$\mathcal{RIQ}$ and $\mathcal{SROIQ}$ are Harder than $\mathcal{SHOIQ}$*

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Abstract

We identify the computational complexity of (finite model) reasoning in the sublanguages of the description logic $\mathcal{SROIQ}$—the logic currently proposed as the basis for the next version of the web ontology language OWL. We prove that the classical reasoning problems are N2ExpTime-complete for $\mathcal{SROIQ}$ and 2ExpTime-hard for its sublanguage $\mathcal{RIQ}$. $\mathcal{RIQ}$ and $\mathcal{SROIQ}$ are thus exponentially harder than $\mathcal{SHIQ}$ and $\mathcal{SHOIQ}$. The growth in complexity is due to complex role inclusion axioms of the form $R_1 \circ \cdots \circ R_n \sqsubseteq R$, which are known to cause an exponential blowup in the tableau-based procedures for $\mathcal{RIQ}$ and $\mathcal{SROIQ}$. Our complexity results, thus, also prove that this blowup is unavoidable. We also demonstrate that the hardness results hold already for linear role inclusion axioms of the form $R_1 \circ R_2 \sqsubseteq R_1$ and $R_1 \circ R_2 \sqsubseteq R_2$.

Introduction

In this paper we study the complexity of reasoning in sublanguages of $\mathcal{SROIQ}$—the logic chosen as the basis for the next version of the web ontology language OWL—OWL 2.[1] $\mathcal{SROIQ}$ has been introduced in (Horrocks, Kutz, and Sattler 2006) as an extension of $\mathcal{SRIQ}$ (Horrocks, Kutz, and Sattler 2005), which itself is an extension of $\mathcal{RIQ}$ (Horrocks and Sattler 2003; 2004). For every of these logics a corresponding tableau-based procedure has been provided.

In contrast to sub-languages of $\mathcal{SHOIQ}$ whose computational properties are currently well understood (Tobies 2001), the complexity of languages between $\mathcal{RIQ}$ and $\mathcal{SROIQ}$ has been rather unexplored: it is known that $\mathcal{RIQ}$ and $\mathcal{SRIQ}$ are ExpTime-hard as extensions of $\mathcal{SHIQ}$, and $\mathcal{SROIQ}$ is NExpTime-hard as an extension of $\mathcal{SHOIQ}$. The difficulty in extending the existing complexity proofs to $\mathcal{RIQ}$ and $\mathcal{SROIQ}$ are caused by complex role inclusion axioms of the form $R_1 \circ \cdots \circ R_n \sqsubseteq R$. The unrestricted usage of such axioms easily leads to undecidability of modal and description logics (Baldoni, Giordano, and Martelli 1998; Demri 2001; Horrocks and Sattler 2004), with the notable exception of $\mathcal{EL}^{++}$ (Baader 2003; Baader, Brandt, and Lutz 2005). Therefore, in order to ensure decidability, special syntactic restrictions have been imposed on complex role inclusion axioms in $\mathcal{RIQ}$. In a nutshell, the restrictions ensure that the axioms $R_1 \circ \cdots \circ R_n \sqsubseteq R$ when viewed as production rules of context-free grammars $R \to R_1 \ldots R_n$, induce regular languages—a property that has been used before to characterize a decidable class of multi-modal logic called regular grammar logics (del Cerro and Panttonen 1988; Demri 2001; Demri and de Nivelle 2005). The tableau procedure for $\mathcal{RIQ}$ works with complex role inclusion axioms via the corresponding regular automata for these languages. Unfortunately, the size of the automata can be exponential in the number of axioms, which results in an exponential blowup in the worst-case behaviour of the procedure for $\mathcal{RIQ}$ in comparison to the procedure for $\mathcal{SHIQ}$. It has been an open problem whether this blowup can be avoided (Horrocks and Sattler 2003). In this paper we demonstrate that $\mathcal{RIQ}$ and $\mathcal{SROIQ}$ are indeed exponentially harder than respectively $\mathcal{SHIQ}$ and $\mathcal{SHOIQ}$, which implies that the blowup in the tableau procedures could not be avoided.

This paper is an extended version of (Kazakov 2008) containing new results on linear role inclusion axioms.

Preliminaries

We assume that the reader is familiar with the DL $\mathcal{SHOIQ}$ (Horrocks and Sattler 2007). A $\mathcal{SHOIQ}$ *signature* is a tuple $\Sigma = (C_\Sigma, R_\Sigma, I_\Sigma)$ consisting of the sets of *atomic concepts* C_Σ, *atomic roles* R_Σ and *individuals* I_Σ. A *role* is either some $r \in R_\Sigma$ or an *inverse role* r^-. For each $r \in R_\Sigma$, we set $\mathsf{Inv}(r) = r^-$ and $\mathsf{Inv}(r^-) = r$. A $\mathcal{SHOIQ}$ *RBox* is a finite set $\mathcal{R}$ of *role inclusion axioms* (RIA) $R_1 \sqsubseteq R$, *transitivity axioms* $\mathsf{Tra}(R)$ and *functionality axioms* $\mathsf{Fun}(R)$ where R_1 and R are roles. Let $\sqsubseteq_\mathcal{R}$ be the smallest reflexive transitive relation on roles such that $R_1 \sqsubseteq R \in \mathcal{R}$ implies $R_1 \sqsubseteq_\mathcal{R} R$ and $\mathsf{Inv}(R_1) \sqsubseteq_\mathcal{R} \mathsf{Inv}(R)$. A role S is called *simple w.r.t.* $\mathcal{R}$ if there is no role R such that $R \sqsubseteq_\mathcal{R} S$ and either $\mathsf{Tra}(R) \in \mathcal{R}$ or $\mathsf{Tra}(\mathsf{Inv}(R)) \in \mathcal{R}$. Given an RBox $\mathcal{R}$, the set of $\mathcal{SHOIQ}$ concepts is the smallest set containing $\top$, $\bot$, A, $\{a\}$, $\neg C$, $C \sqcap D$, $C \sqcup D$, $\exists R.C$, $\forall R.C$, $\geqslant n\, S.C$, and $\leqslant n\, S.C$, where A is an atomic concept, a an individual, C and D concepts, R a role, S a simple role w.r.t. $\mathcal{R}$, and n a non-negative integer. A $\mathcal{SHOIQ}$ *TBox* is a finite set $\mathcal{T}$ of *general concept inclusion axioms* (GCIs) $C \sqsubseteq D$ where C and D are concepts. We write $C \equiv D$ as an abbreviation for

*Unless 2ExpTime = NExpTime, in which case just $\mathcal{SROIQ}$ is harder than $\mathcal{SHOIQ}$ because NExpTime $\subsetneq$ N2ExpTime

[1] A.k.a. OWL 1.1: http://www.webont.org/owl/1.1

$C \sqsubseteq D$ and $D \sqsubseteq C$. A $\mathcal{SHOIQ}$ *ABox* is a finite set consisting of *concept assertions* $C(a)$ and *role assertions* $R(a,b)$ where a and b are individuals from I_Σ. A $\mathcal{SHOIQ}$ *ontology* is a triple $\mathcal{O} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$, where $\mathcal{R}$ a $\mathcal{SHOIQ}$ RBox, $\mathcal{T}$ is a $\mathcal{SHOIQ}$ TBox, and $\mathcal{A}$ is a $\mathcal{SHOIQ}$ ABox. $\mathcal{SHIQ}$ is a sub-language of $\mathcal{SHOIQ}$ that does not use *nominals* $\{a\}$.

A $\mathcal{SHOIQ}$ *interpretation* is a pair $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ where $\Delta^\mathcal{I}$ is a non-empty set called the *domain* of $\mathcal{I}$, and $\cdot^\mathcal{I}$ is the *interpretation function*, which assigns to every $A \in C_\Sigma$ a subset $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, to every $r \in R_\Sigma$ a relation $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, and to every $a \in I_\Sigma$, an element $a^\mathcal{I} \in \Delta^\mathcal{I}$. The interpretation $\mathcal{I}$ is *finite* iff $\Delta^\mathcal{I}$ is finite. $\mathcal{I}$ is extended to complex role, complex concepts, axioms, and assertions in the usual way (Horrocks and Sattler 2007). $\mathcal{I}$ is a *model* of a $\mathcal{SHOIQ}$ ontology $\mathcal{O}$, if every axiom and assertion in $\mathcal{O}$ is satisfied in $\mathcal{I}$. $\mathcal{O}$ is *(finitely) satisfiable* if there exists a (finite) model $\mathcal{I}$ for $\mathcal{O}$. A concept C is *(finitely) satisfiable* w.r.t. $\mathcal{O}$ if $C^\mathcal{I} \neq \emptyset$ for some (finite) model $\mathcal{I}$ of $\mathcal{O}$. The problem of (concept) satisfiability is ExpTime-complete for $\mathcal{SHIQ}$, and NExpTime-complete for $\mathcal{SHOIQ}$ (see, e.g., Tobies 2000; 2001).[2]

$\mathcal{RIQ}$ (Horrocks and Sattler 2004) extends $\mathcal{SHIQ}$ with *complex RIAs* in RBoxes of the form $R_1 \circ \cdots \circ R_n \sqsubseteq R$ which are interpreted as ${R_1}^\mathcal{I} \circ \cdots \circ {R_n}^\mathcal{I} \subseteq R^\mathcal{I}$, where $\circ$ is the usual composition of binary relations. A *regular order on roles* is an irreflexive transitive binary relation $\prec$ on roles such that $R_1 \prec R_2$ iff $\mathsf{Inv}(R_1) \prec R_2$. A RIA $R_1 \circ \cdots \circ R_n \sqsubseteq R$ is said to be *$\prec$-regular*, if either: (i) $n = 2$ and $R_1 = R_2 = R$, or (ii) $n = 1$ and $R_1 = \mathsf{Inv}(R)$, or (iii) $R_i \prec R$ for $1 \leq i \leq n$, or (iv) $R_1 = R$ and $R_i \prec R$ for $1 < i \leq n$, or (v) $R_n = R$ and $R_i \prec R$ for $1 \leq i < n$.[3] A $\mathcal{RIQ}$ RBox $\mathcal{R}$ is *regular* if there exists a regular order on roles $\prec$ such that each RIA from $\mathcal{R}$ is $\prec$-regular. A $\mathcal{RIQ}$ ontology can contain only a regular RBox $\mathcal{R}$. The notion of simple role is extended in $\mathcal{RIQ}$ as follows. Let $\sqsubseteq_\mathcal{R}$ be the smallest relation such that $R_1 \circ \cdots \circ R_n \sqsubseteq_\mathcal{R} R$ if either $n = 1$ and $R_1 = R$, or there exist $1 \leq i \leq j \leq n$ and R' such that $R_1 \circ \cdots \circ R_{i-1} \circ R' \circ R_{j+1} \cdots \circ R_n \sqsubseteq_\mathcal{R} R$ and $R_i \circ \ldots \circ R_j \sqsubseteq R' \in \mathcal{R}$ or $\mathsf{Inv}(R_j) \circ \ldots \circ \mathsf{Inv}(R_i) \sqsubseteq R' \in \mathcal{R}$. A role S is *simple w.r.t.* $\mathcal{R}$ if there are no roles $R_1, \ldots, R_n$ with $n \geq 2$ such that $R_1 \circ \cdots \circ R_n \sqsubseteq_\mathcal{R} S$.

$\mathcal{SRIQ}$ (Horrocks, Kutz, and Sattler 2005) further extends $\mathcal{RIQ}$ with: (1) the *universal role* U, which is interpreted as the total relation: $U^\mathcal{I} = \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, and cannot occur in complex RIAs, (2) *negative role assertions* $\neg R(a,b)$, (3) the concept constructor $\exists S.\mathsf{Self}$ interpreted as $\{x \in \Delta^\mathcal{I} \mid \langle x, x\rangle \in S^\mathcal{I}\}$ where S is a simple role, (4) the new role axioms $\mathsf{Sym}(R)$, $\mathsf{Ref}(R)$, $\mathsf{Asy}(S)$, $\mathsf{Irr}(S)$, $\mathsf{Disj}(S_1, S_2)$ where $S_{(i)}$ are simple roles, which restrict $R^\mathcal{I}$ to be *symmetric* or *reflexive*, $S^\mathcal{I}$ to be *asymmetric* or *irreflexive*, or ${S_1}^\mathcal{I}$ and ${S_2}^\mathcal{I}$ to be *disjoint*. $\mathcal{SROIQ}$ (Horrocks, Kutz, and Sattler 2006) extends $\mathcal{SRIQ}$ with nominals like in $\mathcal{SHOIQ}$.

[2] For further information and references on complexities of DLs see `http://www.cs.man.ac.uk/~ezolin/dl/`

[3] The original definition of $\mathcal{RIQ}$ (Horrocks and Sattler 2003), admits only RIAs $R_1 \circ \cdots \circ R_n \sqsubseteq R$ with $n \leq 2$; in this paper we assume the definition for $\mathcal{RIQ}$ from (Horrocks and Sattler 2004)

The Exponential Blowup for Regular RIAs

In this section we discuss the main reason why the tableau procedures for $\mathcal{RIQ}$, $\mathcal{SRIQ}$, and $\mathcal{SROIQ}$ in (Horrocks and Sattler 2004; Horrocks, Kutz, and Sattler 2005; 2006) incur an exponential blowup.

Given an RBox $\mathcal{R}$ containing complex RIAs and a role R, let $L_\mathcal{R}(R)$ be the language defined by:

$$L_\mathcal{R}(R) := \{R_1 R_2 \ldots R_n \mid R_1 \circ \cdots \circ R_n \sqsubseteq_\mathcal{R} R\} \qquad (1)$$

It has been shown in (Horrocks and Sattler 2004) that for every regular RBox $\mathcal{R}$ and every role R the language $L_\mathcal{R}(R)$ is regular. The tableau procedures for $\mathcal{RIQ}$ and $\mathcal{SROIQ}$, utilize non-deterministic finite automata (NFA) corresponding to $L_\mathcal{R}(R)$ to ensure that only finitely many states are produced by the tableau expansion rules. Unfortunately, the NFA for $L_\mathcal{R}(R)$ can be exponentially large in the size of $\mathcal{R}$, which results in exponential blowup in the number of states produced in the worst case by the procedure for $\mathcal{RIQ}$ and $\mathcal{SROIQ}$ compared to the procedures for $\mathcal{SHIQ}$ and $\mathcal{SHOIQ}$. It was conjectured in (Horrocks, Kutz, and Sattler 2006) that without further restrictions on RIAs such blowup is unavoidable. In Example 1, we demonstrate that minimal automata for regular RBoxes can be exponentially large.

Example 1. Let $\mathcal{R}$ be an RBox consisting of the RIA (2).

$$r \circ v \circ r \sqsubseteq v \qquad (2)$$

The RIA (2) is not $\prec$-regular regardless of the ordering $\prec$. Indeed, (2) does not satisfy conditions (i)–(ii) of $\prec$-regularity since $n = 3$, and it does not satisfy conditions (iii)–(iv) since $v = R_2 \not\prec R = v$. It is easy to see that $L_\mathcal{R}(s) = \{r^i v r^i \mid i \geq 0\}$, where r^i denotes the word consisting of i letters r. Thus the language $L_\mathcal{R}(v)$ is non-regular, which can be shown, e.g., by using the pumping lemma for regular languages (see, e.g., Sipser 2005).

As an example of a regular RBox, consider the RIAs (3) over the atomic roles $v_0, \ldots, v_n$.

$$v_i \circ v_i \sqsubseteq v_{i+1} \qquad 0 \leq i < n \qquad (3)$$

It is easy to see that these axioms satisfy condition (iii) of $\prec$-regularity for every ordering $\prec$ such that $v_i \prec v_j$, for every $0 \leq i < j \leq n$. By induction on i, it is easy to show that $L_\mathcal{R}(v_i)$ consist of finitely many words, and hence, are all regular. It is also easy to show that $v_0^j \in L_\mathcal{R}(v_i)$ iff $j = 2^i$. Let $Q(v_i)$ be an NFA for $L_\mathcal{R}(v_i)$ and $q_0, \ldots, q_j$ a run in $Q(v_i)$ accepting v_0^j for $j = 2^i$. Then all states in this run are different, since otherwise there is a cycle, which would mean that $Q(v_i)$ accepts infinitely many words. Hence $Q(v_i)$ has at least $j + 1 = 2^i + 1$ states.

Although Example 1 does not demonstrate the usage of the conditions (i), (ii), (iv) and (v) for $\prec$-regularity of RIAs, as will be shown in the next section, axioms that satisfy just the condition (iii) already make reasoning in $\mathcal{RIQ}$ and $\mathcal{SROIQ}$ hard.

The Lower Complexity Bounds

In this section, we present two hardness results for fragments of $\mathcal{SROIQ}$. First, we prove that reasoning in $\mathcal{R}$—a fragment of $\mathcal{RIQ}$ that does not use counting and inverse roles—is 2ExpTime-hard. The proof is by reduction from the word

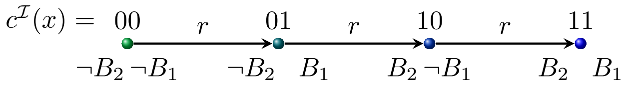

Figure 1: Encoding exponentially long chains

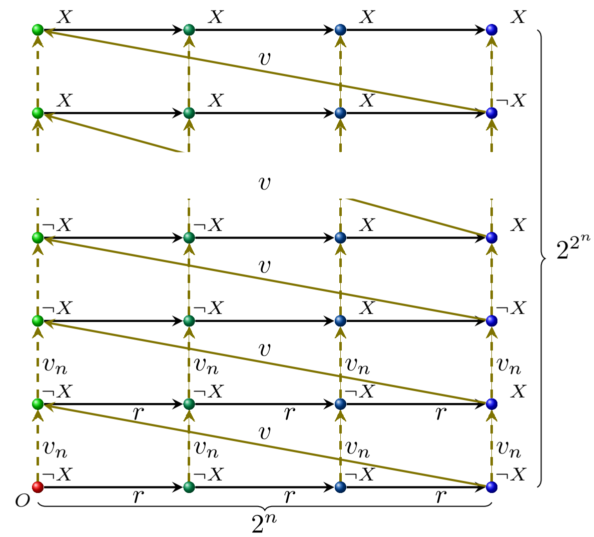

Figure 2: Encoding a double-exponentially long chain

problem for an exponential-space alternating Turing machine. Second, we demonstrate that reasoning in $\mathcal{ROIF}$—the extension of $\mathcal{R}$ with nominals, inverse roles and functional roles—is N2ExpTime-hard. The proof of this result is by reduction from the doubly-exponential Domino tiling problem.

The main idea of our reductions is to enforce double-exponentially long chains using axioms in the DL $\mathcal{R}$. Single-exponentially long chains can be enforced using a well-known "integer counting" technique (Tobies 2000). A *counter* $c^{\mathcal{I}}(x)$ is an integer between 0 and $2^n - 1$ assigned to an element x of the interpretation $\mathcal{I}$ using n atomic concepts $B_1, \ldots, B_n$ as follows: the k^{th} bit of $c^{\mathcal{I}}(x)$ is equal to 1 if and only if $x \in {B_k}^{\mathcal{I}}$. It is easy to see that axioms (4)–(8) induce an exponentially long r-chain by initializing the counter and incrementing it over the role r (see Figure 1).

$$Z \equiv \neg B_1 \sqcap \cdots \sqcap \neg B_n \tag{4}$$
$$E \equiv B_1 \sqcap \cdots \sqcap B_n \tag{5}$$
$$\neg E \sqsubseteq \exists r.\top \tag{6}$$
$$\top \equiv (B_1 \sqcap \forall r.\neg B_1) \sqcup (\neg B_1 \sqcap \forall r.B_1) \tag{7}$$
$$B_{k-1} \sqcap \forall r.\neg B_{k-1} \equiv (B_k \sqcap \forall r.\neg B_k) \sqcup (\neg B_k \sqcap \forall r.B_k) \quad 1 < k \leq n \tag{8}$$

Axiom (4) is responsible for initializing the counter to zero using the atomic concept Z. Axiom (5) can be used to detect whether the counter has reached the final value $2^n - 1$, by checking whether E holds. Thus, using axiom (6), we can express that every element whose counter has not reached the final value has an r-successor. Axioms (7) and (8) express how the counter is incremented over r: axiom (7) expresses that the lowest bit of the counter is always flipped; axioms (8) express that any other bit of the counter is flipped if and only if the lower bit is changed from 1 to 0.

Lemma 2. *Let $\mathcal{O}$ be an ontology containing axioms* (4)–(8)*. Then for every model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of $\mathcal{O}$ and $x \in Z^{\mathcal{I}}$ there exist $x_i \in \Delta^{\mathcal{I}}$ with $0 \leq i < 2^n$ such that $x = x_0$, $\langle x_{i-1}, x_i \rangle \in r^{\mathcal{I}}$ when $1 \leq i < 2^n$, and $c^{\mathcal{I}}(x_i) = i$.*

Proof. We construct the required $x_i \in \Delta^{\mathcal{I}}$ with $0 \leq i < 2^n$ by induction on i and simultaneously show that $c^{\mathcal{I}}(x_i) = i$. For the base case $i = 0$ we take $x_0 := x$. Since $\mathcal{I}$ is a model of (4) and $x \in Z^{\mathcal{I}}$, we have $c^{\mathcal{I}}(x_0) = 0$. For the induction step, assume that we have constructed x_i with $1 \leq i < 2^n - 1$ and $c^{\mathcal{I}}(x_i) = i$. We construct x_{i+1} and prove that $c^{\mathcal{I}}(x_{i+1}) = i + 1$. Consider $X_{i+1} = \{x \mid \langle x_i, x \rangle \in r^{\mathcal{I}}\}$. Since $\mathcal{I}$ is a model of (6) and $c^{\mathcal{I}}(x_i) \neq 2^n - 1$, we have that $x_i \notin E^{\mathcal{I}}$, and therefore there exists some $x_{i+1} \in X_{i+1}$. Now we demonstrate that $c^{\mathcal{I}}(x) = i+1$ for every $x \in X_{i+1}$, and in particular for $x = x_{i+1}$.

By induction on k with $1 \leq k \leq n$, we prove that for every $x \in X_{i+1}$, the k^{th} bits of $c^{\mathcal{I}}(x_i)$ and of $c^{\mathcal{I}}(x)$ differ if and only if the $(k-1)^{th}$ bit of $c^{\mathcal{I}}(x_i)$ is 1 and of $c^{\mathcal{I}}(x)$ is 0. Note that, in particular, the induction hypothesis implies that the values for the k^{th} bits of $c^{\mathcal{I}}(x)$ are the same for all $x \in X_{i+1}$.

The base case $k = 1$ of induction holds since $\mathcal{I}$ is a model of (7), and therefore, for every $x \in X_{i+1}$ the lowest bits of $c^{\mathcal{I}}(x_i)$ and of $c^{\mathcal{I}}(x)$ differ. The induction step hols because $\mathcal{I}$ is a model of (8) which implies that the $(k-1)^{th}$ bit of $c^{\mathcal{I}}(x_i)$ is 1 and of $c^{\mathcal{I}}(x)$ is 0 for all $x \in X_{i+1}$ if and only if for every $x \in X_{i+1}$, the k^{th} bit of $c^{\mathcal{I}}(x_i)$ and of $c^{\mathcal{I}}(x)$ differ. ☐

Now we use similar ideas to enforce double-exponentially long chains in the model. This time, however, we cannot use just atomic concepts to encode the bits of the counter since there are exponentially many bits. Therefore, we assign a counter not to elements but to exponentially long r-chains induced by axioms (4)–(8) using one atomic concept X: the i^{th} bit of the counter corresponds to the value of X at the i^{th} element of the chain. In Figure 2 we have depicted a doubly-exponential chain formed for the sake of presentation as a zig-zag that we are going to induce using $\mathcal{R}$ axioms. The chain consists of 2^{2^n} r-chains, each having exactly 2^n elements, that are joined together using a role v—the last element of every r-chain, except for the final chain, is v-connected to the first element of the next r-chain. The tricky part is to ensure that the counters corresponding to r-chains are properly incremented. This is achieved by using the regular role inclusion axioms from (3), which allow us to propagate information using a role v_n across chains of exactly 2^n roles. The structure in Figure 2 is enforced using axioms (9)–(18) in addition to axioms (3)–(8).

$$O \sqsubseteq Z \sqcap Z_v \tag{9}$$
$$Z_v \sqsubseteq \neg X \sqcap \forall r.Z_v \tag{10}$$
$$Z \sqsubseteq E_v \qquad E_v \sqcap X \sqsubseteq \forall r.E_v \tag{11}$$
$$\neg(E_v \sqcap X) \sqsubseteq \forall r.\neg E_v \tag{12}$$

$$E \sqcap \neg(E_v \sqcap X) \sqsubseteq \exists v.Z \tag{13}$$

$$r \sqsubseteq v_0 \qquad v \sqsubseteq v_0 \tag{14}$$

$$E \sqcup \exists r.(X \sqcap X^f) \sqsubseteq X^f \tag{15}$$

$$\exists r.\neg(X \sqcap X^f) \sqsubseteq \neg X^f \tag{16}$$

$$X^f \sqsubseteq (X \sqcap \forall v_n.\neg X) \sqcup (\neg X \sqcap \forall v_n.X) \tag{17}$$

$$\neg X^f \sqsubseteq (X \sqcap \forall v_n.X) \sqcup (\neg X \sqcap \forall v_n.\neg X) \tag{18}$$

The atomic concept O corresponds to the origin of our structure. Axiom (9) expresses that O starts a 2^n-long r-chain because of the atomic concept Z and axioms (4)–(8). This chain is initialized to "zero" using Z_v and axiom (10). In order to identify the final chain, we use the atomic concept E_v which should hold on an element of an r-chain iff X holds on all the preceding elements of this r-chain. Axioms (11) say that E_v holds at the first element of every r-chain and propagates the positive values of E_v. Axiom (12) propagates the negative values of E_v. Now, axiom (13) says that the last element of every non-final r-chain has a v-successor which initializes a new r-chain.

Axioms (14)–(18), together with axioms (3) are responsible for incrementing the counter between r-chains. Recall that axioms (3) imply $(v_0)^i \sqsubseteq v_n$ if and only if $i = 2^n$, where $(v_0)^i$ denotes i compositions of the role v_0. Using axioms (14) we now make sure that exactly the corresponding elements of the consequent r-chains are connected with v_n. Axioms (15)–(18) express the transformation of bits analogously to axioms (7) and (8). We introduce the concept X^f to indicate that the current bit should be flipped. Axioms (15) and (16) express that the bit is flipped iff it is either the last bit, or its previous bit is flipped from 1. Axioms (17) and (18) implement the bit flipping using the role v_n.

For convenience, let us denote by $j[i]_2$ the i^{th} bit of j in binary coding (the lowest bit of j is $j[1]_2$).

Lemma 3. *Let $\mathcal{O}$ be an ontology containing axioms* (3)–(8) *and* (12)–(18)*, and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ a model of $\mathcal{O}$. Let $x_i \in \Delta^{\mathcal{I}}$ with $0 \le i < 2^n$ be such that (a) $\langle x_{i-1}, x_i\rangle \in r^{\mathcal{I}}$ when $i > 1$, (b) $c^{\mathcal{I}}(x_i) = i$, and (c) there exists an integer $j < 2^{2^n} - 1$ such that $x_i \in X^{\mathcal{I}}$ iff $j[2^n - i]_2 = 1$. Then there exist elements $y_i \in \Delta^{\mathcal{I}}$ with $0 \le i < 2^{2^n}$ such that (i) $\langle x_{2^n-1}, y_0\rangle \in v^{\mathcal{I}}$, (ii) $\langle y_{i-1}, y_i\rangle \in r^{\mathcal{I}}$ when $i > 1$, (iii) $c^{\mathcal{I}}(y_i) = i$, and (iv) $y_i \in X^{\mathcal{I}}$ iff $(j+1)[2^n - i]_2 = 1$.*

Proof. Since by condition (c) $j < 2^{2^n} - 1$, there exists i' with $0 \le i' < 2^n$ such that $j[2^n - i']_2 = 0$, and therefore, by condition (c) we have $x_{i'} \notin X^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of (12) and by condition (a) $\langle x_{i-1}, x_i\rangle \in r^{\mathcal{I}}$ when $i > 1$, it is easy to see that $x_i \notin (E_v \sqcap X)^{\mathcal{I}}$ for all $i \ge i'$. In particular, $x_{2^n-1} \notin (E_v \sqcap X)^{\mathcal{I}}$. Since by condition (b) $c^{\mathcal{I}}(x_{2^n-1}) = 2^n - 1$ and $\mathcal{I}$ is a model of (5), we have $x_{2^n-1} \in E^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of (13), there exists an element $y_0 \in \Delta^{\mathcal{I}}$ such that $\langle x_{2^n-1}, y_0\rangle \in v^{\mathcal{I}}$ and $y_0 \in Z^{\mathcal{I}}$, which proves the claim (i) of the lemma. Since $y_0 \in Z^{\mathcal{I}}$, by Lemma 2 there exist elements $y_i \in \Delta^{\mathcal{I}}$ with $1 \le i < 2^n$ such that $\langle y_{i-1}, y_i\rangle \in r^{\mathcal{I}}$ and $c^{\mathcal{I}}(y_i) = i$. This proves claims (ii) and (iii) of the lemma. It remains thus to prove claim (iv).

Since $\mathcal{I}$ is a model of (15) and $x_{2^n-1} \in E^{\mathcal{I}}$, we have $x_{2^n-1} \in X^{f^{\mathcal{I}}}$. Furthermore, since $\mathcal{I}$ is a model of (15) and (16), for every i with $1 \le i < 2^n$, we have $x_{i-1} \in X^{f^{\mathcal{I}}}$ if and only if $x_i \in (X \sqcap X^f)^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of (3) and (14), it is easy to see that $\langle x_i, y_i\rangle \in v_n{}^{\mathcal{I}}$ for every i with $0 \le i < 2^n$. Therefore, axioms (17) and (18) ensure that $x_i \in X^{\mathcal{I}}$ and $y_i \in X^{\mathcal{I}}$ or $x_i \notin X^{\mathcal{I}}$ and $y_i \notin X^{\mathcal{I}}$ if and only if $i < 2^n - 1$ and $x_{i+1} \notin X^{\mathcal{I}}$ or $y_{i+1} \in X^{\mathcal{I}}$. Now claim (iv) easily follows from the condition (c). $\square$

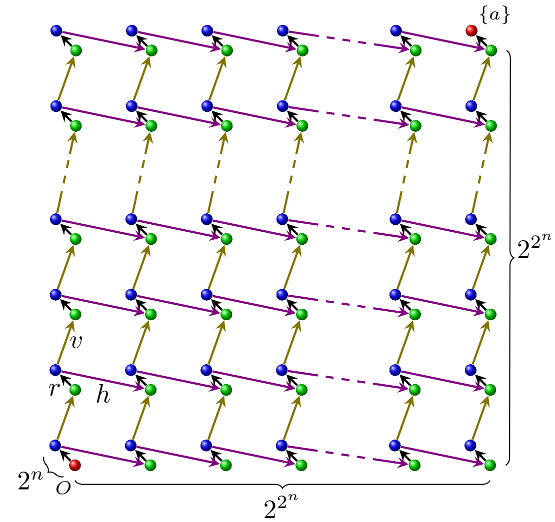

Figure 3: Encoding a double-exponentially large grid

$\mathcal{ROIF}$ is N2ExpTime-hard

Now we demonstrate that using $\mathcal{ROIF}$ axioms one can express the grid-like structure in Figure 3. The main idea of our construction is taken from the hardness proof for $\mathcal{ALCOIQ}$ (Tobies 2000) where a pair of counters is used to encode the coordinates of the grid and a nominal with inverse functionality to join the elements with the same coordinates. The only difference is that for $\mathcal{ROIF}$ we use the counters up to 2^{2^n} instead of just up to 2^n.

The grid-like structure in Figure 3 consists of $2^{2^n} \times 2^{2^n}$ 2^n-long r-chains which are joined vertically using the role v and horizontally using the role h in the same way as in Figure 2. Every r-chain stores information about two counters. The first counter uses the concept name X and corresponds to the vertical coordinate of the r-chain; the second counter uses Y and corresponds to the horizontal coordinate of the r-chain. The axioms (3)–(18) are now used to express that the vertical counter for r-chains is initialized in O and is incremented over v. A copy of these axioms (19)–(29) expresses the analogous property for the horizontal counter.

$$O \sqsubseteq Z \sqcap Z_h \tag{19}$$

$$Z_h \sqsubseteq \neg Y \sqcap \forall r.Z_h \tag{20}$$

$$Z \sqsubseteq E_h \qquad E_h \sqcap Y \sqsubseteq \forall r.E_h \tag{21}$$

$$\neg(E_h \sqcap Y) \sqsubseteq \forall r.\neg E_h \tag{22}$$

$$E \sqcap \neg(E_h \sqcap Y) \sqsubseteq \exists h.Z \tag{23}$$

$$r \sqsubseteq h_0 \qquad h \sqsubseteq h_0 \tag{24}$$

$$h_i \circ h_i \sqsubseteq h_{i+1}, \quad 0 \le i < n \tag{25}$$

$$E \sqcup \exists r.(Y \sqcap Y^f) \sqsubseteq Y^f \quad (26)$$

$$\exists r.\neg(Y \sqcap Y^f) \sqsubseteq \neg Y^f \quad (27)$$

$$Y^f \sqsubseteq (Y \sqcap \forall h_n.\neg Y) \sqcup (\neg Y \sqcap \forall h_n.Y) \quad (28)$$

$$\neg Y^f \sqsubseteq (Y \sqcap \forall h_n.Y) \sqcup (\neg Y \sqcap \forall h_n.\neg Y) \quad (29)$$

The grid structure in Figure 3 is now enforced by adding axioms (30)–(33).

$$\top \sqsubseteq (X \sqcap \forall h_n.X) \sqcup (\neg X \sqcap \forall h_n.\neg X) \quad (30)$$

$$\top \sqsubseteq (Y \sqcap \forall v_n.Y) \sqcup (\neg Y \sqcap \forall v_n.\neg Y) \quad (31)$$

$$E \sqcap E_v \sqcap X \sqcap E_h \sqcap Y \sqsubseteq \{a\} \quad (32)$$

$$\mathsf{Fun}(r^-) \qquad \mathsf{Fun}(v^-) \qquad \mathsf{Fun}(h^-) \quad (33)$$

Axioms (30) and (31) express that the values of the vertical / horizontal counters are copied across h_n / respectively v_n. Axiom (32) expresses that the last element of the r-chain with the final coordinates is unique. Together with axiom (33) expressing that the roles r, v, and h are inverse functional, this ensures that no two different r-chains have the same coordinates. Note that the roles r, v, and h are simple because they do not occur at the right hand side of RIAs (3), (14), (24), and (25). The following analog of Lemma 2 claims that the models of our axioms that satisfy O correspond to the grid in Figure 3.

Lemma 4. *For every model $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ of every ontology $\mathcal{O}$ containing axioms* (3)–(33)*, and every $x \in O^\mathcal{I}$ there exist $x_{i,j,k} \in \Delta^\mathcal{I}$ with $0 \le i < 2^n$, $0 \le j,k < 2^{2^n}$ such that (i) $x = x_{0,0,0}$, (ii) $\langle x_{i-1,j,k}, x_{i,j,k}\rangle \in r^\mathcal{I}$ when $i \ge 1$, (iii) $\langle x_{2^n-1,j-1,k}, x_{0,j,k}\rangle \in v^\mathcal{I}$ when $j \ge 1$, and (iv) $\langle x_{2^n-1,j,k-1}, x_{0,j,k}\rangle \in h^\mathcal{I}$ when $k \ge 1$.*

Proof. By induction on $j + k$ with $0 \le j,k < 2^{2^n}$, we construct non-empty sets of elements $X_{i,j,k} \subseteq \Delta^\mathcal{I}$ for $0 \le i < 2^n$ such that (a) $x \in X_{0,0,0}$, (b) $\forall x_{i-1,j,k} \in X_{i-1,j,k}\, \exists x_{i,j,k} \in X_{i,j,k}$ and $\forall x_{i,j,k} \in X_{i,j,k}\, \exists x_{i-1,j,k} \in X_{i-1,j,k}$ such that $\langle x_{i-1,j,k}, x_{i,j,k}\rangle \in r^\mathcal{I}$ when $i \ge 1$, (c) $\forall x_{2^n-1,j-1,k} \in X_{2^n-1,j-1,k}\, \exists x_{0,j,k} \in X_{0,j,k}$ such that $\langle x_{2^n-1,j-1,k}, x_{2^n-1,j,k}\rangle \in v^\mathcal{I}$ when $j \ge 1$, and (d) $\forall x_{2^n-1,j,k-1} \in X_{2^n-1,j,k-1}\, \exists x_{0,j,k} \in X_{0,j,k}$ such that $\langle x_{2^n-1,j,k-1}, x_{0,j,k}\rangle \in h^\mathcal{I}$ when $k \ge 1$. We also prove by induction that for every $x \in X_{i,j,k}$, we have (e) $c^\mathcal{I}(x) = i$, (f) $x \in X^\mathcal{I}$ iff $j[2^n - i]_2 = 1$, and (g) $x \in Y^\mathcal{I}$ iff $k[2^n - i]_2 = 1$. After that, we demonstrate that every set $X_{i,j,k}$ contains exactly 1 element which we define by $x_{i,j,k}$.

For the base case $j = k = 0$, we construct sets $X_{i,0,0}$ as follows. Since $\mathcal{I}$ is a model of (9), we have $x \in O^\mathcal{I} \subseteq Z^\mathcal{I}$. By Lemma 2, there exist elements $x_i \in \Delta^\mathcal{I}$ with $0 \le i < 2^n$ such that $c^\mathcal{I}(x_i) = i$, $x = x_0$, and $\langle x_{i-1}, x_i\rangle \in r^\mathcal{I}$ when $i \ge 1$. We define $X_{i,0,0} := \{x_i\}$ for $0 \le i \le n$. It is easy to see that conditions (a), (b), and (e) are satisfied for the constructed sets. Since $\mathcal{I}$ is a model of (9), (10), (19), and (20), we have $x_i \notin X^\mathcal{I}$ and $x_i \notin Y^\mathcal{I}$ for every i with $0 \le i < 2^n$, and therefore the conditions (f) and (g) are satisfied for $X_{i,0,0}$.

For the induction step $j + k > 0$, we construct $X_{i,j,k}$ provided we have constructed all $X_{i,j',k'}$ with $j'+k' < j+k$ and $0 \le i < 2^n$. We first initialize $X_{i,j,k}$ to the empty set, and then add new elements as described below.

If $j \ge 1$, by the induction hypothesis (b), for every element $x_{2^n-1,j-1,k} \in X_{2^n-1,j-1,k}$ there exist elements $x_{i,j-1,k} \in X_{i,j-1,k}$ with $0 \le i < 2^n - 1$ such that $\langle x_{i-1,j-1,k}, x_{i,j-1,k}\rangle \in r^\mathcal{I}$ when $i \ge 1$. By the induction hypothesis (e) we have $c^\mathcal{I}(x_{i,j-1,k}) = i$. Since $j - 1 < 2^{2^n} - 1$, by Lemma 3, there exist elements $x_{i,j,k}$ with $0 \le i < 2^n - 1$ such that $\langle x_{2^n-1,j-1,k}, x_{0,j,k}\rangle \in v^\mathcal{I}$, $\langle x_{i-1,j,k}, x_{i,j,k}\rangle \in r^\mathcal{I}$ when $i > 1$, $c^\mathcal{I}(x_{i,j,k}) = i$, and $x_{i,j,k} \in X^\mathcal{I}$ iff $j[2^n - i]_2 = 1$. Since $\mathcal{I}$ is a model of (3), (14), and (31), it is also easy to show that $x_{i,j,k} \in Y^\mathcal{I}$ iff $x_{i,j-1,k} \in Y^\mathcal{I}$ iff $k[2^n - i]_2 = 1$. We add every constructed element $x_{i,j,k}$ with $0 \le i < 2^n$ to the corresponding set $X_{i,j,k}$. We have demonstrated that the properties (b), and (e)–(g) hold for each of these elements.

Similarly, if $k \ge 1$, by the induction hypothesis (b), for every element $x_{2^n-1,j,k-1} \in X_{2^n-1,j,k-1}$, there exist elements $x_{i,j,k-1} \in X_{i,j,k-1}$ with $0 \le i < 2^n - 1$ such that $\langle x_{i-1,j,k-1}, x_{i,j,k-1}\rangle \in r^\mathcal{I}$ when $i \ge 1$. By applying the analog of Lemma 3 where v is replaced with h, we construct elements $x_{i,j,k}$ with $0 \le i < 2^n$ such that $\langle x_{2^n-1,j,k-1}, x_{0,j,k}\rangle \in h^\mathcal{I}$, and the properties (d)–(g) are satisfied. We add every constructed element $x_{i,j,k}$ to the corresponding set $X_{i,j,k}$. Note that since either $j \ge 1$ and $X_{i,j-1,k}$ is non-empty, or $k \ge 1$ and $X_{i,j,k-1}$ is non-empty, the constructed set $X_{i,j,k}$ is non-empty as well.

Now after all sets $X_{i,j,k}$ are constructed, it is easy to see that the conditions (c) and (d) are satisfied as well. It remains thus to prove that every set $X_{i,j,k}$ contains exactly one element. Fist, consider the set $X_{i',j',k'}$ for $i' = 2^n - 1$ and $j' = k' = 2^{2^n} - 1$. By condition (b), for every element $x_{i',j',k'} \in X_{i',j',k'}$ there exist elements $x_{i,j',k'} \in X_{i,j',k'}$ with $0 \le i < 2^n - 1$ such that $\langle x_{i-1,j',k'}, x_{i,j',k'}\rangle \in r^\mathcal{I}$ when $i \ge 1$ and $c^\mathcal{I}(x_{i,j',k'}) = i$. Since $\mathcal{I}$ is a model of (11) and (21), it can be shown using condition (f) and (g) that $x_{i',j',k'} \in (E \sqcap E_v \sqcap X \sqcap E_h \sqcap Y)^\mathcal{I}$. Now, since $\mathcal{I}$ is a model of (32), we have $x_{i',j',k'} = a^\mathcal{I}$, and therefore $X_{i',j',k'}$ contains exactly one element. Since $\mathcal{I}$ is a model of (33), using conditions (b), (c), and (d), it is easy to show that each set $X_{i,j,k}$ with $0 \le i < 2^n$ and $0 \le j,k < 2^{2^n}$ contains at most one element. $\square$

Our complexity result for $\mathcal{ROIF}$ is obtained by a reduction from the bounded domino tiling problem. A *domino system* is a triple $\mathcal{D} = (T, V, H)$, where $T = \{1, \dots, p\}$ is a finite set of *tiles* and $H, V \subseteq T \times T$ are *horizontal* and *vertical matching relations*. A *tiling* of $m \times m$ for a domino system $\mathcal{D}$ with *initial condition* $c^0 = \langle t^0_1, \dots, t^0_n\rangle$, $t^0_i \in T$, $1 \le i \le n$, is a mapping $t : \{1, \dots, m\} \times \{1, \dots, m\} \to T$ such that $\langle t(i-1, j), t(i, j)\rangle \in V$, $1 < i \le m$, $1 \le j \le m$, $\langle t(i, j-1), t(i, j)\rangle \in H$, $1 < i \le m$, $1 \le j \le m$, and $t(1, j) = t^0_j$, $1 \le j \le n$. It is well known (Börger, Grädel, and Gurevich 1997) that there exists a domino system $\mathcal{D}_0$ that is $\mathsf{N2ExpTime}$-complete for the following decision problem: given an initial condition c^0 of the size n, check if $\mathcal{D}_0$ admits the tiling of $2^{2^n} \times 2^{2^n}$ for c^0. Axioms

(34)–(41) in addition to axioms (3)–(33) provide a reduction from this problem to the problem of concept satisfiability in $\mathcal{ROIF}$.

$$\top \sqsubseteq D_1 \sqcup \cdots \sqcup D_p \tag{34}$$

$$D_i \sqcap D_j \sqsubseteq \bot \qquad 1 \le i < j \le p \tag{35}$$

$$D_i \sqsubseteq \forall r.D_i \qquad 1 \le i \le p \tag{36}$$

$$D_i \sqcap \exists v.D_j \sqsubseteq \bot \qquad \langle i,j\rangle \notin V \tag{37}$$

$$D_i \sqcap \exists h.D_j \sqsubseteq \bot \qquad \langle i,j\rangle \notin H \tag{38}$$

$$O \sqsubseteq I_1 \tag{39}$$

$$I_k \sqsubseteq D_{t_k^0} \sqcap \forall r.I_k \qquad 1 \le k \le n \tag{40}$$

$$I_k \sqsubseteq \forall h.I_{k+1} \qquad 1 \le k < n \tag{41}$$

The atomic concepts $D_1, \ldots, D_p$ correspond to the tiles of the domino system $\mathcal{D}_0$. Axioms (34) and (35) express that every element in the model is assigned with a unique tile D_i. Axiom (36) expresses that the elements of the same r-chain are assigned with the same tile. Axioms (37) and (38) express the vertical and horizontal matching properties. Finally, axioms (39)–(41) expresses that the initial condition holds for the first row. It is easy to see that this reduction is polynomial in n since $\mathcal{D}_0$ is fixed.

Theorem 5. *Let c^0 be an initial condition of size n for the domino system $\mathcal{D}_0$ and $\mathcal{O}$ an ontology consisting of the axioms (3)–(41). Then $\mathcal{D}_0$ admits the tiling of $2^{2^n} \times 2^{2^n}$ for c^0 if and only if O is (finitely) satisfiable in $\mathcal{O}$.*

Proof. ($\Rightarrow$) Let $t : 2^{2^n} \times 2^{2^n} \to T$ be a tiling for the domino system $\mathcal{D}_0 = (T, V, H)$ with the initial condition c^0. We use t to build a finite model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ for $\mathcal{O}$ that satisfies O.

We define $\Delta^{\mathcal{I}} := \{x_{i,j,k} \mid 0 \le i < 2^n,\, 0 \le j,k < 2^{2^n}\}$. The interpretation of the roles r, v, and h is defined by $r^{\mathcal{I}} = \{\langle x_{i-1,j,k}, x_{i,j,k}\rangle \mid i \ge 1\}$, $v^{\mathcal{I}} = \{\langle x_{2^n-1,j-1,k}, x_{0,j,k}\rangle \mid j \ge 1\}$, $h^{\mathcal{I}} = \{\langle x_{2^n-1,j,k-1}, x_{0,j,k}\rangle \mid k \ge 1\}$. The roles v_ℓ and h_ℓ for $0 \le \ell \le n$ are interpreted as the smallest relations that satisfy axioms (3), (14), (24), and (25). It is easy to see that ${v_n}^{\mathcal{I}} = \{\langle x_{i,j-1,k}, x_{i,j,k}\rangle \mid j \ge 1\}$ and ${h_n}^{\mathcal{I}} = \{\langle x_{i,j,k-1}, x_{i,j,k}\rangle \mid k \ge 1\}$. We interpret concepts B_k with $1 \le k \le n$ that determine the bits of the counter $c^{\mathcal{I}}(x)$ in such a way that $c^{\mathcal{I}}(x_{i,j,k}) = i$. Thus $Z^{\mathcal{I}} = \{x_{0,j,k}\}$, $Z^{\mathcal{I}} = \{x_{2^n-1,j,k}\}$. We define $X^{\mathcal{I}} = \{x_{i,j,k} \mid j[2^n - i]_2 = 1\}$, and $Y^{\mathcal{I}} = \{x_{i,j,k} \mid k[2^n - i]_2 = 1\}$. Finally, we define ${D_\ell}^{\mathcal{I}} = \{x_{i,j,k} \mid t(j+1, k+1) = \ell\}$. Other concepts such as Z_v, Z_h, E_v, E_h, X^f, Y^f, I_k are interpreted in a clear way. It is straightforward to check that $\mathcal{I}$ satisfies all axioms in $\mathcal{O}$. In particular, $\mathcal{I}$ satisfies (37) and (38), since t satisfies the matching conditions V and H of $\mathcal{D}_0$, and the roles v and h connect only the corresponding consequent r-chains.

($\Leftarrow$) Let $\mathcal{I}$ be a model of $\mathcal{O}$ that satisfies O. By Lemma 4, there exist $x_{i,j,k} \in \Delta^{\mathcal{I}}$ with $0 \le i < 2^n$, $0 \le j,k < 2^{2^n}$ that satisfy the conditions (i)–(iv) of Lemma 4. Let us define a function $t : 2^{2^n} \times 2^{2^n} \to \{1, \ldots, p\}$ by setting $t(j,k) = \ell$ if and only if $x_{0,j-1,k-1} \in {D_\ell}^{\mathcal{I}}$. This function is defined correctly because $\mathcal{I}$ satisfies axioms (34) and (35). We demonstrate that t is a tiling for $\mathcal{D}_0 = (T, H, V)$ with the initial condition c^0.

In order to prove that t satisfies the initial condition c^0, we show by induction on k that $x_{0,0,k-1} \in {I_k}^{\mathcal{I}}$ for all k with $1 \le k \le n$. Since $\mathcal{I}$ is a model of (40), it follows then that $x_{0,0,k-1} \in {D_{t_k^0}}^{\mathcal{I}}$, and hence $t(1,k) = t_k^0$ by definition of $t(j,k)$. The base case of induction $k = 1$ holds since $\mathcal{I}$ is a model of (39) and by condition (i) of Lemma 4 we have $x_{0,0,0} = x \in O^{\mathcal{I}} \subseteq {I_1}^{\mathcal{I}}$. For the induction step, assume that $x_{0,0,k-1} \in {I_k}^{\mathcal{I}}$ for some k with $1 \le k < 2^n$, and let us show that $x_{0,0,k} \in {I_{k+1}}^{\mathcal{I}}$. By condition (ii) of Lemma 4, $\langle x_{i-1,0,k-1}, x_{i,0,k-1}\rangle \in r^{\mathcal{I}}$ for all i with $1 \le i < 2^n$. Therefore, since $\mathcal{I}$ is a model of (40), we have $x_{i,0,k-1} \in {I_k}^{\mathcal{I}}$ for all i with $0 \le i < 2^n$ and, in particular, $x_{2^n-1,0,k-1} \in {I_k}^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of (41), and $\langle x_{2^n-1,0,k-1}, x_{0,0,k}\rangle \in h^{\mathcal{I}}$ by condition (iv) of Lemma 4, we have $x_{0,0,k} \in {I_{k+1}}^{\mathcal{I}}$ what was required to show.

Finally we prove that t satisfies the matching conditions H and V of $\mathcal{D}_0$. If $t(j-1,k) = \ell_1$ and $t(j,k) = \ell_2$ for some $j > 1$, $1 \le j,k < 2^{2^n}$, then by definition of $t(j,k)$, we have $x_{0,j-2,k-1} \in {D_{\ell_1}}^{\mathcal{I}}$ and $x_{0,j-1,k-1} \in {D_{\ell_2}}^{\mathcal{I}}$. Furthermore, since $\mathcal{I}$ is a model of (36) and by condition (ii) of Lemma 4, $\langle x_{i-1,j-2,k-1}, x_{i,j-2,k-1}\rangle \in r^{\mathcal{I}}$ when $i \ge 1$, we have $x_{i,j-2,k-1} \in {D_{\ell_1}}^{\mathcal{I}}$ for every i with $0 \le i < 2^n$, and in particular $x_{2^n-1,j-2,k-1} \in {D_{\ell_1}}^{\mathcal{I}}$. By condition (iii) of Lemma 4, we have $\langle x_{2^n-1,j-2,k-1}, x_{0,j-1,k-1}\rangle \in v^{\mathcal{I}}$. Since $x_{2^n-1,j-2,k-1} \in {D_{\ell_1}}^{\mathcal{I}}$, $x_{0,j-1,k-1} \in {D_{\ell_2}}^{\mathcal{I}}$, and $\mathcal{I}$ is a model of (37), we have $\langle t(j-1,k), t(j,k)\rangle = \langle \ell_1, \ell_2\rangle \in V$. Therefore t satisfies the vertical matching condition. Analogously using condition (iv) of Lemma 4 and axiom (38) it is easy to show that t satisfies the horizontal matching condition. □

Corollary 6. *The problem of (finite) concept satisfiability in the DL $\mathcal{ROIF}$ is N2ExpTime-hard (and so are all the standard reasoning problems).*

$\mathcal{R}$ is 2ExpTime-hard

In this section, we prove that (finite model) reasoning in the DL $\mathcal{R}$ is 2ExpTime-hard. The proof is by reduction from the word problem of an exponential-space alternating Turing machine. The main idea of our reduction is to use the zig-zag-like structures in Figure 2 to simulate a computation of an alternating Turing machine.

An *alternating Turning machine* (ATM) is a tuple $M = (\Gamma, \Sigma, Q, q_0, \delta_1, \delta_2)$ where Γ is a finite *working alphabet* containing a *blank symbol* $\boxdot$; $\Sigma \subseteq \Gamma \setminus \{\boxdot\}$ is the *input alphabet*; $Q = Q_\exists \uplus Q_\forall \uplus \{q_a\} \uplus \{q_r\}$ is a finite set of *states* partitioned into *existential states* $Q_\exists$, *universal states* $Q_\forall$, an *accepting state* q_a and a *rejecting state* q_r; $q_0 \in Q_\forall$ is the *starting state*, and $\delta_1, \delta_2 : (Q_\exists \cup Q_\forall) \times \Gamma \to Q \times \Gamma \times \{L, R\}$ are *transition functions*. A *configuration* of M is a word $c = w_1 q w_2$ where $w_1, w_2 \in \Gamma^*$ and $q \in Q$. An *initial configuration* is $c^0 = q_0 w^0$ where $w^0 \in \Sigma^*$. The *size* $|c|$ of a configuration c is the number of symbols in c. The *successor configurations* $\delta_1(c)$ and $\delta_2(c)$ of a configuration $c = w_1 q w_2$ with $q \neq q_a, q_r$ over the transition functions δ_1 and δ_2 are defined like for deterministic Turing machines (see, e.g., Sipser 2005). The sets $C_a(M)$ of *accepting configurations* and $C_r(M)$ of *rejecting configurations* of M are the smallest sets such that (i) $c = w_1qw_2 \in C_a(M)$ if either $q = q_a$, or $q \in Q_\forall$ and $\delta_1(c), \delta_2(c) \in C_a(M)$, or $q \in Q_\exists$ and $\delta_1(c) \in C_a(M)$ or $\delta_2(c) \in C_a(M)$, and (ii) $c = w_1qw_2 \in C_r(M)$ if either $q = q_r$, or $q \in Q_\exists$ and $\delta_1(c), \delta_2(c) \in C_r(M)$, or $q \in Q_\forall$ and $\delta_1(c) \in C_r(M)$ or $\delta_2(c) \in C_r(M)$. The set of configurations *reachable from* an initial configuration c^0 in M is the smallest set $M(c^0)$ such that $c^0 \in M(c^0)$ and $\delta_1(c), \delta_2(c) \in M(c^0)$ for every $c \in M(c^0)$. A *word problem* for an ATM M is to decide given an initial configuration c^0 whether $c^0 \in C_a(M)$. M is $g(n)$ *space bounded* if for every initial configuration c^0 we have: (i) $c^0 \in C_a(M) \cup C_r(M)$, and (ii) $|c| \leq g(|c^0|)$ for every $c \in M(c^0)$. It follows from a classical complexity result $\mathsf{AExpSpace} = \mathsf{2ExpTime}$ (Chandra, Kozen, and Stockmeyer 1981) that there exists a 2^n space bounded ATM M_0 for which the word problem is $\mathsf{2ExpTime}$-complete.

In order to reduce the word problem of M_0 to reasoning problems in $\mathcal{R}$, we introduce an auxiliary notion of a computation of an ATM that is more convenient to deal with when determining accepting computations. Let us denote by $\{0,1\}^*$ the set of all finite words over the letters 0 and 1, by ϵ the empty word, and for every $b \in \{0,1\}^*$, by $b \cdot 0$ and $b \cdot 1$ a word obtain by appending 0 and 1 to b. A subset $B \subseteq \{0,1\}^*$ is *prefix-closed* if $b \cdot 1 \in B$ or $b \cdot 0 \in B$ implies $b \in B$. A *computation of an ATM M from c^0* is a pair $P = (B, \pi)$, where $B \subseteq \{0,1\}^*$ is a prefix-closed set, and $\pi : B \to M(c^0)$ a mapping from words to configurations reachable from c^0, such that (i) $\pi(\epsilon) = c^0$, and for every $b \in B$ with $\pi(b) = c = w_1qw_2$ we have (ii) $q \neq q_r$, (iii) $q \in Q_\forall$ implies $\{b \cdot 0, b \cdot 1\} \subseteq B$, $\pi(b \cdot 0) = \delta_1(c)$, and $\pi(b \cdot 1) = \delta_2(c)$, and (iv) $q \in Q_\exists$ implies $b \cdot 0 \in B$ and $\pi(b \cdot 0) = \delta_1(c)$, or $b \cdot 1 \in B$ and $\pi(b \cdot 1) = \delta_2(c)$. A computation is *finite* if B is finite. It is easy to see that for any $g(n)$ space bounded ATM M, we have $c^0 \in C_a(M)$ iff there exists a finite computation of M from c^0.

Let c^0 be an initial configuration of M_0 and $n = |c^0|$ (w.l.o.g., $n \geq 3$). In order to decide whether $c^0 \in C_a(M_0)$, we introduce axioms expressing the existence of a computation of M_0 from c^0. The axioms induce a tree-like structure depicted in Figure 4 that stores configurations of M_0 on 2^n-long r-chains. The r-chain starting from the root element stores the initial configuration c^0; every configuration, depending on its state has up to two successor configurations stored on r-chains reachable by roles v and h—an r-chain corresponding to c is connected to r-chains corresponding to $\delta_1(c)$ and $\delta_2(c)$ via the roles v and h in a similar way as in Figure 2. For encoding configurations, we introduce an atomic concept A_s for every s from the set of states Q and the working alphabet Γ of M_0. We also introduce two concepts $S_\exists$ and $S_\forall$ that are used to mark configurations having existential and universal states. The underlying tree-like structure of the computation is induced by axioms (42)–(50).

$$O \sqsubseteq Z \tag{42}$$

$$O \sqsubseteq A_{c_1^0} \sqcap \forall r.(A_{c_2^0} \sqcap \cdots (\forall r.A_{c_n^0} \sqcap \forall r.Z_{\boxdot}) \cdots) \tag{43}$$

$$Z_{\boxdot} \sqsubseteq A_{\boxdot} \sqcap \forall r.Z_{\boxdot} \tag{44}$$

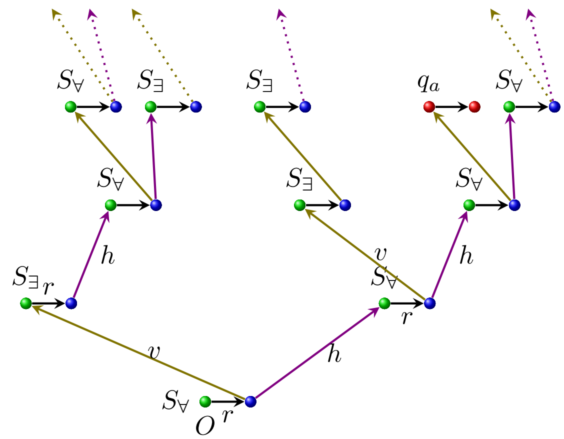

Figure 4: Encoding a computation of an ATM

$$A_q \sqsubseteq S_\exists \qquad q \in Q_\exists \tag{45}$$

$$A_q \sqsubseteq S_\forall \qquad q \in Q_\forall \tag{46}$$

$$S_\exists \sqsubseteq \forall r.S_\exists \quad S_\forall \sqsubseteq \forall r.S_\forall \tag{47}$$

$$A_{q_r} \sqsubseteq \bot \tag{48}$$

$$E \sqcap S_\exists \sqsubseteq \exists v.Z \sqcup \exists h.Z \tag{49}$$

$$E \sqcap S_\forall \sqsubseteq \exists v.Z \sqcap \exists h.Z \tag{50}$$

Axioms (42)–(44) initialize the configuration c^0 on the r-chain starting from the origin O of the structure. Axioms (45)–(46) determine the universal and existential types of configurations from their states. Axioms (47) then propagate these types until the end of the r-chain. Axiom (48) forbids rejecting configuration in the computation. Finally, axioms (49) and (50) express the existence of successor configurations depending on the types of the configuration.

In order to express that the successor configurations are obtained by the transition functions δ_1 and δ_2, we are going to use the roles v_n and h_n that connect the corresponding elements of successor r-chains thanks to axioms (3), (14), (24), and (25). It is a well-known property of the transition functions of Turing machines that the symbols c_i^1 and c_i^2 at the position i of $\delta_1(c)$ and $\delta_2(c)$ are uniquely determined by the symbols c_{i-1}, c_i, c_{i+1}, and c_{i+2} of c at the positions $i-1$, i, $i+1$, and $i+2$.[4] We assume that this correspondence is given by the (partial) functions λ_1 and λ_2 such that $\lambda_1(c_{i-1}, c_i, c_{i+1}, c_{i+2}) = c_i^1$ and $\lambda_2(c_{i-1}, c_i, c_{i+1}, c_{i+2}) = c_i^2$. To encode these functions, for every quadruple of symbols $s_1, s_2, s_3, s_4 \in Q \cup \Gamma$, we introduce a concept $S_{s_1s_2s_3s_4}$ that expresses the "neighborhood" of an element in an r-chain—it expresses that the current element is assigned with s_2, its r-predecessor with s_1 and its next two r-successors with s_3 and s_4 (s_1, s_3, and s_4 are $\boxdot$ if there are no such elements). Axioms (51)–(56) below express the required properties of the transition functions.

$$Z \sqcap A_{s_2} \sqcap \exists r.(A_{s_3} \sqcap \exists r.A_{s_4}) \ \sqsubseteq S_{\boxdot s_2s_3s_4} \tag{51}$$

[4]If any of the indexes $i-1$, $i+1$, or $i+2$ are out of range for the configuration c, we assume that the corresponding symbols c_{i-1}, c_{i+1}, or c_{i+2} are the blank symbol $\boxdot$

$$A_{s_1} \sqcap \exists r.(A_{s_2} \sqcap \exists r.(A_{s_3} \sqcap \exists r.A_{s_4})) \sqsubseteq \forall r.S_{s_1 s_2 s_3 s_4} \quad (52)$$

$$A_{s_1} \sqcap \exists r.(A_{s_2} \sqcap \exists r.(A_{s_3} \sqcap E)) \sqsubseteq \forall r.S_{s_1 s_2 s_3 \boxdot} \quad (53)$$

$$A_{s_1} \sqcap \exists r.(A_{s_2} \sqcap E) \sqsubseteq \forall r.S_{s_1 s_2 \boxdot\boxdot} \quad (54)$$

$$S_{s_1 s_2 s_3 s_4} \sqsubseteq \forall v_n.A_{\lambda_1(s_1,s_2,s_3,s_4)} \quad (55)$$

$$S_{s_1 s_2 s_3 s_4} \sqsubseteq \forall h_n.A_{\lambda_2(s_1,s_2,s_3,s_4)} \quad (56)$$

Axioms (51)–(54) initialize concepts $S_{s_1 s_2 s_3 s_4}$. Axioms (55)–(56) express that the corresponding symbols in the successor r-chains are computed using the functions λ_1 and $\lambda 2$.

Theorem 7. *Let $\mathcal{O}$ be an ontology consisting of axioms* (3)–(8)*,* (14)*,* (24)*,* (25)*, and* (42)–(56)*. Then $c^0 \in C_a(M_0)$ if and only if O is (finitely) satisfiable in $\mathcal{O}$.*

Proof. ($\Rightarrow$) Assume that $c^0 \in C_a(M_0)$. Since M_0 is 2^n space bounded, there exists a finite computation $P = (B, \pi)$ of M_0 from c^0 such that $|\pi(b)| \leq 2^n$ for every $b \in B$. We will use this computation in order to guide the construction of a finite model $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ for $\mathcal{O}$ that satisfies O.

We define $\Delta^\mathcal{I} := \{x_{b,i} \mid b \in B, 0 \leq i < 2^n\}$. The interpretation of roles r, v, and h is defined by $r^\mathcal{I} = \{\langle x_{b,i-1}, x_{b,i}\rangle \mid i \geq 1\}$, $v^\mathcal{I} = \{\langle x_{b,2^n-1}, x_{b\cdot 0,0}\rangle \mid b \cdot 0 \in B\}$, $h^\mathcal{I} = \{\langle x_{b,2^n-1}, x_{b\cdot 1,0}\rangle \mid b \cdot 1 \in B\}$. The roles v_k and h_k for $0 \leq k \leq n$ are interpreted as the smallest relations that satisfy axioms (3), (14), (24), and (25). It is easy to see that ${v_n}^\mathcal{I} = \{\langle x_{b,i}, x_{b\cdot 0,i}\rangle \mid b \cdot 0 \in B\}$ and ${h_n}^\mathcal{I} = \{\langle x_{b,i}, x_{b\cdot 1,i}\rangle \mid b \cdot 1 \in B\}$. We interpret concepts B_k with $1 \leq k \leq n$ that determine the bits of the counter $c^\mathcal{I}(x)$ in such a way that $c^\mathcal{I}(x_{b,i}) = i$. Thus $Z^\mathcal{I} = \{x_{b,0} \mid b \in B\}$, $E^\mathcal{I} = \{x_{b,2^n-1} \mid b \in B\}$. For every $s \in Q \cup \Gamma$ we define ${A_s}^\mathcal{I} = \{x_{b,i} \mid \pi(b)_i = s\}$, where $\pi(b)_i$ denotes the i^{th} symbol in the configuration $\pi(b)$. We set ${S_\exists}^\mathcal{I} = \{x_{b,i} \mid \exists j : \pi(b)_j \in Q_\exists\}$ and ${S_\forall}^\mathcal{I} = \{x_{b,i} \mid \exists j : \pi(b)_j \in Q_\forall\}$. Other concepts such as $Z_\boxdot$ and $S_{s_1 s_2 s_3 s_4}$ are interpreted in a clear way. It is straightforward to check that $\mathcal{I}$ satisfies all axioms in $\mathcal{O}$. In particular, $\mathcal{I}$ satisfies (55) and (56), since v_n and h_n connect only the corresponding elements of r-chains.

($\Leftarrow$) Assume that $\mathcal{I}$ is a model of $\mathcal{O}$. We build a computation $P = (B, \pi)$ of M_0 from c^0 witnessed by $\mathcal{I}$. The elements $b \in B$ and the values $\pi(b)$ are built inductively on $|b|$ for $b \in B$, together with elements $x_{b,i} \in \Delta^\mathcal{I}$ with $0 \leq i < 2^n$. We demonstrate by induction that $c^\mathcal{I}(x_{b,i}) = i$, $(x_{b,i-1}, x_{b,i}) \in r^\mathcal{I}$ when $i \geq 1$, and the property $(*)$: $\pi(b)_i = s$ implies $x_{b,i} \in {A_s}^\mathcal{I}$ for $0 \leq i < 2^n$, where as before, $\pi(b)_i$ denotes the i^{th} symbol of the configuration $\pi(b)$ (we assume that $\pi(b)_i = \boxdot$ if $i > |\pi(b)|$).

For the base case $b = \epsilon$, we define $x_{\epsilon,0} := x \in O$ and $\pi(\epsilon) := c^0$. Since $\mathcal{I}$ is a model of (42), we have $x_{\epsilon,0} \in Z^\mathcal{I}$. Since $\mathcal{I}$ is a model of (4)–(8), by Lemma 2, there exist elements $x_{\epsilon,i} \in \Delta^\mathcal{I}$ with $1 \leq i < 2^n$ such that $\langle x_{\epsilon,i-1}, x_{\epsilon,i}\rangle \in r^\mathcal{I}$ and $c^\mathcal{I}(x_{\epsilon,i}) = i$. The property $(*)$ for $b = \epsilon$ holds since $\mathcal{I}$ is a model of (43) and (44).

Now assume that we have constructed some $b \in B$, all elements $x_{b,i} \in \Delta^\mathcal{I}$ with $1 \leq i < 2^n$, and the value of $\pi(b)$. Let $\pi(b)_j = q \in Q$ be the state of the configuration $\pi(b)$ occurring at the position j. By the property $(*)$, we have $x_{b,j} \in {A_q}^\mathcal{I}$. If $q \in Q_\exists$, then since $\mathcal{I}$ is a model of (45) and (47), we have $x_{b,2^n-1} \in {S_\exists}^\mathcal{I}$. Since $\mathcal{I}$ is a model of (49), there exists either $x_{b\cdot 0,0} \in Z^\mathcal{I}$ such that $\langle x_{b,2^n-1}, x_{b\cdot 0,0}\rangle \in v^\mathcal{I}$, or $x_{b\cdot 1,0} \in Z^\mathcal{I}$ such that $\langle x_{b,2^n-1}, x_{b\cdot 1,0}\rangle \in h^\mathcal{I}$. In either case we add the respective elements $b \cdot 0$ or $b \cdot 1$ to B. If $q \in Q_\forall$ then similarly, since $\mathcal{I}$ is a model of (46), (47), and (50), there exist $x_{b\cdot 0,0}, x_{b\cdot 1,0} \in Z^\mathcal{I}$such that $\langle x_{b,2^n-1}, x_{b\cdot 0,0}\rangle \in v^\mathcal{I}$ and $\langle x_{b,2^n-1}, x_{b\cdot 1,0}\rangle \in h^\mathcal{I}$. In this case, we add both elements $b\cdot 0$ and $b\cdot 1$ to B. Note that it is not possible that $q = q_r$ since $\mathcal{I}$ is a model of (48). If we add element $b\cdot 0$ to B then we define $\pi(b\cdot 0) := \delta_1(\pi(b))$. Since $x_{b\cdot 0,0} \in Z^\mathcal{I}$ and $\mathcal{I}$ is a model of (4)–(8), by Lemma 2, there exist elements $x_{b\cdot 0,i} \in \Delta^\mathcal{I}$ with $1 \leq i < 2^n$ such that $\langle x_{b\cdot 0,i-1}, x_{b\cdot 0,i}\rangle \in r^\mathcal{I}$ and $c^\mathcal{I}(x_{b\cdot 0,i}) = i$. Similarly, if we add $b \cdot 1$ to B then we define $\pi(b \cdot 1) := \delta_2(\pi(b))$ and find elements $x_{b\cdot 1,i} \in \Delta^\mathcal{I}$. It remains thus to show the property $(*)$ for the new elements in B. If $b \cdot 0 \in B$ then since $\langle x_{b,2^n-1}, x_{b\cdot 0,0}\rangle \in v^\mathcal{I}$ and $\mathcal{I}$ is a model of (3) and (14), we have $\langle x_{b,i}, x_{b\cdot 0,i} \in {v_n}^\mathcal{I}\rangle$ for every i with $0 \leq i < 2^n$. Since $\mathcal{I}$ is a model of (51)–(55) and function λ_1 correspond to the transition function δ_1, we obtain $(*)$ for $b \cdot 0$. Similarly, if $b \cdot 1 \in B$ then since $\langle x_{b,2^n-1}, x_{b\cdot 1,0}\rangle \in h^\mathcal{I}$ and $\mathcal{I}$ is a model of (24), (25), (51)–(54), and (56) we have $(*)$ for $b \cdot 1$. ☐

Corollary 8. *The problem of (finite) concept satisfiability in the DL $\mathcal{R}$ is* 2ExpTime*-hard (and so are all the standard reasoning problems).*

Hardness Results for Linear RIAs

In this section we sharpen our hardness results for the *linear* RIAs of the form $R_1 \circ R_2 \sqsubseteq R_2$ (right-linear) or $R_2 \circ R_1 \sqsubseteq R_2$ (left-linear) that were considered in the original definition of $\mathcal{RIQ}$ (Horrocks and Sattler 2003). We say that an RBox $\mathcal{R}$ is *linear regular* if $\mathcal{R}$ is regular and every complex RIA in $\mathcal{R}$ is linear. Since linear regular RBoxes can already provide many desirable features for modeling of bio-medical ontologies (e.g., propagation of properties over properties) but still cause an exponential blowup in the size of regular automata, the question about the exact computational complexity of $\mathcal{RIQ}$ and $\mathcal{SROIQ}$ with linear RIAs becomes apparent. In this section we demonstrate how our hardness proofs for $\mathcal{RIQ}$ and $\mathcal{SROIQ}$ can be adapted for linear RIAs.

Linear regular RBoxes are strictly less expressive than the general ones: for every linear RBox $\mathcal{R}$, whenever $R_1 \circ \cdots \circ R_n \sqsubseteq_\mathcal{R} R$ holds for some $n \geq 2$, then either $R_1 \circ R_1 \circ \cdots \circ R_n \sqsubseteq_\mathcal{R} R$ or $R_1 \circ \cdots \circ R_n \circ R_n \sqsubseteq_\mathcal{R} R$ holds as well. In particular, the key property "$(v_0)^i \sqsubseteq_\mathcal{R} v_n$ *iff* $i = 2^n$" used in our construction cannot be expressed using linear RIAs. Therefore we use a slightly more complicated construction in Figure 5 to connect the corresponding elements of r-chains. Instead of connecting r-chains using a single role v, we now connect them using an exponential v-chain. Moreover, the r-chains and the v-chains are now composed of several roles $r_0, \ldots, r_{n-1}$ and $v_0, \ldots, v_n$ respectively. Intuitively, a role r_k (v_k) connects elements when the $(k+1)^{th}$ bit of the counter is changed from 1 to 0. The chains are created using axioms (57)–(64) that replace

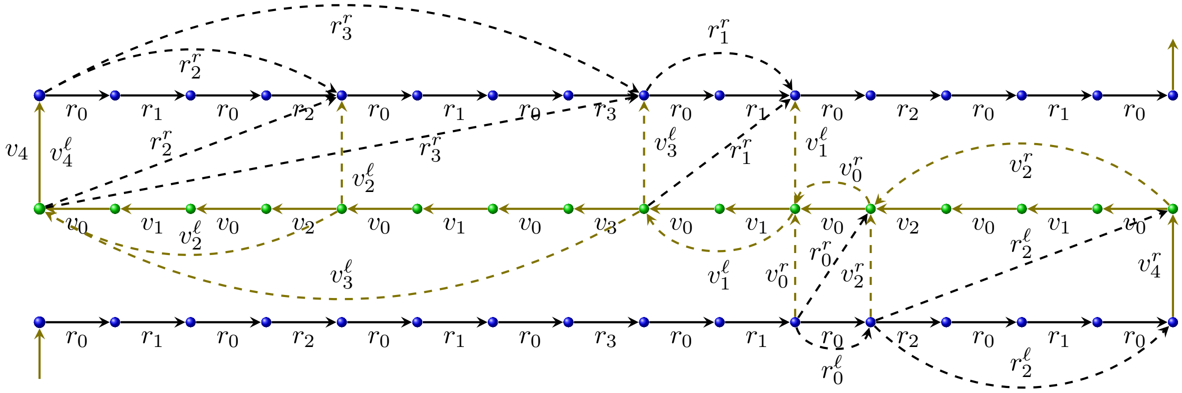

Figure 5: Connecting the corresponding elements of r-chains using linear RIAs

axioms (6), (13), and (14).

$$O \sqsubseteq \neg V \tag{57}$$

$$\neg V \sqcap \neg B_k \sqcap \textstyle\bigsqcap_{\ell<k} B_\ell \sqsubseteq \exists r_{k-1}.\neg V \qquad 1 \le k \le n \tag{58}$$

$$V \sqcap \neg B_k \sqcap \textstyle\bigsqcap_{\ell<k} B_\ell \sqsubseteq \exists v_{k-1}.V \qquad 1 \le k \le n \tag{59}$$

$$\neg V \sqcap E \sqcap \neg(E_v \sqcap X) \sqsubseteq \exists v_n.(Z \sqcap V) \tag{60}$$

$$V \sqcap E \sqsubseteq \exists v_n.(Z \sqcap \neg V) \tag{61}$$

$$r_k \sqsubseteq r \qquad v_k \sqsubseteq v \qquad 0 \le k < n \tag{62}$$

$$\top \equiv (B_1 \sqcap \forall v.\neg B_1) \sqcup (\neg B_1 \sqcap \forall v.B_1) \tag{63}$$

$$B_{k-1} \sqcap \forall v.\neg B_{k-1} \equiv (B_k \sqcap \forall v.\neg B_k) \sqcup (\neg B_k \sqcap \forall v.B_k) \qquad 1 < k \le n \tag{64}$$

We use a new concept V to distinguish elements in v-chains from the elements of r-chains. Axiom (57) expresses that the origin of our zig-zag-like structure belongs to an r-chain. Axioms (58) and (59) construct the successor elements of the chains, whereby (58) replaces (6). Axioms (60) and (61) create successor chains and replace axiom (13). Axiom (62) initializes the roles r and v which are used to increment the counters in (7)–(8) and (63)–(64). To connect the corresponding elements of the chains, we use RIAs (65)–(70).

$$r_k \sqsubseteq r_k^\ell \qquad r_k \sqsubseteq r_k^r \qquad v_k \sqsubseteq v_k^\ell \qquad v_k \sqsubseteq v_k^r \tag{65}$$

$$v_k^\ell \circ v_{k'} \sqsubseteq v_k^\ell \qquad v_{k'} \circ v_k^r \sqsubseteq v_k^r \qquad k' < k \tag{66}$$

$$r_k^\ell \circ r_{k'} \sqsubseteq r_k^\ell \qquad r_{k'} \circ r_k^r \sqsubseteq r_k^r \qquad k' < k \tag{67}$$

$$r_{k'}^\ell \circ v_k^r \sqsubseteq r_{k'}^\ell \qquad v_k^\ell \circ r_{k'}^r \sqsubseteq r_{k'}^r \qquad k' < k \tag{68}$$

$$v_k^\ell \circ r_k^r \sqsubseteq v_k^\ell \qquad r_k^\ell \circ v_k^r \sqsubseteq v_k^r \tag{69}$$

$$v_k^r \sqsubseteq v' \qquad v_k^\ell \sqsubseteq v' \tag{70}$$

We have introduced new atomic roles r_k^ℓ, r_k^r, $v_{k'}^\ell$ and $v_{k'}^r$ with $0 \le k < n$ and $0 \le k' \le n$. These roles are implied by r_k and v_k using axioms (65), and are propagated using left- and right-linear RIAs (66) and (67) over $r_{k'}$ and $v_{'k}$ with smaller indexes. Thus, every element of an r-chain (v-chain) has at most one r_k^ℓ (v_k^ℓ)-successor and at most one r_k^r (v_k^r)-predecessor for every k (see Figure 5). RIAs (68) and (69) are used to connect the corresponding elements of r- and v-chains. Intuitively, this is done by first going via r_k^ℓ (r_k^r) in the r-chain and then going via the corresponding v_k^r (v_k^ℓ) in the v-chain and connecting the resulting elements. These axioms together with (70) ensure that v' connects only those elements of successor chains that correspond. It is easy to see that RIAs (65)–(70) are $\prec$-regular for any ordering such that $v_0 \prec r_0 \prec \cdots \prec v_n \prec r_n \prec r_n^\ell \prec r_n^r \prec v_n^\ell \prec v_n^r \prec \cdots \prec r_0^\ell \prec r_0^r \prec v_0^\ell \prec v_0^r \prec v'$. Now to complete the construction, we replace in the remaining axioms every concept of the form $\forall v_n.C$ with $V \sqcup \forall v'.\forall v'.(V \sqcup C)$, which says that C holds at every $v' \circ v'$-successor of an element when both elements belong to r-chains (and hence, correspond). Our modified construction proves the following theorem:

Theorem 9. *The standard reasoning problems in $\mathcal{R}$ and $\mathcal{ROIF}$ are respectively 2ExpTime-hard and N2ExpTime-hard even for linear regular RBoxes.*

$\mathcal{SROIQ}$ is in N2ExpTime

In this section we prove the matching upper complexity bound for reasoning in $\mathcal{SROIQ}$ using an exponential-time translation into the two variable fragment with counting $\mathcal{C}^2$.

Let $\mathcal{O}$ be a $\mathcal{SROIQ}$ ontology for which we need to test satisfiability. By Theorem 9 from (Horrocks, Kutz, and Sattler 2006), w.l.o.g., we can assume that $\mathcal{O}$ does not contain concept and role assertions, the universal role, and axioms of the form $\mathsf{Irr}(S)$, $\mathsf{Tra}(R)$, and $\mathsf{Sym}(R)$. We can also assume that $\mathcal{O}$ contains no $\mathsf{Ref}(R)$ or $\mathsf{Asy}(S)$: we replace every $\mathsf{Ref}(R)$ with $s \sqsubseteq R$ and $\top \sqsubseteq \exists s.\mathsf{Self}$ for a fresh (simple) atomic role s, and replace every $\mathsf{Asy}(S)$ with $\mathsf{Disj}(S, \mathsf{Inv}(S))$. Next, we convert $\mathcal{O}$ into the *simplified form* which contains only axioms of the form 1–10 in Table 1, where $A_{(i)}$ and $B_{(j)}$ are atomic concepts, $r_{(i)}$ atomic roles, $s_{(i)}$ simple atomic roles, and v non-simple atomic roles. The transformation can be done in polynomial time using the standard structural transformation which iteratively introduces definitions for compound sub-concept and sub-roles (see, e.g., Kazakov and Motik 2008). For example, the axiom $A \equiv \exists r^-.\{a\}$ is replaced with the axioms $A \sqsubseteq \geqslant 1\, s.A_a$, $s \sqsubseteq r^-$, $A_a \equiv \{a\}$, and $A_a \sqsubseteq \forall r.A$ where s is a fresh (simple) atomic role introduced for r^-, and A_a a fresh atomic concepts introduced for $\{a\}$.

After the transformation, we eliminate RIAs of the form 10 using a technique from (Demri and de Nivelle 2005).

$\mathcal{SROIQ}$ axiom	first-order translation
1. $A \sqsubseteq \forall r.B$	$\forall x.(A(x) \to \forall y.[r(x,y) \to B(y)])$
2. $A \sqsubseteq \geqslant n\, s.B$	$\forall x.(A(x) \to \exists^{\geq n} y.[s(x,y) \wedge B(y)])$
3. $A \sqsubseteq \leqslant n\, s.B$	$\forall x.(A(x) \to \exists^{\leq n} y.[s(x,y) \wedge B(y)])$
4. $A \equiv \exists s.\mathsf{Self}$	$\forall x.(A(x) \leftrightarrow s(x,x))$
5. $A_a \equiv \{a\}$	$\exists^{=1} y.A_a(y)$
6. $\bigsqcap A_i \sqsubseteq \bigsqcup B_j$	$\forall x.(\bigvee \neg A_i(x) \vee \bigvee B_j(x))$
7. $\mathsf{Disj}(s_1, s_2)$	$\forall xy.(s_1(x,y) \wedge s_2(x,y) \to \bot)$
8. $s_1 \sqsubseteq s_2$	$\forall xy.(s_1(x,y) \to s_2(x,y))$
9. $s_1 \sqsubseteq s_2^-$	$\forall xy.(s_1(x,y) \to s_2(y,x))$
10. $r_1 \circ \cdots \circ r_n \sqsubseteq v,\ n \geq 1$	

Table 1: Translation of simplified $\mathcal{SROIQ}$ axioms into $\mathcal{C}^2$

Such axioms can cause unsatisfiability of $\mathcal{O}$ only through axioms of the form 1, since other axioms do not contain non-simple roles. In particular, for every $r_1 \circ \cdots \circ r_n \sqsubseteq_{\mathcal{R}} v$, the axiom $A \sqsubseteq \forall v.B$ implies axiom (71) below:

$$A \sqsubseteq \forall r_1 . \ldots \forall r_n .B \qquad (71)$$

The main idea of the transformation is to use the NFA for $L_{\mathcal{R}}(v)$ to express all properties of the form (71) using additional non-compositional axioms.

Let $B_{\mathcal{R}}(v)$ be an NFA for $L_{\mathcal{R}}(v)$ with the set of states Q, starting state $q_0 \in Q$, accepting states $F \subseteq Q$, and transition relation $\delta \subseteq Q \times R_{\Sigma} \times Q$. Given $B_{\mathcal{R}}(v)$, we replace every axiom $A \sqsubseteq \forall v.B$ of the form 1 with axioms (72)–(74) where A^v_q is a fresh atomic concept for every $q \in Q$.

$$A \sqsubseteq A^v_q \qquad q = q_0 \qquad (72)$$
$$A^v_{q_1} \sqsubseteq \forall s.A^v_{q_2} \qquad \langle q_1, s, q_2\rangle \in \delta \qquad (73)$$
$$A^v_q \sqsubseteq B \qquad q \in F \qquad (74)$$

Lemma 10. *Let $\mathcal{O}$ be an ontology consisting of axioms of the form* 1–10 *from Table 1, and $\mathcal{O}'$ obtained from $\mathcal{O}$ by replacing every axiom $A \sqsubseteq \forall v.B$ of the form* 1 *with axioms* (72)–(74) *and removing all axioms of the form* 10. *Then* (1) *every model of $\mathcal{O}$ can be expanded to a model of $\mathcal{O}'$ by interpreting A^v_q, and* (2) *every model of $\mathcal{O}'$ can be expanded to a model of $\mathcal{O}$ by interpreting the non-simple roles in $\mathcal{O}$.*

Proof. (1) Let $\mathcal{I}$ be a model of $\mathcal{O}$ and $\mathcal{I}'$ an expansion of $\mathcal{I}$ satisfying all axioms of the form (72) and (73) in $\mathcal{O}'$ that interprets A^v_q as smallest possible sets—$\mathcal{I}'$ can be constructed as a fixed point of the transformation that expands the interpretation according to these axioms. We prove that $\mathcal{I}'$ also satisfies all axioms of the form (74), and, therefore, is a model of $\mathcal{O}'$. Assume, to the contrary, $\mathcal{I}'$ does not satisfy some axiom $A^v_q \sqsubseteq B$ of the form (74), that is, there exists $x \in \Delta^{\mathcal{I}'}$ such that $x \in (A^v_q)^{\mathcal{I}'}$ but $x \notin B^{\mathcal{I}'}$. Since $\mathcal{I}'$ is obtained by a fixed point construction, there exists a sequence of elements $x_0, \ldots, x_n = x$ in $\Delta^{\mathcal{I}}$ such that $x_0 \in A^{\mathcal{I}}$, $\langle x_{i-1}, x_i\rangle \in s_i{}^{\mathcal{I}}$ for $1 \leq i \leq n$, and the word $s_1 \ldots s_n$ is accepted by $B_{\mathcal{R}}(v)$. Since $B_{\mathcal{R}}(v)$ is an automaton for $L_{\mathcal{R}}(v)$, we have $s_1 \ldots s_n \in L_{\mathcal{R}}(v)$, and so $s_1 \circ \cdots \circ s_n \sqsubseteq_{\mathcal{R}} v$ by definition of $L_{\mathcal{R}}(v)$, which implies that $\langle x_0, x_n\rangle \in v^{\mathcal{I}}$. Since $x_0 \in A^{\mathcal{I}}$ and $\mathcal{I} \models A \sqsubseteq \forall v.B$, we obtain $x_n \in B^{\mathcal{I}}$ which contradicts the assumption $x \notin B^{\mathcal{I}'}$ since $x = x_n$ and $B^{\mathcal{I}'} = B^{\mathcal{I}}$. The proof by contradiction implies that $\mathcal{I}'$ is a model of $\mathcal{O}'$.

(2) Let $\mathcal{I}'$ be a model of $\mathcal{O}'$ and $\mathcal{I}$ an expansion of $\mathcal{I}'$ that interprets all non-simple roles v as smallest possible relations that satisfy all axioms in $\mathcal{O}$ of the form 10—$\mathcal{I}$ can be constructed by a fixed point from $\mathcal{I}'$. We claim that $\mathcal{I}$ is a model of $\mathcal{O}$. Indeed, $\mathcal{I}$ is a model of all axioms in $\mathcal{O}$ that do not contain non-simple roles (in particular those of the form 2–9). It remains thus to demonstrate that $\mathcal{I}$ satisfies all axioms $A \sqsubseteq \forall v.B$ of the form 1 where v is a non-simple role. Let $x \in A^{\mathcal{I}}$ and $\langle x, y\rangle \in v^{\mathcal{I}}$; we need to prove that $y \in B^{\mathcal{I}}$. By the construction of $v^{\mathcal{I}}$, there exists a sequence of elements $x = x_0, \ldots, x_n = y$ in $\Delta^{\mathcal{I}}$ such that $\langle x_{i-1}, x_i\rangle \in s_i{}^{\mathcal{I}'}$ for some simple role s_i, $1 \leq i \leq n$, and $s_1 \circ \cdots \circ s_n \sqsubseteq_{\mathcal{R}} v$. The last implies that $s_1 \ldots s_n$ is accepted by some run of $q_0, q_1, \ldots, q_n \in F$ of the automaton $B_{\mathcal{R}}(v)$. Since $\mathcal{O}'$ contains the corresponding axioms of the form (72)–(74) that are satisfied by $\mathcal{I}$, it is easy to show by induction on i that $x_i \in (A^r_{q_i})^{\mathcal{I}}$, which implies that $y = x_n \in (A^r_{q_n})^{\mathcal{I}} \subseteq B^{\mathcal{I}}$, which was required to prove. □

Theorem 11. *The problem of (finite) satisfiability of $\mathcal{SROIQ}$ ontologies is solvable in* N2ExpTime *(and so are all the standard reasoning problems).*

Proof. The input $\mathcal{SROIQ}$ ontology $\mathcal{O}$ can be translated in exponential time (due to the sizes of the automata) preserving (finite) satisfiability into a simplified ontology containing only axioms of the form 1–9 in Table 1, which in turn can be translated into the two variable fragment with counting quantifiers $\mathcal{C}^2$ according to the second column of Table 1. Since (finite) satisfiability of $\mathcal{C}^2$ is NExpTime-complete (Pratt-Hartmann 2005), our reduction proves that (finite) satisfiability of $\mathcal{SROIQ}$ is in N2ExpTime. □

Conclusions

In this paper we have identified the computational complexity of (finite model) reasoning in $\mathcal{SROIQ}$ to be N2ExpTime-complete, and in $\mathcal{RIQ}$ and $\mathcal{SRIQ}$ to be 2ExpTime-hard—that is, exponentially harder then for $\mathcal{SHOIQ}$ and $\mathcal{SHIQ}$ respectively. The complexity blowup is due to complex role inclusion axioms, and in particular due to their ability to “chain” a fixed exponential number of roles. The blowup occurs even when no other new constructors in $\mathcal{SROIQ}$ are used, such as (as)symmetric, (ir)reflexive, disjoint, universal roles, or “self” constructor, and for $\mathcal{RIQ}$, even without number restrictions and inverse roles. Moreover, we have demonstrated that our hardness results hold already for complex role inclusion axioms of the form $R_1 \circ R_2 \sqsubseteq R_1$ and $R_1 \circ R_2 \sqsubseteq R_2$ that have been originally introduced in $\mathcal{RIQ}$.

Several conclusions can be immediately drawn from our complexity results. First, they demonstrate that the exponential blowup in the existing tableau-based procedures for $\mathcal{RIQ}$ and $\mathcal{SROIQ}$ (Horrocks and Sattler 2004; Horrocks, Kutz, and Sattler 2006) are unavoidable. That is, there is essentially no better way of dealing with complex RIAs other

<table>
<tr><th colspan="3">$\mathcal{SH}[\mathcal{O}]\mathcal{IQ}$</th><th colspan="3">$\mathcal{SR}[\mathcal{O}]\mathcal{IQ}$</th></tr>
<tr><td>ABox</td><td>TBox</td><td>RBox</td><td>ABox</td><td>TBox</td><td>RBox</td></tr>
<tr><td>NP[5]</td><td></td><td></td><td>NP[5]</td><td></td><td></td></tr>
<tr><td colspan="2">[N]ExpTime</td><td></td><td colspan="2">[N]ExpTime[6]</td><td></td></tr>
<tr><td colspan="3">[N]ExpTime</td><td colspan="3">[N]2ExpTime[6]</td></tr>
</table>

Table 2: Parametrized complexity of $\mathcal{SRIQ}$ and $\mathcal{SROIQ}$

than representing them by (possibly exponential) automata. Second, the complexity results imply that $\mathcal{SROIQ}$, and therefore OWL 2 provide for strictly richer expressivity than $\mathcal{SHOIN}$ and respectively OWL DL—up to now there were no formal evidences that the new constructors in $\mathcal{SROIQ}$ could not be polynomially expressed in $\mathcal{SHOIN}$.

Finally, it remains, as usual, to point out that the high worst-case complexity of the logic tells little about the typical behaviour of the reasoning procedures. As has been demonstrated, only RIAs are responsible for the exponential blowup. In particular, the complexity of $\mathcal{SROIQ}$ is N2ExpTime in the size of the ontology but only NExpTime in the size of the TBox (see Table 2). Since in existing ontologies the size of the RBox is typically much smaller than the size of the TBox, the impact on the complexity of the procedures might not be as dramatic as it sounds. In our proofs, the worst-case situations are achieved by a rather artificial usage of RIAs which is also unlikely to occur in real ontologies. In fact, as discussed in (Horrocks and Sattler 2004) the exponential blowup does not take place in many natural cases, e.g., when the length of every sequence $R_1 \prec R_2 \prec \cdots$ is bounded. The existing tableau-based procedures are already "aware" of these cases and therefore exhibit a "pay as you go" behavior.

There are several interesting questions and problems that can be considered for future work. First, we did not obtain the upper complexity bound for $\mathcal{RIQ}$ and $\mathcal{SRIQ}$. We think that a matching 2ExpTime decision procedure can be easily obtained by the elimination of complex RIAs as described in this paper followed by the modification of of the ExpTime automaton for $\mathcal{SHIQ}$ (Tobies 2001) to include other new constructors of $\mathcal{SRIQ}$, such as $\exists r.\mathsf{Self}$. Second, there is a gap between regular RBoxes and regular grammars that needs to be filled in. For example, the axioms $R_1 \circ R_1 \sqsubseteq R_2$, $R_2 \circ R_2 \sqsubseteq R_2$, $R_1 \circ R_2 \sqsubseteq R_1$, and $R_2 \circ R_1 \sqsubseteq R_1$ correspond to a regular grammar, but cannot be captured by a regular RBox. Finally, it would be interesting to identify other practically relevant restrictions of regular RBoxes which do not result in the exponential blowup. For example, it is not clear from our results whether $\mathcal{RIQ}$ or $\mathcal{SROIQ}$ are still exponentially harder than $\mathcal{SHIQ}$ and $\mathcal{SHOIQ}$ when the complex RIAs are only left-linear or only right-linear.

[5]The data complexity of $\mathcal{SHOIQ}$ is currently not known but is generally believed to be NP-complete; we conjecture that the data complexity of $\mathcal{SRIQ}$ and $\mathcal{SROIQ}$ is also NP-complete

[6]For $\mathcal{SRIQ}$ only hardness results are proved in this paper

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