Risk minimizing portfolios and HJBI equations for stochastic differential games

Sure Mataramvura[1,2], Bernt Øksendal[3,4]

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Abstract

In this paper we consider the problem to find a market portfolio that minimizes the convex risk measure of the terminal wealth in a jump diffusion market. We formulate the problem as a two player (zero-sum) stochastic differential game. To help us find a solution, we prove a theorem giving the HJBI conditions for a general zero-sum stochastic differential game in a jump diffusion setting. We then use the theorem to study particular risk minimization problems. Finally, we extend our approach to cover general stochastic differential games (not necessarily zero-sum), and we obtain similar HJBI equations for the Nash equilibria of such games.

Key words: convex measure of risk, monetary utility function, optimal max-min control, stochastic differential game, HJBI equation, jump diffusion market, Nash equilibrium.

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1   Introduction

We study the problem of minimizing convex risk measures in a Lévy market. The paper [2] introduced the concept of a coherent risk measures as a measure which is sub-additive, positive homogeneous, monotonic and translation invariant. However, in [8] and [6], the idea was extended to relax the subadditivity and positive homogeneity and substitute these with the convex property. It was observed in [19] that the negative of a convex risk measure can be interpreted

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[1]Department of Mathematics and Applied Mathematics, University of KwaZulu-Natal, Westville Campus, P.Bag X54001,Durban 4000, South Africa.

[2]Department of Mathematics, University of Zimbabwe, Box MP 167, Mount Pleasant, Harare, Zimbabwe. *E-mail*: mataramvuras@ukzn.ac.za

[3]Center of Mathematics for Applications (CMA), Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, N–0316 Oslo, Norway. *E-mail*: oksendal@math.uio.no

[4]Norwegian School of Economics and Business Administration, Helleveien 30, N–5045 Bergen, Norway

as a special utility function termed monetary utility function. As a result, risk minimization leads to a max-min problem whose solution we are interested to find.

This problem then extends to solving a zero-sum stochastic differential game between an agent and a market. In order to simplify the max-min problem that we get in the process, we formulate and prove a general Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation for two player stochastic differential games.

Our paper is designed as follows: In the next section we describe the jump diffusion market model we want to deal with, and we formulate the risk minimizing problem as a zero-sum stochastic differential game problem (Problem 2.4). Then in Section 3 we state and prove an HJBI equation for such games (Theorem 3.2). In Section 4 we apply Theorem 3.2 to study some risk minimizing portfolio examples. Finally, in Section 5 we extend our approach to cover general stochastic differential games, not necessarily of zero-sum type, and we prove an HJBI equation for determination of the Nash equilibria of such.

2 The market model

Let $\eta(t) = \eta(t, \omega)$; $(t, \omega) \in [0, \infty) \times \Omega$ be a Lévy process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ satisfying the usual conditions. Let $N(dt, dz)$ be the jump measure of $\eta(\cdot)$ and let

$$\nu(V) = E[N((0, 1], V)]; \qquad V \subset \mathbb{R} \setminus \{0\} \text{ Borel set}$$

be the Lévy measure of $\eta(\cdot)$, where $E[\cdot]$ denotes expectation with respect to P. Define

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt,$$

the compensated Poisson random measure of $\eta(\cdot)$. We assume for simplicity that

$$E[\eta^2(t)] < \infty \quad \text{for all } t \geq 0.$$

Then by the Lévy-Itô decomposition theorem $\eta(\cdot)$ has the form

$$(2.1) \qquad \eta(t) = at + bB(t) + \int_0^t \int_{\mathbb{R}} z\tilde{N}(ds, dz); \qquad t \geq 0$$

where a, b are constants and $B(t)$ is a standard Brownian motion. In view of this we see that when we deal with models involving integration with respect to $\eta(t)$ it is natural to consider linear combinations of dt-integrals, $dB(t)$-integrals (i.e. classical Itô integrals) and $\tilde{N}(dt, dz)$-integrals. This leads to *Lévy-Itô processes* $X(t)$ of the form

$$(2.2) \qquad dX(t) = \alpha(t, \omega)dt + \beta(t, \omega)dB(t) + \int_{\mathbb{R}} \gamma(t, z)\tilde{N}(dt, dz)$$

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where α, β and γ are $\mathcal{F}_t$-predictable processes satisfying

$$\int\limits_0^T \left\{ |\alpha(t)| + \beta^2(t) + \int\limits_{\mathbb{R}} \gamma^2(t,z)\nu(dz) \right\} dt < \infty \qquad \text{for all } T < \infty.$$

We refer to [1] for more information about the stochastic calculus of Lévy processes. See also [25, Chapter 1].

We now consider such a model in finance: Let α, β and γ be as above, fix $T > 0$ and let $r(t) = r(t,\omega)$ be adapted with $\int_0^T |r(t)|dt < \infty$ a.s. Suppose there are two investment possibilities:

(i) a *risk free* asset (e.g. a bond), with unit price $S_0(t)$ at time t given by

$$(2.3) \qquad\qquad dS_0(t) = r(t)S_0(t)dt; \qquad S_0(0) = 1$$

(ii) a *risky* asset (e.g. a stock), with unit price $S_1(t)$ given by

$$(2.4) \quad dS_1(t) = S_1(t^-)\left[\alpha(t)dt + \beta(t)dB(t) + \int\limits_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz)\right]; \qquad S_1(0) > 0$$

where we assume that $\gamma(t,z) > -1$ for a.a. t, z, a.s.

Let $\pi(t)$ be a portfolio, giving the *proportion* of the total wealth $V(t)$ invested in the stocks at time t. Then the dynamics of the corresponding wealth process $V(t) = V^{(\pi)}(t)$ is

$$(2.5) \quad dV^{(\pi)}(t) = V^{(\pi)}(t^-)\Big[\{(1-\pi(t))r(t) + \pi(t)\alpha(t)\}dt$$
$$+ \pi(t)\beta(t)dB(t) + \pi(t^-)\int\limits_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz)\Big]; \quad V^{(\pi)}(0) > 0.$$

We require that $\pi(t^-)\gamma(t,z) > -1$ for a.a. t, z, a.s. and that

$$\int\limits_0^T \left\{ |(1-\pi(t))r(t)| + |\pi(t)\alpha(t)| + \pi^2(t)\beta^2(t) + \pi^2(t)\int\limits_{\mathbb{R}} \gamma^2(t,z)\nu(dz) \right\} dt < \infty \;\; \text{a.s.}$$

We now pose the following problem:

(2.6) *Find the portfolio $\pi(t)$ that minimizes* the risk *of the terminal wealth* $V^{(\pi)}(T)$.

As a measure of risk we will use the *convex risk measure*, introduced in [8] and [6]. It is a generalization of the concept of a *coherent risk measure*, introduced in [2].

Definition 2.1 ([8], [6]) *Let $\mathbb{F}$ be the family of all lower bounded $\mathcal{F}_T$-measurable random variables. A* convex risk measure *on $\mathbb{F}$ is a map $\rho : \mathbb{F} \to \mathbb{R}$ such that*

 (i) *(convexity)* $\quad \rho(\lambda X + (1-\lambda)Y) \le \lambda \rho(X) + (1-\lambda)\rho(Y)$ *for all $X, Y \in \mathbb{F}$,*
    $\lambda \in (0, 1)$,

 (ii) *(monotonicity)* $\quad$ *If $X \le Y$, $X, Y \in \mathbb{F}$, then $\rho(X) \ge \rho(Y)$,*

 (iii) *(translation invariance)* $\quad$ *If $X \in \mathbb{F}$ and $m \in \mathbb{R}$ then*

$$\rho(X + m) = \rho(X) - m.$$

In view of the general representation formula for convex risk measures (see e.g. [8, Theorem 9]), we will assume that the risk measure ρ that we consider, is of the following type:

$$(2.7) \qquad \rho(X) = \sup_{Q \in \mathcal{M}_a} \{E_Q[-X] - \zeta(Q)\}$$

(where E_Q denotes expectation with respect to Q), for some family $\mathcal{M}_a$ of measures Q which are absolutely continuous with respect to P and for some "penalty" function $\zeta : \mathcal{M}_a \to \mathbb{R}$. (See also [2], [4], [6] and [9].)

One possible choice of penalty function $\zeta(Q)$ is the *relative entropy of Q with respect to P*, defined by

$$(2.8) \qquad \zeta(Q) := I(Q; P) := E\left[\frac{dQ}{dP} \log \frac{dQ}{dP}\right].$$

(See e.g. [13].)

A natural choice of the family $\mathcal{M}_a$ is the set of all measures $Q = Q_\theta$ of Girsanov transformation type, i.e.

$$(2.9) \qquad dQ_\theta(\omega) = Z_\theta(T)dP(\omega) \qquad \text{on } \mathcal{F}_T$$

where $\theta = (\theta_0, \theta_1)$ and

$$(2.10) \quad dZ_\theta(t) = Z_\theta(t^-)\left[-\theta_0(t)dB(t) - \int_{\mathbb{R}} \theta_1(t, z)\tilde{N}(dt, dz)\right]; \qquad Z_\theta(0) = 1.$$

Here $\theta_0(t)$ and $\theta_1(t, z)$ are $\mathcal{F}_t$-adapted processes, $\theta_1(t, z) < 1$ for a.a. t, z and

$$(2.11) \qquad \int_0^T \left\{\theta_0^2(t) + \int_{\mathbb{R}} \theta_1^2(t, z)\nu(dz)\right\}dt < \infty \quad \text{a.s.}$$

Then by the Itô formula the solution of the stochastic differential equation (2.10)

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is

$$Z_\theta(t) = \exp\Big[-\int_0^t \theta_0(s)dB(s) - \tfrac{1}{2}\int_0^t \theta_0^2(s)ds$$

$$+ \int_0^t \int_{\mathbb{R}} \log(1 - \theta_1(s,z))\tilde{N}(ds,dz)$$

$$(2.12) \qquad + \int_0^t \int_{\mathbb{R}} \{\log(1 - \theta_1(s,z)) + \theta_1(s,z)\}\nu(dz)ds; \qquad 0 \le t \le T.$$

Remark 2.2 In particular, if $\theta_0(t), \theta_1(t,z)$ are such that

$$(2.13) \qquad\qquad\qquad E[Z(T)] = 1$$

and

$$(2.14) \qquad \beta(t)\theta_0(t) + \int_{\mathbb{R}} \gamma(t,z)\theta_1(t,z)\nu(dz) = \alpha(t) - r(t) \qquad \text{for a.a. } t,$$

then $dQ_\theta(\omega) = Z_\theta(T)dP(\omega)$ is an *equivalent local martingale measure* (see e.g. [25, Ch. 1]), but we do not assume this here.

In view of (2.7) we can now make our risk minimizing portfolio problem more precise:

(2.15)   Given a convex risk measure ρ of the form (2.7) find the portfolio π which minimizes
$$\sup_{Q \in \mathcal{M}_a} \{E_Q[-V^{(\pi)}(T)] - \zeta(Q)\}.$$

A related concept is the following (see e.g. [19]):

Definition 2.3 *A* monetary utility function *is a map* $U : \mathbb{F} \to \mathbb{R}$ *with the following properties:*

(i)′ *(concavity)* $U(\lambda X + (1-\lambda)Y) \ge \lambda U(X) + (1-\lambda)U(Y)$, *for all* $X, Y \in \mathbb{F}$, $\lambda \in (0,1)$.

(ii)′ *(monotonicity)* *If* $X \le Y$, $X, Y \in \mathbb{F}$, *then* $U(X) \le U(Y)$.

(iii)′ *(translation invariance)* *If* $X \in \mathbb{F}$ *and* $m \in \mathbb{R}$ *then* $U(X + m) = U(X) + m$.

It is easily seen that if ρ is a convex risk measure, then $U(X) := -\rho(X)$ is a monetary utility function and conversely. Therefore, in view of (2.7) we have the following general representation of a monetary utility function:

$$(2.16) \qquad\qquad U(X) = \inf_{Q \in \mathcal{M}_a} \{E_Q[X] + \zeta(Q)\}.$$

Using this our risk minimizing portfolio problem (2.15) becomes equivalent to the following *monetary utility maximizing problem*:

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Problem 2.4 Find

(2.17)
$$\Phi := \sup_{\pi} \left(\inf_{Q \in \mathcal{M}_a} \{ E_Q[V^{(\pi)}(T)] + \zeta(Q) \} \right)$$

and find the optimal π^* and Q^* such that

$$\Phi = E_{Q^*}\left[V^{(\pi^*)}(T)\right] + \zeta(Q^*).$$

Remark 2.5 We may regard this as a game theoretic problem: The measure Q is the control of player number 1 (the "market"), while the portfolio $\pi(t)$ is the control of player number 2 (the representative agent/trader).

Problem 2.4 is a special case of a stochastic zero-sum differential game. In the next sections we present a general formulation of such a game and we give a general solution method in terms of a Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation. See Theorem 3.2 (for the zero-sum case) and Theorem 5.2 (for the Nash equilibrium case), which together with Example 4.1 and Theorem 5.4 are our main new results. To the best of our knowledge this is the first time differential games of this type are studied in a jump diffusion context. For *deterministic* systems similar differential games have been studied by many authors. See e.g. [16], [5], [21] and [7] and the references therein. See also [11], [14], [15], [22], [28] and [31] for papers of related interest.

A nice presentation of games in a general Markov process context is given in [18]. For information on the existence of a value function in zero-sum stochastic differential games see [10].

Problem 2.4 is related to what is also known as a *robust utility maximization* problem. See e.g. [12] or [30] for a duality approach to such problems. A stochastic control approach (different from ours) is presented in [3]. Another paper of related interest is [20], where certain *worst case crash scenarios* are studied using a stochstic control approach.

Risk measures can also be represented by *g-expectations* and hence by solutions of backward stochastic differential equations. For more information we refer to [26] and the references therein.

3 An HJBI equation for zero-sum stochastic differential games

Suppose the equation for the state $Y(t) = Y^{(u)}(t)$ at time t has the form

(3.1)
$$dY(t) = b(Y(t), u_0(t))dt + \sigma(Y(t), u_0(t))dB(t)$$
$$+ \int_{\mathbb{R}^k} \gamma(Y(t^-), u_1(t^-, z), z)\tilde{N}(dt, dz); \qquad Y(0) = y \in \mathbb{R}^k,$$

Here $b : \mathbb{R}^k \times K \to \mathbb{R}^k$, $\sigma : \mathbb{R}^k \times K \to \mathbb{R}^{k \times k}$ and $\gamma : \mathbb{R}^k \times K \times \mathbb{R}^k \to \mathbb{R}^{k \times k}$ are given functions, $B(t)$ is a k-dimensional Brownian motion, $\tilde{N}(\cdot, \cdot) =$

$(\tilde{N}_1(\cdot,\cdot),\ldots,\tilde{N}_k(\cdot,\cdot))$ are k-independent compensated Poisson random measures and K is a given subset of $\mathbb{R}^p$.

The process $u_0(t) = u_0(t,\omega)$ and $u_1(t,z) = u_1(t,z,\omega)$ are our *control processes*, assumed to be cadlag and adapted to the filtration $\mathcal{F}_t$ generated by the driving processes $B(\cdot)$ and $\tilde{N}(\cdot,\cdot)$ (as usual augmented with all the P-null sets), and with values $u_0(t) \in K$, $u_1(t,z) \in K$ for a.a. t,z, a.s. We put $u = (u_0, u_1)$ and we call $Y^{(u)}(\cdot)$ a *controlled jump diffusion*. We refer to [25] for more information about stochastic control of jump diffusions.

Let $f : \mathbb{R}^k \times K \to \mathbb{R}$ and $g : \mathbb{R}^k \to \mathbb{R}$ be given functions, called the *profit rate* and *bequest function*, respectively. We assume that we are given a family $\mathcal{A}$ of *admissible controls*, contained in the set of controls $u(\cdot)$ such that (3.1) has a unique strong solution and such that

$$(3.2) \qquad E^y\left[\int_0^{\tau_S} |f(Y(t), u_0(t))|dt + |g(Y(\tau_S))|\right] < \infty$$

for all $y \in \mathcal{S}$, where $\mathcal{S} \subset \mathbb{R}^k$ is a given open set (called the *solvency region*),

$$(3.3) \qquad \tau_S = \inf\{t > 0; Y(t) \notin \mathcal{S}\}$$

is the *bankruptcy time* and E^y denotes expectation given that $Y(0) = y \in \mathbb{R}^k$.

For $u \in \mathcal{A}$ we define the *performance functional* $J^u(y)$ by

$$(3.4) \qquad J^u(y) = E^y\left[\int_0^{\tau_S} f(Y(t), u_0(t))dt + g(Y(\tau_S))\right].$$

Here, and in (3.2), we interpret $g(Y(\tau_S))$ as 0 if $\tau_S = \infty$.

Now suppose that the controls $u_0(t)$ and $u_1(t,z)$ have the form

$$(3.5) \qquad u_0(t) = (\theta_0(t), \pi_0(t)) ; \qquad t \geq 0$$

$$(3.6) \qquad u_1(t,z) = (\theta_1(t,z), \pi_1(t,z)) ; \qquad (t,z) \in [0,\infty) \times \mathbb{R}^k$$

Just as in the example discussed in Section 2 we may regard $\theta := (\theta_0, \theta_1)$ as the control of player number 1, while $\pi := (\pi_0, \pi_1)$ is the control of player number 2.

Let Θ and Π be given families of admissible controls $\theta = (\theta_0, \theta_1)$ and $\pi = (\pi_0, \pi_1)$, respectively. Let K_1, K_2 be two sets such that $\theta(t,z) \in K_1$ and $\pi(t,z) \in K_2$ for a.a. t,z,ω. The *general zero-sum stochastic differential game problem* is the following

Problem 3.1 Find $\Phi(y)$ and $(\theta^*, \pi^*) \in \Theta \times \Pi$ such that

$$(3.7) \qquad \Phi(y) = \sup_{\pi \in \Pi} \left(\inf_{\theta \in \Theta} J^{\theta,\pi}(y) \right) = J^{(\theta^*,\pi^*)}(y).$$

The function Φ is called *the value function* and (θ^*, π^*) is called an *optimal control* (if it exists).

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In classical stochastic control theory it is well-known that under mild conditions *Markov* controls can give just as good performance as the more general adapted controls. See e.g. [23, Theorem 11.2.3]. In view of this we will restrict ourselves to consider only Markov controls in Problem 3.1. Hence we assume that

(3.8) $\theta_0(t) = \bar{\theta}_0(Y(t)), \quad \pi_0(t) = \bar{\pi}_0(Y(t)), \quad \theta_1(t, z) = \bar{\theta}_1(Y(t), z)$
    and $\quad \pi_1(t, z) = \bar{\pi}_1(Y(t), z)$

for some function $\bar{\theta}_0 : \mathcal{S} \to K_1$, $\bar{\pi}_0 : \mathcal{S} \to K_2$, $\bar{\theta}_1 : \mathcal{S} \times \mathbb{R}^k \to K_1$ and $\bar{\pi}_1 : \mathcal{S} \times \mathbb{R}^k \to K_2$.

As customary we do not distinguish notationally between θ_0 and $\bar{\theta}_0$, π_0 and $\bar{\pi}_0$, θ_1 and $\bar{\theta}_1$ and π_1 and $\bar{\pi}_1$. Thus our controls can simply be identified with (deterministic) functions $\theta_0(y)$, $\pi_0(y)$, $\theta_1(y, z)$ and $\pi_1(y, z)$; $y \in \mathcal{S}$, $z \in \mathbb{R}^k$ (sometimes also called *feedback* controls).

When the control $u = (\theta, \pi) \in \Theta \times \Pi$ is Markovian, the corresponding system (3.1) becomes a jump diffusion, whose generator $A^{\theta,\pi}$ is given by

$$(3.9) \qquad A^{\theta,\pi}\varphi(y) = \sum_{i=1}^{k} b_i(y, \theta_0(y), \pi_0(y)) \frac{\partial \varphi}{\partial y_i}(y)$$

$$+ \frac{1}{2} \sum_{i,j=1}^{k} (\sigma\sigma^T)_{ij}(y, \theta_0(y), \pi_0(y)) \frac{\partial^2 \varphi}{\partial y_i \partial y_j}(y)$$

$$+ \sum_{j=1}^{k} \int_{\mathbb{R}} \{\varphi(y + \gamma^{(j)}(y, \theta_1, (y, z_j), \pi_1(y, z_j), z_j)) - \varphi(y)$$

$$- \nabla\varphi(y) \cdot \gamma^{(j)}(y, \theta_1(y, z_j), \pi_1(y, z_j), z_j)\}\nu_j(dz_j);$$

where $\varphi \in C_0^2(\mathbb{R}^k)$ (the twice continuously differentiable functions with compact support on $\mathbb{R}^k$). Here $\nabla\varphi = \left(\frac{\partial \varphi}{\partial y_1}, \ldots, \frac{\partial \varphi}{\partial y_k}\right)$ is the gradient of φ and $\gamma^{(j)}$ is the j'th column of the $k \times k$ matrix γ.

We let $\mathcal{T}$ denote the set of all $\mathcal{F}_t$-stopping times $\tau \leq \tau_{\mathcal{S}}$.

We can now state the first main result of this paper:

Theorem 3.2 (An HJBI equation for zero-sum differential games for jump diffusions)   *Suppose there exists a function $\varphi \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$ and a Markov control $(\hat{\theta}(y), \hat{\pi}(y)) \in \Theta \times \Pi$ such that*

(i) $A^{\theta,\hat{\pi}(y)}\varphi(y) + f(y, \theta, \hat{\pi}(y)) \geq 0 \quad$ *for all $\theta \in K_1$, $y \in \mathcal{S}$*

(ii) $A^{\hat{\theta}(y),\pi}\varphi(y) + f(y, \hat{\theta}(y), \pi) \leq 0 \quad$ *for all $\pi \in K_2$, $y \in \mathcal{S}$*

(iii) $A^{\hat{\theta}(y),\hat{\pi}(y)}\varphi(y) + f(y, \hat{\theta}(y), \hat{\pi}(y)) = 0 \quad$ *for all $y \in \mathcal{S}$*

(iv) $Y^{\theta,\pi}(\tau_{\mathcal{S}}) \in \partial\mathcal{S}$ *a.s. on* $\{\tau_{\mathcal{S}} < \infty\} \quad$ *and*
    $\lim_{t \to \tau_{\mathcal{S}}^-} \varphi(Y^{\theta,\pi}(t)) = g(Y^{\theta,\pi}(\tau_{\mathcal{S}}))\chi_{\{\tau_{\mathcal{S}} < \infty\}}$ *a.s. for all* $(\theta, \pi) \in \Theta \times \Pi$, $y \in \mathcal{S}$

(v)   *the family $\{\varphi(Y^{\theta,\pi}(\tau))\}_{\tau \in \mathcal{T}}$ is uniformly integrable,*
    for all $y \in \mathcal{S}$, $(\theta, \pi) \in \Theta \times \Pi$.

Then

$$\varphi(y) = \Phi(y) = \sup_{\pi \in \Pi} \left(\inf_{\theta \in \Theta} J^{\theta,\pi}(y) \right) = \inf_{\theta \in \Theta} \left(\sup_{\pi \in \Pi} J^{\theta,\pi}(y) \right)$$

$$(3.10) \qquad = \sup_{\pi \in \Pi} J^{\hat{\theta},\pi}(y) = \inf_{\theta \in \Theta} J^{\theta,\hat{\pi}}(y) = J^{\hat{\theta},\hat{\pi}}(y); \qquad y \in \mathcal{S}$$

and

$$(3.11) \qquad (\hat{\theta}(y), \hat{\pi}(y)) \quad \text{is an optimal (Markov) control.}$$

PROOF. Choose $(\theta, \pi) \in \Theta \times \Pi$. Then by the Dynkin formula for jump diffusions (see e.g. [25, Theorem 1.24]) we have

$$(3.12) \qquad E^y\left[\varphi(Y(\tau_{\mathcal{S}}^{(N)}))\right] = \varphi(y) + E^y\left[\int_0^{\tau_{\mathcal{S}}^{(N)}} A^{\theta,\pi}\varphi(Y(t))dt \right]$$

where $Y(t) = Y^{\theta,\pi}(t)$ and

$$\tau_{\mathcal{S}}^{(N)} = \tau_{\mathcal{S}} \wedge N \wedge \inf\{t > 0; |Y(t)| \geq N\}, \qquad N = 1, 2, \ldots$$

(I) If we apply (3.12) to $\theta, \hat{\pi}$ and use (i) for all $y = Y(t)$ we get

$$E^y\left[\varphi(Y(\tau_{\mathcal{S}}^{(N)}))\right] \geq \varphi(y) - E^y\left[\int_0^{\tau_{\mathcal{S}}^{(N)}} f(Y(t), \theta(Y(t)), \hat{\pi}(Y(t))dt \right]$$

or

$$\varphi(y) \leq E^y\left[\int_0^{\tau_{\mathcal{S}}^{(N)}} f(Y(t), \theta(Y(t)), \hat{\pi}(Y(t)))dt + \varphi(Y(\tau_{\mathcal{S}}^{(N)})) \right].$$

Letting $N \to \infty$ and using (iv) and (v) we obtain

$$(3.13) \qquad \varphi(y) \leq J^{\theta,\hat{\pi}}(y).$$

Since this holds for all $\theta \in \Theta$ we deduce that

$$(3.14) \qquad \varphi(y) \leq \inf_{\theta \in \Theta} J^{\theta,\hat{\pi}}(y).$$

Hence

$$(3.15) \qquad \varphi(y) \leq \sup_{\pi \in \Pi} \left(\inf_{\theta \in \Theta} J^{\theta,\pi}(y) \right) = \Phi(y).$$

(II) Next, if we apply (3.12) to $\hat{\theta}, \pi$, with $\pi \in \Pi$, and use (ii) for all $y = Y(t)$ we get

$$E^y[\varphi(Y(\tau_{\mathcal{S}}^{(N)}))] \leq \varphi(y) - E^y\left[\int_0^{\tau_{\mathcal{S}}^{(N)}} f(Y(t), \hat{\theta}(Y(t)), \pi(Y(t)))dt \right]$$

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or

$$\varphi(y) \geq E^y\Big[\int\limits_0^{\tau_{\mathcal{S}}^{(N)}} f(Y(t), \hat{\theta}(Y(t)), \pi(Y(t)))dt + \varphi(Y(\tau_{\mathcal{S}}^{(N)}))\Big].$$

Letting $N \to \infty$ and using (iv) and (v) we obtain

(3.16) $$\varphi(y) \geq J^{\hat{\theta},\pi}(y) \geq \inf_{\theta \in \Theta} J^{\theta,\pi}(y).$$

Since this holds for all $\pi \in \Pi$ we deduce that

(3.17) $$\varphi(y) \geq \sup_{\pi \in \Pi} \Big(\inf_{\theta \in \Theta} J^{\theta,\pi}(y)\Big) = \Phi(y).$$

(III) Finally, we apply (3.12) to $\hat{\theta}, \hat{\pi}$ and proceed as above. Then we end up with

(3.18) $$\varphi(y) = J^{\hat{\theta},\hat{\pi}}(y).$$

Combining (3.17), (3.18) and (3.15) we conclude that

$$\Phi(y) \leq \varphi(y) = J^{\hat{\theta},\hat{\pi}}(y) \leq \Phi(y),$$

Combining (3.16) and (3.13) we get

(3.19) $$\inf_{\theta \in \Theta} \Big(\sup_{\pi \in \Pi} J^{\theta,\pi}(y)\Big) \leq \sup_{\pi \in \Pi} J^{\hat{\theta},\pi}(y) \leq \varphi(y) \leq \inf_{\theta \in \Theta} J^{\theta,\hat{\pi}}(y)$$
$$\leq \sup_{\pi \in \Pi} \Big(\inf_{\theta \in \Theta} J^{\theta,\pi}(y)\Big) = \Phi(y).$$

On the other hand, we always have

(3.20) $$\sup_{\pi \in \Pi} \Big(\inf_{\theta \in \Theta} J^{\theta,\pi}(y)\Big) \leq \inf_{\theta \in \Theta} \Big(\sup_{\pi \in \Pi} J^{\theta,\pi}(y)\Big).$$

Combining (3.19) and (3.20) with (3.18) we get (3.10) and (3.11). $\qquad\square$

Remark 3.3 In the case when $dY(t) = (dt, X(t)), Y(0) = y = (s, x) \in [0, T] \times \mathbb{R}^n$ it suffices to assume that $\varphi \in C^{1,2}((0, T) \times \mathbb{R}^n) \cap C(\bar{\mathcal{S}})$. This applies to Examples 4.1, 4.2 and 5.3.

Remark 3.4 In this paper we do not discuss the (important) question to what extent the value function Φ is indeed a C^2 function (or $C^{1,2}$ function) under some general conditions. However, we verify that this is the case in some specific examples (Example 4.1 and Example 5.3). It is known that under some conditions the value function Φ is a *viscosity solution* of the corresponding HJBI equation. See [17].

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4 Examples

Example 4.1 Let us return to the market (2.3), (2.4) in Section 2. However, as suggested by [27], let us assume that the mean rate of return of the stock, $\alpha(.)$, is not given a priori, but is the consequence of the portfolio choice $\pi(.) \in \Pi$ of a "representative" trader. The trader tries to maximize the expected utility of her terminal wealth by choosing her portfolio optimally, while the "market" tries to minimize this maximum expected utility by choosing $\alpha(t)$ accordingly. This leads to the min-max problem

$$(4.1) \qquad \inf_{\alpha \in \mathfrak{A}} \left(\sup_{\pi \in \Pi} E[U_0(V^{(\pi)}(T))] \right)$$

where $U_0 \in C^2(0, \infty)$ is a given utility function (in the usual sense, not necessarily monetary) and $\mathfrak{A}, \Pi$ are given families of admissible processes $\alpha(t), \pi(t)$. For simplicity we assume that $\gamma(t, z) = z > -1$ a.s. ν and that β is constant. We assume that if $(\alpha, \pi) \in \mathfrak{A} \times \Pi$ then

$$(4.2) \qquad E\left[\int_0^T \{|(1 - \pi(t))r(t)| + |\pi(t)\alpha(t)| + \pi^2(t) \int_{\mathbb{R}} z^2 \nu(dz)\} dt \right] < \infty,$$

which guarantees that $V^{(\pi)}(t) > 0$ for all t.

To put this problem into the framework of Section 3 we define the process $Y(t) = (Y_0(t), Y_1(t))$ by

$$(4.3) \qquad dY_0(t) = dt; \qquad Y_0(0) = y_0 = s \in \mathbb{R}$$

and (see (2.5))

$$(4.4) \qquad dY_1(t) = dV^{(\pi)}(t) = Y_1(t^-)\Big[\{(1 - \pi(t))r(t) + \pi(t)\alpha(t)\}dt$$
$$+ \pi(t)\beta\, dB(t) + \pi(t^-) \int_{\mathbb{R}} z\, \tilde{N}(dt, dz)\Big]; \qquad Y_1(0) = y_1 = x > 0.$$

Then problem (4.1) can be formulated as follows:

Find $(\alpha^*, \pi^*) \in \mathfrak{A} \times \Pi$ and $\Phi(y) = \Phi(s, x)$ such that

$$(4.5) \qquad \Phi(s, x) = \inf_{\alpha \in \mathfrak{A}} \left(\sup_{\pi \in \Pi} E^{s,x}[U_0(Y_1^{(\alpha, \pi)}(T))] \right) = E^{s,x}[U_0(Y_1^{(\alpha^*, \pi^*)}(T))].$$

In this case the generator $A^{\alpha, \pi}$ has the form

$$(4.6) \qquad A^{\alpha, \pi}\varphi(s, x) = \frac{\partial \varphi}{\partial s} + x[(1 - \pi)r(s) + \pi\alpha]\frac{\partial \varphi}{\partial x} + \tfrac{1}{2}\beta^2\pi^2 x^2 \frac{\partial^2 \varphi}{\partial x^2}$$
$$+ \int_{\mathbb{R}} \Big\{ \varphi(s, x + \pi\, xz) - \varphi(s, x) - \frac{\partial \varphi}{\partial x}(s, x) \cdot \pi\, xz \Big\} \nu(dz)$$

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and the HJBI equation for this problem can be written (in compact form)

(4.7)
$$\begin{cases} \inf_{\alpha \in K_1} \left(\sup_{\pi \in K_2} \{ A^{\alpha,\pi} \varphi(s,x) \} \right) = 0; \quad s < T \\ \varphi(T,x) = U_0(x) \end{cases}$$

Note that we only need to have $\varphi \in C^{1,2}((0,T) \times (0,\infty))$ in this case. See Remark 3.3.

Let us try the following function:

$$\varphi(s,x) = U_0 \left(x \exp \left(\int_s^T r(t)dt \right) \right).$$

This gives

(4.8) $A^{\alpha,\pi} \varphi(s,x) = -U_0'(x \exp(\int_s^T r(t)dt)) x \exp(\int_s^T r(t)dt) r(s)$

$$+ x[(1-\pi)r(s) + \pi\alpha] U_0'(x \exp(\int_s^T r(t)dt)) \exp(\int_s^T r(t)dt)$$

$$+ \frac{1}{2}\beta^2 \pi^2 x^2 U_0''(x \exp(\int_s^T r(t)dt)) \exp(2 \int_s^T r(t)dt)$$

$$+ \int_{\mathbb{R}} \{ U_0((x + \pi xz) \exp(\int_s^T r(t)dt)) - U_0(x \exp(\int_s^T r(t)dt))$$

$$- U_0'(x \exp(\int_s^T r(t)dt)) \exp(\int_s^T r(t)dt)\pi xz \} \nu(dz)$$

Maximizing over π we get the following first order condition for the maximum point $\hat{\pi} = \hat{\pi}(\alpha)$:

(4.9) $(\alpha - r(s)) U_0'(x \exp(\int_s^T r(t)dt))$

$$+ \beta^2 \hat{\pi} x U_0''(x \exp(\int_s^T r(t)dt)) \exp(\int_s^T r(t)dt)$$

$$+ \int_{\mathbb{R}} \{ U_0'((x + \hat{\pi} xz) \exp(\int_s^T r(t)dt)) - U_0'(x \exp(\int_s^T r(t)dt)) \} z\nu(dz) = 0$$

Substituting $\pi = \hat{\pi}(\alpha)$ into (4.7) and minimizing over α gives the first order

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condition for the minimum point $\alpha = \hat{\alpha}$:

$$(4.10) \qquad \left[-r(s)\hat{\pi}'(\alpha) + \hat{\pi}(\alpha) + \alpha\hat{\pi}'(\alpha) \right] U_0'(x \exp(\int_s^T r(t)dt))$$

$$+ \beta^2 \hat{\pi}(\alpha)\hat{\pi}'(\alpha) x U_0''(x \exp(\int_s^T r(t)dt)) \exp(\int_s^T r(t)dt)$$

$$+ \int_{\mathbb{R}} \{ U_0'((x + \hat{\pi}(\alpha)xz) \exp(\int_s^T r(t)dt))$$

$$- U_0'(x \exp(\int_s^T r(t)dt)) \} \hat{\pi}(\alpha) z \nu(dz) = 0$$

Combining (4.9) with (4.8) we get

$$(4.11) \qquad\qquad\qquad \hat{\pi}(\hat{\alpha}) = 0.$$

Substituting $\alpha = \hat{\alpha}$ and $\hat{\pi}(\hat{\alpha}) = 0$ in (4.8) we obtain

$$(4.12) \qquad\qquad\qquad \hat{\alpha}(s) = r(s).$$

With these values of $\hat{\alpha}, \hat{\pi}(\hat{\alpha})$, we easily verify that

$$(4.13) \qquad\qquad\qquad A^{\hat{\alpha},\hat{\pi}(\hat{\alpha})}\varphi(s,x) = 0$$

which proves that the value function is indeed

$$(4.14) \qquad\qquad \Phi(s,x) = \varphi(s,x) = U_0\left(x \exp(\int_s^T r(t)dt) \right)$$

We conclude that in this virtual game between the trader and market, the "market" reacts to the trader's optimal portfolio choice by choosing

$$\alpha = \hat{\alpha}(t) = r(t) \qquad \text{for all } t \in [0,T].$$

This corresponds to changing the original probability measure P to some equivalent martingale measure Q, which is in agreement with the principle of arbitrage free option pricing. See [27] for more discussion on this. The result above is an extension of the result in [27] to (Markovian) stochastic processes $r(s), \alpha(s), \beta(s)$ and to the jump diffusion case.

Example 4.2 We now use Theorem 3.2 to study Problem 2.4. First, let us assume for simplicity that in the market (2.3)–(2.4) we have

$$(4.15) \quad r(t) = 0, \quad \alpha(t) = \alpha \text{ (constant)}, \quad \beta(t) = \beta \text{ (constant)}, \quad \gamma(t,z) = 0$$

i.e. the market is a classical Black-Scholes market.

Let $\mathcal{M}_a$ consist of all probability measure $Q = Q_\theta$ of the form

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$$dQ_\theta = \exp\Big(-\int\limits_0^T \theta_0(s)dB(s) - \tfrac{1}{2}\int\limits_0^T \theta_0^2(s)ds\Big)dP(\omega) \qquad \text{on } \mathcal{F}_T$$

where $\theta_0(s)$ is $\mathcal{F}_s$-adapted, $E\big[\exp\big(\tfrac{1}{2}\int\limits_0^T \theta_0^2(s)ds\big)\big] < \infty$.

We put Problem 2.4 into the framework of Section 3 by defining

$$(4.16) \qquad dY(t) = (dY_0(t), dY_1(t), dY_2(t)); \qquad Y(0) = y = (s, y_1, y_2),$$

where

$$dY_0(t) = dt; \qquad Y_0(0) = s \in \mathbb{R}$$
$$dY_1(t) = dV^{(\pi)}(t) = Y_1(t)[\alpha\pi(t)dt + \beta\pi(t)dB(t)]; \qquad Y_1(0) = y_1 > 0$$
$$dY_2(t) = -\theta(t)Y_2(t)dB(t); \qquad Y_2(0) = y_2 > 0$$

and where we have put $\theta_0 = \theta$, $\pi_0 = \pi$, for simplicity of notation.

Assume that the penalty function has the form

$$\zeta(Q_\theta) = E\Big[\zeta_0\Big(\frac{dQ_\theta}{dP}\Big)\Big] = E[\zeta_0(Y_2(T))] \quad \text{for some function } \zeta_0 : \mathbb{R} \to \mathbb{R}.$$

Then Problem 2.4 gets the form

$$(4.17) \qquad \Phi(y) = \Phi(s, y_1, y_2) = \sup_{\pi \in \Pi} \Big(\inf_{\theta \in \Theta} E^{s, y_1, y_2}[Y_1(T)Y_2(T) + \zeta_0(Y_2(T))]\Big).$$

In this case the generator of $Y(\cdot)$ becomes

$$(4.18) \qquad A^{\theta, \pi}\varphi(y) = \frac{\partial\varphi}{\partial s} + y_1\alpha\pi\frac{\partial\varphi}{\partial y_1} + \tfrac{1}{2}y_1^2\beta^2\pi^2\frac{\partial^2\varphi}{\partial y_1^2}$$
$$- y_1 y_2\beta\theta\pi\frac{\partial^2\varphi}{\partial y_1 \partial y_2} + \tfrac{1}{2}y_2^2\theta^2\frac{\partial^2\varphi}{\partial y_2^2}; \qquad \varphi \in C_0^2(\mathbb{R}^3).$$

Then the corresponding HJBI equation becomes

$$(4.19) \qquad \begin{cases} \sup\limits_{\pi \in K_2}\Big(\inf\limits_{\theta \in K_1} A^{\theta, \pi}\varphi(s, y_1, y_2)\Big) = 0; & s < T \\ \varphi(T, y_1, y_2) = y_1 y_2 + \zeta_0(y_2) \end{cases}$$

To study this equation we first fix $\pi \in K_2$ and minimize

$$h(\theta) := -y_1 y_2\beta\theta\pi\varphi_{12} + \tfrac{1}{2}y_2^2\theta^2\varphi_{22}$$

with respect to θ. (Here $\varphi_{12} = \frac{\partial^2\varphi}{\partial y_1 \partial y_2}(y)$ etc.) The minimum is attained at

$$(4.20) \qquad \theta = \hat{\theta}(y) = \frac{y_1\beta\pi\varphi_{12}}{y_2\varphi_{22}} \qquad (\text{if } \varphi_{22} > 0).$$

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Substituting this into (4.18) we get to maximize

$$k(\pi) := y_1 \alpha \pi \varphi_1 + \tfrac{1}{2} y_1^2 \beta^2 \pi^2 \left(\varphi_{11} - \frac{\varphi_{12}^2}{\varphi_{22}} \right)$$

with respect to π. If $\varphi_{11} < \frac{\varphi_{12}^2}{\varphi_{22}}$ the maximum is attained when

$$(4.21) \qquad \pi = \hat{\pi}(y) = \frac{\alpha \varphi_1 \varphi_{22}}{y_1 \beta^2 (\varphi_{12}^2 - \varphi_{11} \varphi_{22})} \, .$$

Substituting (4.19) and (4.20) into (4.18) we get the following non-linear partial differential equation for the unknown value function φ:

$$(4.22) \qquad \begin{cases} \frac{\partial \varphi}{\partial s} + \frac{\alpha^2 \varphi_1^2 \varphi_{22}}{2\beta^2 (\varphi_{12}^2 - \varphi_{11} \varphi_{22})} = 0 \, ; & s < T \\ \varphi(T, y_1, y_2) = y_1 y_2 + \zeta_0(y_2) \end{cases}$$

If the solution φ of this equation exists and the corresponding $\hat{\theta}(y), \hat{\pi}(y)$ given by (4.19)–(4.20) satisfies all the requirements of Theorem 3.2, then $\varphi = \Phi$ and $(\theta^*, \pi^*) := (\hat{\theta}, \hat{\pi})$ is an optimal (Markov) control.

Example 4.3 Let us again use Theorem 3.2 to study Problem 2.4, but in this case assuming that in the market (2.3)–(2.4) we have

$$(4.23) \qquad r(t) = 0, \quad \alpha(t) = \alpha \text{ (constant)}, \quad \beta = 0 \quad \text{and} \quad \gamma(t, z) = z.$$

Now let $\mathcal{M}_a$ consist of all probability measures $Q = Q_\theta$ of the form

$$dQ_\theta(\omega) = Z_{\theta_1}(T) dP(\omega) \qquad \text{on } \mathcal{F}_T,$$

where

$$(4.24) \qquad Z_{\theta_1}(t) = \exp \left[\int_0^t \int_{\mathbb{R}} \log(1 - \theta_1(s, z)) \tilde{N}(ds, dz) \right.$$

$$\left. + \int_0^t \int_{\mathbb{R}} \{ \log(1 - \theta_1(s, z)) + \theta_1(s, z) \} \nu(dz) ds; \qquad 0 \le t \le T. \right.$$

Then, by the Itô formula,

$$(4.25) \qquad dZ_{\theta_1}(t) = -Z_{\theta_1}(t^-) \int_{\mathbb{R}} \theta_1(t, z) \tilde{N}(dt, dz).$$

Therefore, to put Problem 2.4 into the framework of Section 3 we define

$$(4.26) \qquad Y(t) = (Y_0(t), Y_1(t), Y_2(t)), \qquad Y(0) = (s, y_1, y_2)$$

where

$$dY_0(t) = dt; \qquad Y_0(0) = s \in \mathbb{R}$$

$$dY_1(t) = dV^{(\pi)}(t) = Y_1(t^-)\Big[\alpha\pi(t)dt + \pi(t^-)\int_{\mathbb{R}} z\tilde{N}(dt, dz)\Big]; \qquad Y_1(0) > 0$$

$$dY_2(t) = -Y_2(t^-)\int_{\mathbb{R}} \theta_1(t, z)\tilde{N}(dt, dz); \qquad Y_2(0) = y_2 > 0.$$

The generator of $Y(\cdot)$ then becomes (see (3.9))

$$(4.27) \qquad A^{\theta_1, \pi}\varphi(s, y_1, y_2) = \frac{\partial\varphi}{\partial s} + y_1\alpha\pi\frac{\partial\varphi}{\partial y_1}$$

$$+ \int_{\mathbb{R}} \Big\{ \varphi(s, y_1 + y_1\pi z, y_2 - y_2\theta_1(z)) - \varphi(s, y_1, y_2)$$

$$- y_1\pi z\frac{\partial\varphi}{\partial y_1} + y_2\theta_1(z)\frac{\partial\varphi}{\partial y_2} \Big\}\nu(dz).$$

Therefore, to solve the corresponding HJBI equation, we first fix $\pi \in K_2$ and minimize

$$h(\theta_1(\cdot)) := \int_{\mathbb{R}} \Big\{ \varphi(s, y_1 + y_1\pi z, y_2 - y_2\theta_1(z)) - \varphi(s, y_1, y_2)$$

$$- y_1\pi z\frac{\partial\varphi}{\partial y_1} + y_2\theta_1(z)\frac{\partial\varphi}{\partial y_2} \Big\}\nu(dz)$$

over all functions $\theta_1(z)$.

We can minimize this pointwise for each z and get a minimum point

$$(4.28) \qquad \hat{\theta}_1(z) = \hat{\theta}_1(y, z) \in \operatorname{Argmin}\psi(\theta_1)$$

where

$$\psi(\theta_1) = \varphi(s, y_1(1 + \pi z), y_2(1 - \theta_1)) + y_2\theta_1\frac{\partial\varphi}{\partial y_2}.$$

The first order condition for a minimum $\hat{\theta}_1$ of ψ is

$$(4.29) \qquad \frac{\partial\varphi}{\partial y_2}(s, y_1(1 + \pi z), y_2(1 - \hat{\theta}_1)) = \frac{\partial\varphi}{\partial y_2}(s, y_1, y_2).$$

This value $\hat{\theta}_1 = \hat{\theta}_1(y, z, \pi)$ is substituted into (4.26), and the resulting expression

$$(4.30) \qquad k(\pi) := y_1\alpha\pi\frac{\partial\varphi}{\partial y_1} + \int_{\mathbb{R}} \Big\{ \varphi(s, y_1(1 + \pi z), y_2(1 - \hat{\theta}_1(s, z, \pi)))$$

$$- y_1\pi z\frac{\partial\varphi}{\partial y_1} + y_2\hat{\theta}(y, z, \pi)\frac{\partial\varphi}{\partial y_2} \Big\}\nu(dz)$$

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is then maximized with respect to π. Call this maximum point $\hat{\pi} = \hat{\pi}(y)$.

Then the pair $(\hat{\theta}_1(y, z, \hat{\pi}(y)), \hat{\pi}(y))$ is our candidate for the optimal control, and our candidate $\varphi(y)$ for the value function $\Phi(y)$ is the solution of the corresponding integro-differential equation

$$(4.31) \qquad \frac{\partial \varphi}{\partial s} + y_1 \alpha \hat{\pi}(y) \frac{\partial \varphi}{\partial y_1} + \int_{\mathbb{R}} \Big\{ \varphi(s, y_1(1 + \hat{\pi}(y)z), y_2(1 - \hat{\theta}_1(y, z, \hat{\pi}(y))))$$

$$- \varphi(s, y_1, y_2) - y_1 \hat{\pi}(y) z \frac{\partial \varphi}{\partial y_1}$$

$$+ y_2 \hat{\theta}_1(y, z, \hat{\pi}(y)) \frac{\partial \varphi}{\partial y_2} \Big\} \nu(dz) = 0 \qquad \text{for } s < T,$$

with boundary values

$$(4.32) \qquad\qquad\qquad \varphi(T, y_1, y_2) = y_1 y_2 + \zeta_0(y_2).$$

5 An HJBI equation for Nash equilibria of general stochastic differential games

Let $Y(t) = Y^u(t)$ be a controlled jump diffusion, as described in (3.1). Let $u = (\theta, \pi)$, where $\theta = (\theta_0, \theta_1) \in \Theta$ and $\pi = (\pi_0, \pi) \in \Pi$ are admissible controls for player 1 and 2, respectively, as in (3.5) and (3.6). Let us now assume that there are *two* performance functionals of the form (with $\mathcal{S}$ and $\tau_{\mathcal{S}}$ as before)

$$(5.1) \qquad J_i^u(y) = E^y \Big[\int_0^{\tau_s} f_i(Y(t), u(t)) dt + g_i(Y(\tau_{\mathcal{S}})) \Big]; \qquad i = 1, 2.$$

We may regard $J_i^u(y)$ as the *payoff* to player number i if the players use the controls (θ_0, θ_1) and (π_0, π_1), respectively.

Definition 5.1 *A pair $(\theta^*, \pi^*) \in \Theta \times \Pi$ is called a* Nash equilibrium *for the stochastic differential game (3.1), (5.1) if the following holds:*

$$(5.2) \qquad\qquad J_1^{\theta, \pi^*}(y) \le J_1^{\theta^*, \pi^*}(y) \quad \text{for all} \quad \theta \in \Theta, \ y \in \mathcal{S}$$

$$(5.3) \qquad\qquad J_2^{\theta^*, \pi}(y) \le J_2^{\theta^*, \pi^*}(y) \quad \text{for all} \quad \pi \in \Pi, \ y \in \mathcal{S}$$

Condition (5.2) states that if player 2 uses the control π^*, then it is optimal for player 1 to use the control θ^*. Similarly, condition (5.3) states that if player 1 uses θ^*, then it is optimal for player 2 to use π^*. Thus (θ^*, π^*) is an equilibrium point in the sense that there is no reason for each individual player to deviate from it, as long as the other player does not.

As in Section 3 we restrict ourselves to Markov controls, and we use the same notation for $A^{\theta, \pi}$ and $\mathcal{T}$ as there. The following result may be regarded as a generalization of Theorem 3.2:

Theorem 5.2 (HJBI equations for Nash equilibria)
Suppose there exists functions $\varphi_i \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}}); i = 1, 2$ and a Markov control $(\hat{\theta}(y), \hat{\pi}(y)) \in \Theta \times \Pi$ such that

(i) $A^{\theta, \hat{\pi}} \varphi_1(y) + f_1(y, \theta, \hat{\pi}(y))$
$\leq A^{\hat{\theta}, \hat{\pi}} \varphi_1(y) + f_1(y, \hat{\theta}(y), \hat{\pi}(y)) = 0$ *for all $\theta \in K_1$, $y \in \mathcal{S}$*

(ii) $A^{\hat{\theta}, \pi} \varphi_2(y) + f_2(y, \hat{\theta}(y), \pi)$
$\leq A^{\hat{\theta}, \hat{\pi}} \varphi_2(y) + f_2(y, \hat{\theta}(y), \hat{\pi}(y)) = 0$ *for all $\pi \in K_2$, $y \in \mathcal{S}$*

(iii) $A^{\theta, \pi}(\tau_{\mathcal{S}}) \in \partial \mathcal{S}$ *a.s. on $\{\tau_{\mathcal{S}} < \infty\}$ and*
$\lim_{t \to \tau_{\mathcal{S}}^-} \varphi_i(Y^{\theta, \pi}(t)) = g_i(Y^{\theta, \pi}(\tau_{\mathcal{S}})) \chi_{\{\tau_{\mathcal{S}} < \infty\}}$ *a.s.   for $i = 1, 2$ and for all $(\theta, \pi) \in \Theta \times \Pi$.*

(iv) *the families $\{\varphi_i(Y^{\theta, \pi}(\tau))\}_{\tau \in \mathcal{T}}$ are uniformly integrable for all $y \in \mathcal{S}$, $(\theta, \pi) \in \Theta \times \Pi$, $i = 1, 2$.*

Then $(\hat{\theta}, \hat{\pi})$ is a Nash equilibrium for the game (3.1), (5.1) and

$$\tag{5.4} \varphi_1(y) = \sup_{\theta \in \Theta} J_1^{\theta, \hat{\pi}}(y) = J_1^{\hat{\theta}, \hat{\pi}}(y)$$

$$\tag{5.5} \varphi_2(y) = \sup_{\pi \in \Pi} J_2^{\hat{\theta}, \pi}(y) = J_2^{\hat{\theta}, \hat{\pi}}(y).$$

PROOF.   We proceed as in the proof of Theorem 3.2. With $Y = Y^{\theta, \hat{\pi}}(y)$ we get by the Dynkin formula, using (i),

$$E^y[\varphi_1(Y(\tau_{\mathcal{S}}^{(N)}))] = \varphi(y) + E^y\left[\int_0^{\tau_{\mathcal{S}}^{(N)}} A^{\theta, \hat{\pi}} \varphi_1(Y(t)) dt \right]$$

$$\leq \varphi_1(y) - E^y\left[\int_0^{\tau_{\mathcal{S}}^{(N)}} f_1(Y(t), \theta(Y(t)), \hat{\pi}(Y(t)) dt \right].$$

Hence

$$\varphi_1(y) \geq E^y\left[\int_0^{\tau_{\mathcal{S}}^{(N)}} f_1(Y(t), \theta(Y(t)), \hat{\pi}(Y(t))) dt + \varphi_1(Y(\tau_{\mathcal{S}}^{(N)})) \right]$$

Letting $N \to \infty$ we obtain
$$\varphi_1(y) \geq J_1^{\theta, \hat{\pi}}(y).$$

Since this holds for all $\theta \in \Theta$ we have

$$\tag{5.6} \varphi_1(y) \geq \sup_{\theta \in \Theta} J_1^{\theta, \hat{\pi}}(y).$$

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Similarly, applying the argument above to the control $(\hat{\theta}, \hat{\pi})$ we get *equality* in (5.6), i.e.

$$(5.7) \qquad \varphi_1(y) = J_1^{\hat{\theta}, \hat{\pi}}(y).$$

Combining (5.6) and (5.7) we get

$$\varphi_1(y) = \sup_{\theta \in \Theta} J_1^{\theta, \hat{\pi}}(y) = J_1^{\hat{\theta}, \hat{\pi}}(y),$$

which is (5.4).

Statement (5.5) is proved in the same way. These two statements imply that $(\hat{\theta}, \hat{\pi})$ is a Nash equilibrium and the proof is complete. $\qquad \square$

Example 5.3 To illustrate Theorem 5.2 we use it to solve a jump diffusion version of a problem which was solved in [24] in the continuous diffusion case (by a different method):

Suppose the states $X_1(t), X_2(t)$ of two companies at time t are described by

$$(5.8) \qquad \begin{cases} dX_1(t) = \theta(t)dt + \sigma_{11}X_1(t)d\eta_1(t) + \sigma_{12}X_1(t)d\eta_2(t) \\ X_1(0) = x_1 \in \mathbb{R} \end{cases}$$

and

$$(5.9) \qquad \begin{cases} dX_2(t) = \pi(t)dt + \sigma_{21}X_2(t)d\eta_1(t) + \sigma_{22}X_2(t)d\eta_2(t) \\ X_2(0) = x_2 \in \mathbb{R} \end{cases}$$

Here $\theta(t), \pi(t)$ are the controls to be used by company 1 and 2, respectively. We assume that σ_{ij} are constants and $\eta_i(t) = \int\limits_0^t \int\limits_{\mathbb{R}} z\tilde{N}_i(ds, dz)$ are independent pure jump Lévy martingales.

The payoff functionals to company 1 and company 2 are assumed to have the form

$$(5.10) \qquad J_1^{\theta, \pi}(s, x_1, x_2) = E^{x_1, x_2}\left[\int\limits_0^{T-s} -\alpha_1 \theta^2(t) X_2^2(t) dt + \gamma_1 X_1^2(T) X_2^2(T) \right]$$

and

$$(5.11) \qquad J_2^{\theta, \pi}(s, x_1, x_2) = E^{x_1, x_2}\left[\int\limits_0^{T-s} -\alpha_2 \pi^2(t) X_1^2(t) dt + \gamma_2 X_1^2(T) X_2^2(T) \right],$$

respectively. Here α_i and γ_i are positive constants, $i = 1, 2$.

A possible interpretation of this is that $\theta(t), \pi(t)$ represent investment rates in company 1 and 2, respectively. But for each company the size of the other

heats up the economy in such a way that both the terminal payoff and the energy cost rate is proportional to the squared size of the other.

To find a possible Nash equilibrium $(\hat{\theta}, \hat{\pi}) \in \Theta \times \Pi$ for this game, we first note that the generator of the process $Y(t) := (s + t, X_1(t), X_2(t))$ is given by, with $y = (s, x_1, x_2)$,

$$
A^{\theta,\pi}\varphi(y) = \frac{\partial \varphi}{\partial s} + \theta \frac{\partial \varphi}{\partial x_1} + \pi \frac{\partial \varphi}{\partial x_2} + \int_{\mathbb{R}} \Big\{ \varphi(s, x_1 + x_1\sigma_{11}z, x_2 + x_2\sigma_{21}z)
$$

$$
- \varphi(s, x_1, x_2) - x_1\sigma_{11}z \frac{\partial \varphi}{\partial x_1} - x_2\sigma_{21}z \frac{\partial \varphi}{\partial x_2} \Big\} \nu_1(dz)
$$

$$
+ \int_{\mathbb{R}} \Big\{ \varphi(s, x_1 + x_1\sigma_{12}z, x_2 + x_2\sigma_{22}z)
$$

$$
\text{(5.12)} \qquad - \varphi(s, x_1, x_2) - x_1\sigma_{12}z \frac{\partial \varphi}{\partial x_1} - x_2\sigma_{22}z \frac{\partial \varphi}{\partial x_2} \Big\} \nu_2(dz)
$$

where $\nu_i(dz)$ are the Lévy measures of $\eta_i(\cdot)$, respectively, $i = 1, 2$.

We also note that in this problem we have

$$
f_1(s, x_1, x_2, \theta, \pi) = -\alpha_1 \theta^2 x_2^2, \quad g_1(s, x_1, x_2) = \gamma_1 x_1^2 x_2^2
$$

and

$$
f_2(s, x_1, x_2, \theta, \pi) = -\alpha_2 \pi^2 x_1^2, \quad g_2(s, x_1, x_2) = \gamma_2 x_1^2 x_2^2.
$$

We first fix $\pi \in \mathbb{R}$, $(s, x_1, x_2) \in \mathbb{R}^3$ and maximize

$$
\text{(5.13)} \qquad A^{\theta,\pi}\varphi_1(s, x_1, x_2) + f_1(s, x_1, x_2, \theta, \pi)
$$

with respect to θ, for a given function φ_1 (to be determined). This amounts to maximizing

$$
h_1(\theta) := \frac{\theta \partial \varphi_1}{\partial x_1} - \alpha_1 \theta^2 x_2^2; \qquad \theta \in \mathbb{R}.
$$

The maximum is attained at

$$
\text{(5.14)} \qquad \theta = \hat{\theta} = \tfrac{1}{2} \alpha_1^{-1} x_2^{-2} \cdot \frac{\partial \varphi_1}{\partial x_1}.
$$

For this choice of θ and with $\pi = \hat{\pi}$ (to be determined) we are required to have

$$
A^{\hat{\theta},\hat{\pi}}\varphi_1(y) + f_1(y, \hat{\theta}(y), \hat{\pi}(y)) = 0
$$

i.e.

$$
\frac{\partial \varphi_1}{\partial s} + \tfrac{1}{2} \alpha_1^{-1} x_2^{-2} \Big(\frac{\partial \varphi_1}{\partial x_1} \Big)^2 + \hat{\pi} \frac{\partial \varphi_1}{\partial x_2} + \sum_{j=1}^{2} \int_{\mathbb{R}} \Big\{ \varphi_1(s, x_1
$$

$$
+ x_1\sigma_{1j}z, x_2 + x_2\sigma_{2j}z) - \varphi_1(s, x_1, x_2) - x_1\sigma_{1j}z \frac{\partial \varphi_1}{\partial x_1}
$$

$$
\text{(5.15)} \qquad - x_2\sigma_{2j}z \frac{\partial \varphi_1}{\partial x_2} \Big\} \nu_j(dz) - \tfrac{1}{4} \alpha_1^{-1} x_2^{-2} \Big(\frac{\partial \varphi_1}{\partial x_1} \Big)^2 = 0.
$$

Let us try functions φ_i of the form

(5.16) $$\varphi_i(s, x_1, x_2) = k_i(s)x_1^2 x_2^2; \qquad i = 1, 2$$

where $k_i(s)$ are functions to be determined. Then

(5.17) $$\hat{\theta} = \alpha_1^{-1} k_1(s) x_1$$

and (5.15) gets the form

(5.18) $$k_1'(s) + \alpha_1^{-1} k_1^2(s) + 2k_1(s)\frac{\hat{\pi}}{x_2} + ak_1(s) = 0,$$

where

(5.19)
$$a = \sum_{j=1}^{2} \int_{\mathbb{R}} z^2 \{ 4\sigma_{1j}\sigma_{2j} + \sigma_{1j}^2 + \sigma_{2j}^2 \\ + z(2\sigma_{1j}\sigma_{2j}^2 + 2\sigma_{2j}\sigma_{1j}^2) + z^2\sigma_{1j}^2\sigma_{2j}^2 \} \nu_j(dz).$$

Similarly, if we fix θ and maximize $A^{\theta,\pi}\varphi_2(s, x_1, x_2) + f_2(s, x_1, x_2, \theta, \pi)$ with respect to π we find the maximum point

$$\pi = \hat{\pi} = \tfrac{1}{2}\alpha_2^{-1} x_1^{-2} \frac{\partial \varphi_2}{\partial x_2} = \alpha_2^{-1} k_2(s) x_2$$

and the requirement

$$A^{\hat{\theta},\hat{\pi}}\varphi_2(y) + f_2(y, \hat{\theta}(y), \hat{\pi}(y)) = 0$$

leads to the equation

(5.20) $$k_2'(s) + 2\alpha_1^{-1} k_1(s)k_2(s) + \alpha_2^{-1} k_2^2(s) + ak_2(s) = 0.$$

Combining (5.18) and (5.20) we get the 2-dimensional Riccati equation

(5.21) $$k_1'(s) + \alpha_1^{-1} k_1^2(s) + 2\alpha_2^{-1} k_1(s)k_2(s) + ak_1(s) = 0; \qquad s < T$$

(5.22) $$k_2'(s) + \alpha_2^{-1} k_2^2(s) + 2\alpha_1^{-1} k_1(s)k_2(s) + ak_2(s) = 0; \qquad s < T$$

(5.23) $$k(T) = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

These equations can be solved explicitly using standard methods for solving Riccati equations, on the condition that our coefficients are such that k_1 and k_2 do not explode. See e.g [29] for information about Riccati equations.

Based on Theorem 5.2 and Remark 3.3 we can summarize the above as follows:

Theorem 5.4 *Let* $k(s) = \begin{bmatrix} k_1(s) \\ k_2(s) \end{bmatrix}$ *be the solution of the Riccati equation (5.21)–(5.23). Then the pair* $(\hat{\theta}, \hat{\pi})$ *given by*

$$(5.24) \qquad \hat{\theta}(s, x_1, x_2) = \alpha_1^{-1} k_1(s) x_1, \qquad \hat{\pi}(s, x_1, x_2) = \alpha_2^{-1} k_2(s) x_2$$

is a Nash equilibrium of the jump diffusion game (5.8)–(5.11). The corresponding equilibrium performances are

$$(5.25) \qquad \varphi_i(s, x_1, x_2) = J_i^{\hat{\theta}, \hat{\pi}}(s, x_1, x_2) = k_i(s) x_1^2 x_2^2; \qquad i = 1, 2.$$

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