Risk, Ambiguity, and the Rank-Dependence Axioms

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Choice problems in the spirit of Ellsberg (1961) suggest that rank-dependent ("Choquet expected utility") preferences over subjective gambles might be subject to the same difficulties that Ellsberg's earlier examples posed for subjective expected utility. These difficulties stem from event-separability properties that rank-dependent preferences partially retain from expected utility, and suggest that nonseparable models of preferences might be better at capturing features of behavior that lead to these paradoxes. (JEL D81)

Consider an urn holding 101 balls, each marked with a number from 1 through 4. You don't know the number of balls of each type, but you do know that exactly 50 are marked with either a 1 or a 2, and 51 are marked with either a 3 or a 4. This is a variation on the classic urns of Daniel Ellsberg (1961). Given its information structure, you don't know the probability of any given number being drawn, but you do know there's an exact 50/101 chance of it being either a 1 or a 2, and a 51/101 chance of it being either a 3 or a 4.

Say you were offered the following pair of bets on this urn. Which one would you choose? (Of course, you could be indifferent between the two bets.)

TABLE 1—50–51 EXAMPLE

(*First pair of bets*)

	50 balls		51 balls	
	E_1	E_2	E_3	E_4
$f_1(\cdot)$	$8,000	$8,000	$4,000	$4,000
$f_2(\cdot)$	$8,000	$4,000	$8,000	$4,000

Say you were instead offered the following bets. In this case, which would you choose?

TABLE 2—50–51 EXAMPLE

(*Second pair of bets*)

	50 balls		51 balls	
	E_1	E_2	E_3	E_4
$f_3(\cdot)$	$12,000	$8,000	$4,000	$0
$f_4(\cdot)$	$12,000	$4,000	$8,000	$0

Call this urn and the bets above the *50:51 example.*

The purpose of this paper is to explore how this and similar examples may pose difficulties for the well-known *rank-dependent* or *Choquet expected utility* model of choice under subjective uncertainty, similar to those posed by Ellsberg's original counterexamples to the classical subjective expected utility hypothesis. The following section recalls some of Ellsberg's examples and the approach taken by the Choquet model to address them. Section II shows how (typical?) choices in the above and similar problems may pose Ellsberg-type difficulties for the Choquet model. Section III discusses the sources of these difficulties and their implications for modeling choice under subjective uncertainty.

I. Savage, Ellsberg, and Choquet

The classic *subjective expected utility* (SEU) model of choice under uncertainty, as axiomatized by Leonard Savage (1954), involves bets of the form $f(\cdot) = [x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n]$ for some mutually exclusive and exhaustive collection of events $\{E_1, \ldots, E_n\}$ and (not necessarily distinct) outcomes $x_1, \ldots, x_n$. Savage's axioms imply the existence of a cardinal function $U(\cdot)$ over outcomes and a subjective probability measure $\mu(\cdot)$ over events, such that the individual evaluates such bets according to an ordinal preference function of the form $W_{SEU}(f(\cdot)) \equiv W_{SEU}(x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n) \equiv \sum_{i=1}^{n} U(x_i)\mu(E_i)$. This model of risk preferences and beliefs has seen widespread application in the economics and decision theory literature. A key aspect of the model, which can be seen from the additive structure of $W_{SEU}(\cdot)$, is that preferences are *separable across mutually exclusive events*.[1]

The classic counterexample to the SEU model is the well-known *Ellsberg Paradox* (Daniel Ellsberg 1961), which involves the following pairs of bets on a 90-ball urn:

TABLE 3—THREE-COLOR ELLSBERG PARADOX

	30 balls	60 balls		
	red	black	yellow	
$f_1^*(\cdot)$	$100	$0	$0	
$f_2^*(\cdot)$	$0	$100	$0	∨
$f_3^*(\cdot)$	$100	$0	$100	
$f_4^*(\cdot)$	$0	$100	$100	∧

When faced with these choices, most individuals express a strict preference for $f_1^*(\cdot)$ over $f_2^*(\cdot)$ and for $f_4^*(\cdot)$ over $f_3^*(\cdot)$, as indicated to the right of the table. However, these preferences violate the SEU functional form $\sum_{i=1}^{n} U(x_i)\mu(E_i)$, since they imply the inconsistent inequalities $U(\$100)\mu(\text{red}) + U(\$0)\mu(\text{black}) > U(\$0)\mu(\text{red}) + U(\$100)\mu(\text{black})$ and $U(\$100)\mu(\text{red}) + U(\$0)\mu(\text{black}) < U(\$0)\mu(\text{red}) + U(\$100)\mu(\text{black})$. More specifically, they violate event-separability, since individuals' ranking of the subacts [$100 on red; $0 on black] versus [$0 on red; $100 on black] depends upon whether the mutually exclusive event yellow yields a payoff of $0 or $100.

The intuition behind these choices is clear: $f_1^*(\cdot)$ offers the $100 prize on an objective ⅓-probability event, whereas $f_2^*(\cdot)$ offers it on one element of an informationally symmetric but subjective partition ({black, yellow}) of a ⅔-probability event. Similarly, $f_4^*(\cdot)$ offers the prize on an objective ⅔-probability event, whereas $f_3^*(\cdot)$ offers it on the union of a ⅓-probability event

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[1] Axiomatically, event-separability follows from Savage's Axiom P2 (1954, 23), termed the *Sure-Thing Principle*.

and the other element of that subjective partition. In each case, the purely objective bet is preferred to its informationally equivalent[2] subjective counterpart.

Though the urn described above is the most well known of Ellsberg's examples, it is not the only one. Another example (1961, 654, n. 4), suggested to Ellsberg by Kenneth Arrow, involves bets on the following four-color urn. Again, the objective bets $f_1^{**}(\cdot)$ and $f_4^{**}(\cdot)$ are preferred to their informationally equivalent subjective counterparts $f_2^{**}(\cdot)$ and $f_3^{**}(\cdot)$.

TABLE 4—FOUR-COLOR ELLSBERG PARADOX

	100 balls		50 balls	50 balls	
	red	black	green	yellow	
$f_1^{**}(\cdot)$	$100	$100	$0	$0	
$f_2^{**}(\cdot)$	$100	$0	$100	$0	∨
$f_3^{**}(\cdot)$	$0	$100	$0	$100	
$f_4^{**}(\cdot)$	$0	$0	$100	$100	∧

These examples and others (e.g., Ellsberg 1961, 2001) suggest that individuals' preferences depart from the classic SEU model by exhibiting a systematic preference for objective over subjective bets, a phenomenon known as *ambiguity aversion*.

Such examples have spurred the development of alternatives to subjective expected utility, most notably the *expected utility with rank-dependent subjective probabilities* or *Choquet expected utility* model of preferences over subjective bets. Axiomatized by Itzhak Gilboa (1987) and David Schmeidler (1989), it posits a preference function $W_{CEU}(\cdot)$ of the form[3]

$$(1) \qquad W_{CEU}(f(\cdot)) \equiv W_{CEU}(x_1^{\geqq} \text{ on } E_1; \ldots; x_n^{\geqq} \text{ on } E_n) \equiv \sum_{i=1}^{n} U(x_i^{\geqq})[C(\cup_{j=1}^{i} E_j) - C(\cup_{j=1}^{i-1} E_j)]$$

for some cardinal utility function $U(\cdot)$ and nonadditive measure or *capacity* $C(\cdot)$, defined over $f(\cdot)$'s *decumulative events* $\cup_{j=1}^{i} E_j$ and satisfying $C(\varnothing) = 0$ and $C(\cup_{j=1}^{n} E_j) = 1$, where the outcomes $x_1^{\geqq}, \ldots, x_n^{\geqq}$ must be labeled in order of weakly decreasing preference (outcomes labeled in this manner are denoted by the notation $x_1^{\geqq}, \ldots, x_n^{\geqq}$). Since it replaces subjective probabilities $\mu(E_i)$ by weights of the form $[C(\cup_{j=1}^{i} E_j) - C(\cup_{j=1}^{i-1} E_j)]$, $W_{CEU}(\cdot)$ no longer exhibits event-separability. However the telescoping nature of these weights ensures that, like subjective probabilities, they sum to unity.[4] There are several alternative axiomatic derivations of this model, but each involves some form of the following property:[5]

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[2] If $\{E_1, \ldots, E_n\}$ is an informationally symmetric partition of an objective event E with probability p, we will say that any k-element union of its events is *informationally equivalent* to any objective event with probability pk/n. Acts are said to be informationally equivalent if they assign their respective payoffs to informationally equivalent events.

[3] In the following formulas, unions of the form $\cup_{j=1}^{0}$ are taken to equal $\varnothing$, and sums $\sum_{j=i}^{i-1}$ are taken to equal zero.

[4] The property of summing to unity prevents the nonmonotonicity exhibited by preference functions of the form $W(f(\cdot)) = U(x_1^{\geqq})C(E_1) + \cdots + U(x_n^{\geqq})C(E_n)$. Since the telescoping property also implies $U(x)[C(\cup_{j=1}^{i} E_j) - C(\cup_{j=1}^{i-1} E_j)] + U(x)[C(\cup_{j=1}^{i+1} E_j) - C(\cup_{j=1}^{i} E_j)] = U(x)[C(\cup_{j=1}^{i+1} E_j) - C(\cup_{j=1}^{i-1} E_j)] = U(x)[C((\cup_{j=1}^{i-1} E_j) \cup (E_i \cup E_{i+1})) - C(\cup_{j=1}^{i-1} E_j)]$, equal outcomes $x_i^{\geqq} = x_{i+1}^{\geqq}$ and their events can be combined, and an individual outcome's event can be split.

[5] This property is similar, though not quite equivalent, to P2* of Gilboa (1987) and Axiom (ii) of Schmeidler (1989). See also the treatments of Yutaka Nakamura (1990), Rakesh Sarin and Peter P. Wakker (1992), Soo Hong Chew and Edi Karni (1994), Chew and Wakker (1996), Wakker (1996), Mohammed Abdellaoui and Wakker (2005), and the review of Veronika Köbberling and Wakker (2003).

Comonotonic Sure-Thing Principle: If $[x_1^{\geqq} \text{ on } E_1; \ldots; x_n^{\geqq} \text{ on } E_n] \geqq [y_1^{\geqq} \text{ on } E_1; \ldots; y_n^{\geqq} \text{ on } E_n]$,

$$\text{if} \quad x_i^{\geqq} = y_i^{\geqq} \quad \text{and} \quad \hat{x}_i^{\geqq} = \hat{y}_i^{\geqq} \qquad \text{for all } i \in \mathcal{I} \subseteq \{1, \ldots, n\},$$

$$\text{and if} \quad x_i^{\geqq} = \hat{x}_i^{\geqq} \quad \text{and} \quad y_i^{\geqq} = \hat{y}_i^{\geqq} \qquad \text{for all } i \notin \mathcal{I},$$

$$\text{then} \quad [\hat{x}_1^{\geqq} \text{ on } E_1; \ldots; \hat{x}_n^{\geqq} \text{ on } E_n] \geqq [\hat{y}_1^{\geqq} \text{ on } E_1; \ldots; \hat{y}_n^{\geqq} \text{ on } E_n].$$

That is to say, if two acts have the same set of decumulative events $E_1, E_1 \cup E_2, E_1 \cup E_2 \cup E_3, \ldots$ and common outcomes over some subfamily of events $\{E_i \mid i \in I\}$, then the outcomes they actually have over that subfamily should not affect their ranking. This implies a partial form of event-separability, which we term *tail-separability,* and which is apparent from the following decomposition of $W_{CEU}(\cdot)$ into terms involving its upper-tail, middle, and lower-tail decumulative events:

$$(2) \qquad W_{CEU}(x_1^{\geqq} \text{ on } E_1; \ldots; x_n^{\geqq} \text{ on } E_n) \equiv \sum_{i=1}^{i'} U(x_i^{\geqq}) [C(\cup_{j=1}^{i} E_j) - C(\cup_{j=1}^{i-1} E_j)]$$

$$+ \sum_{i=i'+1}^{i''} U(x_i^{\geqq}) [C(\cup_{j=1}^{i} E_j) - C(\cup_{j=1}^{i-1} E_j)]$$

$$+ \sum_{i=i''+1}^{n} U(x_i^{\geqq}) [C(\cup_{j=1}^{i} E_j) - C(\cup_{j=1}^{i-1} E_j)].$$

This model can represent the ambiguity-averse rankings of $f_1^*(\cdot) \succ f_2^*(\cdot)$ and $f_3^*(\cdot) \prec f_4^*(\cdot)$ in the three-color Ellsberg example, since any capacity satisfying $C(\text{red}) = \frac{1}{3} > C(\text{black})$ and $C(\text{red} \cup \text{yellow}) < \frac{2}{3} = C(\text{black} \cup \text{yellow})$ will imply $W_{CEU}(f_1^*(\cdot)) = \frac{1}{3} \times U(\$100) + \frac{2}{3} \times U(\$0) > W_{CEU}(f_2^*(\cdot))$ and $W_{CEU}(f_3^*(\cdot)) < \frac{2}{3} \times U(\$100) + \frac{1}{3} \times U(\$0) = W_{CEU}(f_4^*(\cdot))$. The modal preferences in the four-color Ellsberg urn can be modeled in a similar manner. As shown by Gilboa, Schmeidler, and others, Choquet expected utility preferences retain much of the structure and predictive power of classical subjective expected utility, but are sufficiently more general to be able to accommodate the various Ellsberg Paradoxes and other systematic departures from expected utility and probabilistic beliefs.

II. An Ellsberg Paradox for Choquet Expected Utility?

The Choquet expected utility model of subjective act preferences successfully captures the type of ambiguity aversion displayed in Ellsberg's examples, and has received widespread attention in the literature. It is not clear, however, how it performs in other cases, such as the 50:51 example above. From the tables, it is clear that acts $f_3(\cdot)$ and $f_4(\cdot)$ are obtained from $f_1(\cdot)$ and $f_2(\cdot)$ by a pair of common-outcome tail shifts, namely \$8,000 up to \$12,000 in event E_1 and \$4,000 down to \$0 in E_4. By tail-separability, a Choquet expected utility maximizer would prefer $f_1(\cdot)$ to $f_2(\cdot)$ if and only if he or she prefers $f_3(\cdot)$ to $f_4(\cdot)$. A pair of reversed rankings, such as $f_1(\cdot) \succ f_2(\cdot)$ and $f_3(\cdot) \prec f_4(\cdot)$, would contradict tail-separability, and hence contradict Choquet.

Whether individuals (including the reader) actually exhibit such reversed rankings is of course an empirical matter. But there is a strong Ellsberg-like argument why they might. Though neither pair consists of informationally equivalent acts as in Ellsberg's examples, each involves a trade-off between objective and subjective uncertainty.

On the one hand, the lower act in each pair differs from the upper act only in its placement of the outcomes \$4,000 and \$8,000 between the events E_2 and E_3. Since the urn has a 50:51 ball count, in each pair the lower act has a slight "objective advantage" over the upper act.

On the other hand, the lower act in each pair is more ambiguous than the upper act, though the extent of the ambiguity difference is not the same for the two pairs. In the first pair, $f_2(\cdot)$ is distinctly more ambiguous than $f_1(\cdot)$: whereas $f_1(\cdot)$'s payoffs are perfectly correlated with the objec-

tive partition $\{E_1\cup E_2, E_3\cup E_4\}$, $f_2(\cdot)$'s payoffs are not correlated at all. Thus an ambiguity-averse individual may well feel that $f_2(\cdot)$'s ambiguity more than offsets its slight objective advantage, and choose $f_1(\cdot)$. But in the second pair, $f_4(\cdot)$ is only somewhat more ambiguous than $f_3(\cdot)$: while $f_4(\cdot)$'s outcomes are somewhat less correlated with the partition $\{E_1\cup E_2, E_3\cup E_4\}$ than are $f_3(\cdot)$'s, the difference in these correlations is less stark than between $f_1(\cdot)$ and $f_2(\cdot)$. The individual may feel that this smaller ambiguity difference does not offset $f_4(\cdot)$'s objective advantage, and choose $f_4(\cdot)$.

III. Ambiguity Aversion as Event-Nonseparability

How is it that rank-dependent preferences, which successfully capture ambiguity-averse behavior in Ellsberg's original examples, are violated by what would seem to be similar behavior in our modified version of these examples? The answer lies in a property that rank-dependent preferences retain from classical expected utility preferences, namely, tail-separability.

In the three-color Ellsberg urn, the acts $f_3^*(\cdot)$ and $f_4^*(\cdot)$ are obtained from $f_1^*(\cdot)$ and $f_2^*(\cdot)$ by a common-outcome shift of $0 up to $100 in the event yellow. By full event-separability, this should not alter the preference ranking of the upper versus lower act within each pair. But this shift reverses the ambiguity properties of the upper versus lower act, from $f_1^*(\cdot)$ being fully objective and $f_2^*(\cdot)$ subjective, to the other way around for $f_3^*(\cdot)$ and $f_4^*(\cdot)$. Thus, an ambiguity-averse individual would exhibit the reversed rankings $f_1^*(\cdot)\succ f_2^*(\cdot)$ and $f_3^*(\cdot)\prec f_4^*(\cdot)$. In other words, models that exhibit full event-separability—such as subjective expected utility—are incompatible with cross-event effects due to attitudes toward ambiguity.

Models that drop full event-separability but retain *tail-separability*—such as rank-dependent subjective expected utility—retain this type of incompatibility. In the 50:51 example, the common-outcome tail shifts that convert acts $f_1(\cdot)$ and $f_2(\cdot)$ to acts $f_3(\cdot)$ and $f_4(\cdot)$ have similar cross-event effects on the ambiguity properties of the upper versus lower acts. These shifts do not *reverse* the ambiguity properties as they do for Ellsberg. They still *affect* them, however, by reducing the extent of the ambiguity difference between the upper and lower act. If ambiguity matters to the decision maker, the magnitude of this ambiguity difference can determine whether it does or does not offset some other difference between the acts, such as an objective probability difference (as in our example) or, alternatively, an outcome difference.[6]

Common-outcome tail shifts do not affect only ambiguity properties. Consider a choice between $f_5(\cdot)$ and $f_6(\cdot)$ below. It is not clear which of these acts is more ambiguous: $f_6(\cdot)$ has a payoff difference of $8,000 riding on the subjective subpartition $\{E_3, E_4\}$, whereas $f_5(\cdot)$ divides this stake, with $4,000 riding on each of the subpartitions $\{E_1, E_2\}$ and $\{E_3, E_4\}$. Thus, $f_5(\cdot)$ gives a 100 percent chance of $4,000 riding on the unknown composition of the urn, whereas $f_6(\cdot)$ gives a 50 percent chance of $8,000 riding on this subjective uncertainty.[7] As in the 50:51 example, an ambiguity averter's choice may well depend on their trade-off rate between this objective and subjective uncertainty, and their attitudes toward the different payoff levels involved. If their personal trade-off rate exactly matched that of the problem, they could be indifferent.

Say the individual does have a strict preference, $f_5(\cdot)\succ f_6(\cdot)$. Since $f_7(\cdot)$ and $f_8(\cdot)$ differ from $f_5(\cdot)$ and $f_6(\cdot)$ by an ordered sequence of common-outcome tail shifts,[8] tail-separability implies that this ranking should extend to $f_7(\cdot)\succ f_8(\cdot)$. But as seen in Table 5, $f_8(\cdot)$ is an informationally symmetric left-right reflection of $f_5(\cdot)$, and $f_7(\cdot)$ is a left-right reflection of $f_6(\cdot)$. Surely, anyone

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[6] To create an example of the latter, change the ball count to 50:50, and change the E_3 payoff in acts $f_1(\cdot)$ and $f_3(\cdot)$ from $4,000 up to $4,010.

[7] Neither act can be said to have a greater correlation of its outcomes with the objective partition $\{E_1\cup E_2, E_3\cup E_4\}$.

[8] First, shift $0 up to $4,000 in event E_4, then shift $4,000 down to $0 in E_1.

TABLE 5—REFLECTION EXAMPLE

	50 balls			50 balls	
	E_1	E_2		E_3	E_4
$f_5(\cdot)$	$4,000	$8,000		$4,000	$0
$f_6(\cdot)$	$4,000	$4,000		$8,000	$0
$f_7(\cdot)$	$0	$8,000		$4,000	$4,000
$f_8(\cdot)$	$0	$4,000		$8,000	$4,000

with the ranking $f_5(\cdot) > f_6(\cdot)$ should have the "reflected" ranking $f_7(\cdot) < f_8(\cdot)$. Besides affecting the relative ambiguity properties of acts, tail-shifts can also affect their symmetry properties, again leading to Ellsberg-like difficulties for the Choquet model. Call this urn and bets the *reflection example*.

Of course, there is no reason why an individual need have a strict preference in either pair of acts, in which case the reflections cause no problem. Indeed, one might argue that since the decumulative events of $f_5(\cdot)$ and $f_6(\cdot)$ are either identical or informationally symmetric to each other, a Choquet individual *must* be indifferent between them, and similarly between $f_7(\cdot)$ and $f_8(\cdot)$, so there is no reflection paradox. This is not a counterargument to the point being made here, so much as its contrapositive: if no Choquet individual would have a strict preference in either pair, then anyone who does have a strict preference cannot be Choquet. It is much like arguing that Ellsberg's three-color urn does not violate SEU, since no SEU individual would ever exhibit the reversed rankings $f_1^*(\cdot) > f_2^*(\cdot)$ and $f_3^*(\cdot) < f_4^*(\cdot)$. In all such choice problems, the issue is not how individuals *ought* to choose, but rather, how they *do* choose. In recent experimental tests, Olivier L'Haridon and Lætitia Placido (forthcoming) found that over 90 percent of subjects expressed strict preference in choice problems with the above structure, and that roughly 70 percent violated tail-separability by exhibiting reversed rankings of the form $f_5(\cdot) > f_6(\cdot)$ and $f_7(\cdot) < f_8(\cdot)$, or the form $f_5(\cdot) < f_6(\cdot)$ and $f_7(\cdot) > f_8(\cdot)$.

If there is a general lesson to be learned from Ellsberg's examples and the examples here, it is that the phenomenon of ambiguity aversion is intrinsically one of nonseparable preferences across mutually exclusive events, and that models that exhibit full—or even partial—event-separability cannot capture all aspects of this phenomenon. This suggests that the study of choice under uncertainty should proceed in a manner similar to standard consumer theory, which develops theoretical results like the Slutsky equation—which *do not require* separability across individual commodities—and models empirical phenomena like inferior goods—which *are not compatible* with this property.[9]

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[9] Ehud Lehrer (2008) demonstrates that the *partially specified probabilities* model of Lehrer (2007) and the *concave integral* preference function of Lehrer (forthcoming), neither of which are event-separable or tail-separable, are each consistent with the rankings $f_1(\cdot) > f_2(\cdot)$ and $f_3(\cdot) < f_4(\cdot)$ in the 50:51 example, as well as with $f_5(\cdot) > f_6(\cdot)$ and $f_7(\cdot) < f_8(\cdot)$ in the reflection example. Jacob Sagi (private correspondence) has shown that the nonseparable models of Peter Klibanoff, Massimo Marinacci, and Sujoy Mukerji (2005) and Chew and Sagi (2008) are similarly consistent with $f_5(\cdot) > f_6(\cdot)$ and $f_7(\cdot) < f_8(\cdot)$. Marciano Siniscalchi (2008a,b) shows that the *vector expected utility* model of Siniscalchi (2008a) is consistent with the plausible preference pattern in the reflection example, and explores further implications for the notion of ambiguity aversion. Aurélien Baillon, L'Haridon, and Placido (2008) explore conditions under which multiple-priors models such as the *maxmin expected utility* model of Gilboa and Schmeidler (1989) are/are not consistent with the 50:51 example and the reflection example, and Kin Chung Lo (2007) has shown how a modified version of the reflection example can be used to test the model of Klibanoff (2001).

The arguments and examples of this paper have involved theoretical properties of models and hypothesized properties of preferences. As with any such questions, the issues must ultimately be decided by the data.[10]

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