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# Risk Aversion, Price Uncertainty, and Irreversible Investments

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**Abstract**

This paper generalizes the theory of irreversible investment under uncertainty by allowing for risk averse investors in the absence of complete markets. Until now this theory has only been developed in the cases of risk neutrality, or risk aversion in combination with complete markets. Within a general setting, we prove the existence of a unique critical output price that distinguishes price regions in which it is optimal for a risk averse investor to invest and price regions in which one should refrain from investing. We use a class of utility functions that exhibit non-increasing absolute risk aversion to examine the effects of risk aversion, price uncertainty, and other parameters on the optimal investment decision. We find that risk aversion reduces investment, particularly if the investment size is large. Moreover, we find that a rise in price uncertainty increases the value of deferring irreversible investments. This effect is stronger for high levels of risk aversion. In addition, we provide, for the first time, closed-form comparative statics formulas for the risk neutral investor.

*Keywords*: investment under uncertainty, real options, risk averse investor, incomplete markets
*JEL codes*: C61, D81, G31

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# 1 Introduction

How should investors decide whether and when to invest in uncertain, irreversible projects in the case of incomplete markets? And what is the effect of risk aversion on investment behavior? This paper addresses these questions in the context of the real options theory developed by McDonald and Siegel (1985, 1986). They show that the conventional net present value rule to decide whether or not to invest in some uncertain project ignores the option value of postponing the investment.

Dixit and Pindyck (1994) give a textbook treatment of this new investment theory. They describe two closely related but essentially different mathematical tools to model investment decisions: dynamic programming and contingent claims analysis. The latter endogenously determines an investor's discount rate as an implication of the overall capital market equilibrium. Both risk neutrality and risk aversion can be dealt with within the contingent claims approach, but the approach requires the existence of a sufficiently rich set of markets of risky assets so that a dynamic portfolio of traded assets exactly replicates the payoff of the investment that is to be valued. This assumption of complete markets is in reality quite strong, especially for investments in non-traded assets such as investments in marketing or advertising, or the development of new products (see, e.g., Magill and Quinzii (1995)). Dynamic programming, however, makes no such demand; if risk cannot be traded in markets, the investor's objective function can simply reflect the decision maker's valuation of risk. Until now, dynamic programming has only been applied to the problem of irreversibility under the assumption of risk neutrality.

In this paper we consider the economically relevant problem faced by risk averse investors who contemplate an irreversible investment in an asset whose payoff cannot be replicated by a dynamic portfolio of traded securities. Hence, in this (realistic) situation of incomplete markets, we are not able to use contingent claims analysis as a tool to solve the investment problem. Instead, we apply dynamic programming to an objective function that reflects risk aversion.

The purpose of this paper is to generalize the approach of McDonald and Siegel (1986) and Dixit and Pindyck (1994) by allowing for risk aversion in an environment of incomplete markets. Our aim is to find out how the optimal investment decision is affected by risk aversion, investment size, price uncertainty, and other parameters.

Our main results are the following. First of all we prove that, within a general setting, a unique critical price level exists for which the risk averse

investor is indifferent between investing and not investing. Second, we introduce a class of utility functions with the desirable property of non-increasing absolute risk aversion to examine the comparative statics of this critical price level with respect to risk aversion, investment size, price uncertainty, and other parameters. We find that risk aversion reduces investment, particularly if the investment size is large. Moreover, we find that a rise in uncertainty increases the value of deferring irreversible investments. This effect is stronger for high levels of risk aversion.

The remainder of the paper is organized as follows. Section 2 formulates the investment problem. Section 3 describes the general solution of the investment problem. In Section 4 we introduce a class of utility functions which exhibit the desirable property of non-increasing absolute risk aversion. This class of utility functions allows us to numerically examine the comparative statics of the critical price level under risk aversion in Section 5. In addition, we provide analytical comparative statics formulas for the risk neutral investor. Section 6 concludes.

## 2 The Investment Problem

We use a set-up along the lines of Dixit and Pindyck (1994, pp. 185–186). Consider an infinitely-lived investor contemplating an irreversible, discrete investment opportunity with sunk cost $I > 0$. For simplicity we assume that once the investment is made, it produces one unit of output flow into the indefinite future with no variable costs of production. The output price $P_t$ is assumed to follow a geometric Brownian motion,

$$dP_t = \alpha P_t dt + \sigma P_t dz_t, \tag{1}$$

where $\sigma > 0$ and $z_t$ is a standard Wiener process. Let $P_0 = P \geq 0$ denote the current output price. The required amount of money $I$ is borrowed at an instantaneous riskless rate of interest $r > 0$ which we assume to be constant and larger than $\alpha$. Thus, if the investor decides to invest at time $t = 0$, then the instantaneous net cash flow accruing from the project at any time $t \geq 0$ is

$$ncf_t \equiv P_t - rI.$$

Note that since $P \geq 0$, the range of possible values for $ncf$ is $[-rI, \infty)$.

We assume that the investor's preferences are intertemporally additive, and that they can be represented by an increasing, twice differentiable von Neumann-Morgenstern utility function $u(\cdot)$ which is defined over the instantaneous net cash flows and independent of time, $u : [-rI, \infty) \to \mathbb{R}$.

Furthermore, we assume that utility flows are discounted at the riskless rate of return $r$. We shall consider both situations in which $u$ reflects risk neutrality and situations in which $u$ reflects risk aversion.

Our goal is to determine whether and when the investor should invest in the project. In making this investment decision it is important to not only take into account the expected utility of the net cash flows produced by the project, but also the real option value embedded in its irreversible nature. Once the investment has been made, it cannot be undone should prospects change for the worse. By deferring the investment, however, the investor can await new information that affects the desirability of the expenditure.

## 3 Valuing the investment opportunity

If the investor decides to invest at $t = 0$, the expected utility of the net cash flows produced by the project is given by

$$V(P) = E \int_{t=0}^{\infty} e^{-rt} u(ncf_t) dt.$$

As indicated by the notation, $V$ depends on the current output price $P$ of the project. According to the classical net present value ($npv$) rule, the investor would have to invest at $t = 0$ if $P$ were such that $V(P)$ is positive, and refrain from investing otherwise. However, this approach disregards the option value of postponing the irreversible investment at time $t = 0$. Let $C(P)$ denote this option value. It is determined by the following Bellman equation:

$$C(P) = u(0)dt + e^{-rdt} E\left\{C(P + dP_t)\right\}, \tag{2}$$

that is, the option value of deferring the investment is equal to the sum of the utility of waiting during a time interval $[0, dt]$ in which no cash flow occurs, and the discounted expected future utility of waiting.

Without loss of generality we assume that $u(0) = 0$, thereby in effect associating net cash inflows with positive utility levels, and net cash outflows with negative utility levels. Using this convention, we apply Itô's Lemma to rewrite the right-hand side of (2) as[1]

$$C(P) + \left[\frac{1}{2}\sigma^2 P^2 C''(P) + \alpha P C'(P) - rC(P)\right] dt + o(dt).$$

Substitution of this expression into (2), dividing by $dt$, and letting $dt$ approach zero yields a second-order differential equation which is solved by

[1] A quantity is said to be $o(dt)$ if $o(dt)/dt \to 0$ as $dt \downarrow 0$.

$C(P) = A_1 P^{\beta_1} + A_2 P^{\beta_2}$, where $A_1$ and $A_2$ are integration constants, and $\beta_1 > 1$ and $\beta_2 < 0$ are the roots of the quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - r = 0$. Clearly, the option to postpone the investment is worthless if the current output price is zero, i.e., $C(0) = 0$. Therefore $A_2$ must be zero, and hence,

$$C(P) = A_1 P^{\beta_1}. \tag{3}$$

Note that $C(P)$ is increasing and convex in $P$.

We can now characterize the optimal investment decision. The investor should undertake the investment if the expected utility of the cash flows accruing from the project exceeds the value of delaying it; otherwise, he should postpone the investment. Let $P^*$ be the output price for which the investor is indifferent between investment and delay. Then

$$V(P^*) = C(P^*). \tag{4a}$$

Eq. (4a) is referred to as the value-matching condition. Furthermore, $V$ and $C$ should meet tangentially at $P^*$, that is,

$$V'(P^*) = C'(P^*), \tag{4b}$$

where $V'$ and $C'$ denote the partial derivatives of $V$ and $C$ with respect to $P$, respectively. Eq. (4b) is called the high-order contact or smooth-pasting condition. See Dixit and Pindyck (1994, pp. 130–132) for a discussion on smooth pasting.

Concerning existence and uniqueness of $P^*$, we were able to prove the following proposition.

**Proposition 1** *Consider an investor who is either risk neutral or risk averse within the model outlined above. Then it holds that:*

1. *If there exists an output price $P^*$ satisfying (4a) and (4b), it is unique.*
2. *Existence of $P^*$ is guaranteed if the utility function is unbounded.*

**Proof 1** *See Appendix A.*

Proposition 1 states that the existence of a critical output price implies its uniqueness. Hence, if there if a critical output price, the optimal investment decision is tantamount to a simple investment rule: invest if $P > P^*$ and wait if $P < P^*$. If no critical output price exists, it is optimal never to invest, however high the current price level.

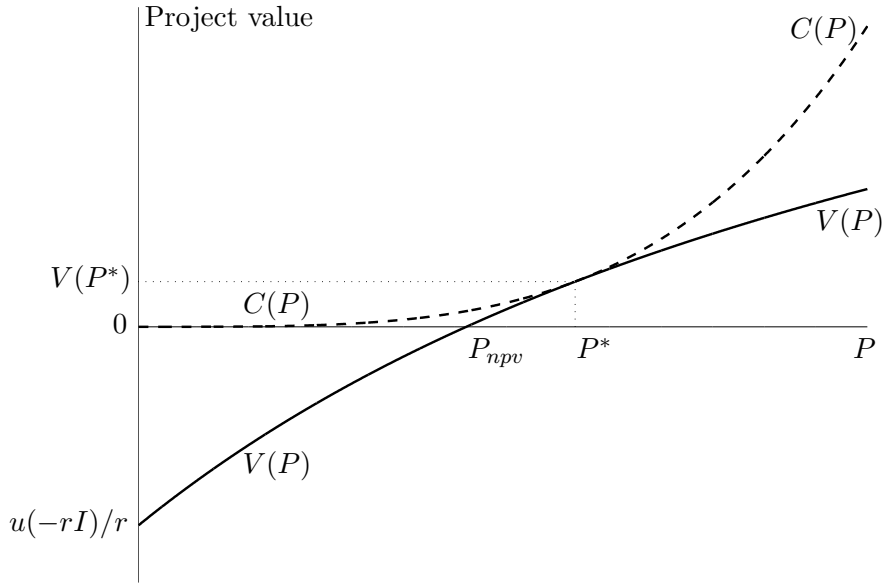


Figure 1: Graphical illustration of the optimal investment decision. The solid graph depicts $V$ as a function of $P$. The dashed curve is $C$ as a function of $P$. The critical output price $P^*$ is located at the point where $V$ and $C$ are tangent and intersect. The *npv* critical price is located at the point where $V$ intersects the $P$-axis.

Figure 1 illustrates the investment decision graphically. It depicts $V$ and $C$ as functions of $P$. The critical output price is located at the point where $V$ and $C$ intersect. The functions are also tangent at this point. If the current output price is below this threshold, the investor defers the investment, and its value is equal to the option value. If the current output price exceeds the threshold, the investment will be made, and its value is equal to the expected utility of its net cash flows.

Note that the *npv* critical output price ($P_{npv}$) is located at the point where the expected utility of the net cash flows produced by the project is equal to zero, i.e., $V(P_{npv}) = 0$. In fact, this is the relevant threshold if the investment project were reversible or when the investment decision is a now-or-never option. Clearly, the *npv* threshold is always smaller than the critical output price under irreversibility.

## 4 An example

In order to analyze the effects of changes in investors' attitudes toward risk on the optimal investment decision, we introduce the following utility function:

$$u(x) = (s - \eta)x + \eta(1 - e^{-x}), \tag{5}$$

where $s > 0$ and $\eta \in [0, s]$. This utility function is constructed as a linear combination of a risk neutral utility function and a constant absolute risk aversion (CARA) utility function with unit Arrow-Pratt measure (see, e.g., Mas-Colell, Whinston, and Green (1995)). It is increasing and concave (for $\eta \neq 0$), and it meets the imposed normalization $u(0) = 0$. Moreover, it has the attractive feature that it incorporates risk neutrality as a special case (for $\eta = 0$). Hence, it allows us to compare the case of risk neutrality to the case of risk aversion.

Another important property of the utility function considered is that it exhibits non-increasing absolute risk aversion. The hypothesis of non-increasing absolute risk aversion was already propounded by Arrow (1970). It is supported by the empirical observation that the willingness to take small bets increases as individuals get wealthier. For $\eta \neq 0$, the Arrow-Pratt measure of absolute risk aversion is given by

$$R_A(x) = -\frac{u''(x)}{u'(x)} = \frac{1}{1 + \frac{s-\eta}{\eta}e^x}, \tag{6}$$

which is indeed decreasing in $x$. Another consequence of Eq. (6) is that the parameter $\eta$ may be interpreted as a measure of the degree of risk aversion of the investor, since $R_A(x)$ is increasing in $\eta$ for all $x$.

Under this specification, the expected utility of net cash flows resulting from investing is given by

$$V(P) = (s - \eta)\left[\frac{P}{r - \alpha} - I\right] + \eta\left[\frac{1}{r} - e^{rI}G(P)\right],$$

where

$$G(P) \equiv E\int_{t=0}^{\infty} e^{-rt}e^{P_t}dt.$$

An explicit expression for $G(P)$ can be obtained by writing down the dynamic programming-like recursion expression (cf. Dixit and Pindyck (1994, pp. 315–316)):

$$\begin{aligned} G(P) &= e^{-P}dt + e^{-rdt}E\left\{G(P + dP_t)\right\} \\ &= G(P) + \left[\frac{1}{2}\sigma^2 P^2 G''(P) + \alpha P G'(P) - rG(P) + e^{-P}\right]dt + o(dt), \end{aligned}$$

which implies a second-order differential equation whose solution reads

$$G(P) = \frac{1}{r} \times \frac{P^{\beta_1}\Psi_1(P) + P^{\beta_2}\Psi_2(P)}{1/\beta_1 - 1/\beta_2},$$

where

$$\begin{aligned} \Psi_1(P) &\equiv \int_{\nu=P}^{D_1} \nu^{-\beta_1-1} e^{-\nu} d\nu \\ \Psi_2(P) &\equiv \int_{\nu=D_2}^{P} \nu^{-\beta_2-1} e^{-\nu} d\nu, \end{aligned}$$

with integration constants $D_1 \geq 0$ and $D_2 \geq 0$. In Appendix B it is shown that $D_1 = \infty$ and $D_2 = 0$.

While the utility function considered allows for an explicit expression for $V(P)$, the corresponding critical output price $P^*$ cannot be solved for analytically. However, it can easily be computed numerically by means of traditional search algorithms given the parameters of the model. Note in particular that if $\eta = 0$, that is if the investor is risk neutral, then $P^*$ is equal to the investment threshold discussed in Dixit and Pindyck (1994, p. 186):

$$P^* = \frac{\beta_1}{\beta_1 - 1}(r - \alpha)I. \tag{7}$$

# 5 Comparative statics

In this section we examine the influence of the parameters of the model on the investment decision. First we derive the comparative statics for the risk neutral case. Subsequently we analyze the comparative statics for the utility function introduced in Section 4.

## 5.1 Risk neutrality

We start by examining the effect of a change in the investment cost $I$ on the critical output price under risk neutrality. Recall that the critical output price is given by (7) in the risk neutral case. A first, trivial observation is that $P^*$ is proportionally increasing with the investment cost $I$. Next, consider the other parameters of the model: $\alpha$, $r$, and $\sigma^2$. The partial derivative of $P^*$ with respect to $x \in \{\alpha, r, \sigma^2\}$ is equal to

$$\frac{\partial P*}{\partial x} = \frac{I}{\beta_1 - 1}\left[\beta_1 \frac{\partial(r-\alpha)}{\partial x} - \frac{r-\alpha}{\beta_1 - 1}\frac{\partial \beta_1}{\partial x}\right].$$

Using $\frac{1}{2}\sigma^2\beta_i(\beta_i-1)=r-\alpha\beta_i$ for $i=1,2$, we find

$$\frac{\partial\beta_1}{\partial x}=\begin{cases} -\frac{\beta_1}{\frac{1}{2}\sigma^2(\beta_1-\beta_2)} & \text{for } x=\alpha \\ \frac{1}{\frac{1}{2}\sigma^2(\beta_1-\beta_2)} & \text{for } x=r \\ -\frac{r-\alpha\beta_1}{\frac{1}{2}\sigma^4(\beta_1-\beta_2)} & \text{for } x=\sigma^2, \end{cases}$$

and, hence, the comparative statics in the risk neutral case are given by

$$\begin{aligned} \frac{\partial P^*}{\partial\alpha} &= -\frac{\beta_1}{\beta_1-\beta_2}I \\ \frac{\partial P^*}{\partial r} &= \frac{1+\beta_1-\beta_2}{\beta_1-\beta_2}I \\ \frac{\partial P^*}{\partial\sigma^2} &= \frac{1}{2}\beta_1\frac{1-\beta_2}{\beta_1-\beta_2}I. \end{aligned}$$

To the best of our knowledge, this is the first time these comparative statics results have been analytically derived; Dixit and Pindyck only compute them numerically for certain sets of parameter values.

In particular, we find the following bounds on the partial derivatives:

$$\begin{array}{ccccccc} & & -I & < & \frac{\partial P^*}{\partial\alpha} & < & 0 \\ 0 & < & I & < & \frac{\partial P^*}{\partial r} & < & \frac{1+\beta_1}{\beta_1}I \\ 0 & < & \frac{1}{2}I & < & \frac{\partial P^*}{\partial\sigma^2}. & & \end{array}$$

Thus, in the risk neutral model, an increase in the drift term $\alpha$ always reduces the critical output price. That is, the utility of postponing the investment always decreases because its growth rate is higher. In contrast, an increase in the interest or discount rate raises the critical output price. Apparently, the discouraging effects of a rise in interest payments and a reduction in present value dominate the accelerating effect of higher impatience on investment. Moreover, the effect of a change in the interest rate on the critical output price is always greater than on the *npv* critical price. Furthermore, an increase in the volatility also raises the investment threshold: uncertainty adds to the value of waiting.

### 5.2 Risk aversion

We now analyze the comparative statics under risk aversion using the utility function defined in Section 4. In order to assess the impact of a change of a parameter $x$ on the threshold price, it is useful to define $\varphi(P)\equiv\beta_1V(P)-$

$PV'(P)$. From (3), (4a), and (4b) we have $\varphi(P^*) = 0$. Total differentiation of $\varphi(P^*) = 0$ gives

$$\frac{\partial \varphi}{\partial P}\frac{\partial P^*}{\partial x} + \frac{\partial \varphi}{\partial x} = 0,$$

where all partial derivatives are evaluated at $P^*$. This implies that the influence of a change in parameter $x$ on $P^*$ is measured by

$$\frac{\partial P^*}{\partial x} = -\frac{1}{\varphi'(P^*)} \left.\frac{\partial \varphi(P)}{\partial x}\right|_{P=P^*}.$$

As pointed out in Appendix A, the function $\varphi$ is strictly increasing on $[0, \infty)$, so $\varphi'(P^*) > 0$. Hence, the sign of the partial derivative of $P^*$ with respect to $x$ is opposite to the sign of the partial derivative of $\varphi$ with respect to $x$ evaluated at $P^*$.

As in the risk neutral case, we start by analyzing the effect of a change in the investment cost on the threshold price. Monotonicity and concavity of $u$ are sufficient to show that—not surprisingly—an increase in $I$ raises the threshold price, while a decrease in $I$ reduces it:

$$\frac{\partial \varphi(P)}{\partial I} = -rE\int_{t=0}^{\infty} e^{-rt}[\beta_1 u'(P_t - rI) - P_t u''(P_t - rI)]dt < 0,$$

and, hence, $\partial P^* / \partial I > 0$.

Such general statements are not possible with respect to the other parameters in the model, not even in the case the utility function in Section 4. Therefore, we conduct a number of numerical analyses to find out the influence of these parameters on the optimal investment decision using this utility function. In particular, we are interested in the influence of a change in risk aversion.

Unless mentioned otherwise, we set $\alpha = 0$, $r = 0.05$, $\sigma = 0.1$, and $s = 1$. Figure 2 shows the threshold price to interest payment ratio $P^* / rI$ as a function of the risk aversion parameter $\eta$ for different values of the investment cost. For $\eta = 0$ the risk neutral case applies. In this case the threshold price is proportional to the investment cost. This implies that, for a given interest rate, the fraction $P^* / rI$ is constant for different levels of $I$, which explains that in Figure 2 all curves coincide at $\eta = 0$. Figure 2 further shows that, for given $I$, $P^*$ increases with $\eta$. This means that, the more risk averse the investor is, the higher must be the output price for investment to be optimal. We conclude that a risk averse investor is more cautious to invest. Moreover, this effect is reinforced by the size of the investment. This can be explained by the fact that concavity of the utility

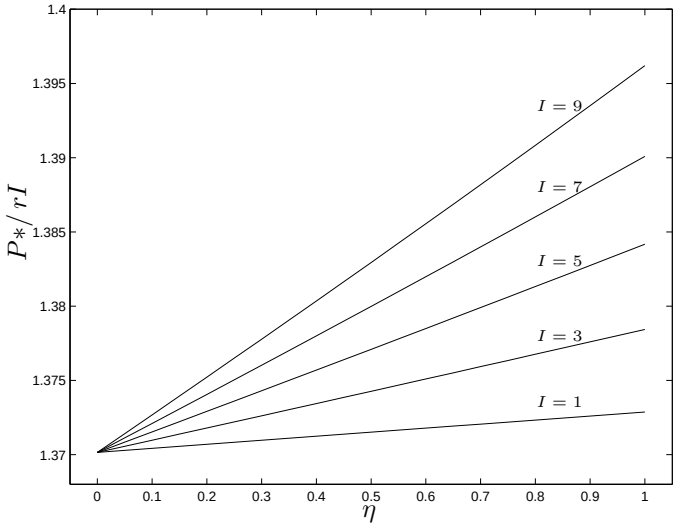


Figure 2: $P^*/rI$ as a function of $\eta$ for $I \in \{1, 3, 5, 7, 9\}$, $\alpha = 0$, $r = 0.05$, $\sigma = 0.1$, and $s = 1$.

function implies that as the investment cost goes up, the disutility of a large negative cash flow becomes more and more important. Consequently, the larger the investment outlay, the more the investor needs to be compensated for by a higher critical output price relative to interest payments.

Figure 3 demonstrates that the wedge between $P^*$ and the *npv* critical output price decreases with $\eta$. This means that the difference between the optimal investment decision and the decision based on the *npv* criterion shrinks the more risk averse the investor is. Hence, the error made by applying the *npv* rule, or, equivalently, the importance of irreversibility, becomes smaller under risk aversion. Again, the larger the investment cost, the stronger this effect becomes. The reason is that the large disutility of large investments plays a major role in case of a concave utility function, and this dominant factor affects $P^*$ and $P_{npv}$.

Figure 4 shows $P^*$ as a function of $r$. From this figure it can be concluded that—as in the risk neutral case—the critical output price increases with $r$ implying that it is less attractive to invest if the discount rate is larger. Another thing that emerges from Figure 4 is that risk aversion reinforces the influence of $r$ on $P^*$. The reason is that, similarly to the dependence of the investor's utility on $I$, the disutility of large net cash outflows becomes more and more important for higher values of $\eta$.

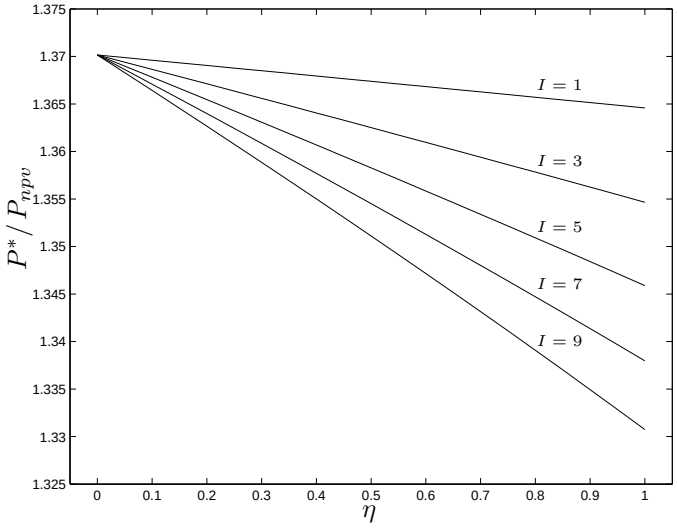


Figure 3: $P^* / P_{npv}$ as a function of $\eta$ for $I \in \{1, 3, 5, 7, 9\}$, $\alpha = 0$, $r = 0.05$, $\sigma = 0.1$, and $s = 1$.

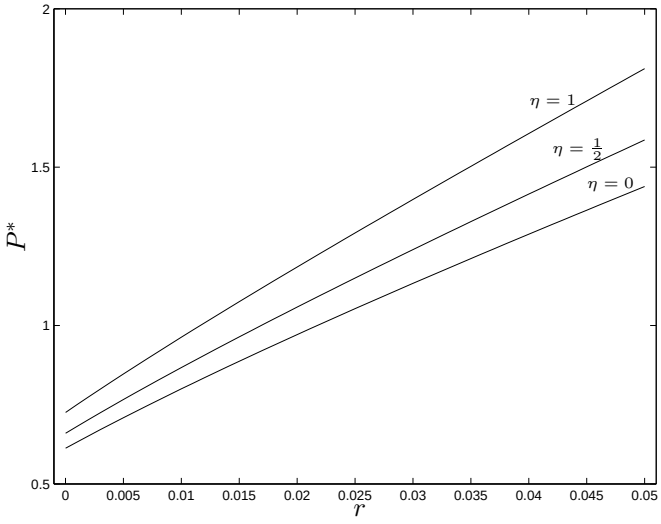


Figure 4: $P^*$ as a function of $r$ for $\eta \in \{0, \frac{1}{2}, 1\}$, $\alpha = 0$, $\sigma = 0.1$, $s = 1$, and $I = 1$.

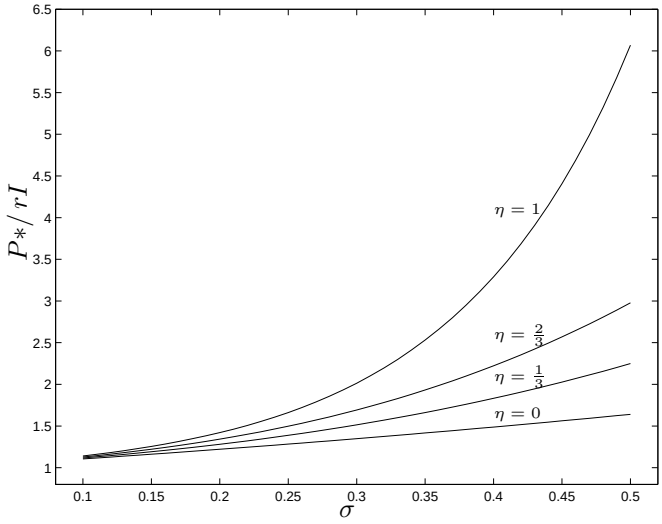


Figure 5: $P^*/rI$ as a function of $\sigma$ for $\eta \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, $\alpha = 0$, $r = 0.05$, $s = 1$, and $I = 1$.

Finally, we examine the effect of a change in price uncertainty on the investment decision. Figure 5 plots $P^*/rI$ as a function of $\sigma$ for different levels of risk aversion. Clearly, an increase in the volatility of the output price causes the threshold price to grow. After all, the more uncertain the future revenues are, the more it pays to wait for more information concerning the development of output prices. Figure 5 shows that this effect intensifies under risk aversion. Interestingly, the effect becomes huge for high levels of risk aversion. Figure 6 demonstrates that the wedge between $P^*$ and the *npv* critical output price widens with an increase in $\sigma$. Hence, the error made by deciding according to the *npv* rule exacerbates as price uncertainty rises. The figure shows that this effect can be quite substantial, but, as we already concluded from Figure 3, the effect is weaker when the level of risk aversion is higher.

# 6 Conclusion

In this paper we generalize the theory of irreversible investment under uncertainty by allowing for risk averse investors in a situation of incomplete markets. The model we use is similar to that of Dixit and Pindyck (1994,

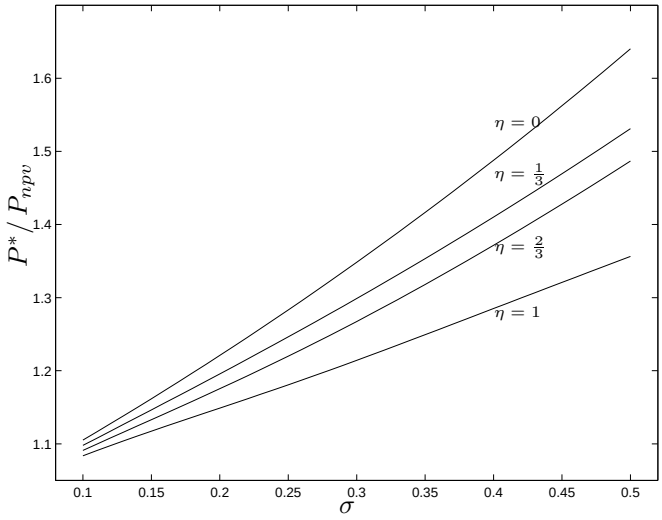


Figure 6: $P^* / P_{npv}$ as a function of $\sigma$ for $\eta \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, $\alpha = 0$, $r = 0.05$, $s = 1$, and $I = 1$.

pp. 185–186), the only difference being that in their set-up the investment expenditure is immediately incurred, whereas in our model there is a flow of interest payments over the lifespan of the project. It is this adaptation that allows us to extend their model beyond risk neutrality using utility functions.

We have introduced a class of utility functions with non-increasing absolute risk aversion to examine the effects of risk aversion, price uncertainty, and other parameters on the optimal investment decision. We find that risk aversion reduces investment, particularly if the investment size is large. Moreover, we find that a rise in uncertainty increases the value of deferring irreversible investments, especially for high levels of risk aversion. Furthermore, we find that applying the net present value rule leads to better (although not optimal) decisions when the level of risk aversion is high. In addition, we provide closed-form comparative statics formulas for the risk neutral investor.

Finally, departing from the realistic situation of risk averse firms operating in an incomplete market setting, we list some ideas for further research. One of our main results is that risk aversion reduces the gap between the optimal decision and investing according to the net present value rule. Traditional real options theory shows that the gap is there because the option to

wait for more information is valuable. Apparently, this option value is of less importance under risk aversion. It would be interesting to find out whether the gap shrinks even more when the behavior of competitors is taken into account, so that the incentive to preempt rivals will also play a role.

A second topic relates to investments in R&D. Since R&D investments create options (e.g., to produce cheaper or to commercialize patents), and option values increase with uncertainty, it is known from the real options literature (e.g., Dixit and Pindyck (1995)) that the incentive to invest in R&D goes up with uncertainty. It would be interesting to see to what extent this result still holds within our framework of risk aversion combined with incomplete markets.

# A Proof of Proposition 1

In this appendix we show that there is a single, strictly positive critical output price (if it exists at all), whether the investor is risk neutral or risk averse. Risk aversion corresponds to a concave utility function; see, e.g., Mas-Colell, Whinston, and Green (1995, p. 187). This is equivalent to $u'' < 0$ since $u$ is twice differentiable. Risk neutrality corresponds to a linear utility function. In that case, $u'' = 0$. In either case, we have $u' > 0$ and $u'' \leq 0$.

Assume there is a $P^* \in [0, \infty)$ such that (4a) and (4b) hold. Define $\varphi : [0, \infty) \to \mathbb{R}$, $\varphi(P) \equiv \beta_1 V(P) - PV'(P)$. Then, by construction, $\varphi(P^*) = 0$. Note that $\varphi$ is strictly increasing on $[0, \infty)$:

$$\begin{aligned} \varphi'(P) &= (\beta_1 - 1)V'(P) - PV''(P) \\ &= E\int_{t=0}^{\infty} e^{-rt}[(\beta_1 - 1)u'(ncf_t) - P_t u''(ncf_t)]\frac{P_t}{P}dt \end{aligned}$$

is positive since $\beta_1 > 1$, $u' > 0$, and $u'' \leq 0$. Note furthermore, that $\varphi(0) = \beta_1 u(-rI)/r < \beta_1 u(0)/r = 0$ since $u' > 0$. Then, by continuity of the function $\varphi$, $P^* > 0$ and unique.

A sufficient condition for existence of $P^*$ is unboundedness of the utility function. To see this, note that

$$\frac{\varphi(P)}{P} = E\int_{t=0}^{\infty} e^{-rt}\left[\beta_1 \frac{u(ncf_t)}{P_t} - u'(ncf_t)\right]\frac{P_t}{P}dt.$$

As $P \to \infty$, this ratio converges to $\frac{\beta_1 - 1}{r - \alpha}\lim_{x\to\infty} u'(x)$ which is positive if $u$ is unbounded from above. Consequently, $\lim_{P\to\infty}\varphi(P) > 0$, which, together with $\varphi(0) < 0$, ensures there exists a $P^* \in (0, \infty)$ such that $\varphi(P^*) = 0$.

# B Determination of $D_1$ and $D_2$

To determine the integration constant $D_1 \geq 0$, first note from the definition of $G$ that $\lim_{P\to\infty} G(P) = 0$. This implies

$$\lim_{P\to\infty}\left[P^{\beta_1}\Psi_1(P) + P^{\beta_2}\Psi_2(P)\right] = 0. \tag{8}$$

We make the following observations: $\lim_{P\to\infty} P^{\beta_2} = 0$ since $\beta_2 < 0$, and $\lim_{P\to\infty}\Psi_2(P) = \int_{\nu=D_2}^{\infty}\nu^{-\beta_2-1}e^{-\nu}d\nu \leq \int_{\nu=0}^{\infty}\nu^{-\beta_2-1}e^{-\nu}d\nu = \Gamma(-\beta_2)$ which

is finite because $\beta_2 < 0$.[2] As a consequence, $\lim_{P\to\infty} P^{\beta_2}\Psi_2(P) = 0$. Therefore, in view of (8), it should hold that

$$\lim_{P\to\infty} P^{\beta_1}\Psi_1(P) = 0. \tag{9}$$

Suppose $D_1 < \infty$. Then $\lim_{P\to\infty}\Psi_1(P) = -\int_{\nu=D_1}^{\infty}\nu^{-\beta_1-1}e^{-\nu}d\nu < 0$. Also, $\lim_{P\to\infty} P^{\beta_1} = \infty$ since $\beta_1 > 0$. Hence, $\lim_{P\to\infty} P^{\beta_1}\Psi_1(P) = -\infty$, which contradicts (9). Therefore, a necessary condition for (9) to hold is that $D_1 = \infty$. To see that this condition is also sufficient, consider

$$\lim_{P\to\infty} P^{\beta_1}\Psi_1(P) = \lim_{P\to\infty}\frac{\int_{\nu=P}^{\infty}\nu^{-\beta_1-1}e^{-\nu}d\nu}{P^{-\beta_1}}.$$

We can apply l'Hôpital's rule to this limit, for $\lim_{P\to\infty}\int_{\nu=P}^{\infty}\nu^{-\beta_1-1}e^{-\nu}d\nu = 0$ and $\lim_{P\to\infty} P^{-\beta_1} = 0$:

$$\begin{aligned}\lim_{P\to\infty} P^{\beta_1}\Psi_1(P) &= \lim_{P\to\infty}\frac{-P^{-\beta_1-1}e^{-P}}{-\beta_1 P^{-\beta_1-1}}\\ &= 0,\end{aligned}$$

so that, indeed, $D_1 = \infty$ is sufficient for (9), and thus (8) holds.

As for the other integration constant $D_2 \geq 0$, a similar reasoning holds. First, observe from the definition of $G$ that $\lim_{P\downarrow 0} G(P) = \frac{1}{r}$. This implies

$$\lim_{P\downarrow 0}\left[P^{\beta_1}\Psi_1(P) + P^{\beta_2}\Psi_2(P)\right] = \frac{1}{\beta_1} - \frac{1}{\beta_2}. \tag{10}$$

Now that we know $D_1 = \infty$, consider

$$\lim_{P\downarrow 0} P^{\beta_1}\Psi_1(P) = \lim_{P\downarrow 0}\frac{\int_{\nu=P}^{\infty}\nu^{-\beta_1-1}e^{-\nu}d\nu}{P^{-\beta_1}},$$

to which we can apply l'Hôpital's rule, because of the fact that $\lim_{P\downarrow 0} P^{-\beta_1}$ and $\lim_{P\downarrow 0}\int_{\nu=P}^{\infty}\nu^{-\beta_1-1}e^{-\nu}d\nu = \int_{\nu=0}^{\infty}\nu^{-\beta_1-1}e^{-\nu}d\nu = \Gamma(-\beta_1)$ are both equal to $\infty$ since $\beta_1$ is positive:

$$\begin{aligned}\lim_{P\downarrow 0} P^{\beta_1}\Psi_1(P) &= \lim_{P\downarrow 0}\frac{-P^{-\beta_1-1}e^{-P}}{-\beta_1 P^{-\beta_1-1}}\\ &= \frac{1}{\beta_1}.\end{aligned}$$

[2] $\Gamma(\cdot)$ denotes the Euler gamma function, $\Gamma(a) \equiv \int_{\nu=0}^{\infty}\nu^{a-1}e^{-\nu}d\nu$, $a \in \mathbb{R}$.

Therefore, in view of (10), it should hold that

$$\lim_{P\downarrow 0} P^{\beta_2}\Psi_2(P) = -\frac{1}{\beta_2}. \tag{11}$$

Suppose $D_2 > 0$. Then $\lim_{P\downarrow 0}\Psi_2(P) = -\int_{\nu=0}^{D_2}\nu^{-\beta_2-1}e^{-\nu}d\nu < 0$. In addition, $\lim_{P\downarrow 0} P^{\beta_2} = \infty$. Hence, $\lim_{P\downarrow 0} P^{\beta_2}\Psi_2(P) = -\infty$, which contradicts (11). Therefore, a necessary condition for (11) to hold is that $D_2 = 0$. To see that this is also sufficient, consider

$$\lim_{P\downarrow 0} P^{\beta_2}\Psi_2(P) = \lim_{P\downarrow 0}\frac{\int_{\nu=0}^{P}\nu^{-\beta_2-1}e^{-\nu}d\nu}{P^{-\beta_2}}.$$

Again, l'Hôpital's rule can be applied, as $\lim_{P\downarrow 0}\int_{\nu=0}^{P}\nu^{-\beta_2-1}e^{-\nu}d\nu = 0$ and $\lim_{P\downarrow 0} P^{-\beta_2} = 0$:

$$\begin{aligned}\lim_{P\downarrow 0} P^{\beta_2}\Psi_2(P) &= \lim_{P\downarrow 0}\frac{P^{-\beta_2-1}e^{-P}}{-\beta_2 P^{-\beta_2-1}}\\ &= -\frac{1}{\beta_2},\end{aligned}$$

so that, indeed, $D_2 = 0$ is sufficient for (11), and thus (10) holds.

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