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RINGS WITH THE CONTRACTION PROPERTY

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ABSTRACT. A ring R (not necessarily commutative or with unit) has the contraction property iff every ideal of every subring of R is a contracted ideal.

It is shown that R is a primitive ring with the contraction property iff R is an absolutely algebraic field. This result, together with the fact that the Jacobson Radical of a ring with the contraction property is nil, shows that a nil semisimple ring with the contraction property is a subdirect sum of absolutely algebraic fields (and is therefore commutative).

It is shown that if R is a torsion free nil ring with the contraction property then $R^2=(0)$. It follows that any torsion free ring with the contraction property is the extension of a zero ring and a subdirect sum of absolutely algebraic fields. Also, if R is a nil ring with the contraction property then R^2 is torsion as an additive group.

1. Rings with the contraction property.

DEFINITION 1.1. A ring R (not necessarily commutative or with unit) has the contraction property iff every ideal of every subring of R is a contracted ideal; that is, if S is a subring of R and I an ideal of S, we have $I=T\cap S$ for some ideal T of R [6, pp. 218–221]).

Hereafter, we call R a c-ring iff it satisfies the condition of Definition 1.1. The ring of integers is a simple example of a c-ring. Another example is any absolutely algebraic field F [3, p. 147], for if F is any absolutely algebraic field and S is any subring of F, it is easy to check that S is actually a subfield of F. Thus, the only ideals of S are (0) and S which are contracted ideals.

THEOREM 1.1. *The class of all c-rings is closed under homomorphisms, direct sums, if each summand is a ring with unit, and is hereditary for subrings.*

PROOF. (1) Let A be a c-ring and B a homomorphic image of A via a homomorphism φ. Let S be a subring of B, I an ideal of S. $\varphi^{-1}(I)$ is an ideal in $\varphi^{-1}(S)$, a subring of A, thus $\varphi^{-1}(I)=T\cap\varphi^{-1}(S)$, with

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T an ideal of A. Let $T' = (T + \text{Ker } \varphi)$. It is easy to check that $I = \varphi(T') \cap S$.

(2) Let $\{A_\alpha | \alpha \in V\}$ be a set of c-rings with unit and $A = \sum_{\alpha \in V} \oplus A_\alpha$. Let S be a subring of A, I an ideal in S. Let π_α be the projection of A onto $\overline{A}_\alpha$, the subring of A naturally isomorphic to the ring A_α. For each α $\pi_\alpha(I)$ is an ideal in $\pi_\alpha(S)$, a subring of the c-ring $\overline{A}_\alpha$. Thus, for all α we have $\pi_\alpha(I) = \pi_\alpha(S) \cap \overline{I}_\alpha$, where $\overline{I}_\alpha$ is an ideal in $\overline{A}_\alpha$. Let $\overline{I} = \sum_{\alpha \in V} \oplus \overline{I}_\alpha$. It is now easy to check, as each A_α is a ring with unit, that $\overline{I}$ is an ideal in A with $\overline{I} \cap S = I$.

(3) Let B be a subring of a c-ring A, S be a subring of B, I be an ideal in S. As S is also a subring of A, we have $I = \overline{I} \cap S$ for $\overline{I}$ an ideal of A. But then, clearly, $I = (\overline{I} \cap B) \cap S$, so B is also a c-ring.

2. c-rings without nil ideals.

We next consider c-rings without nil ideals. First we classify all primitive c-rings.

THEOREM 2.1. *R is a primitive c-ring iff R is an absolutely algebraic field.*

PROOF. We have already noted that any absolutely algebraic field is a c-ring.

Now let R be any primitive c-ring. Let M be an irreducible R module with centralizer Γ. We first note that dim $(M:\Gamma) = 1$, otherwise we could find a subring $S \subset R$ which could be mapped homomorphically onto $(\Gamma)_{2\times 2}$—the ring of all 2×2 matrices with entries from Γ, [**4**, p. 33, Theorem 3]. But this, together with Theorem **1.1**, would imply that $(\Gamma)_{2\times 2}$ is a c-ring—a contradiction, since

$$\left\{\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}\right\} \quad \text{is an ideal in the subring} \quad \left\{\begin{pmatrix} b & 0 \\ a & c \end{pmatrix}\right\}$$

which is not a contracted ideal. As dim $(M:\Gamma) = 1$ we have $R = \Gamma$, Γ a division ring.

We next note that Γ has characteristic $p \neq 0$, since otherwise Z, the ring of integers, would be a subring of Γ. As Γ is simple, no proper ideal of Z could be contracted—a contradiction.

Let Z_p denote the prime field of Γ. For all $x \in \Gamma$, x must be algebraic over Z_p, since otherwise the polynomial ring $Z_p[x]$ would be a subring of Γ in which no proper ideal was contracted. Finally, as $Z_p[x]$ is a finite field, for all $x \in \Gamma$ we have $x^{p^k} = x$ where $k = \dim(Z_p[x]:Z_p)$. But this means that Γ must be commutative [**2**, p. 72]. Thus, $R = \Gamma$ is an absolutely algebraic field.

In [**5**, Theorem 6], we proved that the Jacobson Radical of any c-ring is nil. This, together with 2.1, gives the following result.

THEOREM 2.2. *Let R be a c-ring without nil ideals. Then R is a subdirect sum of absolutely algebraic fields (and is therefore commutative).*

3. Nil c-rings.

First we consider nilpotent c-rings.

THEOREM 3.1. *Let R be a torsion free nilpotent c-ring. Then $R^3=(0)$.*

PROOF. Let R be a nilpotent c-ring of index $k\geqq 4$. Choose $x_1, x_2, \cdots, x_{k-1}$ in R such that $x_1x_2\cdots x_{k-1}\neq 0$. The set of all integral multiples of $x_1\cdots x_{k-2}$ is a subgroup of R^2 as $k\geqq 4$. As $R^k=(0)$ this subgroup is moreover an ideal in R^2. As R is a c-ring this subgroup must actually be an ideal in R. Thus, we have $x_1\cdots x_{k-2}\cdot x_{k-1} = mx_1\cdots x_{k-2}$ for some integer m. This yields $mx_1\cdots x_{k-1}=0$, a contradiction since R is torsion free.

THEOREM 3.2. *Let R be a torsion free nil c-ring. Then $R^3=(0)$.*

PROOF. As R is a torsion free nil c-ring, for all $x\in R$, $\langle x\rangle$, the subring generated by x, is a torsion free nilpotent c-ring. Thus $x\in R$ implies $x^3=0$. We now can conclude that R is locally nilpotent [**1**, p. 130]. Let u, v, w be any elements of R. S, the subring generated by u, v, w is a torsion free nilpotent c-ring. Applying 3.1 we have $S^3=(0)$. Thus, $uvw=0$ and we have $R^3=(0)$.

THEOREM 3.3. *Let R be a torsion free nil c-ring. Then $R^2=(0)$.*

PROOF. First we note that $x^2=0$ for every $x\in R$. To see this, let $\langle x\rangle$ be the subring generated by x and m an integer with $m>1$. $I=\{tmx+lm^2x^2 \mid t,\ l\in Z\}$ is an ideal of $m\langle x\rangle$. As $\langle x\rangle$ is a c-ring and $m\langle x\rangle$ is an ideal in $\langle x\rangle$ we see that I must be an ideal in $\langle x\rangle$. Thus $mx^2=tmx+lm^2x^2$ for some $t, l\in Z$. Multiplying by x we have $tmx^2=0$. If $t\neq 0$, as R is torsion free, we have $x^2=0$. If $t=0$ then $(lm^2-m)x^2=0$ and $lm^2-m\neq 0$ so $x^2=0$.

Now let u, v be arbitrary elements of R. We have $uv+vu=(u+v)^2-u^2-v^2=0$, so $uv=-uv$. Let S be the subring of R generated by u and v. Now $L=\{tu+suv \mid t,\ s\in Z\}$ is an ideal in S since $S^3=(0)$ and $uv=-vu$. Also, $K=\{tu \mid t\in Z\}$ is an ideal in L since $KL=LK=(0)$. As S is a c-ring, we have K is an ideal in S. Thus $uv=tu$ for some $t\in Z$. If $t=0$ then $uv=0$; if $t\neq 0$ multiplying by v yields $tuv=0$ and again $uv=0$ as R is torsion free. As u, v were arbitrary we have $R^2=(0)$.

COROLLARY. *Let R be a torsion free c-ring. Then R is the extension of a zero ring and a subdirect sum of absolutely algebraic fields.*

COROLLARY. *Let R be a nil c-ring. Then R^2 is torsion as an additive group.*

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