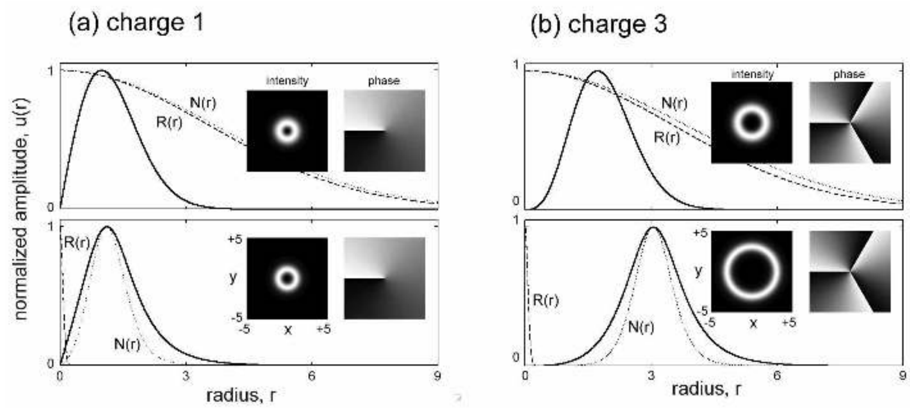

Fig. 1. Examples of ring vortex solitons in a nonlocal medium with charge m=1 (a) and m=3 (b). Solid line - normalized amplitude profile, dotted line - refractive index profile, dashed line - profile of the nonlocal response function. Top row - strongly nonlocal regime ($\sigma_0 = 5,\ \sigma = 1, \Lambda = 26.0783$). Bottom row - weak nonlocality ($\sigma_0 = 0.1,\ \sigma = 1, \Lambda = 2.0198$).

It has been shown that as long as the response function is real, symmetric, positive definite and monotonically decaying, the physical properties do not depend strongly on its shape [32]. For convenience in the variational calculations described below, we therefore choose to work with the following Gaussian response

$$R(\mathbf{r}) = \frac{1}{\pi\sigma_0^2}\exp(-|\mathbf{r}|^2/\sigma_0^2). \tag{3}$$

Except for certain special cases [35, 36], stationary solutions to Eq. (2) cannot be found analytically. As we are interested in circularly symmetric solutions (vortex rings) we look for the stationary solution in a cylindrical coordinate system

$$\psi(\mathbf{r},z) = \psi(r,\phi,z) = u(r)\exp(im\phi)\exp(i\Lambda z) \tag{4}$$

where r and ϕ are the radial and angular coordinates, $u(r)$ represents the radial structure of the solution, Λ is the propagation constant, and m denotes the vorticity (charge). In this case, Eq. (2) assumes the following form

$$\partial_r^2 u(r) + \frac{1}{r}\partial_r u(r) - \left(\frac{m^2}{r^2} + \Lambda\right) u(r) + u(r)\int_0^{2\pi}\int_0^{\infty} R(|\mathbf{r}-\mathbf{r}'|)|u(r')|^2 r'\,\mathrm{d}r'\mathrm{d}\phi = 0 \tag{5}$$

In a first approach we find approximate vortex soliton solutions using the variational technique [37]. It can be shown that the nonlocal propagation equation can be derived from the following Lagrangian density

$$\mathscr{L} = -\Lambda|u(r)|^2 - |\nabla u(r)|^2 + |u(r)|^2\int R(|\mathbf{r}-\mathbf{r}'|)|u(r')|^2\,\mathrm{d}^2\mathbf{r}'. \tag{6}$$

As a trial function, we choose a functional form which closely represents a typical single ring vortex soliton, viz.

$$u(r) = Ar\exp(-r^2/(2\sigma^2)). \tag{7}$$

Inserting the trial function (7) into the Lagrangian $L = \int \mathscr{L}\,\mathrm{d}^2\mathbf{r}$ and performing the spatial integration, we obtain the effective Lagrangian $L = L(A,\sigma)$ depending only on the parameters A and σ. From the Euler-Lagrange equations we then find A and σ as a function of Λ.

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To find exact stationary solutions to the nonlocal equation (5), we resort to an iterative numerical procedure. At each iteration we first calculate the convolution integral $N(|u|^2)$ using the function found in the previous iteration. This is done in the frequency domain employing a fast Hankel transform algorithm [38]. This newly calculated nonlocal term is then fed back into the original Eq. (5), whose new solution is obtained by solving the following *linear* equation

$$\Lambda u_{n+1}(r) - \partial_r^2 u_{n+1}(r) - \frac{1}{r}\partial_r u_{n+1}(r) + \frac{m^2}{r^2}u_{n+1}(r) = u_n(r)N(|u_n(r)|^2) \qquad (8)$$

The effectiveness of this iterative procedure depends strongly on the accuracy of the initial guess for the function $u(r)$. We found that use of approximate variational solutions ensured fast convergence to the exact solutions.

Illustrative results of the soliton-finding algorithm are presented in Fig.1 showing amplitude profiles of ring solitons and their corresponding contour plots with charge $m = 1$ (a) and $m = 3$ (b) for strongly (top) and weakly (bottom) nonlocal regimes. These plots also show the profile of the nonlocal response function $R(r)$ (dashed line) and the nonlinearity-induced index profile $N(I)$ (dotted line). Note that in the strongly nonlocal regime the index profile is very broad and resembles the shape of the nonlocal response function rather than the intensity profile of the vortex (as in the local case).

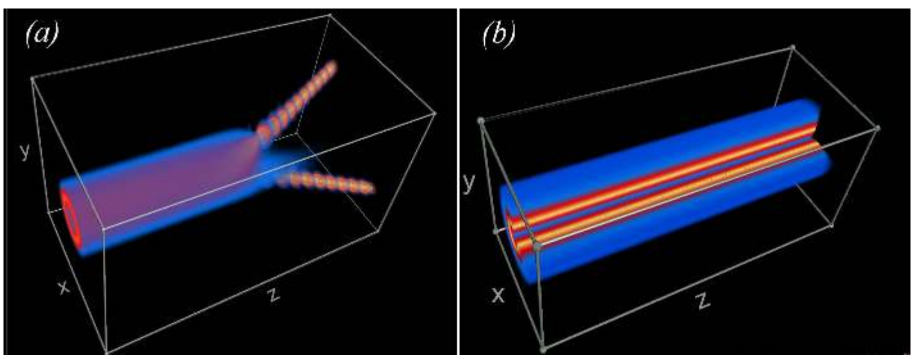

Fig. 2. Propagation of nonlocal charge $m = 1$ ring vortex solitons. (a) unstable propagation in the weakly nonlocal case with $\sigma_0/\sigma = 0.1$ and $\sigma = 1$, $\Lambda = 2.0198$, $(x,y) \in [-10,10] \times [-10,10]$ and $z \in [0,5]$. (b) stable propagation in the highly nonlocal case with $\sigma_0/\sigma = 10$, $\Lambda = 101.0199$ and $\sigma = 1$, $(x,y) \in [-30,30] \times [-30,30]$ and $z \in [0,50]$.

In order to investigate the stability of the vortex soliton solutions we simulate their propagation by directly solving the original nonlocal evolution Eq. (2) using the numerically-obtained exact solutions as initial conditions. By perturbing these solutions with a small amount of random complex noise containing a mixture of all possible unstable modes and evolving the input profiles to large z (typically between 50 and 100, but in the most extreme cases up to 900) we have determined the conditions necessary to propagate practically (i.e. experimentally viable) stable vortex rings. The simulations have been conducted using the split-step beam propagation method on a regular 512×512 mesh. Since the nonlocal term has the form of a convolution, it is evaluated in the frequency domain.

Our simulations reveal that the stability of vortex solitons is determined by the degree of nonlocality of the nonlinearity. In the weakly and moderately nonlocal regime all vortex ring solitons experience azimuthal instability and break into filaments after a certain propagation distance, as seen in Fig. 2(a). The nonlocality decreases the maximum growth-rate of the instability and thereby increases the length over which the solitons remain stable, but is too weak

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to remove the instability completely [29]. However, when the nonlocality becomes large the resulting vortex ring propagates in a stable fashion exhibiting only small-scale oscillations of its amplitude and radius due to an initial destabilizing perturbation. These oscillations are manifestations of the so-called internal modes of the nonlocal solitons discussed by Krepostnov *et al.* [39]. An illustrative example of stable propagation is displayed in Fig. 2(b). Importantly, stabilization of the vortex ring is not restricted to charge $m = 1$ but was also observed for higher charges (up to $m = 5$).

It should be emphasized that the stabilization mechanism reported here is markedly different from that discussed in the context of competing nonlinearities. In the latter the stabilization occurred when the nonlinearity changed its character from focusing to defocusing. Here, the stabilizing mechanism is more closely related to the propagation of vortex vector solitons [14], where the fundamental beam co-propagating with the vortex component acts as a confining potential. In nonlocal media the role of confining potential is played by the self-induced nonlocal waveguide structure $N(|u(r)|^2)$. As the nonlocality tends to average and smooth out spatial variations of the beam intensity, soliton perturbations exert a reduced effect on the nonlinearity-induced potential which, being sufficiently broad and deep (see top graphs in Fig. 1), confines the vortex ring and inhibits its decomposition.

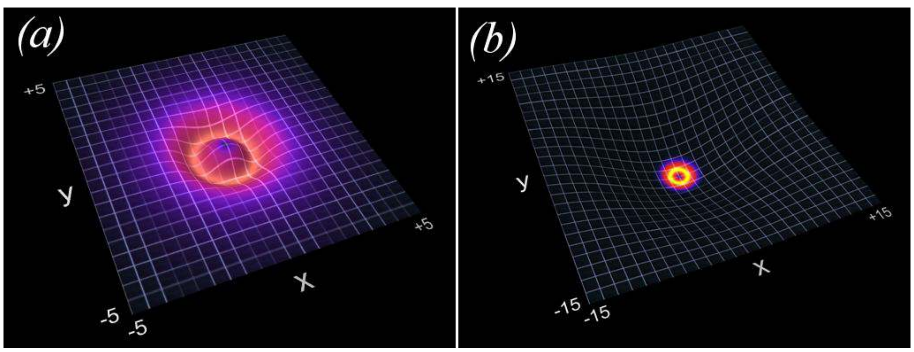

Fig. 3. Transverse structure of the nonlinearity-induced potential nesting the ring vortex soliton from Fig. 2. (a) Movie (3.2MB) depicting spatial evolution of the potential in a weakly nonlocal case (unstable propagation) (Click here for high resolution movie - 15MB); (b) Transverse structure of potential for strong nonlocality (stable propagation of the vortex soliton). Simulation parameters are the same as in Fig. 2.

In Fig. 3 we illustrate the 2-dimensional structure of the self-induced potential represented by the nonlocal refractive index $N(I)$ with the superimposed vortex beam. It is evident that for weak nonlocality (Fig. 3(a)) the potential mimics the structure of the vortex and hence its evolution is sensitive to perturbations of the vortex itself. On the other hand, in a strongly nonlocal case, Fig. 3(b), the nonlinear potential no longer resembles the vortex intensity distribution and, because of the averaging character of nonlocality, becomes insensitive to beam perturbation providing the stabilizing factor in the vortex dynamics.

In fact, one could anticipate the stabilization effect of nonlocality by considering the case of strong nonlocality ($\sigma_0/\sigma \gg 1$). In this limit, the nonlocal term in Eq. (2) can be represented as $N(|u|^2) = P_0 R(r)$ where $P_0 = 2\pi \int |u(r)|^2 r \, dr$ is the total power of the beam [36]. The plots depicting $R(r)$ and $N(r)$ in the top graphs in Fig. 1 show that this is indeed a very good approximation. Hence, in this limit, the nonlocal NLS equation becomes actually *linear and local*. Further, the dynamics of waves is then equivalent to that of a particle in an external potential represented by the function $P_0 R(r)$. This regime corresponds to the case of the so-called "accessible soliton" discussed by Snyder and Mitchell [27]. For certain forms of the potential

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$R(r)$ the ensuing eigenvalue problem can be treated analytically. Here the response function is Gaussian and Eq. (2) cannot be solved analytically. However, by considering only the lower order solutions corresponding to states located near the bottom of the potential well represented by the function $R(r)$, one can use the harmonic approximation $R(r) \approx R(0)(1 - r^2/\sigma_0^2)$. Eigensolutions to the Schrödinger equation with harmonic potential have the form of Laguerre-Gauss functions. In fact, the simplest solution with a nontrivial phase is

$$\psi(r,\phi) = 2\sigma_0^2\sigma^{-4}r^{|m|}\exp(-r^2/(2\sigma^2))\exp(im\phi), \tag{9}$$

which is just a bright vortex ring (σ is a free parameter). Since the nonlocal equation is to a good approximation linear in this strongly nonlocal regime, it is to be expected that the solution (9) is structurally stable, or at worst weekly unstable with a growth-rate so small that for any practical/experimental situation, it must be regarded as being stable.

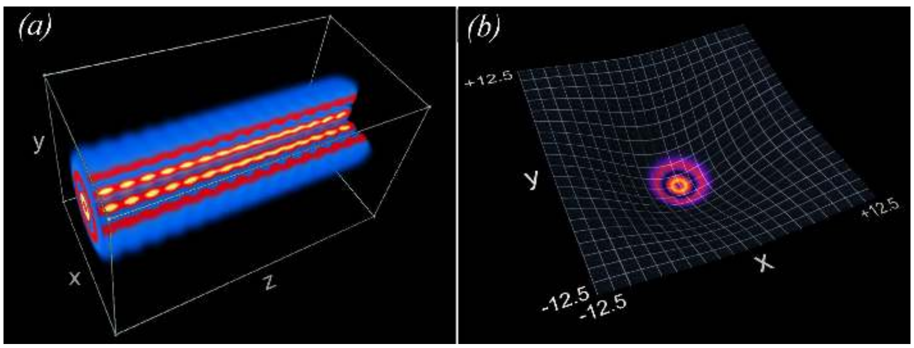

Fig. 4. Stable propagation of a double ring nonlocal vortex soliton with charge m=1, $\sigma_0 = 9$, $\sigma = 1$, $(x,y) \in [-25,25]\times[-25,25]$ and $z \in [0,25]$. (a) Transverse intensity structure. (b) Movie (1.5MB) illustrating dynamics of the vortex nested in the self-induced nonlocal potential. Note the difference in oscillations of the inner and outer rings. (Click here for high resolution movie - 10MB).

Finally, in Fig. 4 we illustrate the dynamical behavior of the higher order single charge vortex soliton, which, in the highly nonlocal regime is approximated by the function

$$\psi(r,\phi) = 4\sqrt{2}\sigma_0^2\sigma^{-4}r(1 - r^2/(2\sigma^2))\exp(-r^2/(2\sigma^2))\exp(i\phi). \tag{10}$$

This structure has a form of two out-of phase bright rings. As the plot shows, this two-ring vortex soliton propagates in a stable manner exhibiting only internal oscillations caused by the fact that the initial condition was only an approximate solution to the original nonlocal equation. Interestingly, as the plots show, the inner and outer rings oscillate with different frequencies, which may indicate the simultaneous excitation of different internal modes.

3. Conclusions

We studied the properties of bright vortex solitons in a nonlinear medium with a spatially nonlocal nonlinearity. We demonstrated that the spatial averaging mechanism of the nonlocality generates a robust confining potential. This potential prevents instability of the solitons allowing them to propagate as stable objects even in the presence of relatively strong perturbations. While this result does not constitute proof of the absolute stability of vortex rings, it is nevertheless an unambiguous indication of the feasibility of their experimental observation As the physical mechanism of this stabilizing process is generic, our results are applicable to the general class of nonlinear models with spatially nonlocal nonlinearity. In particular, we predict the

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possible observation of stable vortex ring solitons in Bose-Einstein condensates with a pronounced long-range attractive interparticle interaction, such as those formed by molecules with a significant dipole moment.

An independent study of stable vortex solitons propagating in media with a nonlocal thermal nonlinearity has been reported recently by A. Yakimenko, Y. Zaliznyak and Y. Kivshar [40].

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