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RINGS WHOSE MODULES HAVE MAXIMAL SUBMODULES

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Dedicated to Laci Fuchs on his 70th birthday

Abstract

A ring R is a **right max ring** if every right module $M \neq 0$ has at least one maximal submodule. It suffices to check for maximal submodules of a single module and its submodules in order to test for a max ring; namely, any cogenerating module E of mod-R; also it suffices to check the submodules of the injective hull $E(V)$ of each simple module V (Theorem 1). Another test is transfinite nilpotence of the radical of E in the sense that $\operatorname{rad}^\alpha E = 0$; equivalently, there is an ordinal α such that $\operatorname{rad}^\alpha(E(V)) = 0$ for each simple module V. This holds iff each $\operatorname{rad}^\beta(E(V))$ has a maximal submodule, or is zero (Theorem 2). If follows that R is right max iff every nonzero (subdirectly irreducible) quasi-injective right R-module has a maximal submodule (Theorem 3.3). We characterize a right max ring R via the endomorphism ring Λ of any injective cogenerator E of mod-R; namely, Λ/L has a minimal submodule for any left ideal $L = \operatorname{ann}_\Lambda M$ for a submodule (or subset) $M \neq 0$ of E (Theorem 8.8). Then Λ/L_0 has socle $\neq 0$ for: (1) any finitely generated left ideal $L_0 \neq \Lambda$; (2) each annihilator left ideal $L_0 \neq \Lambda$; and (3) each proper left ideal $L_0 = L + L'$, where $L = \operatorname{ann}_\Lambda M$ as above (e.g. as in (2)) and L' finitely generated (Corollary 8.9A).

HAMSHER MODULES

A module M is a **Hamsher module** provided each submodule $S \neq 0$ has a maximal submodule.[1]

[1]Hamsher modules are called max modules by Shock [**S**].

1. One-Module Theorem. *A ring R is a right max ring iff R has a cogenerating right Hamsher module E. A n.a.s.c. for this is that the injective hull $E(V)$ of each simple right R-module V is a Hamsher module.*

Proof: A module E cogenerates the category mod-R of all right R-modules iff for every module $M \neq 0$, there is a nonzero map $h : M \to E$ ([**F1**, pp. 91, 148 & 165]). Then $h(M) = M'$ is a nonzero submodule of E. Thus, when E is a Hamsher module, then M' has a maximal submodule M'', so $h^{-1}(M'')$ is a maximal submodule of M.

This proves the first statement in Theorem 1. Next let $E = \oplus E(V)$, as V range over all simple R-modules. Then E is a cogenerator module for mod-R ([**F1**, p. 167, prop. 3.55]). Let P_V be the projection $E \to E(V)$. Then, in the above, $0 \neq M' = h(M) \subseteq E$ implies $0 \neq P_V h(M) = M_V \subseteq E(V)$ is a nonzero submodule of $E(V)$ for some V, and so M has a maximal submodule, as before, whenever $E(V)$ is a Hamsher module for all V. ■

Note. $E(V)$ is direct summand of any cogenerator E of mod-R, hence the Hamsher condition on $E(V)$ is a consequence of that on E in Theorem 1. Moreover, this is sufficient for E to be Hamsher.

1.1. Corollary. *If R is a ring such that each simple module V has Noetherian injective hull $E(V)$, then R is a right max ring.*

To illustrate when $E(V)$ is not only Noetherian, but simple we will cite a theorem of Kaplansky, but first we recall some terminology:

R is **right V-ring** in case R has the equivalent properties. (See [**F1**, p. 356, 7.32A].)

(V1) Every simple right R-module V is injective, that is, $E(V)$ is simple.

(V2) $\operatorname{rad} M = 0$ for each right R-module M.

(V3) Every right ideal $I \neq R$ is the intersection of maximal right ideals, that is, $\operatorname{rad}(R/I)_R = 0$.

Note. A right V-ring is a right max ring since $\operatorname{rad} M \neq M$ for every $M \neq 0$.

Kaplansky's Theorem. [2] *A commutative ring R is a V-ring iff R is Von Neuman regular (= VNR).*

[2] According to my inquiry of Professor Kaplansky, "It worked its way into the public domain" (Letter of October 12, 1994).

Let $J = \operatorname{rad} R$. Then J is **left vanishing** (= T**-nilpotent in** [**B**], [**H**]) if for every sequence $\{a_n\}_{n=1}^{\infty}$ of elements of A, there is an $n \geq 1$ so that $a_n \cdots a_1 = 0$, that is the left-hand partial product $a_n \cdots a_1$ vanishes.

First Max Theorem ([**H**], [**K**]). *A commutative ring R is a max ring iff R/J is VNR and $J = \operatorname{rad} R$ is vanishing.*

Expressed otherwise: R is a max ring iff R/J is a V-ring, and J is vanishing. The **radical series** $\operatorname{rad}^{\alpha}(M)$ is defined inductively for each ordinal α in the usual way, where $\operatorname{rad}(M)$ is the intersection of all maximal submodules of M, $\operatorname{rad}^{\alpha+1}(M) = \operatorname{rad}(\operatorname{rad}^{\alpha}(M))$ for any ordinal, and

$$\operatorname{rad}^{\beta}(M) = \bigcap_{\alpha \in \beta} \operatorname{rad}^{\alpha}(M)$$

for each limit ordinal β.

Second Max Theorem ([**H**], [**K**]). *A ring R is right max iff R/J is right max and J is left vanishing.*

We next show that the modules in the radical series are test submodules for a Hamsher module.

2. Theorem. [3] *The f.a.e.c.'s on a right R-module M.*

(1) *M is Hamsher.*
(2) $\operatorname{rad}^{\beta}(M)$ *has a maximal submodule, or is* 0, *for every ordinal* β.
(3) $\operatorname{rad}^{\alpha}(M) = 0$ *for some* α.

Proof: (1) ⇒ (2) is obvious, and (2) ⇒ (3) follows by cardinal number theory for any α of cardinal greater than that of R. (3) ⇒ (1). If $S \neq 0$ is a submodule of M, then $S \nsubseteq \operatorname{rad}^{\lambda}(M)$ for least ordinal $\lambda < \alpha$, and obviously λ is not a limit ordinal, so $S \subseteq \operatorname{rad}^{\lambda-1}(M)$. If $S = \operatorname{rad}^{\lambda-1} M$, then S has a maximal submodule since $\operatorname{rad} S = \operatorname{rad}^{\lambda}(M) \neq S$. And if $S \neq \operatorname{rad}^{\lambda-1}(M)$, then S is not contained in a maximal submodule M' of $\operatorname{rad}^{\lambda-1}(M)$, hence $S \cap M'$ is a maximal submodule of S. This proves that M is a Hamsher module. ■

3.1. Corollary. *Let E be a right cogenerator module for R. The R is right max iff E has transfinite nilpotent radical. A n.a.s.c. for*

[3]The equivalence (1) ⇔ (3) is a theorem of Shock [**S**] who also proved that every semi-Artinian Hamsher module is Noetherian.

this is that $E(V)$ have transfinite nilpotent radical for each simple right R-module V.

3.2. Lemma. *If M is a quasi-injective right R-module, then so is every fully invariant submodule, in particular, so is $\operatorname{rad}^\alpha(M)$, for each ordinal α.*

Proof: A theorem of Wong-Johnson ([**W-J**]) characterizes a quasi-injective module as the fully invariant submodules of their injective hulls (see, e.g. [**F2**, p. 63, Prop. 19.2]). For example, if $E = E(M)$ has endomorphism Λ, then M is quasi-injective iff $\lambda(M) \subseteq M\ \forall\lambda \in \Lambda$. Now let M_0 be a fully invariant submodule of M. Since $E_0 = E(M) \subseteq E$, and since E is injective, then every element $\lambda_0 \in \Lambda_0 = \operatorname{End} E_0$ is induced by an element $\lambda \in \Lambda$. Since λ induces an endomorphism $\bar{\lambda}$ in M, and since $\bar{\lambda}(M_0) \subseteq M_0$ by the hypothesis that M_0 is fully invariant in M, then $\lambda_0(M_0) \subseteq M_0$ for each $\lambda_0 \in \Lambda_0$, that is, M_0 is fully invariant in $E(M_0)$, hence is quasi-injective.

It follows that $\operatorname{rad}^{\alpha+1}(M)$ is quasi-injective for all α, since $\operatorname{rad}^{\alpha+1}(M)$ is fully invariant in $\operatorname{rad}^\alpha(M)$ which by an inductive hypothesis may be assumed to be quasi-injective. Furthermore, $\operatorname{rad}^\beta(M)$ is fully invariant hence quasi-injective for each limit ordinal β, since it is the intersection of fully invariant submodules of M. ■

3.3. Theorem. *For a ring R, the f.a.e.c.'s:*

(1) *R is right max.*
(2) *Every nonzero quasi-injective module has a maximal submodule.*
(3) *Every nonzero subdirectly irreducible quasi-injective module has a maximal submodule.*

Proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial, and (3) $\Rightarrow$ (1) is an immediate consequence of Theorem 2, Corollary 3.1 and Lemma 3.2. ■

4. Corollary. *If a right R module M is faithful and has transfinite nilpotent radical, then R has transfinite nilpotent radical J.*

Proof: One shows inductively that $\operatorname{rad}^\alpha(M) \supseteq MJ^\alpha$, where $J = \operatorname{rad} R$. ■

Note. Let R be a commutative Noetherian ring. Then $J^\omega = 0$ by the Krull intersection Theorem and if R is a domain, then $I^\omega = 0$ for any ideal $I \neq R$ ([**Z-S**, p. 216, Theorem 12 and Corollary]). Thus, J is transfinite but not T-nilpotent when R is e.g., a Noetherian local domain not a field.

LOEWY SERIES AND TRANSFINITE SEMISIMPLE MODULES

A **descending** or **dual Loewy series** for a module M is descending chain $\{M_\alpha\}_{\alpha\in\Lambda}$ of submodules indexed by an ordinal Λ such that $M_0 = M$, and $M_\alpha/M_{\alpha+1}$ is semisimple

$$M_\beta = \cap_{\alpha\in\beta} M_\alpha$$

for any limit ordinal $\beta \in \Lambda$. We say that M is **transfinitely semisimple** if there is a descending Loewy series $\{M_\alpha\}$ with $M_\alpha = 0$ for some $\alpha \in \Lambda$.

5. Theorem. *Any transfinitely semisimple module M is a Hamsher module.*

Proof: By transfinite induction,

$$M_\alpha \supseteq \operatorname{rad}^\alpha(M)$$

for each M_α as defined above, hence $\operatorname{rad}^\alpha(M) = 0$ for some ordinal α, and Theorem 1 applies: M is Hamsher module. ∎

By Theorem 1, we also have the following:

5.1. Corollary. *If $E(V)$ is transfinite semisimple for each simple right R-module V, then R is right max.*

BASS MODULES

Recall that a module M is a **Bass module** ([**F2**]) if every submodule $M' \neq M$ is contained in a maximal submodule of M.

6. Theorem. *Let E be an quasi-injective right R-module that contains a copy of each simple image of E and $\Lambda = \operatorname{End} E_R$. If E is a Bass module, then Λ has essential left socle, $\operatorname{soc}_\ell \Lambda$.*

Proof: By the Harada-Ishii ([**H-I**]) double annihilator condition (= *DAC*) for a quasi-injective modules,

$$\operatorname{ann}_\Lambda \operatorname{ann}_E I = I$$

for finitely generated left ideals of Λ, one can show that each such $I \neq 0$ contains a minimal left ideal L. For if E' is a maximal submodule, containing $\operatorname{ann}_E I$ the fact $V = E/E' \hookrightarrow E$ yields $\lambda \in \Lambda$ such that $\lambda E \approx V$, hence $L = \Lambda\lambda$ is a minimal left ideal contained in I. Thus, $\operatorname{soc}_\ell \Lambda$ is an essential left ideal of Λ. ∎

In the next corollary, we see what happens to Λ when E is Noetherian.

6.1. Corollary. *If E is a Noetherian quasi-injective right module over R, then $\Lambda = \operatorname{End} E_R$ is a right perfect ring, hence a right max ring.*

Proof: By the Harada-Ishii DAC cited in the proof of Theorem 6, E_R Noetherian implies that Λ satisfies the DAC on finitely generated left ideals, hence Λ is right perfect ([**B**]). ■

DOUBLE ANNIHILATOR CONDITIONS FOR COGENERATORS

It is known that any cogenerator F satisfies the double annihilator conditions (DAC)

$$I = \operatorname{ann}_R \operatorname{ann}_F I$$

(see, e.g. [**F1**]). We next prove another DAC for F.

7. Dac Theorem. [4] *If F is any right cogenerator of R, and I and M are submodules of R_R and F_R respectively, then they satisfy the DAC's:*

(a) $$I = \operatorname{ann}_R \operatorname{ann}_F I$$

(b) $$M = \operatorname{ann}_F \operatorname{ann}_\Omega M$$

where $\Omega = \operatorname{End} F_R$.

Proof:

(1) Since F is a cogenerator then $R/I \hookrightarrow F^\alpha$ for some cardinal α, and if (x_i) is the image in F of the coset $1 + I$ in R/I, one sees that

$$I = \operatorname{ann}_R\{x_i\},$$

so (a) follows.

(2) F/M embeds in a direct product F^α of copies of F, and hence there is a map $h : F \to F^\alpha$ that has $\ker h = M$. Then, if $p_\alpha : F^\alpha \to F$ is the α-th projection, it follows that $\omega_\alpha = p_\alpha \circ h \in \Omega$ and that

(3) $$M = \cap_\alpha \ker \omega_\alpha.$$

Then,

(4) $$M = \operatorname{ann}_F L,$$

where $L = \Sigma_\alpha \Omega \omega_\alpha$.

Since (4) $\Longrightarrow$ (b), the proof is complete. ■

[4] After this was written, I found Kurata's report [**Ku**] where (b) is stated without proof in greater generality.

INJECTIVE COGENERATORS

If any cogenerator of mod-R is a Hamsher module, then R is a right max ring. In this section we list two conditions on a minimal injective cogenerator E that are each necessary and sufficient in order that R be a right V-ring: (1) $\operatorname{rad} E = 0$. (Theorem 8.1) and (2) E_R is a Bass module, and $\Lambda = \operatorname{End} E_R$ has zero Jacobson radical (Theorem 8.2).

8.1. Theorem. *Let E be a minimal injective cogenerator of R, and W the direct sum of a complete set of non-isomorphic simple right R-modules. (Thus, E is the injective hull of W, and W is the socle of E.) Then, the f.a.e.c.'s:*

(1) *R is a right V-ring.*
(2) $\operatorname{rad} E = 0$.

Proof: (1) $\Rightarrow$ (2). As stated, (1) $\Leftrightarrow$ $\operatorname{rad} M = 0$ for every right R-module M.

(2) $\Rightarrow$ (1). If V is a simple submodule of E, then (2) implies that there exists a maximal submodule M of E not containing V. Then since $V \cap M = 0$, and $V + M \supset M$, we see that $E = V \oplus M$, so V is injective. Since every simple right R-module embeds in E, then R is a right V-ring. ■

8.2. Theorem. *If the right minimal injective cogenerator E of a ring R is a Bass Module, and if $\Lambda = \operatorname{End} E_R$ has zero Jacobson radical, then R is a right V-ring (and E is semisimple).*

Proof: Let $W = \operatorname{soc} E$, the sum of all simple module, one for each isomorphy class. If $W = E$, then every submodule of E is a direct summand, hence is injective, so R is right V-ring. We may therefore assume that $E \neq W$, and hence by our Bass module assumption that there is a maximal submodule M of E that contains W. Since $V = E/M \hookrightarrow W$, there is an endomorphism λ of E such that $\ker \lambda = M$. Since M is an essential submodule of E, then $\lambda \in J = J(\Lambda)$ by a theorem of Utumi (e.g. [**F2**, p. 76, Theorem 19.27(a)]) contradicting the $J = 0$ assumption, and completing the proof. ■

8.3. Proposition. *If S is any semisimple right R-module with injective hull $E = E(S)$, then the endomorphism ring Λ has radical*

$$J(\Lambda) = \{\lambda \in \wedge \ker \lambda \supseteq S\}, \quad (1)$$

and moreover,

$$J(\Lambda) = \operatorname{ann}_\Lambda S. \tag{2}$$

Furthermore,

$$\overline{\Lambda} = \Lambda/J(\Lambda) = \operatorname{End} S_R \tag{3}$$

is a full product $= \Pi_{i\in A} L_i$ *of full linear rings, where* $L_i = \operatorname{End} W_{Di}$*, and* W_i *is a vector space over a sfield* D_i, $\forall i \in A$.

Proof: By Utumi's theorem cited above (proof of 8.2), (2) has the description (1) above. Since a submodule M of $E = E(S)$ is essential iff $M \supseteq S$, this shows that (2) holds. Furthermore since E is injective, any element of $\operatorname{End} S_R$ is induced by some $\lambda \in \Lambda$, so (2) $\Rightarrow$ (3). Finally, $\bar{\Lambda}$ is a product as described by classical ring theory. ■

8.4. Corollary. *If* E *is a minimal injective cogenerator of* mod-R*, and* $\Lambda = \operatorname{End} E_R$*, then* $\overline{\Lambda} = \Lambda/J(\Lambda)$ *is product* $\Pi_{i\in A} D_i$ *of sfields* $D_i = \operatorname{End}(V_i)_R$*, one for each isomorphy class* $[V_i]$ *of simple modules. Consequently,* $\overline{\Lambda}$ *is a* V*-ring.*

Proof: Follows from 8.3. $\bar{\Lambda}$ is thus abelian VNR (=strongly regular), hence is a right and left V-ring. ■

8.5. Corollary. *If (in Theorem 8.3)* E *is a minimal injective cogenerator, then* $E = E(S)$*, where* $S = \oplus V_i$*, exactly one simple module* V_i *of each isomorphy class, and*

$$\bar{\Lambda} = \Lambda/J(\Lambda) = \Pi_{i\in A} D_i$$

where $D_i = \operatorname{End} V_i$*, one for each* V_i.

Furthermore, $\bar{\Lambda}$ *is a right and left* V*-ring. Finally,* Λ *is a right (left) max ring iff* $J(\Lambda)$ *is left (right) vanishing. Moreover,* Λ *is right max iff* E_R *satisfies the acc on kernels of finite products* $\{j_n \cdots j_2 j_1\}$ *of elements of* $J(\Lambda)$.

Proof: Follows from Corollary 8.4, the Harada-Ishii theorem, and the Second Max Theorem. ■

8.6. Corollary. *If the minimal injective cogenerator E of* mod-R *satisfies the acc on essential submodules (equivalently, $E/\operatorname{soc} E$ is Noetherian), then $\Lambda = \operatorname{End} E_R$ is a right max ring.*

Proof: Since $\Lambda/J(\Lambda)$ is a V-ring (both sides) hence a max ring, then by Hamsher's theorem, Λ is right max iff $J(\Lambda)$ is left vanishing. But this follows from Corollary 8.5 and the Harada-Ishi Theorem as in the proof of Theorem 6. (Since $\operatorname{soc} E$ is the intersection of all essential submodule by a theorem of Kasch-Sandomierski, the parenthetical equivalence holds.) ■

Remark 8.6A. The condition of Corollary 8.6 implies that $E(V)$ is Noetherian for any simple module V, and by Corollary 1.1, this is also a sufficient condition for R to be right max.

8.7. Theorem (Partial Converse of Theorem 6)**.** *If E is an injective cogenerator for* mod-R*, and if $\Lambda = \operatorname{End} E_R$ has essential left socle then E is a Bass module.*

Proof: The proof is a straightforward application of the Harada-Ishii theorem. For if M is a proper submodule of E, the fact that E is an injective cogenerator yields $\hom(E/M, E) \neq 0$, hence some $\lambda \in \Lambda$ with $\ker \lambda \supseteq M$. Then, if $\Lambda\lambda_0$ is a minimal left ideal of Λ contained in $\Lambda\lambda$, by the Harada-Ishii theorem, $E_0 = \ker \lambda_0$ is a maximal submodule containing $\ker \lambda$, hence M. ■

In the proof of the next theorem, we let $\ker L = \cap_{\lambda \in L} \ker \lambda$.

8.8. Theorem. *For a ring R, right injective cogenerator E, and $\Lambda = \operatorname{End} E_R$ the f.a.e.c.'s:*

(1) *R is right max.*
(2) *E is a Hamsher module.*
(3) *Λ/L has nonzero socle for any left ideal $L = \operatorname{ann}_\Lambda M$, where M is a nonzero submodule of E.*

Proof: (1) ⇔ (2) by Theorem 1. (2) ⇒ (3). By the *DAC* Theorem 6.2, if $L = \operatorname{ann}_\Lambda M$, then $M = \ker L$, hence, since E is Hamsher module, M has a maximal submodule M_0. Since $\hom_R(M/M_0, E) \neq 0$ and E is injective, then there exists $\lambda_0 \in \Lambda$ such that $\lambda_0 M_0 = 0$ and $\lambda_0 M \neq 0$. Moreover, if $L_0 = \operatorname{ann}_\Lambda M_0$, then by the *DAC* Theorem 7, $\operatorname{ann}_E L_0 = M_0$, and since $M \cap (\ker \lambda_0) = M_0$, then:

$$\operatorname{ann}_E(L + \Lambda\lambda_0) = (\ker L) \cap (\ker \lambda_0) = M \cap (\ker \lambda_0) = M_0 = \operatorname{ann}_E L_0. \quad ■$$

By the Harada-Ishii theorem, $L + \Lambda\lambda_0$ satisfies the DAC, hence

$$L + \Lambda\lambda_0 = \operatorname{ann}_\Lambda \operatorname{ann}_E(L + \Lambda\lambda_0) = \operatorname{ann}_\Lambda M_0 = L_0.$$

Moreover, the same argument shows that

$$L_0 = L + \Lambda\lambda' \quad \text{for all } \lambda' \in L_0\backslash(L)$$

that is, necessarily $\operatorname{ann}_E(L + \Lambda\lambda') = M_0$ so $L + \Lambda\lambda' = \operatorname{ann}_\Lambda M_0 = L$. Thus $L_0\backslash L$ is a minimal submodule of $\Lambda\backslash L$, so (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2). Let $L = \operatorname{ann}_\Lambda M$. Then, by the DAC Theorem 6.2, $M = \operatorname{ann}_E L$. Let L_0/L be a minimal submodule of Λ/L, and let $M_0 = \operatorname{ann}_E L_0$. Since $L_0 = L + \Lambda\lambda$ for any $\lambda \in L_0\backslash L$, then by the Harada-Ishii DAC, necessarily $L_0 = \operatorname{ann}_\Lambda M_0$. If $M' \neq M$ is a submodule of M containing M_0, then by simplicity of L_0/L, necessarily $\operatorname{ann}_\Lambda M' = L_0$ whence by the DAC Theorem 6.2,

$$M' = \operatorname{ann}_E \operatorname{ann}_\Lambda M' = \operatorname{ann}_E L_0 = M_0$$

so M_0 is a maximal submodule of M. Thus, (3) $\Rightarrow$ (2). ■

8.9A. Corollary. *If R is right max, E an injective cogenerator, and $\Lambda = \operatorname{End} E_R$, then Λ/I has nonzero socle for each proper left ideal I of the (3) types:*

(0) *L_0 finitely generated left ideal of Λ.*
(1) *L_1 an annihilator left ideal of Λ.*
(2) *$L_2 = L + L_0$, where L_0 is finitely generated and $L = \operatorname{ann}_\Lambda M$ for a submodule M of E.*

In particular, $L_1 = \operatorname{ann}_\Lambda M_1$, where $M_1 = L_1^\perp E$, so L can have the form L_1 in (2).

Proof: By the Harada-Ishii DAC, any left ideal L_2 of the form (2) satisfies the DAC, hence $L_2 = \operatorname{ann}_\Lambda M_2$, where $M_2 = \operatorname{ann}_\Lambda L_2 = \ker L_2$, so Theorem 8.8 applies.

Furthermore, if L_1 is the left annihilator ${}^\perp X$ in Λ of a subset X of Λ, then $L_1 ={}^\perp (L_1^\perp)$ so

$$L_1 = \operatorname{ann}_\Lambda({}^\perp L_1 E)$$

is the annihilator of an R-submodule of E. ■

8.9B. Corollary. *If E is an injective cogenerator of* mod-R *with left Loewy (equivalently, left semiartinian) endomorphism ring Λ, then R is right max and Λ is right perfect. Moreover, R has just finitely many simple right modules.*

Proof: If Λ is left Loewy, then Λ/L has nonzero socle for all left ideals $L \neq \Lambda$, so Theorem 8.8 applies to establish that R is right max. Since $\overline{\Lambda} = \Lambda/J(\Lambda)$ is also left Loewy and right self-injective (see, e.g. (3) of Prop. 8.3), then $\overline{\Lambda}$ is semisimple Artinian and $J = J(\Lambda)$ is left vanishing, hence Λ is right perfect. (See, for example, the discussion in [**C-P**, esp. Lemma 1 and the proof of Proposition 2].) Furthermore, since $\overline{\Lambda}$ is semisimple and isomorphic to the endomorphism ring of the socle S of E (see the proof of 8.3), then S has finite length. This shows that the isomorphy set of simple right R-modules is finite. ■

8.10. Corollary. *If E is an injective cogenerator of* mod-R, *and $\Lambda = \operatorname{End} E_R$, then R is right max iff $J = \operatorname{rad} R$ left vanishing, and Λ/L has nonzero left socle for any left ideal $L = \operatorname{ann}_\Lambda M$, where M is a nonzero R-submodule of E annihilated by J.*

Proof: One knows that $F = \operatorname{ann}_E J$ is an injective cogenerator of mod-R/J (F is injective as an R/J-module and contains a copy of each simple R-module). Moreover, F is a fully invariant R-submodule of E, hence, by injectivity of E,

$$\bar{\Lambda} = \Lambda/\operatorname{ann}_\Lambda F \approx \operatorname{End} F_R.$$

The corollary now follows from Hamsher's Second Theorem and Theorem 8.8. ■

8.11. Example. Let M be any bimodule over a right max ring A. Then the split-null or trivial extension $R = (A, M)$ is a right max ring.

Proof: Let $J(A)$ be the (left vanishing) radical of A. Then $J(R) = (J(A), M)$ and

$$R/J(R) \approx A/J(A)$$

is a right max ring, so R is right max iff $J(R)$ is left vanishing. But

$$J(R)/(0, M) \approx J(A)$$

is left vanishing and $(0, M)^2 = 0$, and then an easy computation shows that $J(R)$ is left vanishing. ■

REMARKS ON THE LITERATURE

A module M is quotient finite dimensional (= *q.f.d.*) provided that all factor modules have finite Goldie dimension, i.e., contain no infinite direct sums. Generalizing a theorem of Shock [**S**], Camillo [**C1**] proved that an R-module M is q.f.d. iff every submodule N contains a finitely generated submodule K with N/K having no maximal submodules. This implies that a q.f.d. module M is Noetherian iff every factor module M/K is Hamsher. Since linearly compact modules are q.f.d., then by duality theory [**M**] one shows that a Morita ring R(= R has a Morita duality) is right max iff left Loewy (= semi-Artinian and iff R is right and left Artinian.

Results of Camillo and Fuller [**C-F1**], [**C-F2**] and Nastasescu and Popescu [**N-P**] are germane here: A left Loewy ring R of finite Loewy length is right max ([**C-F1**], [**N-P**]). More generally, any left Loewy ring with acc on primitive ideals is right max ([**C-F2**]). The example of a right but not left V-ring R of the author's in [**F4**] is a VNR of left Loewy length 2 hence left max.

As an application of Theorem 1, we prove in [**F3**] that for a commutative ring R that the f.e.c.'s : (1) R is locally a perfect ring (= R_m is perfect at each maximal ideal m); (2) R_m is a max ring for each maximal ideal m; (3) R is a max ring.

QUESTIONS

(1) If $\Lambda = \operatorname{End} E_R$ is a right max ring, for a minimal injective cogenerator E of mod-R, is R right max?

(2) If R is right max, is Λ?

In [**C2**], Camillo proves that a right max right and left $PID\,R$ is simple, and that given two maximal right ideals, pR and qR, either R/pqR or R/qpR is semisimple.

(3) Characterize when a PID ring R is right (or left) max. It is of course if R/aR (or R/Ra) is semisimple for any $0 \neq a \in R$. (See [**C2**].)

(4) (Hamsher [**H**]) When is a full linear ring right or left max? (Regarding the corresponding question for V-rings, see Osofsky [**0**].)

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References

[A-F] E. P. ARMENDARIZ AND J. W. FISHER, Regular PI-rings, *Proc. Amer. Math. Soc.* **39** (1973), 247–251.

[B] H. BASS, Finitistic dimension and a homological generalization of semiprimary rings, *Trans. Amer. Math. Soc.* **95** (1960), 466–488.

[C1] V. CAMILLO, Modules whose quotients have finite Goldie dimension, *Pac. J. Math.* **69** (1977), 337–338.

[C2] V. CAMILLO, On some rings whose modules have maximal submodules, *Proc. Amer. Math. Soc.* **50** (1975), 97–100.

[C-F1] V. CAMILLO AND K. FULLER, On Loewy length of rings, *Pac. J. Math.* **53** (1974), 347–354.

[C-F2] V. CAMILLO AND K. FULLER, A note on Loewy rings and chain conditions on primitive ideals, in "*Module Theory*," Lecture Notes in Math. **700**, Springer-Verlag Berlin, Heidelberg and New York, 1979, pp. 75–85.

[C-Y] V. CAMILLO AND M. YOUSIF, Semi-V-modules, *Comm. Alg.* **17** (1989), 165–177.

[C-P] J. CLARK AND P. F. SMITH, On semi-artinian modules and injectivity conditions, preprint (1994).

[D-S] N. V. DUNG AND P. F. SMITH, On semi-artinian V-modules, *J. Pure & Appl. Algebra* **82** (1992), 27–37.

[F1] C. FAITH, "*Algebra I: Rings, Modules, and Categories*," Grundl. der Math. Wiss. **190**, Springer-Verlag, Berlin, Heidelberg, and New York, 1973 (rev. 1981).

[F2] C. FAITH, Algebra II: Ring theory, Grundl. der Math. Wiss. **191**, Springer-Verlag, Berlin, Heidelberg, and New York.

[F3] C. FAITH, Locally perfect commutative rings are those whose modules have maximal submodules, *Abstracts of the Amer. Math. Soc.* **116** (1995), p. 466.

[F4] C. FAITH, Modules finite over endomorphism ring, in "*Lectures on Rings and Modules*," Lecture Notes in Math. **246**, 1972, pp. 145–189.

[H] R. Hamsher, Commutative rings over which every module has a maximal submodule, *Proc. Amer. Math. Soc.* **18** (1967), 1133–1137.

[H-I] M. Harada and Y. Ishii, On endomorphism rings of Noetherian quasi-injective modules, *Osaka J. Math.* **9** (1972), 217–223.

[K] L. A. Koifman, Rings over which each module has a maximal submodule, *Mat. Zametki* **7** (1970), 359–367; (transl.) Math. Notes of the Acad. Sci. **7** (1970), 215–219.

[Ku] Y. Kurata, Note on dual bimodules,, Proc. First China-Japan Symp. on Ring Theory (1991), Okayama U., Okayama, Japan.

[M] B. Müller, Linear compactness and Morita duality, *J. Algebra* **4** (1966), 373–387.

[N-P] C. Nastasescu and N. Popescu, Anneaux semi-artiniens, *Bull. de la Soc. Math. de France* **96** (1968), 357–368.

[O] B. Osofsky, Cyclic injective modules of full linear rings, *Proc. Amer. Math. Soc.* **17** (1966), 247–253.

[S] R. Shock, Dual generatizations of the Artinian and Noetherian conditions, *Pacific. J. Math.* **54** (1974), 227–235.

[W-J] E. T. Wong and R. E. Johnson, Quasi-injective modules and irreducible rings, *J. London Math. Soc.* **16** (1965), 526–528.

[Z-S] O. Zariski, P. Samuel and D. Van Nostrand, "*Commutative Algebra*," I, Springer-Verlag, Berlin-Heidelberg- New York, 1958.

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