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Rings on indecomposable torsion free groups of rank two

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Abstract

Let G be a torsion free abelian group of rank two. The principal purpose of the present paper is to show the structure of a ring on G. We prove that if G is indecomposable with cardinality of the typeset two or three, then all of the rings on G are associative and commutative. It will be presented two examples, one about an associative ring, and other non-associative ring on the torsion-free abelian groups of rank two.

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1. Type and Rank of Torsion Free Groups

Let L be the set of all subgroups of Q(The additive group of rational numbers). Define a relation $\equiv$ on L :
$A \equiv B$ if and only if there is a pair of non-zero integers m and n, such that $mA = nB$.

It is clear that $\equiv$ is an equivalence relation.

Definition 1. Let $A \in L$, then the equivalence class of A is called the *type* of A and denoted by $t(A)$, and

$t(A) \leq t(B)$ means that there exist $0 \neq m \in Z$ such that $mA \subseteq B$.

$t(A) < t(B)$ means $t(A) \leq t(B)$, and $t(A) \neq t(B)$.

If $A, B \in L$, then $AB = \{ab | a \in A, b \in B\}$ is a subgroup of Q(under addition). So we define $t(A)t(B) = t(AB)$. If $[t(A)]^2 = t(A)$, then $t(A)$ is called *idempotent*.

Definition 2. Let G be a torsion-free group. For any $0 \neq x \in G$, let $R(x) = \{r \in Q | rx \in G\}$, is a subgroup of Q. We define $t_G(x) = t[R(x)]$. A *typeset* of

G, $T(G)$ defined by $T(G) = \{t_G(x) | 0 \neq x \in G\}$. The cardinality of $T(G)$ is denoted by $|T(G)|$.

In a group if the nonzero elements be the same type t, is called *homogeneous group*.

Definition 3. A subset $\{a_i | i \in I\}$ of a torsion-free group G is said to be *independent* if for distinct elements $i_1, i_2, \ldots, i_k$ of I, and for integers $n_1, n_2, \ldots, n_k$ of Z, $n_1 a_{i_1} + n_2 a_{i_2} + \ldots + n_k a_{i_k} = 0$ implies that $n_1 = n_2 = \ldots = n_k = 0$.

By the Zorn's lemma, there exists a maximal independent set M in G. The cardinality of M is called the *rank* of G.

2. Lemmas and Main Result

We call a group G a nil group if there is no ring on G other than zero-ring. A. E. Stratton made considerable progress towards classifying the nil, rank two torsion-free groups:

Theorem 1.[5] Let G be a torsion-free group of rank two. If $|T(G)| > 3$, then G is nil.

For every positive integer n, there exist rank two torsion-free groups G with $|T(G)| = n$, by [2], and also there exist rank two torosion-free groups with infinite typeset, (see [4, p.112]). Therefore theorem 1 is not a statement concerning the empty set.

Clearly the classification of nil, rank two torosion-free groups to the case $|T(G)| \leq 3$. It is easy to show that the following are necessary conditions for a rank two torsion-free group to be a non-nil group:

(i) $|T(G)| = 1$, i.e. G is homogeneous, $t(G)$ must be idempotent.

(ii) $|T(G)| = 2$, $T(G)$ must consist of one minimal type and one maximal type.

(iii) $|T(G)| = 3$, $T(G)$ must consist of one minimal type, and two maximal types, where one of the maximal types must be idempotent, by [5].

Lemma 1. *In any ring $(G, *)$ with any non-zero product $x * y$,*

$$R(x * y) \supseteq R(x)R(y) \supseteq R(x), \quad t(x * y) \geq t(x)t(y) \geq t(x).$$

Proof. Let $\frac{m}{n} \in R(x)$ and $\frac{m'}{n'} \in R(y)$, then $w = (\frac{m}{n}x) * (\frac{m'}{n'}y) \in G$, and $nn'w = mm'(x*y)$, so that $\frac{mm'}{nn'}(x*y) = w \in G$. Therefore $R(x)R(y) \subset R(x*y)$. Hence $t(x * y) \geq t(x)t(y) \geq t(x)$.

Lemma 2. *Let G be a torsion-free group and $t \in T(G)$, then $G(t) = \{g \in G | t(g) \geq t\}$ is a subgroup of G.*

Proof. Let $x \in G$ such that $t(x) = t$, i.e. $t[R(x)] = t$, and $g_1, g_2 \in G$ such that $t(g_1) \geq t$ and $t(g_2) \geq t$. Since there are non-zero $m, n \in Z$ such that $mR(x) \subseteq R(g_1)$, and $nR(x) \subseteq R(g_2)$, we have $nmR(x) \subseteq R(g_1)$, and $nmR(x) \subseteq R(g_2)$. Hence

$$t(g_1 + g_2) \geq t(R(g_1) \cap R(g_2)) \geq t[mnR(x)] = t[R(x)] = t.$$

Therefore $g_1 + g_2 \in G(t)$ and $t(x) = t(-x)$ so $-x \in G(t)$. This means $G(t)$ is a subgroup of G.

Theorem 2. Let G be a torsion-free group of rank two and $T(G) = \{t_0, t_1, t_2\}$. Let $x, y \in G$, such that $t(x) = t_1$ and $t(y) = t_2$. Suppose that $t_0 < t_1$ and $t_0 < t_2$, then for any ring on G we have

$$x^2 = ax,\ y^2 = by,\ xy = yx = 0\ for\ some\ a, b \in Q.$$

Proof. By lemma 2, $G(t_1)$ is a subgroup of G. Let $z \in G$ such that $t(z) = t_0$, then $z \notin G(t_1)$. Hence $r(G(t_1)) = 1$, and $t(x^2) \geq t(x) = t_1$, therefore x^2 and x are in $G(t_1)$. Thus x^2 and x are dependent elements, that is $x^2 = ax$ for some $a \in Q$. By the similar way we conclude that $y^2 = by$ for some $b \in Q$. Since $t(yx) \geq t(x)$ implies that $yx, x \in G(t_1)$, we have $yx = ex$ for some $e \in Q$. By the similar way we deduce that $yx = fy$ for some $f \in Q$. If $yx \neq 0$, then $t(x) = t(yx) = t(y)$, this is a contradiction with our hypothesis, therefore $yx = 0$. By the same manner we conclude $xy = 0$. Therefore $x^2 = ax, y^2 = by, xy = yx = 0$, where $a, b \in Q$.

Definition 4. A subgroup H of a group G is *pure* in G if and only if

$$nH = H \cap nG\ for\ every\ n \in Z.$$

We denote by $< x >^*$ the pure subgroup of G generated by x. Let x, y be independent elements of G of rank two. Each element w of G has a unique representation $w = ux + vy$, where u, v are rational numbers. Let

$$U = \{u \in Q | ux + vy \in G,\quad for\ some\ v \in Q\},\ U_0 = \{u_0 \in Q | u_0 x \in G\},$$

$$V = \{v \in Q | ux + vy \in G,\quad for\ some\ u \in Q\},\ V_0 = \{v_0 \in Q | v_0 y \in G\}.$$

Clearly U_0, V_0 are subgroups of U, V respectively, which are isomorphic to the pure subgroups $< x >^*$ and $< y >^*$ of G. We call U, V, U_0, V_0 the groups of rank one belonging to the independent set $\{x, y\}$ of G.

Lemma 3.[3] *Let G be a torsion-free group of rank two. If U, V, U_0, V_0 are the groups of rank one belonging to $\{x, y\}$, then $\frac{U}{U_0} \cong \frac{V}{V_0}$.*

Lemma 4. Let C be a pure subgroup of a torsion-free group A such that

(i) A/C is completely decomposable and homogeneous of type t,

(ii) All the elements in A but not in C are of type t.

Then C is a direct summand of A.
Proof. See [4, p.114].

Lemma 5. *Let G be a torsion-free group of rank two, and $T(G) = \{t_1, t_2\}$ such that $t_1 < t_2$. Let $\{x, y\}$ be independent subset of G such that $t(x) = t_1$ and $t(y) = t_2$. Assume that U, V, U_0, V_0 are the groups of rank one belonging to $\{x, y\}$. If $t(U_0) = t(U)$, then $< y >^*$ is a direct summand of G. In particular, if $kU \subset U_0$ or $kV \subset V_0$ for some integer $k \neq 0$, then G is decomposable.*

Proof. We have $\frac{G}{<y>^*} \cong U$, hence $t(\frac{G}{<y>^*}) \cong t(U)$. Let $a \in G$ and $a \notin < y >^*$, then $t(a) = t_1$. By assumption we have $t(U) = t(U_0) = t_1$, therefore the type of all elements in G but not in $< y >^*$ are equal to $t(U) = t(\frac{G}{<y>^*})$. By lemma 4, $< y >^*$ is a direct summand of G. In particular, if $kU \leq U_0$ or $kV \leq V_0$ for some $k \neq 0$, then because of $\frac{U}{U_0} \cong \frac{V}{V_0}$. We have $t(U) = t(U_0)$ and hence G is decomposable.

Lemma 6. *If R is a finite rank torsion-free ring without zero divisors, then R^+ is homogeneous.*

Proof. Let $x_1, x_2, \ldots, x_k$ be independent subset of R^+. Let x be in R and $x \neq 0$. First we claim that $xx_1, xx_2, \ldots, xx_k$ are independent, otherwise there exist integers $a_1, a_2, \ldots, a_k$ such that

$$a_1xx_1 + a_2xx_2 + \cdots + a_kxx_k = 0, \ i.e. \ x(a_1x_1 + a_2x_2 + \cdots + a_kx_k) = 0$$

But R has no zero divisors, which is a contradiction. Therefore $\{xx_1, xx_2, \ldots, xx_k\}$ is an independent set.
Hence if $x \neq 0$ and $y \neq 0$ belong to R, then

$$my = m_1xx_1 + m_2xx_2 + \cdots + m_kxx_k = x(m_1x_1 + m_2x_2 + \cdots + m_kx_k)$$

implies that $t(y) \geq t(x)$ and similarly

$$nx = n_1yx_1 + n_2yx_2 + \cdots + n_kyx_k = y(n_1x_1 + n_2x_2 + \cdots + n_kx_k)$$

implies that $t(x) \geq t(y)$
Therefore $t(x) = t(y)$, consequently R^+ is homogeneous.

Theorem 3.[1] Let G be a torsion-free indecomposable abelian group of rank two. Let $T(G) = \{t_1, t_2\}$ such that $t_1 < t_2$. If $\{x, y\}$ is an independent set such that $t(x) = t_1$ and $t(y) = t_2$, then all non-trivial rings on G satisfy the following multiplication table:
$x^2 = by, xy = yx = y^2 = 0$, where b is a rational number.

Theorem 4. *Let G be a torsion-free group of rank two. If $|T(G)| = 3$ or $|T(G)| = 2$, and G is indecomposable, then any ring on G is associative and* *commutative.*

Proof. Let $|T(G)| = 3$. Suppose that $T(G) = \{t_0, t_1, t_2\}$ such that $t_0 < t_1$, $t_0 < t_2$ and $\{x, y\}$ be an independent set such that $t(x) = t_1$ and $t(y) = t_2$. By the theorem 2, we have

$$x^2 = ax,\ y^2 = by,\ xy = yx = 0.$$

Therefore any ring on G is associative and commutative.
Let $|T(G)| = 2$, and G is indecomposable. Suppose that $T(G) = \{t_1, t_2\}$ such that $t_1 < t_2$, and $\{x, y\}$ be an independent set such that $t(x) = t_1$ and $t(y) = t_2$. By the theorem 3, we have

$$x^2 = ay,\ y^2 = xy = yx = 0.$$

Hence it is easy to see that

$$\forall g_1, g_2, g_3 \in G,\ (g_1g_2)g_3 = g_1(g_2g_3) = 0,\ g_1g_2 = g_2g_1.$$

3. Examples

Example 1.[1] Let G be the subgroup of $Qx_1 \bigoplus Qx_2$ generated by the set

$$\{\frac{1}{p}x_1, \frac{1}{p^2}x_1 + \frac{1}{p^5}x_2 | p \textit{ is running over } \pi\},$$

where π is the set of all prime numbers.

$$R(x_1) = \{\frac{1}{p} | p \in \pi\}, \quad R(x_2) = \{\frac{1}{p^4} | p \in \pi\}.$$

Hence $T(G) = \{t_1, t_2\}$ and $t_1 = t(x_1)$, $t_2 = t(x_2)$. Any ring $R = (G, *)$ satisfies $x_1^2 = ax_2,\ x_1x_2 = x_2x_1 = x_2^2 = 0$. Consequently any ring on G is associative and commutative.

Example 2. Let $\pi = \{p_1, p_2, p_3, \ldots\}$ be the set of all prime numbers such that $i < j$ implies $p_i < p_j$. Let

$$A =< \frac{1}{p_i} | i = 2k + 1 >,$$

and

$$B =< \frac{1}{p_i^n}, \frac{1}{p_j} | i = 2k + 1, j = 2k, n \in N > .$$

Let $G = Ax \bigoplus By$, it easy to see that $t(A) < t(B)$. We define a ring on G by

$$x^2 = y, xy = y, yx = 2y, y^2 = 0.$$

Then $(xx)y = yy = 0$, $x(xy) = xy = y$. Hence

$$(xx)y \neq x(xy).$$

This is a non-associative and noncommutative ring.

References

[1] A. M. Aghdam, On the strong nilstufe of rank two torsion free groups, *Acta Sci. Math.* **49**(1985), 53-61.

[2] R. A. Beaument, R. S. Pierce, Torsion-free groups of rank two, *Mem. Amer. Math. Soc.* **38** (1961).

[3] R. A. Beaument, R. J. Wisner, Ring with additive group which is a Torsion-free groups of rank two, *Acta Sci. Math.* **20**(1959), 105-116.

[4] L.Fuchs, Infinite abelian group, Vol 2, *Academi Press*, Newyork (1973).

[5] A. E. Stratton, The typeset of torsion-free rings of finite rank, *Comment Math. Unit. St.* **27**(1979), 199-211.

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