PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 2, November 1979

RINGS WHOSE RIGHT MODULES ARE DIRECT SUMS OF INDECOMPOSABLE MODULES

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ABSTRACT. It is shown that, given a module M over a ring with 1, every direct product of copies of M is a direct sum of modules with local endomorphism rings if and only if every direct sum of copies of M is algebraically compact. As a consequence, the rings whose right modules are direct sums of indecomposable modules coincide with those whose right modules are direct sums of finitely generated modules.

1. Introduction and notation. It is well known that the rings whose right modules are direct sums of finitely generated modules are characterized by the fact that their pure exact sequences of right modules split (Gruson and Jensen [9]). According to Auslander [3], Ringel and Tachikawa [**10**], and Fuller and Reiten [7], the rings enjoying this property on both sides are precisely the rings of finite representation type. Moreover, as was shown by Chase [**4**], if a ring R has the condition on the right, then every right R-module possesses an indecomposable decomposition (R_R being artinian). For R commutative, the converse was established by Warfield [**13**] who left the general case open. From our main theorem (Theorem A) it will follow that Warfield's result remains true for arbitrary rings. Evidence that this might be true has recently been provided by Fuller [**8**] who proved that the rings whose right modules are direct sums of finitely generated modules coincide with those whose right modules have decompositions that complement direct summands in the sense of [**1**]. Theorem A also furnishes a more general background for this result.

An intermediate step in the proof of Theorem A deserves emphasis: Every Σ-algebraically compact module M (i.e. all direct sums of copies of M are algebraically compact) has the exchange property which in turn implies a certain cancellation property for M (Theorem B).

Throughout, R is an associative ring with identity and R-module means unitary right R-module. Recall that M_R is algebraically compact (abbreviated by a.c. in proofs) if each finitely solvable system of equations $\sum_{i\in I} X_i a_{ij} = m_j$ ($j \in J$), where $(a_{ij})_{i\in I, j\in J}$ is a column-finite R-matrix and $m_j \in M$, is solvable (finitely solvable means that for each finite subset J' of J there is an element $(n_i) \in M^I$ with $\sum_{i\in I} n_i a_{ij} = m_j$ for $j \in J'$). More manageable char-

Received by the editors February 24, 1978 and, in revised form, October 5, 1978.
AMS (*MOS*) *subject classifications* (1970). Primary 16A46, 16A64; Secondary 16A52.
Key words and phrases. Indecomposable module, ring of finite representation type, exchange property, cancellation property.

acterizations of (Σ-)algebraically compact modules and large classes of examples can be found in [**14**], [**15**]. We will make constant use of Warfield's observation that algebraically compact = pure-injective [**11**, Theorem 2], as well as of the fact that the endomorphism ring of an indecomposable algebraically compact module is local [**15**, Theorem 9].

The author would like to express her gratitude to Professor C. U. Jensen for a stimulating conversation in which he conjectured a modified version of (1) $\Rightarrow$ (3) of Theorem A.

2. Theorems.

THEOREM A. *For an R-module M the following statements are equivalent*:

(1) *The countably infinite product $M^{\mathbf{N}}$ of copies of M is a direct sum of submodules with local endomorphism rings.*

(2) *Every* (*pure submodule of a*) *direct product of copies of M has a decomposition that complements direct summands.*

(3) *M is Σ-algebraically compact.*

Before giving a proof of Theorem A, we point out several consequences.

COROLLARY 1. *Suppose that M_R is algebraically compact* (*resp., injective*). *Then every* (*countable*) *direct product of copies of M is a direct sum of indecomposable modules iff M is Σ-algebraically compact* (*resp., Σ-injective*).

Note that the second statement includes the well-known theorem of Matlis-Papp, see [**6**].

PROOF. In view of Theorem A the first claim follows from the fact that direct products of a.c. modules are again a.c., and indecomposable a.c. modules have local endomorphism rings. The second is a special case of the first. □

COROLLARY 2. *The following statements about a ring R are equivalent*:

(1) *Every* (*algebraically compact*) *right R-module is a direct sum of indecomposable modules.*

(2) *Every right R-module has a decomposition that complements direct summands.*

(3) *All pure inclusions of right R-modules split.*

PROOF. (3) $\Rightarrow$ (2). Since all pure inclusions split, all right R-modules are (Σ-)a.c. Apply Theorem A. (2) $\Rightarrow$ (1) is clear, since the decompositions in (2) are necessarily indecomposable. (1) $\Rightarrow$ (3). As is well known [**11**, Corollary 6], each module M_R can be embedded as a pure submodule into an a.c. module which by hypothesis and Corollary 1 is even Σ-a.c. But since Σ-algebraic compactness is inherited by pure submodules (see the criterion preceding Lemma 4), M is in turn algebraically compact, which implies (3).

REMARK. As mentioned above, (2) $\Leftrightarrow$ (3) of Corollary 2 is known. Apart from Chase's contribution, the proof given here is completely different from

Fuller's proof [**8**]; in particular, it does not use Harada's results on rings with enough idempotents.

As was discovered by Anderson and Fuller [**1**, Theorems 5, 6], right perfect (resp., semiperfect) rings are characterized by the fact that their projective (resp., finitely generated projective) right modules have decompositions that complement direct summands. We pursue this line of results in the following specialized version of Theorem A.

COROLLARY 3. *For a ring R the following statements are equivalent*:

(1) *Every pure submodule of a direct product of copies of R_R has a decomposition that complements direct summands.*

(2) *$R_R^{\mathbf{N}}$ has a decomposition that complements direct summands.*

(3) *R_R is Σ-algebraically compact.*

REMARK. A right Σ-a.c. ring R need not be left coherent; in particular, the products considered in Corollary 3 need not be projective. For typical examples of Σ-a.c. rings see [**15**].

An important tool in the proof of (1) $\Rightarrow$ (3) of Theorem A is the following adaptation of Chase's [**5**, Theorem 1.2]. To his ideas we only add the concept of a p-functor of Mod R. This is a subfunctor of the forgetful functor Mod $R \to$ Mod $\mathbf{Z}$ that commutes with direct products, see [**14**]. Such a functor automatically commutes with direct sums. The p-functorial subgroups of a module M_R include several important types of subgroups: All subgroups $M\mathfrak{a}$, where $\mathfrak{a}$ is a finitely generated left ideal of R, all annihilators in M of subsets of R and all finitely generated $\mathrm{End}(M_R)$-submodules of M, more generally, all subgroups $\mathrm{Hom}_R(A, M)(a)$ where A is an R-module and $a \in A$.

By [**14**, 3.4], M_R is Σ-algebraically compact iff M satisfies the minimum condition on subgroups PM where P is a (pure left exact) p-functor.

LEMMA 4. *Suppose that the direct product $\prod_{i\in\mathbf{N}} M_i$ of a family $(M_i)_{i\in\mathbf{N}}$ of R-modules is a direct sum of submodules Q_l, $l \in L$. Then, given a descending chain $P_1 \supset P_2 \supset P_3 \supset \cdots$ of p-functors, there exist a natural number n_0 and a finite subset L' of L such that*

$$q_l\Big(P_{n_0}\prod_{i>n_0} M_i\Big) \subset \bigcap_{n\in\mathbf{N}} P_n Q_l$$

for all $l \in L \setminus L'$ ($q_l\colon \bigoplus_{k\in L} Q_k \to Q_l$ is the natural projection).

We sketch the proof, since Chase's argument (see [**4**, Theorem 3.1] and [**5**, Theorem 1.2]) is slightly clarified by the use of p-functors.

PROOF. Assume the conclusion to be false. Then a standard induction yields a sequence $(n_i)_{i\in\mathbf{N}}$ of natural numbers with $n_{i+1} > n_i$ and sequences of pairwise different elements $l_i \in L$ resp., $m_i \in P_{n_i}(\prod_{j>n_i} M_j)$ so that

(1) $q_{l_i}(m_i) \notin P_{n_{i+1}} Q_{l_i}$, and

(2) $q_{l_i}(m_j) = 0$ for $j < i$.

Note that $m = \Sigma_{i \in \mathbf{N}} m_i \in \Pi_{j \in \mathbf{N}} M_j$ is well defined (since the sums of the components are finite in view of $m_i \in \Pi_{j \geqslant n_i} M_j$), and that for $k \in \mathbf{N}$ we have

$$q_{l_k}(m) = q_{l_k}(m_k) + q_{l_k}\left(\sum_{i>k} m_i\right) \neq 0,$$

since the first summand does not lie in $P_{n_{k+1}} Q_{l_k}$ whereas the second does. But this contradicts $m \in \bigoplus_{l \in L} Q_l$. $\square$

The next lemma is considerably stronger than is required for the proof of Theorem A.

LEMMA 5. *If $\Pi_{i \in I} M_i$ is a direct sum of submodules with local endomorphism rings and if $P_1 \supset P_2 \supset P_3 \supset \cdots$ is a chain of p-functors of* Mod R, *there is a natural number n_0 such that for each family $(A_i)_{i \in I}$ of indecomposable direct summands A_i of M_i we have $P_n A_i = P_{n_0} A_i$ for $n \geqslant n_0$ and almost all $i \in I$. In particular, each summand A_j whose isomorphism type occurs an infinite number of times in the family $(A_i)_{i \in I}$ is Σ-algebraically compact.*

PROOF. First, suppose that I is countable and write $I = \mathbf{N}$ for simplicity. Let $M = \Pi_{i \in \mathbf{N}} M_i = \bigoplus_{l \in L} Q_l$ be a decomposition as in the hypothesis. Choose $n_0 \in \mathbf{N}$ and a finite subset L' of L according to Lemma 4, and for $m > |L'|$ let $i_1, \ldots, i_m \geqslant n_0$ be pairwise different natural numbers. Since each A_i has in turn a local endomorphism ring, the finite sum $A_{i_1} \oplus \cdots \oplus A_{i_m}$ has the exchange property by **[12**, Proposition 1], hence there are pairwise different elements $l_1, \ldots, l_m$ of L with

$$M = \bigoplus_{k=1}^{m} A_{i_k} \oplus \bigoplus_{l \in L \setminus \{l_1, \ldots, l_m\}} Q_l$$

or, equivalently, the canonical projection $q\colon M = \bigoplus_{l \in L} Q_l \to \bigoplus_{k=1}^{m} Q_{l_k}$ induces an isomorphism $\bigoplus_{k=1}^{m} A_{i_k} \to \bigoplus_{k=1}^{m} Q_{l_k}$. In order to show that $(P_n)_{n \geqslant n_0}$ is stationary on at least one of the A_{i_k}'s let $l \in \{l_1, \ldots, l_m\} \setminus L'$ (such an l exists, because $m > |L'|$). Adopting the notation of Lemma 4, we have

$$P_{n_0} Q_l = q_l\left(\bigoplus_{k=1}^{m} P_{n_0} Q_{l_k}\right) = q_l q\left(\bigoplus_{k=1}^{m} P_{n_0} A_{i_k}\right) = q_l\left(\bigoplus_{k=1}^{m} P_{n_0} A_{i_k}\right) \subset \bigcap_{n \in \mathbf{N}} P_n Q_l,$$

that is, $P_n Q_l = P_{n_0} Q_l$ for $n \geqslant n_0$. But by the Krull-Schmidt theorem Q_l is isomorphic to one of the A_{i_k}'s.

We have proved $P_n A_j = P_{n_0} A_j$ for $n \geqslant n_0$, provided that A_j is isomorphic to an infinite number of A_i's, $i \in \mathbf{N}$. The same argument as above shows that for $j \geqslant n_0$ each A_j which is not isomorphic to any Q_l, $l \in L'$, also satisfies $P_n A_j = P_{n_0} A_j$ for $n \geqslant n_0$. Thus the number of A_j's violating this equation must be finite.

Now let I be arbitrary. If our claim were false, there would be a family $(A_i)_{i \in I}$, a sequence $(i_k)_{k \in \mathbf{N}}$ of pairwise different elements of I and a strictly increasing sequence $(n_k)_{k \in \mathbf{N}}$ of natural numbers so that $P_{n_k} A_{i_k} \supsetneqq P_{n_{k+1}} A_{i_k}$ for each k. Divide I into pairwise disjoint subsets I_k, $k \in \mathbf{N}$, with $i_k \in I_k$ and define $M_k' = \Pi_{i \in I_k} M_i$. Then each A_{i_k} is a direct summand of M_k', and our claim is also violated for the countable product $M = \prod_{k\in\mathbf{N}} M'_k$ and the family $(A_{i_k})_{k\in\mathbf{N}}$. But this we have already proved to be impossible. $\square$

REMARK. The same arguments as employed in the proof of Lemma 5 yield the following generalization of the implications (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3) of Theorem A (the latter will be deduced directly from Lemma 5 in the sequel): If $\prod_{i\in I} M_i$ possesses a direct-sum decomposition which complements direct summands, or if each product $\prod_{i\in \tilde{I}} M_i$ with $\tilde{I} \subset I$ is a direct sum of modules with local endomorphism rings, then, for each descending chain $P_1 \supset P_2 \supset P_3 \supset \cdots$ of p-functors of Mod R, there is a natural number n_0 so that $P_n M_i = P_{n_0} M_i$ for $n \geqslant n_0$ and almost all $i \in I$.

PROOF OF THEOREM A. (1) $\Rightarrow$ (3). Let $M_i = M^{\mathbf{N}}$ for $i \in \mathbf{N}$. By hypothesis each M_i is a direct sum of modules with local endomorphism rings, and so is $\prod_{i\in\mathbf{N}} M_i \cong M^{\mathbf{N}}$. An application of Lemma 5 shows that $M^{\mathbf{N}}$ is Σ-a.c., hence M has the same property. (2) $\Rightarrow$ (1) follows from [**2**, Proposition 12.10]. For the remaining implication (3) $\Rightarrow$ (2) note that Σ-algebraic compactness is inherited by all pure submodules of direct products of copies of M (see the criterion preceding Lemma 4). Since moreover every Σ-a.c. module has an indecomposable decomposition [**14**, p. 1100], our claim is covered by Theorem B following.

THEOREM B. *Every Σ-algebraically compact module M_R has the exchange property (i.e., $M' \oplus N = \bigoplus_{i\in I} A_i$ with $M' \cong M$ implies the existence of submodules C_i of A_i such that $M' \oplus N = M' \oplus \bigoplus_{i\in I} C_i$).*

PROOF. Suppose that $A = M' \oplus N = \bigoplus_{i\in I} A_i$ with $M' \cong M$. Since M' is again Σ-a.c., we may assume $M' = M$. By Zorn's lemma there is a maximal submodule $C = \bigoplus_{i\in I} C_i$ of A with $C_i \subset A_i$ for all i, so that $M \cap C = 0$ and the canonical monomorphism $M \to A/C \cong \bigoplus_{i\in I} A_i/C_i$ is pure. The canonical image $\overline{M}$ of M in $\overline{A} = A/C$ being also Σ-a.c., we have $\overline{A} = \overline{M} \oplus X$. We wish to show $X = 0$. In order to simplify the notation, we replace $\overline{A}$ by A and $\overline{M}$ by M, i.e. we assume the following situation: $A = M \oplus X = \bigoplus_{i\in I} A_i$ with M Σ-a.c. so that, for $0 \neq A'_i \subset A_i$ with $M \cap A'_i = 0$, the sum $M \oplus A'_i$ is not a direct summand of A (the latter is a consequence of the choice of C).

First, we recognize that in this setting A is Σ-a.c.: M has the finite exchange property by [**15**, p. 87(2)], so, fixing $h \in I$, we have $A = M \oplus A'_h \oplus A'$ with $A'_h \subset A_h$ and $A' \subset \bigoplus_{i\in I\setminus\{h\}} A_i$. It follows that $A'_h = 0$, hence A_h is isomorphic to a direct summand of M, and our first claim is established. In particular, all of the A_i's, as well as X, can be decomposed into submodules with local endomorphism rings, say $A_i = \bigoplus_{l\in L_i} A_{il}$ and $X = \bigoplus_{j\in J} X_j$, see [**14**, p. 1100]. If X were nonzero, i.e. J nonempty, say $k \in J$, then the module $M \oplus \bigoplus_{j\in J\setminus\{k\}} X_j$ would be a maximal direct summand of A. Since by Azumaya's theorem (see, e.g., [**2**, Theorem 12.6]) the decomposition $A = \bigoplus_{i\in I; l\in L_i} A_{il}$ complements maximal direct summands, we would infer

$$A = \Big(M \oplus \bigoplus_{j \in J \setminus \{k\}} X_j\Big) \oplus A_{il}$$

for some i and l, which is incompatible with the initial situation. Thus, we conclude $X = 0$. □

Given a Σ-algebraically compact module M, write $M = \bigoplus_{i \in I} M_i$ with $\mathrm{End}_R(M_i)$ local (by [**14**, p. 1100] such a decomposition always exists). According to [**12**, Proposition 2], each finite sum $M' = \bigoplus_{i \in I'} M_i$ has the cancellation property (i.e., $A \oplus M' \cong B \oplus M'$ implies $A \cong B$), but of course this fails for infinite sums. Before being allowed to cancel M in a relation $A \oplus M \cong B \oplus M$, we must clear A and B of direct summands of M. More precisely, the following is true:

COROLLARY 6. *Let M be Σ-algebraically compact, $M = \bigoplus_{i \in I} M_i$ an indecomposable decompositon. Then every R-module A can be uniquely (up to isomorphism) decomposed in the form $A = A_1 \oplus A_2$ where A_1 is isomorphic to a direct sum of M_i's and A_2 and M have no common nonzero direct summand. In particular, $A \oplus M \cong B \oplus M$ with $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ as above implies $A_2 \cong B_2$ and $A_1 \oplus M \cong B_1 \oplus M$.*

PROOF. Existence: By Zorn's lemma there is a maximal family $(C_j)_{j \in J}$ of submodules of A with the following properties:

(1) each C_j is isomorphic to some M_i;

(2) the sum $\Sigma_{j \in J} C_j$ is direct;

(3) each finite subsum $\Sigma_{j \in J'} C_j$ is a direct summand of A.

Since $A_1 = \bigoplus_{j \in J} C_j$ is Σ-a.c. and a pure submodule of A, we have $A = A_1 \oplus A_2$ for some submodule A_2 of A. The maximality of $(C_j)_{j \in J}$ guarantees that A_2 contains no direct summand isomorphic to one of the M_i's. It follows that M and A_2 have no nonzero isomorphic direct summand at all, because direct summands of M are in turn Σ-a.c. and hence direct sums of indecomposable modules (necessarily isomorphic to certain M_i's).

Uniqueness: Suppose $A = A_1 \oplus A_2 = D_1 \oplus D_2$ are two decompositions of the considered type. By investing the exchange property of A_1 (Theorem B), we obtain an equation $A = A_1 \oplus D_1' \oplus D_2'$ with $D_i' \subset D_i$ for $i = 1, 2$, say $D_i = D_i' \oplus D_i''$. From the property of A_2 we infer $D_1' = 0$, and, in view of $A_1 \cong D_1 \oplus D_2''$, we obtain further $D_2'' = 0$. This means $A_1 \cong D_1$ and $A = A_1 \oplus D_2$. But the latter shows that also $A_2 \cong D_2$. □

We restate Theorem B (resp., Corollary 6) for a special class of Σ-a.c. modules.

COROLLARY 7. *Let $(M_i)_{i \in I}$ be a family of R-modules containing only finitely many isomorphism types and suppose that the modules M_i are artinian over their endomorphism rings. Then every direct summand M of the direct sum or the direct product of the M_i's has the exchange property and the cancellation property of Corollary 7. Moreover, M has a decomposition that complements direct summands.*

For R commutative, this extends [2, Corollary 29.6], for R a Dedekind domain we rediscover classical theorems on bounded R-modules. Moreover, we infer that, for R right Σ-algebraically compact (e.g., for R left artinian), every right R-module A has (uniquely up to isomorphism) a decomposition $A = A_1 \oplus A_2$ where A_1 is projective and A_2 has no projective direct summand; in particular, given an isomorphism of R-modules $A \oplus M \cong B \oplus M$, where M is projective and A, B have no nonzero projective direct summands, M may be cancelled.

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