RISK SENSITIVE CONTROL AND AN OPTIMAL INVESTMENT MODEL (II)

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Abstract. We consider an optimal investment problem proposed by Bielecki and Pliska. The goal of the investment problem is to optimize the long term growth of expected utility of wealth. We consider HARA utility functions with exponent $-\infty < \gamma < 1$. The problem can be reformulated as an infinite time horizon risk sensitive control problem. Some useful ideas and results from the theory of risk sensitive control can be used in the analysis. Especially, we analyze the associated dynamical programming equation. Then an optimal (or approximately optimal) Markovian investment policy can be derived.

1. Introduction.

It is known that some optimal investment models can be reformulated as risk sensitive stochastic control problems. The idea was explored in Fleming(1995). Using this approach, in Fleming and Sheu(1999), we gave a detailed analysis of an investment model in which only one risky and one riskless asset are considered and transaction costs are ignored. In this paper, we consider a more general model proposed by Bielecki and Pliska(1999). In the model, N securities and m economic factors are considered and the transaction costs are ignored. The goal is to maximize the long term exponential growth rate of expected utility of wealth. A special feature of the model is that the stochastic economic factors explicitly affect the mean returns of the securities. In Bielecki and Pliska(1999), they develop a mathematical theory for model that the securities and economic factors have

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independent noise. Here, we remove this condition and give a detailed analysis for the investment problem without constraints on the portfolio chosen.

Similar models are also considered in Bielecki and Pliska(2000), Kuroda and Nagai(2000). To compare ours with Bielecki and Pliska(2000), we can show by a suitable transformation that the assumptions made in Bielecki and Pliska(2000) are equivalent to ours. Moreover, they consider only the cases with negative γ such that $|\gamma|$ is small. See Sect. 2 for the role of γ playing in the study.

In Kuroda and Nagai(2000), they allow the diffusion coefficient matrix of the factor process to be degenerate. They assume that the factor process is ergodic under equivalent minimal martingale measure. The role of equivalent minimal martingale measure playing in the investment problem is still not clear. However, this observation seems interesting. In their analysis, they need to assume that the interest rate of the banking account is constant. In our study, we assume that the diffusion coefficient matrix for the factor process is nondegenerate. This is crucial in our analysis, since we need to consider the investment problem with constraints. There is also a difference on the results obtained. In their paper, they give a condition (see condition (2.30) in Kuroda and Nagai(2000)) such that the portfolio derived from the solution of the Bellman equation (or Ricatti equation in the present situation) is optimal for the investment problem for all γ. As a consequence, the Verification Theorem can be proved for all γ. However, they do not discuss if the Verification Theorem still holds when (2.30) in Kuroda and Nagai(2000) is not assumed. In fact, the portfolio mentioned above may not be optimal anymore for general γ. See some discussion later in this section.

The theory of risk sensitive control has received much attention in recent years because it provides a link between stochastic and deterministic approaches to disturbances in control systems. See Whittle(1990) for a comprehensive introduction. For the mathematical developments, see Fleming and McEneaney(1995), McEneaney(1993) and Nagai(1996). The dynamic programming equation (DPE for short) plays an important role in the development of mathematical theory for risk sensitive control. Our analysis here is also based on the study of the dynamic programming equation for the risk sensitive control problem associated to the optimal investment problem. One fundamental difference between the risk sensitive control problem studied here and the usual one is that the running cost here does not have definite sign. This makes the analysis more difficult.

The organization of the paper is the following. In Section 2 we give the framework of the problem studied here. We reformulate the problem as an infinite time horizon risk sensitive stochastic control problem of the kind considered in Fleming and McEneaney (1995). We consider a HARA utility function of wealth, with exponent $-\infty < \gamma < 1$. The case $\gamma = 0$ corresponds to the log utility function.

In Section 3 , we consider the case that $\gamma < 0$. We show that the DPE has a unique solution $(\Lambda^{(\gamma)}, W^{(\gamma)})$ such that $\Lambda^{(\gamma)}$ is the optimal exponential growth rate of the investment problem using bounded investment policies, where $W^{(\gamma)}$ is quadratic and nonpositive definite. We also consider the investment problem with constraint set $U_r = \{u; |u| \leq r\}$, $r > 0$, which has optimal exponential rate $\Lambda_r^{(\gamma)}$. We show $\Lambda^{(\gamma)} = \inf_{r>0} \Lambda_r^{(\gamma)} = \lim_{r\to\infty} \Lambda_r^{(\gamma)}$. The equation (2.14) with $U = U_r$ has unique solution $\Lambda_r^{(\gamma)}, W_r^{(\gamma)}$ such that $W_r^{(\gamma)}(0) = 0$ and $|\nabla W_r^{(\gamma)}(x)| \leq M_r$ for some constant M_r. We also show that $W_r^{(\gamma)}$ converges to $W^{(\gamma)}$ and $\nabla W_r^{(\gamma)}$ converges to $\nabla W^{(\gamma)}$ uniformly on compact sets as $r \to \infty$. Let $u^{(\gamma)}(x)$ be the *argmin* in (3.1) with $U = R^N, \Lambda = \Lambda^{(\gamma)}, W = W^{(\gamma)}$. We define $u_r^{(\gamma)}(x)$ similarly with $U = U_r, \Lambda = \Lambda_r^{(\gamma)}, W = W_r^{(\gamma)}$. We know $u_r^{(\gamma)}(\cdot)$ is a Markovian optimal investment policy for the investment problem with constraint set U_r. We can show that $u_r^{(\gamma)}$ converges to $u^{(\gamma)}$ uniformly on compact set as $r \to \infty$. Therefore, $u_r^{(\gamma)}, r > 0$, give approximately optimal policies for the investment problem without constraints. In general, when using $u^{(\gamma)}$ as the investment policy, the wealth can become infinite in finite time. In such cases, it can not attain the optimal exponential growth rate. However, when $|\gamma|$ is small, $u^{(\gamma)}$ attains optimal exponential growth rate. Some more interesting results can be found in Kuroda and Nagai(2000).

In Section 4, we consider the case that $0 < \gamma < 1$ and use bounded investment policies. In such cases the optimal long term growth rate $\Lambda^{(\gamma)}$ is not necessarily finite. However, we show that if $\Lambda^{(\gamma)}$ is finite, then the DPE has a solution $(\Lambda^{(\gamma)}, W^{(\gamma)})$ such that $W^{(\gamma)}$ is convex. We do not know if such $W^{(\gamma)}$ satisfying $W^{(\gamma)}(0) = 0$ is unique. The idea is to study the same problem with investment constraint set U_r and let $r \to \infty$. Although we expect that $W^{(\gamma)}$ is quadratic but we can not prove it. However, when γ is small, $W^{(\gamma)}$ is shown to be quadratic. The Ricatti equation (2.21) has a solution $K \geq 0$ which satisfies the property that $D^{(\gamma)} + E^{(\gamma)}K$ is semistable. This result has an interesting consequence if we assume $\Lambda^{(\gamma)}$ is finite for all $0 < \gamma < 1$. Following from this, we show that $\Lambda^{(\gamma)}$ is infinite for some γ if the economic factors and the securities have independent noise. Let denote $u^{(\gamma)}(\cdot)$ the *argmax* in (4.1) with $U = R^N, \Lambda = \Lambda^{(\gamma)}, W = W^{(\gamma)}$. We do not know if using $u^{(\gamma)}(\cdot)$ as the investment policy can attain the optimal exponential growth rate. We show that this is true if $|\gamma|$ is small.

We would like to mention that the results presented here have been reported in Fleming and Sheu(2000). In this paper we provide the details of their proofs.

2. Problem formulation.

We consider an infinite time horizon optimal investment model, with N risky and one riskless assets. Let $V(t)$ be the investor's wealth at time $t \geq 0$, and $u_i(t)$ be the fraction of wealth in the ith risky asset. Then $u_i(t)V(t)$ is the amount in the ith risky asset and $(1 - \sum_{i=1}^N u_i(t))V(t)$ the amount in the riskless asset. Let $U \subset R^N$ be the constraint set for the investor. Then $u(t) = (u_1(t), ..., u_N(t)) \in U$ for all t. We denote by $S_i(t)$ the price per share for the ith risky asset at time t and $r(t)$ the riskless interest rate. Assume that there is no transaction fee and the borrowing rate and interest rate are the same. Then $V(t)$ satisfies.

$$dV(t) = V(t)[r(t)(1 - \sum_i u_i(t))dt + \sum_i u_i(t)\frac{dS_i(t)}{S_i(t)}], \tag{2.1}$$

with initial wealth given by $V(0) > 0$. We wish to maximize the long term exponential growth rate of the expectation of $\gamma^{-1}V(T)^{\gamma}$ as $T \to \infty$ over all investment policies for $-\infty < \gamma < 1$. The case $\gamma = 0$ is to maximize the expectation of the average per unit time of $\log V(T)$.

The following are some of the interesting choices for U. The $U = R^N$ corresponds to no investment control constraints. The $U = \{(u_1, ..., u_N); u_i \geq 0, i = 1, ..., N\}$ corresponds to no shortselling constraint. We may also choose $U = \{(u_1, ..., u_N); m_i \leq u_i \leq M_i, i = 1, ..., N\}$ for some real $m_i, M_i, i = 1, ..., N$. In this paper, we shall focus on the case $U = R^N$.

We now describe the dynamics for $S_i(t), i = 1, ..., N$, which is suggested by a work of Bielecki and Pliska (1999). We assume that there are m economic factors, $x_1(t), ..., x_m(t)$, which determine the performance of the market and evolve according to the following dynamics,

$$dx(t) = b(x(t))dt + dB(t), \tag{2.2}$$

where $B(t)$ is the standard m-dim Brownian motion. We assume

$$b(x) = Dx, x \in R^m \tag{2.3}$$

such that D is a stable matrix. That is,

$$\sum D_{ij} u_i u_j \leq -c_0 |u|^2 \tag{2.4}$$

for all $u = (u_1, ..., u_m) \in R^m$ for some $c_0 > 0$. Here $|\cdot|$ is the Euclidean norm.

The dynamics for $r(t), S_i(t), i = 1, ..., N$, are given by

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(x(t))dt + \sigma_D^{(i)} \cdot dB(t) + \sigma_I^{(i)} \cdot d\bar{B}(t), \tag{2.5}$$

$\bar{B}(t)$ is a $\bar{m}$-dim Brownian motion and is independent of $B(\cdot), \sigma_D^{(i)}, \sigma_I^{(i)}$ are m-dim, $\bar{m}$-dim constant vectors. We assume

$$r(t) = \mu_0(x(t))$$

and

$$\mu_i(x) = A^{(i)} \cdot x + a_i, i = 0, 1, 2, ..., N, \tag{2.6}$$

where $A^{(i)}$ is a m-dim vector and $a_i \in R$ is a constant.

We may consider a more general model, for example, to allow the noise intensity to depend on the factors or to allow the coefficients to be nonlinearly dependent on the factors. Such generalization may be necessary when discussing a practical problem. However, the mathematics for such general model will be much more involved and it will not be discussed here.

From (2.1), (2.5),

$$\begin{aligned} dV(t) =& V(t)[(\mu_0(x(t)) + \sum_i u_i(t)\bar{\mu}_i(x(t)))dt \\ &+ \sum_i u_i(t)\sigma_D^{(i)} \cdot dB(t) + \sum_i u_i(t)\sigma_I^{(i)} \cdot d\bar{B}(t)], \end{aligned}$$

where

$$\begin{aligned} \bar{\mu}_i(x) &= \mu_i(x) - \mu_0(x) = \bar{A}^{(i)} \cdot x + \bar{a}_i, \\ \bar{A}^{(i)} &= A^{(i)} - A^{(0)}, \bar{a}_i = a_i - a_0. \end{aligned} \tag{2.7}$$

By Ito's rule,

$$\begin{aligned} d\log V(t) =& (\mu_0(x(t)) + \sum_i \mu_i(t)\bar{\mu}_i(x(t)) - \frac{1}{2}|\sum_i u_i(t)\sigma^{(i)}|^2)dt \\ &+ \sum_i u_i(t)\sigma_D^{(i)} \cdot dB(t) + \sum_i u_i(t)\sigma_I^{(i)} \cdot d\bar{B}(t), \end{aligned}$$

where

$$\sigma^{(i)} = \begin{pmatrix} \sigma_D^{(i)} \\ \sigma_I^{(i)} \end{pmatrix} \in R^{m+\bar{m}}. \tag{2.8}$$

Therefore,

$$\begin{aligned} E[V(T)^\gamma] =& V(0)^\gamma E[\exp(\int_0^T \gamma \sum_i u_i(t)\sigma_D^{(i)} \cdot dB(t) + \gamma \sum_i u_i(t)\sigma_I^{(i)} \cdot d\bar{B}(t) \\ &+ \int_0^T \gamma(\mu_0(x(t)) + \sum_i u_i(t)\bar{\mu}_i(x(t)) - \frac{1}{2}|\sum_i u_i(t)\sigma^{(i)}|^2)dt)] \\ =& V(0)^\gamma E[\exp(\int_0^T \gamma\ell^{(\gamma)}(x^u(t), u(t))dt)], \end{aligned} \tag{2.9}$$

where

$$\ell^{(\gamma)}(x,u) = -\frac{1}{2}(1-\gamma)|\sum_i u_i\sigma^{(i)}|^2 + \sum_i u_i\bar{\mu}_i(x) + \mu_0(x), \tag{2.10}$$

and

$$\begin{aligned} dx^u(t) &= b^u(t,x^u(t))dt + dB(t), \\ b^u(t,x) &= b(x) + \gamma\sum_i u_i(t)\sigma_D^{(i)}. \end{aligned} \tag{2.11}$$

The last step in (2.9) follows from Girsanov Theorem by changing probability measures. This is valid under some conditions, for example, if $u(t)$ is bounded or if $u(t) = \underline{u}(t, x^u(t))$ when $\underline{u}(t,x)$ is Lipschitz. However, this formal calculation suggests to study the stochastic control problem with exponential cost given by the right side of (2.9) (we may take $V(0) = 1$ which we assume in the following). The state dynamics is given by (2.11). For $0 < \gamma < 1$, we maximize the cost and for $-\infty < \gamma < 0$, we minimize the cost. The control process $u(t)$ is assumed to be U valued, $\mathcal{F}_t$ progressive measurable for a filtration $\{\mathcal{F}_t\}$ such that $B(t)$ is a Brownian motion with respect to $\{\mathcal{F}_t\}$. See Fleming and Soner(1992).

To continue, we fix γ with $0 < \gamma < 1$. For each finite T, we consider the problem of choosing $u(t)$ on $0 \le t \le T$ to maximize the right hand side of (2.9). Let

$$W(T,x) = \log\sup_u E_x[\exp(\gamma\int_0^T \ell^{(\gamma)}(x^u(t),u(t))dt)], \tag{2.12}$$

where $x^u(t)$ satisfies (2.11) with $x^u(0) = x$. We anticipate that, under suitable conditions, $T^{-1}W(T,x)$ tends to a limit Λ as $T \to \infty$. See Fleming and McEneaney(1995). Then Λ can be interpreted as the optimal long term growth rate of expected utility of wealth.

As in Fleming and McEneaney(1995), we use the heuristic

$$W(T,x) \sim \Lambda T + W(x), \quad T \to \infty.$$

Then Λ and $W(x)$ satisfy the following dynamic programming equation

$$\begin{aligned} \Lambda = &\frac{1}{2}\Delta W(x) + \frac{1}{2}|\nabla W(x)|^2 + b(x)\cdot\nabla W(x) \\ &+ \max_{u\in U}[\gamma\sum_i u_i\sigma_D^{(i)}\cdot\nabla W(x) + \gamma\ell^{(\gamma)}(x,u)], \end{aligned} \tag{2.13}$$

Similarly, for the HARA parameter $\gamma,\ \gamma < 0$, we consider $W(T,x)$ defined as in (2.12) but change sup to inf and use the heuristic $W(T,x) \sim \Lambda T + W(x)$ as $T \to \infty$. The dynamic programming equation is

$$\begin{aligned}\Lambda = &\frac{1}{2}\Delta W(x) + \frac{1}{2}|\nabla W(x)|^2 + b(x)\cdot\nabla W(x) \\ &+ \min_{u\in U}[\gamma\sum_i u_i\sigma_D^{(i)}\cdot\nabla W(x) + \gamma\ell^{(\gamma)}(x,u)],\end{aligned} \tag{2.14}$$

For $\gamma = 0$, we consider

$$W(T,x) = \sup_u E_x[\int_0^T \ell^{(0)}(x^u(t),u(t))dt],$$

and $W(T,x) \sim \Lambda T + W(x),\quad T \to \infty$. The dynamic programming equation is

$$\Lambda = \frac{1}{2}\Delta W(x) + b(x)\cdot\nabla W(x) + \sup_{u\in U}[\ell^{(0)}(x,u)]. \tag{2.15}$$

For each case, if $W(\cdot)$ is known, a candidate for the optimal investment policy $u^*(x)$ can be obtained by taking *argmax* (or *argmin*) over U in the equation. However, it is not always easy to see if $u^*(x)$ gives an "admissible policy". Moreover, we need to prove a Verification Theorem which ensures that Λ is the optimal long term growth rate.

If U is a compact, convex set, then these questions can be settled by the argument in [Fleming and McEneaney(1995), Sec. 7]. For this particular case, each equation has a unique solution in the viscosity sense (up to a constant) with bounded first order derivatives; $u^*(x)$ gives an optimal policy. In the following, we shall mainly consider $U = R^N$. We shall give some answer to these questions under various assumptions.

We define

$$\bar{g}_{ij} = \sigma^{(i)}\cdot\sigma^{(j)}, \bar{g} = (\bar{g}_{ij}),$$

$x\cdot y$ is the inner product. We assume that

$$\bar{g} \text{ is invertible.} \tag{2.16}$$

Denote by $g, \sigma^{(D)}$ and $\bar{A}$ the matrices,

$$\sigma_{ij}^{(D)} = (\sigma_D^{(i)})_j,\ \bar{A}_{ij} = \bar{A}_j^{(i)},\ g \text{ is the square root of } \bar{g}. \tag{2.17}$$

For $U = R^N$, (2.13) and (2.14) reduce to the following equation,

$$\begin{aligned}\Lambda = &\frac{1}{2}\Delta W(x) + b(x)\cdot\nabla W(x) + \frac{1}{2}|\nabla W(x)|^2 \\ &+ \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}(\bar{\mu}(x) + \sigma^{(D)}\nabla W(x))|^2 + \gamma\mu_0(x),\end{aligned} \tag{2.18}$$

where $\bar{\mu}(x) = (\bar{\mu}_1(x), \cdots, \bar{\mu}_N(x))$. The equation (2.15) reduces to

$$\Lambda = \frac{1}{2}\Delta W(x) + b(x)\cdot\nabla W(x) + \frac{1}{2}|g^{-1}\bar{\mu}(x)|^2 + \mu_0(x). \tag{2.19}$$

In (2.18), we seek a solution $W(x)$ which is quadratic, i.e.,

$$W(x) = \frac{1}{2}Kx\cdot x + e\cdot x \tag{2.20}$$

with K a $m\times m$ symmetric matrix, then

$$D'K + KD + K^2 + \frac{\gamma}{1-\gamma}(\bar{A}' + K\sigma^{(D)'})g^{-2}(\bar{A} + \sigma^{(D)}K) = 0, \tag{2.21}$$

which can be rewritten as

$$KD^{(\gamma)} + D^{(\gamma)'}K + KE^{(\gamma)}K + Q^{(\gamma)} = 0 \tag{2.22}$$

with

$$\begin{aligned}D^{(\gamma)} &= D + \frac{\gamma}{1-\gamma}\sigma^{(D)'}g^{-2}\bar{A}, \\ E^{(\gamma)} &= I + \frac{\gamma}{1-\gamma}\sigma^{(D)'}g^{-2}\sigma^{(D)}, \\ Q^{(\gamma)} &= \frac{\gamma}{1-\gamma}\bar{A}'g^{-2}\bar{A},\end{aligned} \tag{2.23}$$

where D' is the transpose of D, etc.

$$\begin{aligned}(D^{(\gamma)'} + KE^{(\gamma)})e + \frac{\gamma}{1-\gamma}(\bar{A}' + K\sigma^{(D)'})g^{-2}\bar{a} + \gamma A^{(0)} &= 0, \\ \Lambda = \frac{1}{2}trK + \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}(\bar{a} + \sigma^{(D)}e)|^2 + \gamma a_0.&\end{aligned} \tag{2.24}$$

Lemma 2.1. *We have*

$$\|\sigma^{(D)'} g^{-2} \sigma^{(D)}\| \le 1.$$

Here $\|M\| = \max\{|Mx|; |x| = 1\}$ *for a matrix* M, $|x|$ *is the length of a vector* x. *In particular,* $E^{(\gamma)}$ *is positive for all* $-\infty < \gamma < 1$.

Equation (2.22) is a Riccati equation which has appeared in linear control theory. We recall an interesting theorem on the solutions of (2.22). For the details see Willems(1971).

Theorem 2.2. *The equation (2.22) has a solution if and only if*

$$H(s) = E^{(\gamma)^{-1}} - (-si - D^{(\gamma)'})^{-1} Q^{(\gamma)} (si - D^{(\gamma)})^{-1} \ge 0$$

for all real s. *Here* $i = \sqrt{-1}$.

If this condition holds, then there are unique solutions K^-, K^+ *such that the real part of the eigenvalues of* $D^{(\gamma)} + E^{(\gamma)} K^-$ *(resp.* $D^{(\gamma)} + E^{(\gamma)} K^+$*) are nonpositive (resp. nonnegative). Moreover, every solution satisfies* $K^- \le K \le K^+$.

Proof of Lemma 2.1. Let $x \in R^N$. Consider

$$\sigma^{(D)'} g^{-2} \sigma^{(D)} x \cdot x = g^{-2} \sigma^{(D)} x \cdot \sigma^{(D)} x.$$

It is enough to prove

$$g^{-2} \sigma^{(D)} x \cdot \sigma^{(D)} x \le |x|^2. \tag{2.25}$$

Let

$$\sigma^{(D)} x = y, \quad g^{-2} y = z.$$

Then

$$\begin{aligned} g^{-2} \sigma^{(D)} x \cdot \sigma^{(D)} x = z \cdot g^2 z \ &= \sum_{i,j} z_i z_j \sigma^{(i)} \cdot \sigma^{(j)} \\ &= |\sum_i z_i \sigma^{(i)}|^2. \end{aligned} \tag{2.26}$$

On the other hand,

$$g^{-2}\sigma^{(D)}x\cdot\sigma^{(D)}x = z\cdot\sigma^{(D)}x$$

$$(2.27)\qquad = \sum_i z_i\sigma_D^{(i)}\cdot x$$

$$= \sum_i z_i\sigma^{(i)}\cdot\bar{x}.$$

Here $\bar{x} = (x,0) \in R^{m+\bar{m}}$. The above is equal to $\sum_i z_i\sigma^{(i)}\cdot\bar{x}$ which has absolute value smaller than

$$|\sum_i z_i\sigma^{(i)}||\bar{x}| = |\sum_i z_i\sigma^{(i)}||x|.$$

This and (2.26) $\sim$ (2.27) imply (2.25). This completes the proof.

3. Negative HARA parameters.

In this section, we consider the cases of negative HARA parameter γ. We shall study the solutions of the corresponding dynamic programming equations. In particular, for the case of no constraint ($U = R^N$), we show that the Ricatti equation (2.22) has a unique $K^{(\gamma)}$ such that $K^{(\gamma)}$ is nonpositive definite. The matrix $D^{(\gamma)'} + K^{(\gamma)}E^{(\gamma)}$ is stable. From this, a solution $(W^{(\gamma)}, \Lambda^{(\gamma)})$ of the dynamic programming equation, such that $W^{(\gamma)}$ is quadratic, can be derived. We shall show $\Lambda^{(\gamma)}$ is the optimal growth rate in the sense that

$$\Lambda^{(\gamma)} = \min_{r>0}\Lambda_r^{(\gamma)},$$

where $\Lambda_r^{(\gamma)}$ is the optimal growth rate for the portfolio problem with constraint $U = \{u \in R^N; |u| \le r\}$. A candidate for the Markovian optimal investment policy is given by

$$u^{(\gamma)}(x) = \frac{1}{1-\gamma}g^{-2}(\bar{\mu}(x) + \sigma^{(D)}\nabla W^{(\gamma)}(x)),$$

which is equal to the *argmin* in (2.14) with $U = R^N, \Lambda = \Lambda^{(\gamma)}$ and $W = W^{(\gamma)}$. We note that $u^{(\gamma)}(x)$ is linear. For $-\gamma(>0)$ small enough, it is not difficult to show that this gives an optimal investment policy, $u^{(\gamma)*}(t) = u^{(\gamma)}(x(t))$. However, it is not known if this is still true in general. See the study in Fleming and Sheu (1999) for how the difficulty may occur.

Our main interest is the case $U = R^N$. We shall start with the cases $U = U_r = \{u \in R^N; |u| \le r\}$.

The dynamic programming equation associated to the investment problem is given by (see (2.14))

$$\begin{aligned}\Lambda =& \frac{1}{2}\Delta W(x) + \frac{1}{2}|\nabla W(x)|^2 + b(x)\cdot\nabla W(x) \\ &+ \min_{u\in U}[\gamma \sum u_i \sigma_D^{(i)} \cdot \nabla W(x) + \gamma \ell^{(\gamma)}(x,u)].\end{aligned} \tag{3.1}$$

Theorem 3.1. *Let $\gamma < 0, U = U_r, 0 < r < \infty$. Then there is a unique $(\Lambda_r^{(\gamma)}, W_r^{(\gamma)})$ such that $(\Lambda, W) = (\Lambda_r^{(\gamma)}, W_r^{(\gamma)})$ satisfies (3.1) in classical sense, $W_r^{(\gamma)}(0) = 0$ and $|\nabla W_r^{(\gamma)}(x)|$ is a bounded function. Moreover,*

$$\Lambda_r^{(\gamma)} = \inf J(u)$$

where inf is taken over all the process u which is progressive measurable w.r.t. a filtration $\{\mathcal{F}_t\}, |u(t)| \le r$ for any $t \ge 0$,

$$dx^u(t) = (b(x^u(t)) + \gamma \sum u_i(t)\sigma_D^{(i)})dt + dB(t),$$

$x^u(\cdot)$ is adapted to $\{\mathcal{F}_t\}, B(\cdot)$ is an $\mathcal{F}_t$-Brownian motion and

$$J(u) = \liminf_{T\to\infty} \frac{1}{T} \log E[\exp(\int_0^T \gamma \ell^{(\gamma)}(x^u(t), u(t))dt)].$$

Proof. This follows from the arguments in Fleming and McEneaney(1995). Uniqueness of $W_r^{(\gamma)}$ is proved in Fleming and James(1995).

Let $U = R^N$. Then (3.1) becomes

$$\begin{aligned}\Lambda =& \frac{1}{2}\Delta W(x) + \frac{1}{2}|\nabla W(x)|^2 + b(x)\cdot\nabla W(x) \\ &+ \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}(\bar{\mu}(x) + \sigma^{(D)}\nabla W(x))|^2 + \gamma\mu_0(x).\end{aligned} \tag{3.2}$$

Lemma 3.2. *Assume (Λ, W) is a solution of (3.2) such that $W(\cdot)$ is concave and*

$$|\nabla W(x)| \le c(1+|x|) \quad \textit{for all} \quad x \in R^m$$

for some $c > 0$. Let $x^*(t)$ be the diffusion satisfying

$$dx^*(t) = b^*(x^*(t))dt + dB(t), \tag{3.3}$$

where

$$b^*(x) = b(x) + \frac{\gamma}{1-\gamma}\sigma^{(D)'}g^{-2}\bar{\mu}(x) + E^{(\gamma)}\nabla W(x)$$

(see (2.23) for notations). Then $x^*(t)$ is ergodic. Moreover, there are $\alpha > 0, c > 0$ such that

$$E_x[|x^*(t)|^2] \le c(|x|^2 e^{-\alpha t} + 1). \tag{3.4}$$

Proof. Using (3.2) and applying Ito's rule to $W(x^*(t))$,

$$\begin{aligned} dW(x^*(t)) =& (\Lambda + \frac{1}{2}\nabla W(x^*(t)) \cdot E^{(\gamma)}\nabla W(x^*(t)) - \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{\mu}(x^*(t))|^2 \\ & - \gamma\mu_0(x^*(t)))dt + \nabla W(x^*(t)) \cdot dB(t). \end{aligned}$$

Let $\alpha > 0$, to be determined later. We consider $e^{\alpha t}W(x^*(t))$. The above implies,

$$\begin{aligned} & E_x[\int_0^T e^{\alpha t}(\alpha W(x^*(t)) + \Lambda + \frac{1}{2}\nabla W(x^*(t)) \cdot E^{(\gamma)}\nabla W(x^*(t)) \\ & \qquad - \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{\mu}(x^*(t))|^2 - \gamma\mu_0(x^*(t)))dt] \\ =& E_x[W(x^*(T))]e^{\alpha T} - W(x). \end{aligned} \tag{3.5}$$

On the other hand, we apply Ito's rule to $|x^*(t)|^2$,

$$\begin{aligned} d|x^*(t)|^2 =& (2b(x^*(t)) \cdot x^*(t) + \frac{2\gamma}{1-\gamma}g^{-1}\bar{\mu}(x^*(t)) \cdot g^{-1}\sigma^{(D)}x^*(t) \\ & + 2E^{(\gamma)}\nabla W(x^*(t)) \cdot x^*(t) + m)dt + 2x^*(t) \cdot dB(t). \end{aligned}$$

Then considering $e^{\alpha t}|x^*(t)|^2$, we have

$$
\begin{aligned}
(3.6)\qquad & E_x[|x^*(T)|^2]e^{\alpha T} - |x|^2 \\
=& E_x[\int_0^T e^{\alpha t}(\alpha|x^*(t)|^2 + m + 2b(x^*(t))\cdot x^*(t) \\
& + 2\frac{\gamma}{1-\gamma}g^{-1}\bar{\mu}(x^*(t))\cdot g^{-1}\sigma^{(D)}x^*(t) + 2E^{(\gamma)}\nabla W(x^*(t))\cdot x^*(t))dt] \\
\leq& E_x[\int_0^T e^{\alpha t}((-2c_0+\alpha)|x^*(t)|^2 + c_2 + c_0|x^*(t)|^2 \\
& + c_1(\Lambda - \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{\mu}(x^*(t))|^2 + \frac{1}{2}\nabla W(x^*(t))\cdot E^{(\gamma)}\nabla W(x^*(t)) \\
& - \gamma\mu_0(x^*(t))))dt] \\
=& E_x[\int_0^T e^{\alpha t}((-c_0+\alpha)|x^*(t)|^2 - c_1\alpha W(x^*(t)) + c_2)dt] \\
& + c_1(E_x[W(x^*(T))]e^{\alpha T} - W(x)).
\end{aligned}
$$

Here

$$
\begin{aligned}
c_2 &= m + c_1|\Lambda| + c_1\gamma|a_0| + 3\frac{c_1^2\gamma^2}{c_0}|A^{(0)}|^2, \\
c_1 &= \frac{6}{c_0}(\|g^{-1}\sigma^{(D)}\|^2\frac{-\gamma}{1-\gamma} + \|E^{(\gamma)}\|)
\end{aligned}
$$

choose α such that $\alpha(1+\frac{1}{2}cc_1) < c_0$, where c is the constant such that

$$|\nabla W(x)| \leq c(1+|x|).$$

(3.6) implies

$$
\begin{aligned}
(3.7)\qquad e^{\alpha T}E_x[|x^*(T)|^2 - c_1W(x^*(T))] &\leq c_3e^{\alpha T} + c_1|W(x)| \\
&\leq c_4(|x|^2 + e^{\alpha T}).
\end{aligned}
$$

Concavity of $W(\cdot)$ implies that there is $\bar{c}$ such that

$$W(x) \leq \bar{c}(1+|x|).$$

This and (3.7) imply (3.4). This also implies the ergodicity of $x^*(\cdot)$. See [Khasminskii(1980), Chapter IV, Sec. 4].

We now consider a solution (Λ, W) of (3.2) such that W is quadratic. Let

$$W(x) = \frac{1}{2}Kx \cdot x + e \cdot x, \tag{3.8}$$

K is a symmetric $m \times m$ matrix and $e \in R^m$. Then Λ, K, e satisfy (2.22) and (2.24).

The following result is a consequence of Wonham(1968). The uniqueness follows from the same argument as in the proof of Lemma 3.2. For the convenience of the reader, we still provide an argument for it.

Lemma 3.3. *Let $\gamma < 0$. Then (2.22) has a unique solution K such that K is nonpositive definite. For such K,*

$$D^{(\gamma)} + E^{(\gamma)}K$$

is a stable matrix.

Proof. Assume that K is nonegative definite and is a solution of (2.22). Let ϕ be the solution of

$$\begin{aligned}\frac{d\phi(t)}{dt} &= D^*\phi(t),\\ D^* &= D^{(\gamma)} + E^{(\gamma)}K,\end{aligned}$$

with $\phi(0)$ arbitrary. Then

$$\begin{aligned}\frac{d}{dt}K\phi(t)\cdot\phi(t) &= 2K\phi(t)\cdot(D^{(\gamma)}\phi(t) + E^{(\gamma)}K\phi(t))\\ &= K\phi(t)\cdot E^{(\gamma)}K\phi(t) - Q^{(\gamma)}\phi(t)\cdot\phi(t).\end{aligned} \tag{3.9}$$

Here we use (2.22).

On the other hand, we have

$$\begin{aligned}\frac{d}{dt}|\phi(t)|^2 &= 2D^*\phi(t)\cdot\phi(t)\\ &= 2D\phi(t)\cdot\phi(t) + 2\frac{\gamma}{1-\gamma}g^{-1}\sigma^{(D)}\phi(t)\cdot g^{-1}\bar{A}\phi(t) + E^{(\gamma)}K\phi(t)\cdot\phi(t)\\ &\le -2c_0|\phi(t)|^2 + c_0|\phi(t)|^2 + c(-Q^{(\gamma)}\phi(t)\cdot\phi(t) + K\phi(t)\cdot E^{(\gamma)}K\phi(t))\\ &= -c_0|\phi(t)|^2 + c(-Q^{(\gamma)}\phi(t)\cdot\phi(t) + K\phi(t)\cdot E^{(\gamma)}K\phi(t)).\end{aligned}$$

Here

$$c = \frac{2}{c_0}(\|E^{(\gamma)}\| + \frac{-\gamma}{1-\gamma}\|g^{-1}\bar{A}\|^2).$$

Let $\alpha > 0$ and will be determined later. Considering $e^{\alpha t}|\phi(t)|^2$, using the above relation and (3.9), we have

$$\begin{aligned}
|\phi(T)|^2 e^{\alpha T} - |\phi(0)|^2 \leq & \int_0^T e^{\alpha T}((-c_0+\alpha)|\phi(t)|^2 + c(-\phi(t)\cdot Q^{(\gamma)}\phi(t) \\
& + K\phi(t)\cdot E^{(\gamma)}K\phi(t))dt \\
= & \int_0^T e^{-\alpha T}((-c_0+\alpha)|\phi(t)|^2 - c\alpha\phi(t)\cdot K\phi(t))dt \\
& + c(\phi(T)\cdot K\phi(T)e^{\alpha T} - K\phi(0)\cdot\phi(0)).
\end{aligned}$$

Take α small such that $\alpha(1 + c\|K\|) < c_0$. By the above relation and the condition that K is nonpositive definite,

$$\begin{aligned}
|\phi(T)|^2 e^{\alpha T} &\leq |\phi(0)|^2 - cK\phi(0)\cdot\phi(0) \\
&\leq (1 + c\|K\|)|\phi(0)|^2.
\end{aligned}$$

Since $\alpha > 0$, this implies D^* is a stable matrix.

Now we prove the uniqueness of K. Assume $\widetilde{K}$ is another solution of (2.22) which is nonpositive definite. We substract the relations (2.22) for K and $\widetilde{K}$ to get

$$\begin{aligned}
&(K-\widetilde{K})D^* + D^{*'}(K-\widetilde{K}) - (K-\widetilde{K})E^{(\gamma)}(K-\widetilde{K}) = 0 \\
&D^* = D^{(\gamma)} + E^{(\gamma)}K.
\end{aligned}$$

Let $\phi(t)$ be defined by

$$\frac{d}{dt}\phi(t) = D^*\phi(t).$$

Then

$$\begin{aligned}
\frac{d}{dt}(K-\widetilde{K})\phi(t)\cdot\phi(t) &= 2(K-\widetilde{K})\phi(t)\cdot D^*\phi(t) \\
&= (K-\widetilde{K})\phi(t)\cdot E^{(\gamma)}(K-\widetilde{K})\phi(t) \geq 0.
\end{aligned}$$

i.e.,

$$(K-\widetilde{K})\phi(T)\cdot\phi(T) \geq (K-\widetilde{K})\phi(0)\cdot\phi(0)$$

for all $T \geq 0$. Let $T \to \infty$, the left side tends to 0 by the fact that D^* is stable proved earlier. Therefore,

$$(K - \widetilde{K})\phi(0) \cdot \phi(0) \leq 0,$$

hence

$$(K - \widetilde{K})x \cdot x \leq 0 \quad \text{for all} \quad x.$$

since $\phi(0)$ is arbitrary.

Similarly, we have

$$(\widetilde{K} - K)x \cdot x \leq 0 \quad \text{for all} \quad x.$$

Therefore, $K = \widetilde{K}$, which completes the proof of the uniqueness of the solution.

The existence of a nonpositive definite solution for (2.22) follows from the argument in the proof of Theorem 1, Section 2.3, Brockett(1970). This completes the proof.

Remark 3.4. In Brockett(1970), Section 2.3, it shows that Lemma 3.3. holds if the controllability and observability of the system are assumed. In our case, the controllability means

$$[\sqrt{E^{(\gamma)}}, D^{(\gamma)}\sqrt{E^{(\gamma)}}, ..., (D^{(\gamma)})^{m-1}\sqrt{E^{(\gamma)}}]$$

is of full rank, and the observability means

$$\begin{pmatrix} g^{-1}\bar{A} \\ g^{-1}\bar{A}D^{(\gamma)} \\ \vdots \\ g^{-1}\bar{A}(D^{(\gamma)})^m \end{pmatrix}$$

Under such conditions, K is negative definite. Here, we have controllability condition. But observability condition may fail to hold. Under such situation, the proof of Thm 1, Sec. 2.3, Brockett(1970) gives the existence of K. But the stability of $D^{(\gamma)} + E^{(\gamma)}K$ does not follow immediately from the results in Brockett(1970).

Lemma 3.3 follows from the results in Wonham(1968). It assumes the stability and detectability of the system. In our case, the stability means the existence of K_0 such that

$$D^{(\gamma)} - \sqrt{E^{(\gamma)}}K_0$$

is a stable matrix, and the detectability means the existence of K_1 such that

$$D^{(\gamma)\prime} - \frac{-\gamma}{1-\gamma}^{\frac{1}{2}}\bar{A}' g^{-1} K_1$$

is a stable matrix. Under such conditions, K is nonpositive definite, but may not be negative definite.

We now summarize the results obtained above.

Theorem 3.5. *The equation (3.2) has a unique solution (Λ, W) satisfying the following properties*

$$W(x) = \frac{1}{2}Kx \cdot x + e \cdot x, \tag{3.10}$$

K is nonpositive definite, e, Λ are given by (2.24). Moreover, $D^{(\gamma)} + E^{(\gamma)}K$ is stable.

Remark 3.6. If (Λ, W) is a solution with W given by (3.10), then $b^*(x)$ defined in Lemma 3.2 is linear

$$b^*(x) = D^*x + e^*$$

where $D^* = D^{(\gamma)} + E^{(\gamma)}K$ is stable.

Our aim in the rest is to prove that

$$\Lambda^{(\gamma)} = \min_{r>0} \Lambda_r^{(\gamma)}$$

and the convergence of $W_r^{(\gamma)}$ to $W^{(\gamma)}$ as r tends to infinity, where $(\Lambda_r^{(\gamma)}, W_r^{(\gamma)})$ is the unique solution of (3.1) such that $\nabla W_r^{(\gamma)}$ is bounded.

Theorem 3.7. *The solution* (Λ, W) *of (3.2) satisfying the following properties is unique:* $W(0) = 0, W(x)$ *is concave and*

$$|\nabla W(x)| \leq c(1 + |x|), \quad \forall x \in R^m$$

for some $c > 0$.

Proof. Assume $(\Lambda, W), (\widetilde{\Lambda}, \widetilde{W})$ are solutions of (3.2) satisfying the above properties. Subtract the equations for (Λ, W) and $(\widetilde{\Lambda}, \widetilde{W})$ to get

$$\begin{aligned}
&\bar{\Lambda} = \frac{1}{2}\Delta \bar{W}(x) + b^*(x) \cdot \nabla \bar{W}(x) + \frac{1}{2}\nabla \bar{W}(x) \cdot E^{(\gamma)} \nabla \bar{W}(x), \\
&\bar{W}(x) = \widetilde{W}(x) - W(x), \\
&\bar{\Lambda} = \widetilde{\Lambda} - \Lambda,
\end{aligned}$$

$b^*(\cdot)$ is given in Lemma 3.2, i.e.,

$$b^*(x) = b(x) + \frac{\gamma}{1-\gamma}\sigma^{(D)'} g^{-2} \bar{\mu}(x) + E^{(\gamma)} \nabla W(x).$$

Let $x^*(t)$ be the diffusion process defined by

$$dx^*(t) = b^*(x^*(t))dt + dB(t).$$

By Ito's rule,

$$d\bar{W}(x^*(t)) = (\bar{\Lambda} - \frac{1}{2}\nabla \bar{W}(x^*(t)) \cdot E^{(\gamma)} \nabla \bar{W}(x^*(t)))dt + \nabla \bar{W}(x^*(t)) \cdot dB(t).$$

Then

$$\begin{aligned}
\bar{\Lambda} T = &E_x[\int_0^T \frac{1}{2}\nabla \bar{W}(x^*(t)) \cdot E^{(\gamma)} \nabla \bar{W}(x^*(t))dt] \\
&+ E_x[\bar{W}(x^*(T))] - \bar{W}(x).
\end{aligned} \tag{3.11}$$

Dividing this relation by T and letting $T \to \infty$, we get

$$\bar{\Lambda} \geq 0.$$

Here we use the estimate in Lemma 3.2. Similarly we have $\bar{\Lambda} \le 0$. Therefore $\bar{\Lambda} = 0$.

Dividing (3.11) by T again and letting $T \to \infty$, we now have

$$\int \frac{1}{2} \nabla \bar{W}(x) \cdot E^{(\gamma)} \nabla \bar{W}(x) p^*(x) dx = 0,$$

$p^*(x)$ is the invariant density for $x^*(t)$. This implies $\nabla \bar{W}(x) = 0$ a.e. with respect to dx. Then $\widetilde{W}(x) - W(x)$ is a constant which is equal to $\widetilde{W}(0) - W(0) = 0$. This completes the proof.

Here we shall mention some results given in Bensoussan and Frehse(1992) and Nagai(1996) which relate to Theorem 3.7. These works discuss the similar problem under a general framwork. In order to apply their result, we need to assume the condition that

$$V(x) = -\frac{\gamma}{2(1-\gamma)} |g^{-1}\bar{\mu}(x)|^2 - \gamma \mu_0(x)$$

tends to ∞ as $|x|$ tends to ∞. If this holds, then Lemma 3.2 in Nagai(1996), or Theorem 4.1 in Bensoussan and Frehse(1992), implies the uniqueness of the solution satisfying the condition that $-W(x) \to \infty$ as $|x| \to \infty$. We see without suitable assumption these results can not be directly applied to our case.

Let $\Lambda_r^{(\gamma)}$ be the minimal long term growth rate for the investment problem with constraint $|u| \le r$. By Theorem 3.1, there is a unique $W_r^{(\gamma)}$ such that $W_r^{(\gamma)}(0) = 0, |\nabla W_r^{(\gamma)}(x)|$ is bounded and $(\Lambda, W) = (\Lambda_r^{(\gamma)}, W_r^{(\gamma)})$ is a classical solution of (3.1) with $U = U_r$. Let $(\Lambda^{(\gamma)}, W^{(\gamma)})$ be the solution of (3.2) given in Theorem 3.5. In the following, we shall show that $(\Lambda_r^{(\gamma)}, W_r^{(\gamma)})$ converges to $(\Lambda^{(\gamma)}, W^{(\gamma)})$. We need the following lemmas.

Lemma 3.8. *Let $(\Lambda_r^{(\gamma)}, W_r^{(\gamma)})(\Lambda^{(\gamma)}, W^{(\gamma)})$ be defined as above. Then $\bar{W} = W_r^{(\gamma)} - W^{(\gamma)}$ is convex for any $r > 0$.*

Proof. Denote $\bar{\Lambda} = \Lambda_r^{(\gamma)} - \Lambda^{(\gamma)}, \bar{W} = W_r^{(\gamma)} - W^{(\gamma)}$. Then the equation of $(\Lambda_r^{(\gamma)}, W_r^{(\gamma)})$

can be rewritten as follows,

$$
\begin{aligned}
\Lambda_r^{(\gamma)} =& \frac{1}{2}\Delta(\bar{W}+W^{(\gamma)})(x) + b(x)\cdot\nabla(\bar{W}+W^{(\gamma)})(x) + \frac{1}{2}|\nabla(\bar{W}+W^{(\gamma)})(x)|^2 \\
&\quad + \inf_{|u|\le r}\{\gamma\sum_i u_i\sigma_D^{(i)}\cdot\nabla(\bar{W}+W^{(\gamma)})(x) + \gamma\ell^{(\gamma)}(x,u)\} \\
=& \frac{1}{2}\Delta\bar{W}(x) + (b(x)+\nabla W^{(\gamma)})(x)\cdot\nabla\bar{W}(x) + \frac{1}{2}|\nabla\bar{W}(x)|^2 \\
&\quad + \Lambda^{(\gamma)} - \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}(\bar{\mu}(x)+\sigma^{(D)}\nabla W^{(\gamma)}(x))|^2 - \gamma\mu_0(x) \\
&\quad + \inf_{|u|\le r}\{\gamma\sum u_i\sigma_D^{(i)}\cdot\nabla(\bar{W}+W^{(\gamma)})(x) + \gamma\ell^{(\gamma)}(x,u)\}.
\end{aligned}
$$

That is, $(\bar{\Lambda},\bar{W})$ satisfies,

$$
\begin{aligned}
\bar{\Lambda} = &\frac{1}{2}\Delta\bar{W}(x) + b^{(\gamma)}(x)\cdot\nabla\bar{W}(x) + \frac{1}{2}\bar{W}(x)\cdot E^{(\gamma)}\nabla\bar{W}(x) \\
&+ H_r(\bar{\mu}(x)+\sigma^{(D)}\nabla(W^{(\gamma)}+\bar{W})(x))
\end{aligned}
\tag{3.12}
$$

Here

$$
b^{(\gamma)}(x) = b(x) + \frac{\gamma}{1-\gamma}\sigma^{(D)'}g^{-2}\bar{\mu}(x) + E^{(\gamma)}\nabla W^{(\gamma)}(x),
$$

$E^{(\gamma)}$ is given in (2.23) and

$$
\begin{aligned}
H_r(p) &= -\frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}p|^2 + \inf_{|u|\le r}[\gamma u\cdot p - \frac{1}{2}\gamma(1-\gamma)|gu|^2] \\
&= \inf_{|u|\le r}[-\frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}p-(1-\gamma)gu|^2].
\end{aligned}
$$

Then $H_r(p)$ is convex. Denote $L_r(v)$ the convex conjugate of $H_r(p)$. Then

$$
\begin{aligned}
L_r(v) &= \sup_p\{v\cdot p - H_r(p)\} \\
&= \sup_p\sup_{|u|\le r}\{v\cdot p + \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}p-(1-\gamma)gu|^2\} \\
&= \sup_{|u|\le r}\{-\frac{1}{2}\frac{1-\gamma}{\gamma}|g(v-\gamma u)|^2 + \frac{1}{2}\gamma(1-\gamma)|gu|^2\} \\
&= -\frac{1}{2}\frac{1-\gamma}{\gamma}|gv|^2 + (1-\gamma)\sup_{|u|\le r} gv\cdot gu \\
&= -\frac{1}{2}\frac{1-\gamma}{\gamma}|gv|^2 + (1-\gamma)|gv|^2\frac{r}{|v|}.
\end{aligned}
\tag{3.13}
$$

The following relation holds

$$H_r(p) = \sup_v \{v \cdot p - L_r(v)\}.$$

Therefore,

$$\begin{aligned} (3.14) \quad & H_r(\bar{\mu}(x) + \sigma^{(D)}(\nabla W^{(\gamma)}(x) + \nabla \bar{W}(x))) = \sup_v \{\sigma^{(D)'} v \cdot \nabla \bar{W}(x) + L_r(x, v)\}, \\ & L_r(x, v) = v \cdot (\bar{\mu}(x) + \sigma^{(D)} \nabla W^{(\gamma)}(x)) - L_r(v). \end{aligned}$$

Write also

$$\frac{1}{2} q \cdot E^{(\gamma)} q = \sup_u [u \cdot q - \frac{1}{2} u \cdot E^{(\gamma)-1} u]. \tag{3.15}$$

From (3.12), (3.14) and (3.15), the equation (3.12) is the dynamic programming equation for the following stochastic control problem: Let $(\hat{x}(t), v(t), u(t))$ be a process satisfying

$$d\hat{x}(t) = (b^{(\gamma)}(\hat{x}(t)) + \sigma^{(D)'} v(t) + u(t))dt + dB(t), \tag{3.16}$$

such that $\hat{x}(t), v(t), u(t)$ are progressively measurable w.r.t. a filtration $\{\mathcal{F}_t\}$ and $B(t)$ is a m-dim $\mathcal{F}_t$-Brownian motion. Let

$$\begin{aligned} (3.17) \qquad \hat{J}(v, u) &= \overline{\lim}_{T \to \infty} \frac{1}{T} E[\int_0^T \hat{L}(\hat{x}(t), v(t), u(t)) dt], \\ \hat{L}(x, v, u) &= L_r(x, v) - \frac{1}{2} u \cdot E^{(\gamma)-1} u \\ &= v \cdot (\bar{\mu}(x) + \sigma^{(D)} \nabla W^{(\gamma)}(x)) - L_r(v) - \frac{1}{2} u \cdot E^{(\gamma)-1} u. \end{aligned}$$

The goal is to maximize $\hat{J}(v, u)$ over all bounded processes (v, u). We shall prove that $\Lambda_r^{(\gamma)} - \Lambda^{(\gamma)} = \hat{\Lambda}$ where

$$\hat{\Lambda} = \sup \hat{J}(v, u). \tag{3.18}$$

Let apply Ito's rule to $\bar{W}(\hat{x}(t))$ for $\hat{x}(t)$ satisfying (3.16) and use (3.12)(3.14),

$$\begin{aligned} d\bar{W}(\hat{x}(t)) =& (\frac{1}{2} \Delta \bar{W}(\hat{x}(t)) + (b^{(\gamma)}(\hat{x}(t)) + \sigma^{(D)'} v(t) + u(t)) \cdot \nabla \bar{W}(\hat{x}(t)))dt \\ & + \nabla \bar{W}(\hat{x}(t)) \cdot dB(t) \\ \leq & ((\Lambda_r^{(\gamma)} - \Lambda^{(\gamma)}) - \hat{L}(\hat{x}(t), v(t), u(t)))dt + \nabla \bar{W}(\hat{x}(t)) \cdot dB(t). \end{aligned}$$

Then it is easily seen

$$\hat{J}(v,u) \le \Lambda_r^{(\gamma)} - \Lambda^{(\gamma)}$$

i.e.,

$$\hat{\Lambda} \le \Lambda_r^{(\gamma)} - \Lambda^{(\gamma)}.$$

Since $\hat{J}(0,0) = 0$, we have

$$0 \le \hat{\Lambda} \le \Lambda_r^{(\gamma)} - \Lambda^{(\gamma)}.$$

On the other hand, for each $\bar{r} > 0$, we consider the same control problem with constraint $|\bar{v}(t)| \le \bar{r}$. Then we can show the existence of $(\hat{\Lambda}_{\bar{r}}, \hat{W}_{\bar{r}})$ solving the equation

$$\hat{\Lambda}_{\bar{r}} = \frac{1}{2}\Delta \hat{W}_{\bar{r}}(x) + b^{(\gamma)}(x)\cdot\nabla\hat{W}_{\bar{r}}(x) + \sup_{|v|\le\bar{r},u}\{(\sigma^{(D)'}v+u)\cdot\nabla\hat{W}_{\bar{r}}(x) + \hat{L}(x,v,u)\} \tag{3.19}$$

such that $|\nabla\hat{W}_{\bar{r}}|$ is bounded, $\hat{W}_{\bar{r}}(0) = 0$. Moreover, $\hat{W}_{\bar{r}}$ is convex. This can be proved by approximating the control problem using associated discounted control problem with discount factor $\rho \to 0$. Here the properties that the running cost $\bar{L}(x,v,u)$ is linear in x and the dynamics is linear in x, v, u are used to prove the convexity of the value function $\hat{W}_{\bar{r}}^{(\rho)}(x)$ for the discounted control problem. Then $\hat{W}_{\bar{r}}$ is the limit of $\hat{W}_{\bar{r}}^{(\rho)}(x) - \hat{W}_{\bar{r}}^{(\rho)}(0)$ as $\rho \to 0$. See Fleming and McEneaney(1995) or Fleming and Sheu(1999) for the details of this argument.

By (3.19) ,

$$\begin{aligned}
\hat{\Lambda}_{\bar{r}} =& \frac{1}{2}\Delta\hat{W}_{\bar{r}}(x) + b^{(\gamma)}(x)\cdot\nabla\hat{W}_{\bar{r}}(x) + \frac{1}{2}\nabla\hat{W}_{\bar{r}}(x)\cdot E^{(\gamma)}\nabla\hat{W}_{\bar{r}}(x) \\
&+ \sup_{|v|\le\bar{r}}[\sigma^{(D)'}v\cdot\nabla\hat{W}_{\bar{r}}(x) + L_{\bar{r}}(x,v)] \\
\ge& \frac{1}{2}\Delta\hat{W}_{\bar{r}}(x) + b^{(\gamma)}(x)\cdot\nabla\hat{W}_{\bar{r}}(x) + \frac{1}{2}\nabla\hat{W}_{\bar{r}}(x)\cdot E^{(\gamma)}\nabla\hat{W}_{\bar{r}}(x).
\end{aligned}$$

The convexity of $\hat{W}_{\bar{r}}(x)$ and

$$\hat{\Lambda}_{\bar{r}} \le \hat{\Lambda} \le \Lambda_r^{(\gamma)} - \Lambda^{(\gamma)} = \bar{\Lambda}, \tag{3.20}$$

imply $\nabla \hat{W}_{\bar{r}}(x), \bar{r} > 0$, is bounded on bounded sets of x. Then we can take a subsequence $\bar{r} = \bar{r}_n \to \infty$ such that $\hat{W}_{\bar{r}_n}$ converges to $\hat{W}$ uniformly on compact set and $\hat{\Lambda}_{\bar{r}_n}$ converges to $\hat{\Lambda}$ as $n \to \infty$. The equation (3.12) holds for $(\Lambda, W) = (\hat{\Lambda}, \hat{W})$ and $\hat{W}$ is convex. Since $(\bar{\Lambda}, \bar{W}) = (\Lambda_r^{(\gamma)} - \Lambda^{(\gamma)}, \bar{W})$ is also a solution for (3.12), we expect $\bar{W} = \hat{W}$ which will be proved below.

Denote $\widetilde{W} = \hat{W} + W^{(\gamma)}, \widetilde{\Lambda} = \hat{\Lambda} + \Lambda^{(\gamma)}$. Then

$$\begin{aligned}
\widetilde{\Lambda} &= \frac{1}{2}\Delta \widetilde{W}(x) + b(x) \cdot \nabla \widetilde{W}(x) + \frac{1}{2}|\nabla \widetilde{W}(x)|^2 + G_r(x, \nabla \widetilde{W}(x)), \\
G_r(x,p) &= \inf_{|u| \le r} [\gamma \sum u_i \sigma_D^{(i)} \cdot p + \gamma \ell^{(\gamma)}(x,u)\}.
\end{aligned}$$

Note, this is the same as (3.1) with $U = U_r$. Since $(\Lambda_r^{(\gamma)}, W_r^{(\gamma)})$ satisfies the same equation, we substract these two relations. Then

$$\begin{aligned}
\widetilde{\Lambda} - \Lambda_r^{(\gamma)} =& \frac{1}{2}\Delta(\widetilde{W} - W_r^{(\gamma)})(x) + b(x) \cdot \nabla(\widetilde{W} - W_r^{(\gamma)})(x) \\
&+ \nabla W_r^{(\gamma)}(x) \cdot \nabla(\widetilde{W} - W_r^{(\gamma)})(x) + \frac{1}{2}|\nabla(\widetilde{W} - W_r^{(\gamma)})(x)|^2 \\
&+ G_r(x, \nabla \widetilde{W}(x)) - G_r(x, \nabla W_r^{(\gamma)}(x)).
\end{aligned} \tag{3.21}$$

Since $G_r(x,p)$ is Lipschitz in p, there is a bounded vector field $v(x)$,

$$v(x) \cdot (\nabla \widetilde{W}(x) - \nabla W_r^{(\gamma)}(x)) = G_r(x, \nabla \widetilde{W}(x)) - G_r(x, \nabla W_r^{(\gamma)}(x)).$$

Define

$$\widetilde{b}(x) = b(x) + \nabla W_r^{(\gamma)}(x) + v(x).$$

Let $\widetilde{x}(t)$ be the diffusion process satisfying

$$d\widetilde{x}(t) = \widetilde{b}(\widetilde{x}(t))dt + dB(t).$$

Apply Ito's rule to $\widetilde{W}(\widetilde{x}(t)) - W_r^{(\gamma)}(\widetilde{x}(t))$,

$$\begin{aligned}
d(\widetilde{W} - W_r^{(\gamma)})(\widetilde{x}(t)) =& (-\frac{1}{2}|\nabla(\widetilde{W} - W_r^{(\gamma)})(\widetilde{x}(t))|^2 + \widetilde{\Lambda} - \Lambda_r^{(\gamma)})dt \\
&+ \nabla(\widetilde{W} - W_r^{(\gamma)})(\widetilde{x}(t)) \cdot dB(t).
\end{aligned} \tag{3.22}$$

Since $\nabla W_r^{(\gamma)}(x), v(x)$ are bounded functions, $\widetilde{x}(t)$ can be shown to satisfy

$$E_x[|\widetilde{x}(t)|^2] \leq c(e^{-\alpha t}|x|^2 + 1) \quad \text{for all} \quad x \quad \text{and} \quad t > 0. \tag{3.23}$$

Here c, α are some positive constants. This implies $\widetilde{x}(t)$ is ergodic with invariant density $\widetilde{p}(\cdot)$. Integrating (3.22) over $t \in [0, T]$, taking expectation, dividing both sides by T, then letting $T \to \infty$ and by using an ergodic theorem, we get

$$\int (-\frac{1}{2}|\nabla(\widetilde{W} - W_r^{(\gamma)})|^2(x) + \widetilde{\Lambda} - \Lambda_r^{(\gamma)})\widetilde{p}(x)dx = 0. \tag{3.24}$$

Here we use (3.23) and $|\widetilde{W}(x)| \leq c(1 + |x|)$. By (3.20), we have $\widetilde{\Lambda} - \Lambda_r^{(\gamma)} \leq 0$. Then (3.24) implies $\widetilde{\Lambda} - \Lambda_r^{(\gamma)} = 0$ and $\nabla(\widetilde{W} - W_r^{(\gamma)}) = 0$. Therefore, $\widetilde{W} - W_r^{(\gamma)}$ is constant and is equal to $\widetilde{W}(0) - W_r^{(\gamma)}(0) = 0$. i.e. $\widetilde{W} = W_r^{(\gamma)}$. This implies (3.18) and $\bar{W} = \hat{W}$, therefore, is convex. This completes the proof.

Lemma 3.9. *Let $(\Lambda_r^{(\gamma)}, W_r^{(\gamma)})$ be as in Theorem 3.1. Then $W_r^{(\gamma)}$ is concave.*

Proof. Let fix $\gamma < 0, r > 0$. By the argument in Fleming McEneaney(1995)(Thm 7.1) and Fleming and James (Thm 3.1), for each $\rho > 0$ there is a unique $W^{(\rho)}$ in $C^2(R^m)$ such that

$$\begin{aligned}\rho W^{(\rho)}(x) =& \frac{1}{2}\Delta W^{(\rho)}(x) + \frac{1}{2}|\nabla W^{(\rho)}(x)|^2 + b(x) \cdot \nabla W^{(\rho)}(x) \\ &+ \min_{u \in U_r}[\gamma \sum u_i \sigma_D^{(i)} \cdot \nabla W^{(\rho)}(x) + \gamma \ell^{(\gamma)}(x, u)]\end{aligned}$$

and $|\nabla W^{(\rho)}|$ is bounded. Moreover, $\rho W^{(\rho)}(0)$ converges to $\Lambda_r^{(\gamma)}$ and $W^{(\rho)}(x) - W^{(\rho)}(0)$ converges to $W_r^{(\gamma)}(x)$ uniformly for x in compact sets as ρ tends to 0. Therefore, it is enough to prove that $W^{(\rho)}$ is concave for each ρ. In the following, we write W for $W^{(\rho)}$. Our strategy to prove the concavity of W is to express W as the value function of a discounted stochastic control problem with special feature: the dynamics is linear, the running cost is concave in the state and control variables. This implies the concavity of W by a standard argument.

We rewrite the above equation as follows,

$$
\begin{aligned}
\rho W(x) =& \frac{1}{2}\Delta W(x) + b(x)\cdot\nabla W(x) + \frac{1}{2}\nabla W(x)\cdot E^{(\gamma)}\nabla W(x) \\
&+ \frac{\gamma}{1-\gamma}\sigma^{(D)'}g^{-2}\bar{\mu}(x)\cdot\nabla W(x) + \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{\mu}(x)|^2 + \gamma\mu_0(x) \\
&- \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}(\bar{\mu}(x)+\sigma^{(D)}\nabla W(x))|^2 \\
&+ \inf_{|u|\le r}[\gamma(\bar{\mu}(x)+\sigma^{(D)}\nabla W(x))\cdot u - \frac{1}{2}\gamma(1-\gamma)|gu|^2] \\
=& \frac{1}{2}\Delta W(x) + b^{(\gamma)}(x)\cdot\nabla W(x) + \frac{1}{2}\nabla W(x)\cdot E^{(\gamma)}\nabla W(x) \\
&+ \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{\mu}(x)|^2 + \gamma\mu_0(x) + H_r(\bar{\mu}(x)+\sigma^{(D)}\nabla W(x)), \\
& b^{(\gamma)}(x) = b(x) + \frac{\gamma}{1-\gamma}\sigma^{(D)'}g^{-2}\bar{\mu}(x).
\end{aligned}
\tag{3.25}
$$

Here

$$H_r(p) = \inf_{|u|\le r}\{-\frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}p-(1-\gamma)gu|^2\}.$$

Define

$$
\begin{aligned}
L_r(v) &= \sup_p\{v\cdot p - H_r(p)\} \\
&= -\frac{1}{2}\frac{1-\gamma}{\gamma}|gv|^2 + \sup_{|u|\le r}\{(1-\gamma)gv\cdot gu\} \\
&= -\frac{1}{2}\frac{1-\gamma}{\gamma}|gv|^2 + \bar{L}_r(v).
\end{aligned}
$$

See (3.13). $\bar{L}_r(v) = \sup_{|u|\le r}\{(1-\gamma)gv\cdot gu\}$ is convex. Then $H_r(p) = \sup\{v\cdot p - L_r(v); v\}$.

$$
\begin{aligned}
&\frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{\mu}(x)|^2 + H_r(\bar{\mu}(x)+\sigma^{(D)}\nabla W(x)) \\
=& \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{\mu}(x)|^2 + \sup_v\{(\bar{\mu}(x)+\sigma^{(D)}\nabla W(x))\cdot v + \frac{1}{2}\frac{1-\gamma}{\gamma}|gv|^2 - \bar{L}_r(v)\} \\
=& \sup_v\{\sigma^{(D)'}v\cdot\nabla W(x) - \bar{L}_r(v) + \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{\mu}(x) + \frac{1-\gamma}{\gamma}gv|^2\}.
\end{aligned}
$$

From this, (3.25) becomes,

$$
\begin{aligned}
\rho W(x) =& \frac{1}{2}\Delta W(x) + b^{(\gamma)}(x)\cdot\nabla W(x) + \frac{1}{2}\nabla W(x)\cdot E^{(\gamma)}\nabla W(x) + \gamma\mu_0(x) \\
&+ \sup_v\{\sigma^{(D)'}v\cdot\nabla W(x) - \bar{L}_r(v) + \frac{\gamma}{2(1-\gamma)}|g^{-1}\bar{\mu}(x) + \frac{1-\gamma}{\gamma}gv|^2\} \\
=& \frac{1}{2}\Delta W(x) + b(x)\cdot\nabla W(x) + \frac{1}{2}\nabla W(x)\cdot E^{(\gamma)}\nabla W(x) + \gamma\mu_0(x) \\
&+ \sup_v\{\sigma^{(D)'}v\cdot\nabla W(x) - \bar{L}_r(v - \frac{\gamma}{1-\gamma}g^{-2}\bar{\mu}(x)) + \frac{1-\gamma}{2\gamma}|gv|^2\}
\end{aligned}
\tag{3.26}
$$

Denote $\bar{L}_r(x,v) = \bar{L}_r(v - \frac{\gamma}{1-\gamma}g^{-2}\bar{\mu}(x)) - \frac{1}{2}\frac{1-\gamma}{\gamma}|gv|^2$. Thus, (3.26) is the dynamic programming equation for the following stochastic control problem: Let $(\bar{x}(t), v(t), u(t))$ be a process satisfying

$$d\bar{x}(t) = (b(\bar{x}(t)) + \sigma^{(D)'}v(t) + u(t))dt + dB(t) \tag{3.27}$$

such that $\bar{x}(t), v(t), u(t)$ are progressive measurable w.r.t. a filtration $\{\mathcal{F}_t\}, B(t)$ is an m-dim $\mathcal{F}_t$-Brownian motion. Define

$$\bar{J}(v,u) = E[\int_0^\infty e^{-\rho t}(\gamma\mu_0(\bar{x}(t)) - \bar{L}_r(\bar{x}(t), v(t)) - \frac{1}{2}u(t)\cdot E^{(\gamma)^{-1}}u(t))dt].$$

The goal is to maximize $\bar{J}(v,u)$ over all processes (v,u) such that u and v are bounded. Since ∇W is bounded and $b(\cdot)$ is stable, we can prove by a standard argument that W is the value function of this control problem. Since the drift of the dynamics is linear and the running cost is concave in $(\bar{x}, v, u)$, W is concave. See Fleming and Rishel(1975), p.196. This completes the proof.

Now we can state our main result of this section.

Theorem 3.10. *Let $(\Lambda_r^{(\gamma)}, W_r^{(\gamma)}), (\Lambda^{(\gamma)}, W^{(\gamma)})$ be as in Lemma 3.8. Then $\Lambda_r^{(\gamma)}$ converges to $\Lambda^{(\gamma)}, W_r^{(\gamma)}$ converges to $W^{(\gamma)}$ uniformly on compact sets as $r \to \infty$.*

Proof. By Lemma 3.8 and equation (3.12), $\nabla W_r^{(\gamma)}$ is bounded in r uniformly on compact sets. Therefore, we may consider a limit of $W_r^{(\gamma)}$ through a sequence $r = r_n \to \infty$, denoted as (Λ, W). By Lemma 3.9, W is concave and (Λ, W) is a solution of (3.2). By Theorem 3.7, $(\Lambda, W) = (\Lambda^{(\gamma)}, W^{(\gamma)})$. This completes the proof.

Remark 3.11. By the convexity of $W_r^{(\gamma)} - W^{(\gamma)}$ and the convergence of $W_r^{(\gamma)} - W^{(\gamma)}$ as $r \to \infty$, it is not difficult to show that $\nabla W_r^{(\gamma)} - \nabla W^{(\gamma)}$ converges to 0 uniformly on compact sets as $r \to \infty$. Let denote $u^{(\gamma)}(x)$ the *argmin* in (3.1) with $U = R^N, \Lambda = \Lambda^{(\gamma)}, W = W^{(\gamma)}$. Similarly, $u_r^{(\gamma)}(x)$ is the *argmin* in (3.1) with $U = U_r, \Lambda = \Lambda_r^{(\gamma)}, W = W_r^{(\gamma)}$. Using the above result, we can also show that $u_r^{(\gamma)}$ converges to $u^{(\gamma)}$ uniformly on compact sets as $r \to \infty$.

Theorem 3.12. *If $\gamma < 0$ and $-\gamma$ is small, then the Markovian investment policy $u^{(\gamma)}(x)$ defined by*

$$u^{(\gamma)}(x) = \frac{1}{1-\gamma} g^{-2}(\bar{\mu}(x) + \sigma^{(D)} \nabla W^{(\gamma)}(x))$$

attains the optimal exponential growth rate $\Lambda^{(\gamma)}$.

Proof. The following idea has been used in Fleming and Sheu (1999). Denote $x^*(t) = x^u(t)$ defined by (2.11) with $u(t) = u^{(\gamma)}(x^*(t))$. Since $\Lambda = \Lambda^{(\gamma)}, W = W^{(\gamma)}$ satisfy (3.2) which is equivalent to (3.1) with $U = R^N$, the equation can be rewritten as

$$\begin{aligned} \Lambda^{(\gamma)} =& \frac{1}{2} \Delta W^{(\gamma)}(x) + \frac{1}{2} |\nabla W^{(\gamma)}(x)|^2 + b(x) \cdot \nabla W^{(\gamma)}(x) \\ &+ \gamma \sum u_i^{(\gamma)}(x) \sigma_D^{(i)} \cdot \nabla W^{(\gamma)}(x) + \gamma \ell^{(\gamma)}(x, u^{(\gamma)}(x)). \end{aligned}$$

By applying Ito's differential rule to $W^{(\gamma)}(x^*(t))$ and using the above equation, we have

$$\begin{aligned} &\int_0^T \gamma \ell^{(\gamma)}(x, u^{(\gamma)}(x^*(t))) dt \\ =& \Lambda^{(\gamma)} T - W^{(\gamma)}(x^*(T)) + W^{(\gamma)}(x^*(0)) + \int_0^T \nabla W^{(\gamma)}(x^*(t)) \cdot dB(t) \\ &- \frac{1}{2} \int_0^T |\nabla W^{(\gamma)}(x^*(t))|^2 dt. \end{aligned} \tag{3.28}$$

Let $V(t)$ be the investor's wealth at time t using the investment policy $u^{(\gamma)}(\cdot)$. Then, by (2.9) and (3.28),

$$\begin{aligned} E[V(T)^\gamma] =& \exp(\Lambda^{(\gamma)} T + W^{(\gamma)}(x)) E_x[\exp(-W^{(\gamma)}(x^*(T))) \\ &\cdot \exp(\int_0^T \nabla W^{(\gamma)}(x^*(t)) \cdot dB(t) - \frac{1}{2} \int_0^T |\nabla W^{(\gamma)}(x^*(t))|^2 dt)]. \end{aligned}$$

Now change the probability measures from P to $\hat{P}$, where on $\mathcal{F}_T$

$$\frac{d\hat{P}}{dP} = \exp(\int_0^T \nabla W^{(\gamma)}(x^*(t)) \cdot dB(t) - \frac{1}{2} \int_0^T |\nabla W^{(\gamma)}(x^*(t))|^2 dt).$$

Denote $\hat{E}[\cdots]$ the expectation under $\hat{P}$. Then

$$E[V(T)^\gamma] = \exp(\Lambda^{(\gamma)} T + W^{(\gamma)}(x)) \hat{E}_x[\exp(-W^{(\gamma)}(x^*(T)))]. \tag{3.29}$$

Under $\hat{P}$, $x^*(t)$ satisfies the equation

$$dx^*(t) = b^*(x^*(t))dt + d\hat{B}(t),$$

where $\hat{B}(t)$ is a Brownian motion under $\hat{P}$. See Lemma 3.2 with $W = W^{(\gamma)}$. We shall prove later that

$$0 \le -K^{(\gamma)} \le c|\gamma| I \tag{3.30}$$

for some $c > 0$ and small $|\gamma|$, where I is the identity matrix. Using this and the argument in the proof of Lemma 3.2, we can show that there is $c_1 > 0$, independent of γ if $|\gamma|$ is small, and for all $\alpha > 0$ there is $c_2 > 0$ such that we have

$$\hat{E}_x[\exp(c_1|x^*(T)|^2)] \le c_2 + \exp(-\alpha T)\exp(c_1|x|^2).$$

Using this and (3.30), we can deduce the following,

$$\lim_{T\to\infty} \frac{1}{T} \log \hat{E}_x[\exp(-W^{(\gamma)}(x^*(T)))] = 0.$$

By (3.29), this shows that $\Lambda^{(\gamma)}$ is the exponential growth rate using the policy $u^{(\gamma)}(\cdot)$. The proof is complete.

We now show (3.30). Recall that $K = K^{(\gamma)}$ satisfies (2.22). We use the following relation. For any C, a $m \times m$ matrix, we have

$$K^{(\gamma)} E^{(\gamma)} K^{(\gamma)} \ge -C'(E^{(\gamma)})^{-1} C + C' K^{(\gamma)} + K^{(\gamma)} C.$$

Take

$$C = -\frac{\gamma}{1-\gamma} \sigma^{(D)} g^{-2} \bar{A}.$$

Then (2.20) implies

$$KD + D'K - C'(E^{(\gamma)})^{-1} C + Q^{(\gamma)} \le 0.$$

Let $\phi(t)$ be the solution of

$$\frac{d\phi(t)}{dt} = D\phi(t),\ \ \phi(0) = x.$$

Then

$$\frac{d}{dt} < K\phi(t), \phi(t) > \leq < (C'(E^{(\gamma)})^{-1}C - Q^{(\gamma)})\phi(t), \phi(t) > .$$

That is,

$$< K\phi(T), \phi(T) > - < Kx, x > \leq \int_0^T < (C'(E^{(\gamma)})^{-1}C - Q^{(\gamma)})\phi(t), \phi(t) > dt.$$

Let $T \to \infty$, and use the property that $|\phi(T)| \leq |x| \exp(-c_0 T)$ which is a consequence of (2.4). Then

$$- < Kx, x > \leq \int_0^\infty < (C'(E^{(\gamma)})^{-1}C - Q^{(\gamma)})\phi(t), \phi(t) > dt.$$

Since we have $|C| \leq c|\gamma|, \ |Q^{(\gamma)}| \leq c|\gamma|$ for some $c > 0$, then (3.30) follows easily.

Remark 3.13. In the proof of Theorem 3.12, the diffusion $x^*(t)$ is Gaussian and has the invariant measure which is Gaussian with covariance matrix V,

$$V = \int_0^\infty e^{(D^{(\gamma)} + E^{(\gamma)}K^{(\gamma)})t} e^{(D^{(\gamma)} + E^{(\gamma)}K^{(\gamma)})'t} dt.$$

We note that V also satisfies the equation

$$(D^{(\gamma)} + E^{(\gamma)}K^{(\gamma)})V + V(D^{(\gamma)} + E^{(\gamma)}K^{(\gamma)})' = -I.$$

It is not difficult to show that

$$\lim_{T\to\infty} \frac{1}{T} \log \hat{E}_x[\exp(-W^{(\gamma)}(x^*(T)))] = 0$$

if and only if

$$-K^{(\gamma)} \leq V^{-1}.$$

It is very interesting to see when this holds. See Kuroda and Nagai(2000) for some interesting ideas relating to this.

4. Positive HARA Parameter.

In this section, we consider γ, $0 < \gamma < 1$. We continue to study the equation (2.18) for such γ and its relation to the optimal growth rate of the corresponding long term investment problem. Let denote $\Lambda_r^{(\gamma)}$ the optimal growth rate for long term investment problem with constraint $U = U_r$. Then $\Lambda_r^{(\gamma)}$ is finite for each $\gamma > 0$ and there is a unique $W_r^{(\gamma)}$ such that $(\Lambda, W) = (\Lambda_r^{(\gamma)}, W_r^{(\gamma)})$ satisfies

$$\begin{aligned}\Lambda = &\frac{1}{2}\Delta W(x) + \frac{1}{2}|\nabla W(x)|^2 + b(x) \cdot \nabla W(x) \\ &+ \max_{|u| \le r}[\gamma \sum u_i \sigma_D^{(i)} \cdot \nabla W(x) + \gamma \ell^{(\gamma)}(x, u)],\end{aligned} \tag{4.1}$$

and $W_r^{(\gamma)}(0) = 0, |\nabla W_r^{(\gamma)}|$ is bounded. Here

$$\ell^{(\gamma)}(x, u) = -\frac{1}{2}(1 - \gamma)|\sum u_i \sigma^{(i)}|^2 + \sum u_i \bar{\mu}_i(x) + \mu_0(x).$$

For the notations, see Section 2. We define

$$\Lambda^{(\gamma)} = \sup_{r > 0} \Lambda_r^{(\gamma)},$$

and call it the optimal growth rate of the long term investment problem.

Theorem 4.1. *Assume $\Lambda^{(\gamma)}$ is finite. Then (2.18) has a solution (Λ, W) such that $\Lambda = \Lambda^{(\gamma)}$ and $W(x)$ is convex.*

Proof. As in Theorem 3.1, for each $\gamma > 0$ there is unique $W_r^{(\gamma)}$ in $C^2(R^m)$ such that $(\Lambda, W) = (\Lambda_r^{(\gamma)}, W_r^{(\gamma)})$ satisfies (4.1), the properties that $W_r^{(\gamma)}(0) = 0$ and $\nabla W_r^{(\gamma)}$ is bounded. Equation (4.1) is the dynamic programming equation for an average unit time control problem with state dynamics

$$d\bar{x}(t) = (b(\bar{x}(t)) + \gamma \sum u_i(t) \sigma_D^{(i)} + v(t))dt + dB(t),$$

and the cost criterion

$$\bar{J}(u, v) = \limsup_{T \to \infty} \frac{1}{T} E[\int_0^T (\gamma \ell^{(\gamma)}(\bar{x}(t), u(t)) - \frac{1}{2}|v(t)|^2)dt].$$

Since the dynamics is linear in x, u, v and the running cost is convex in x, by a routine argument, we can show that $W_r^{(\gamma)}$ is convex.

By the equation (4.1) and convexity of $W_r^{(\gamma)}$, we can prove that $\nabla W_r^{(\gamma)}$ is bounded on compact sets uniformly in r. We can take a subsequence $r = r_n \to \infty$ such that $W_{r_n}^{(\gamma)}$ converges uniformly on compact sets to W as $n \to \infty$. Then $(\Lambda, W), \Lambda = \Lambda^{(\gamma)}$, satisfies (2.18) and W is convex. This completes the proof.

Let $\Lambda^{(\gamma)} < \infty$ and (Λ, W) be the solution of (2.18) in Theorem 4.1. We can rewrite the equation (2.18) as

$$\begin{aligned} \Lambda =& \frac{1}{2}\Delta W(x) + b^{(\gamma)}(x) \cdot \nabla W(x) + \frac{1}{2}\nabla W(x) \cdot E^{(\gamma)} \nabla W(x) \\ &+ \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{\mu}(x)|^2 + \gamma\mu_0(x), \end{aligned} \tag{4.2}$$

where

$$b^{(\gamma)}(x) = b(x) + \frac{\gamma}{1-\gamma}\sigma^{(D)'} g^{-2}\bar{\mu}(x) = D^{(\gamma)}x + a^{(\gamma)},$$

with

$$\begin{aligned} D^{(\gamma)} &= D + \frac{\gamma}{1-\gamma}\sigma^{(D)'} g^{-2}\bar{A}, \\ a^{(\gamma)} &= \frac{\gamma}{1-\gamma}\sigma^{(D)'} g^{-2}\bar{a}. \end{aligned}$$

Lemma 4.2. *Let $0 < \gamma < 1$. Assume $\Lambda^{(\gamma)} < \infty$. Then $D^{(\gamma)}$ is a stable matrix.*

Proof. Let $z(t)$ be the diffusion process defined by

$$dz(t) = b^{(\gamma)}(z(t))dt + dB(t).$$

It is enough to prove that there are $c, \alpha > 0$ such that

$$E_x[|z(t)|^2 \le c[|x|^2 e^{-\alpha t} + 1] \tag{4.3}$$

for all $x \in R^m, t \ge 0$.

Let (Λ, W) be the solution of (4.2) given in Theorem 4.1. By Ito's rule,

$$\begin{aligned} dW(z(t)) =&(\Lambda - \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{\mu}|^2(z(t)) - \gamma\mu_0(z(t)) \\ &- \frac{1}{2}\nabla W(z(t)) \cdot E^{(\gamma)}\nabla W(z(t)))dt + \nabla W(z(t)) \cdot dB(t). \end{aligned}$$

Then considering $W(z(t))e^{\alpha t}$ for $\alpha > 0$ to be determined later, we have

$$\begin{aligned} E_x[W(z(T))]e^{\alpha T} =&W(x) + E_x[\int_0^T e^{\alpha t}(\Lambda - \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{\mu}(z(t))|^2 - \gamma\mu_0(z(t)) \\ &- \frac{1}{2}\nabla W(z(t)) \cdot E^{(\gamma)}\nabla W(z(t)) + \alpha W(z(t)))dt]. \end{aligned} \tag{4.4}$$

Also

$$d|z(t)|^2 = (2z(t) \cdot b(z(t)) + 2\frac{\gamma}{1-\gamma}z(t) \cdot \sigma^{(D)'}g^{-2}\bar{\mu}(z(t)) + m)dt + 2z(t) \cdot dB(t),$$

$$\begin{aligned} E_x[|z(T)|^2]e^{\alpha T} \leq&|x|^2 + E_x[\int_0^T e^{\alpha t}((-2c_0 + \alpha)|z(t)|^2 \\ &+ c_0|z(t)|^2 + \frac{1}{c_0}(\frac{\gamma}{1-\gamma})^2\|\sigma^{(D)'}g^{-1}\|^2|g^{-1}\bar{\mu}(z(t))|^2 + m)dt]. \end{aligned} \tag{4.5}$$

Here we use (2.4) and

$$\begin{aligned} &2\frac{\gamma}{1-\gamma}|z(t) \cdot \sigma^{(D)'}g^{-2}\bar{\mu}(z(t))| \\ \leq&c_0|z(t)|^2 + \frac{1}{c_0}(\frac{\gamma}{1-\gamma})^2|\sigma^{(D)'}g^{-2}\bar{\mu}(z(t))|^2 \\ \leq&c_0|z(t)|^2 + \frac{1}{c_0}(\frac{\gamma}{1-\gamma})^2\|\sigma^{(D)'}g^{-1}\|^2|g^{-1}\bar{\mu}((t))|^2. \end{aligned}$$

Taking $\alpha < \frac{1}{2}c_0, c = 2\frac{1}{c_0}\frac{\gamma}{1-\gamma}\|\sigma^{(D)'}g^{-1}\|^2$ and using (4.4) (4.5),

$$\begin{aligned} E_x[|(z(T)|^2 + cW(z(T))]e^{\alpha T} \leq&(|x|^2 + cW(x)) + E_x[\int_0^T e^{\alpha t}(-\frac{1}{2}c_0|z(t)|^2 \\ &- c\gamma\mu_0(z(t)) + c\alpha W(z(t)) + m + c\Lambda)dt] \\ \leq&(|x|^2 + cW(x)) + \widetilde{c}e^{\alpha T}, \end{aligned} \tag{4.6}$$

if α is small enough . Here we use, $|W(x)| \leq c_1(1 + |x|^2)$ by (4.2). The convexity of $W(x)$ implies,

$$W(x) \geq -c_2(1 + |x|).$$

These properties and (4.6) imply (4.3). This completes the proof.

In Section 2, we have seen that if W is quadratic,

$$W(x) = \frac{1}{2}Kx \cdot x + e \cdot x,$$

then K satisfies (2.21), i.e.,

$$D'K + KD + K^2 + \frac{\gamma}{1-\gamma}(\bar{A}' + K\sigma^{(D)'})g^{-2}(\bar{A} + \sigma^{(D)}K) = 0 \tag{4.7}$$

holds. Although, we expect W to be quadratic for the solution (Λ, W) of (4.2) in Theorem 4.1, we could not prove this here. However, we shall prove that (4.7) has a solution K which is nonnegative.

Lemma 4.3. *Assume $0 < \gamma < 1$ and $\Lambda^{(\gamma)}$ is finite. Then (4.7) has a unique solution $K^{(\gamma)}$ such that $K^{(\gamma)}$ is nonnegative definite and $D^{(\gamma)} + E^{(\gamma)}K^{(\gamma)}$ is semistable.*

Proof. Let (Λ, W) be the solution of (2.18) in Theorem 4.1. For $\lambda > 0$, consider

$$\bar{W}_\lambda(x) = \frac{1}{\lambda^2}W(\lambda x).$$

Since $|\nabla W(x)| \leq c(1 + |x|)$, then

$$|\nabla \bar{W}_\lambda(x)| \leq c\frac{1}{\lambda}(1 + |\lambda x|).$$

Therefore, $\bar{W}_\lambda(\cdot), \lambda \geq 1$, is a compact family of functions. We choose a sequence $\lambda_n \to \infty$ such that $\bar{W}_{\lambda_n}$ converges uniformly on compact sets as $n \to \infty$, and we denote $\bar{W}(\cdot)$ for the limit. Then $\bar{W}(\cdot)$ has the following properties:

$$\begin{aligned} &(i) \quad \bar{W} \text{ is convex;} \\ &(ii) \quad |\nabla \bar{W}(x)| \leq c_1|x|; \\ &(iii) \quad 0 \leq \bar{W}(x) \leq c_2|x|^2. \end{aligned} \tag{4.8}$$

Moreover, $\bar{W}$ is a viscosity solution of the following equation:

$$D^{(\gamma)}x \cdot \nabla \bar{W}(x) + \frac{1}{2}\nabla \bar{W}(x) \cdot E^{(\gamma)} \nabla \bar{W}(x) + \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{A}x|^2 = 0. \tag{4.9}$$

That is, for any $x \in R^m, T > 0$,

$$\bar{W}(x) = \sup_v \{\int_0^T (\frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{A}\phi(t)|^2 - \frac{1}{2}v(t) \cdot E^{(\gamma)^{-1}} v(t))dt + \bar{W}(\phi(T))\}, \tag{4.10}$$

where ϕ satisfies

$$\frac{d\phi}{dt} = D^{(\gamma)}\phi(t) + v(t), \phi(0) = x \tag{4.11}$$

and

$$\int_0^T |v(t)|^2 dt < \infty.$$

See McEneaney(1995). Clearly, (4.10) implies a dissipation inequality which has appeared in systems theory,

$$\int_0^T (\frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{A}\phi(t)|^2 - \frac{1}{2}v(t) \cdot E^{(\gamma)^{-1}} v(t))dt \leq \bar{W}(x) - \bar{W}(\phi(T))$$

for ϕ satisfying (4.11). Then results in Willems(1971) can be applied to assert the existence of a quadratic solution $W^+(x)$ of (4.9),

$$W^+(x) = \frac{1}{2}K^{(\gamma)}x \cdot x, \tag{4.12}$$

$K^{(\gamma)} \geq 0$ and

$$D^{(\gamma)^*} = D^{(\gamma)} + E^{(\gamma)}K^{(\gamma)} \tag{4.13}$$

is a semistable matrix (i.e. the real part of the eigenvalues are nonpositive). See Lemma 5, Theorem 7 in Willems(1971). This completes the proof.

Remark 4.4. In Theorem 4.8, we show that $\bar{W}$ is equal to W^+ given in (4.12). It is also important to know if $D^{(\gamma)*}$ in (4.13) is a stable matrix. We shall prove these later if γ is small.

Theorem 4.5. *Let $0 < \gamma < 1$. Assume (4.7) has a solution $K^{(\gamma)} \geq 0$ such that $D^{(\gamma)^*}$ defined in (4.13) is a stable matrix. Define $e^{(\gamma)}, \Lambda^{(\gamma)}$ by (2.24) with $K = K^{(\gamma)}$ and*

$$W^{(\gamma)}(x) = \frac{1}{2}K^{(\gamma)}x \cdot x + e^{(\gamma)} \cdot x.$$

Then the optimal growth rate for the investment problem is finite and is equal to $\Lambda^{(\gamma)}$. Moreover, $(\Lambda^{(\gamma)}, W^{(\gamma)})$ is the solution of (2.18) given in Theorem 4.1. In particular, we have $W_r^{(\gamma)}$ converges to $W^{(\gamma)}$ uniformly on compact sets as $r \to \infty$.

Proof. First, we show that $\Lambda_r^{(\gamma)} \leq \Lambda^{(\gamma)}$ for each $r > 0$, therefore, $\Lambda \leq \Lambda^{(\gamma)}$ with Λ being the optimal growth rate. Then by Theorem 4.1, there exists W, a convex function, such that (Λ, W) satisfies (2.18) and

$$|\nabla W(x)| \leq c(1 + |x|).$$

As mentioned in the beginning of this section, there is unique $W_r^{(\gamma)}$ such that (4.1) holds. Then by a standard argument (see Fleming and McEneaney(1995) that

$$\Lambda_r^{(\gamma)} = \sup_{u,v} \bar{J}(u, v)$$

where the *sup* is taken through stochastic processes $u(\cdot), v(\cdot)$ such that they are progressively measurable with respect to a filtration $\{\mathcal{F}_t\}$ and $|u(t)| \leq r$, $v(t)$ is bounded, where

$$\bar{J}(u, v) = \limsup_{T\to\infty} \frac{1}{T}E[\int_0^T (\gamma\ell^{(\gamma)}(\bar{x}(t), u(t)) - \frac{1}{2}|v(t)|^2)dt],$$

$$d\bar{x}(t) = (b(\bar{x}(t)) + \gamma \sum u_i(t)\sigma_D^{(i)} + v(t))dt + dB(t),$$

$\bar{x}(\cdot)$ is progressively measurable with respect to $\{\mathcal{F}_t\}$ and $B(\cdot)$ is a Brownian motion with respect to $\{\mathcal{F}_t\}$.

Since $(\Lambda^{(\gamma)}, W^{(\gamma)})$ also satisfies (2.18), we have

$$\begin{aligned} \Lambda^{(\gamma)} \geq &\frac{1}{2}\Delta W^{(\gamma)}(x) + b(x) \cdot \nabla W^{(\gamma)}(x) + v \cdot \nabla W^{(\gamma)}(x) - \frac{1}{2}|v|^2 \\ &+ \gamma \sum u_i \sigma_D^{(i)} \cdot \nabla W^{(\gamma)}(x) + \gamma\ell^{(\gamma)}(x, u) \end{aligned} \tag{4.14}$$

for all x, u and v. Let $u(\cdot), v(\cdot)$ be progressively measurable with respect to $\{\mathcal{F}_t\}$ such that $|u(t)| \leq r$, $v(t)$ is bounded and $\bar{x}(\cdot)$ satisfy the above equation. Then by Ito's rule and the relation (4.14),

$$\begin{aligned} dW^{(\gamma)}(\bar{x}(t)) \leq & (\Lambda^{(\gamma)} - \gamma\ell^{(\gamma)}(\bar{x}(t), u(t)) + \frac{1}{2}|v(t)|^2)dt \\ & + \nabla W^{(\gamma)}(\bar{x}(t)) \cdot dB(t). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^T (\gamma\ell^{(\gamma)}(\bar{x}(t), u(t)) - \frac{1}{2}|v(t)|^2)dt \leq & \Lambda^{(\gamma)}T + \int_0^T \nabla W^{(\gamma)}(\bar{x}(t)) \cdot dB(t) \\ & + W^{(\gamma)}(\bar{x}(0)) - W^{(\gamma)}(\bar{x}(T)), \end{aligned}$$

then

$$\begin{aligned} & E_x[\int_0^T (\gamma\ell^{(\gamma)}(\bar{x}(t), u(t)) - \frac{1}{2}|v(t)|^2)dt] \\ \leq & \Lambda^{(\gamma)}T + E_x[-W^{(\gamma)}(\bar{x}(T))] + W^{(\gamma)}(x). \end{aligned} \tag{4.15}$$

Since u, v are bounded, by using the condition (2.4), it is routine to prove that there are $\alpha, \beta, c > 0$, such that

$$E_x[\exp(\beta|\bar{x}(t)|^2)] \leq e^{-\alpha t} e^{\beta|x|^2} + c.$$

Together with (4.15) we can prove $\Lambda_r^{(\gamma)} \leq \Lambda^{(\gamma)}$, hence $\Lambda \leq \Lambda^{(\gamma)}$.

Now we prove $\Lambda = \Lambda^{(\gamma)}$ and $W = W^{(\gamma)}$. We substract the equations for $(\Lambda^{(\gamma)}, W^{(\gamma)})$ and (Λ, W),

$$\begin{aligned} \Lambda^{(\gamma)} - \Lambda = & \frac{1}{2}\Delta(W^{(\gamma)} - W) + (b^{(\gamma)} + E^{(\gamma)}\nabla W^{(\gamma)}) \cdot \nabla(W^{(\gamma)} - W) \\ & - \frac{1}{2}\nabla(W^{(\gamma)} - W) \cdot E^{(\gamma)}\nabla(W^{(\gamma)} - W). \end{aligned} \tag{4.16}$$

Let $x^*(t)$ be the diffusion defined by

$$dx^*(t) = (b^{(\gamma)} + E^{(\gamma)}\nabla W^{(\gamma)})(x^*(t))dt + dB(t).$$

By the condition that $D^{(\gamma)*}$ is stable, $x^*(t)$ is ergodic and has unique invariant probability density $p^*(x)$. Then (4.16) implies

$$\Lambda^{(\gamma)} - \Lambda + \int \frac{1}{2}\nabla(W^{(\gamma)} - W)(y) \cdot E^{(\gamma)}\nabla(W^{(\gamma)} - W)(y)p^*(y) = 0.$$

Therefore, $\Lambda^{(\gamma)} - \Lambda = 0$ and $\nabla(W^{(\gamma)} - W) = 0$ a.e. Then $W^{(\gamma)} - W$ is a constant and is identical to $W^{(\gamma)}(0) - W(0) = 0$. This completes the proof.

Theorem 4.6. *If $0 < \gamma < 1$ and γ is small enough, then (4.7) has a unique solution $K^{(\gamma)}$ satisfying*

$$0 \le K^{(\gamma)} \le c\gamma I$$

for some $c > 0$, where I is the identity matrix. Therefore, $D^{(\gamma)*}$ defined in (4.13) is a stable matrix. $\Lambda^{(\gamma)}$ defined by (2.24) with $K = K^{(\gamma)}$ is the optimal growth rate for the investment problem.

Proof. By Theorem 4.5, it is enough to show the existence of $K^{(\gamma)}$ satisfying the required properties. We first show that there is $c > 0$ such that $W_0(x) = c\gamma|x|^2$ satisfies

$$\int_0^T (\frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{A}\phi(t)|^2 - \frac{1}{2}v(t) \cdot E^{(\gamma)-1}v(t))dt \le W_0(x) - W_0(\phi(T)) \tag{4.17}$$

if ϕ satisfies (4.11) and $\phi(0) = x$. In fact, by (4.11),

$$\frac{d}{dt}|\phi(t)|^2 = 2D\phi(t)\cdot\phi(t) + 2\frac{\gamma}{1-\gamma}\sigma^{(D)'}g^{-2}\bar{A}\phi(t)\cdot\phi(t) + 2\phi(t)\cdot v(t).$$

Then using $Dx \cdot x \le -c_0|x|^2$, we have

$$\begin{aligned}\frac{d}{dt}|\phi(t)|^2 &\le -(c_0 - c_1\gamma)|\phi(t)|^2 + \frac{1}{c_0}|v(t)|^2\\ &\le -\frac{1}{2}c_0|\phi(t)|^2 + \frac{1}{c_0}|v(t)|^2\end{aligned}$$

if $c_0 - c_1\gamma \ge \frac{1}{2}c_0$, i.e. $c_1\gamma \le \frac{1}{2}c_0$. Therefore, for some $c_2 > 0$,

$$\begin{aligned}\int_0^T \frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{A}\phi(t)|^2 dt \le & c_2\gamma\int_0^T |\phi(t)|^2 dt\\ &\le \frac{2c_2}{c_0}\gamma(\frac{1}{c_0}\int_0^T |v(t)|^2 dt + |\phi(0)|^2 - |\phi(T)|^2),\end{aligned}$$

$$\int_0^T (\frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{A}\phi(t)|^2 - \frac{1}{2}v(t)\cdot E^{(\gamma)-1}v(t))dt \le W_0(x) - W_0(\phi(T))$$

if γ small enough and $W_0(x) = c\gamma|x|^2$ with $c = \frac{2c_2}{c_0}$. By Theorem 7 in Willems(1971),

$$V^+(x) = \frac{1}{2}K^{(\gamma)}x\cdot x,$$

for some $K^{(\gamma)} \ge 0$ satisfying (4.7), where

$$V^+(x) = \sup \int_0^\infty (\frac{1}{2}\frac{\gamma}{1-\gamma}|g^{-1}\bar{A}\phi(t)|^2 - \frac{1}{2}v(t)\cdot E^{(\gamma)-1}v(t))dt,$$

where the sup is taken over $\phi(t)$ satisfying (4.11) such that $\phi(t) \to 0$ as $t \to \infty$. By (4.17), $V^+(x) \le W_0(x)$. Therefore,

$$0 \le K^{(\gamma)} \le c\gamma I,$$

This completes the proof.

Let fix $\gamma > 0$ and assume that $\Lambda^{(\gamma)}$ is finite. We recall $K^{(\gamma)}$ a particular solution of (4.7) defined in Lemma 4.3. For each $r > 0$, $\Lambda^{(\gamma)}, W_r^{(\gamma)}$ is the solution of (4.1) mentioned before.

Lemma 4.7. *For each $r > 0$, $W_r^{(\gamma)}(x) - \frac{1}{2}K^{(\gamma)}x\cdot x$ is a concave function. In particular, $W(x) - \frac{1}{2}K^{(\gamma)}x\cdot x$ is a concave function, where W is given in Theorem 4.1.*

Proof. The argument is similar to that used in the proof of Lemma 3.8. We shall sketch it.

Fix γ and $r > 0$ and denote $\bar{W}(x) = W_r^{(\gamma)}(x) - \frac{1}{2}K^{(\gamma)}x\cdot x$. Then

$$\begin{aligned}\Lambda^{(\gamma)} =& \frac{1}{2}\bar{W}(x) + \frac{1}{2}trK^{(\gamma)} + b(x)\cdot(\nabla\bar{W}(x) + K^{(\gamma)}x) + \frac{1}{2}|\nabla\bar{W}(x) + K^{(\gamma)}x|^2 \\ &+ \sup_{|x|\le r}\{\gamma\sigma^{(D)'}u\cdot(\nabla\bar{W}(x) + K^{(\gamma)}x) + \gamma\ell^{(\gamma)}(x,u)\} \\ =& \frac{1}{2}\bar{W}(x) + \frac{1}{2}trK^{(\gamma)} + (b(x) + K^{(\gamma)}x)\cdot\nabla\bar{W}(x) + \frac{1}{2}|\nabla\bar{W}(x)|^2 \\ &+ \sup_{|x|\le r}\{\gamma\sigma^{(D)'}u\cdot\nabla\bar{W}(x) + \bar{L}(x,u)\},\end{aligned}$$

where

$$\bar{L}(x,u) = -\frac{1}{2}\gamma(1-\gamma)|gu - \frac{1}{1-\gamma}g^{-1}(\bar{A}+\sigma^{(D)}K^{(\gamma)})x|^2 + \gamma\bar{a}\cdot u + \gamma\mu_0(x).$$

In the derivation, we use the equation (4.7) for $K = K^{(\gamma)}$. Therefore, we can interpret the above equation as the DPE for a control problem which the running cost $\bar{L}(x,u) - \frac{1}{2}|v|^2$ is concave in (x,u,v) and the dynamics is linear. Then a standard argument gives the concavity of $\bar{W}$. The proof is complete.

Theorem 4.8. *Let W be the function given in Theorem 4.1. Then $W(\lambda x)/\lambda^2$ converges to $\frac{1}{2}K^{(\gamma)}x\cdot x$ uniformly on compact sets as $\lambda\to\infty$. Also, $\nabla W(\lambda x)/\lambda$ converges to $K^{(\gamma)}x$ uniformly on compact sets as $\lambda\to\infty$.*

Proof. As in the proof of Lemma 4.3, denote $\bar{W}$ a limit of $W(\lambda x)/\lambda^2$ along a sequence $\lambda=\lambda_n$ and $\lambda_n\to\infty$. Then (4.10) holds. It implies $\frac{1}{2}K^{(\gamma)}x\cdot x \le \bar{W}(x)$ since

$$\frac{1}{2}K^{(\gamma)}x\cdot x = \sup_v\{\int_0^\infty (\frac{\gamma}{2(1-\gamma)}|g^{-1}\bar{A}\phi(t)|^2 - \frac{1}{2}v(t)\cdot E^{(\gamma)}v(t))dt\}$$

where $\phi(t)$ satisfies (4.11). The *sup* is taken over all $v(\cdot)$ such that $\phi(t)\to 0$ as $t\to\infty$.

On the other hand, Lemma 4.7 implies $\bar{W}(x)\le\frac{1}{2}K^{(\gamma)}x\cdot x$ for all x. Therefore, $\bar{W}(x) = \frac{1}{2}K^{(\gamma)}x\cdot x$ for all x. Thus, we have proved that $\frac{1}{2}K^{(\gamma)}x\cdot x$ is the unique limit of $W(\lambda x)/\lambda^2$, $\lambda\to\infty$. This implies that $W(\lambda x)/\lambda^2$ converges to $\frac{1}{2}K^{(\gamma)}x\cdot x$ as $\lambda\to\infty$. The convergence of $\nabla W(\lambda x)/\lambda$ to $K^{(\gamma)}x$ follows from this and the concavity of $W(\lambda x)/\lambda - \frac{1}{2}K^{(\gamma)}x\cdot x$. See Remark 3.11. This completes the proof.

In the rest, for $0<\gamma<1$, the optimal growth rate is denoted by $\Lambda^{(\gamma)}$ if it is finite.

Theorem 4.9. *Assume $\Lambda^{(\gamma)}<\infty$ for all $0<\gamma<1$. Then there is a nonnegative definite matrix $K^{(1)}$ such that*

$$\begin{aligned} &\bar{A} = -\sigma^{(D)}K^{(1)}, \\ &D'K^{(1)} + K^{(1)}D + (K^{(1)})^2 \le 0. \end{aligned} \tag{4.18}$$

Proof. For $0 < \gamma < 1$, by Lemma 4.3, (4.7) has a solution $K^{(\gamma)}$ such that $K^{(\gamma)} \geq 0$ and $D^{(\gamma)*}$ defined by (4.13) is semistable. See also Theorem 2.2. From (4.7),

$$D'K^{(\gamma)} + K^{(\gamma)}D + (K^{(\gamma)})^2 \leq 0. \tag{4.19}$$

This implies

$$\|K^{(\gamma)}\| \leq 2\|D\|.$$

We can take a sequence $\gamma_n \to 1$ such that $K^{(\gamma_n)} \to K^{(1)}$ as $n \to \infty$.

Again, from (4.7),

$$(\bar{A}' + K^{(\gamma)}\sigma^{(D)'})g^{-2}(\bar{A} + \sigma^{(D)}K^{(\gamma)}) = -\frac{1-\gamma}{\gamma}(D'K^{(\gamma)} + K^{(\gamma)}D + (K^{(\gamma)})^2).$$

The quantity on the right hand side tends to 0 as $\gamma \to 1$ by the boundedness of $K^{(\gamma)}$. In particular, taking $\gamma = \gamma_n$ and letting $n \to \infty$, we get

$$\bar{A} + \sigma^{(D)}K^{(1)} = 0.$$

Also, in (4.19), letting $\gamma = \gamma_n$ and $n \to \infty$, we have

$$D'K^{(1)} + K^{(1)}D + (K^{(1)})^2 \leq 0$$

The proof is complete.

Remark 4.10. Assume (4.7) has a solution $K^{(\gamma)} \geq 0$ such that (4.13) is a stable matrix for each $0 < \gamma < 1$. Define $\Lambda^{(\gamma)}, W^{(\gamma)}$ as in Theorem 4.5. Assume $\Lambda^{(\gamma)}$ is bounded in $0 < \gamma < 1$. Then by (2.18)

$$\bar{\mu}(x) + \sigma^{(D)}(K^{(\gamma)}x + e^{(\gamma)}) \to 0 \quad \text{as} \quad \gamma \to 1 \tag{4.20}$$

and $K^{(\gamma)}x + e^{(\gamma)}$ is bounded in γ for x in bounded sets. Therefore, we may take $\gamma = \gamma_n \to 1$ as $n \to \infty$ such that

$$K^{(\gamma_n)} \to K^{(1)}, \; e^{(\gamma_n)} \to e^{(1)} \quad \text{as } n \to \infty.$$

Then (4.20) implies

$$\bar{\mu}(x) + \sigma^{(D)}(K^{(1)}x + e^{(1)}) = 0,$$

i.e.,

$$\bar{A} = -\sigma^{(D)}K^{(1)},$$

$$\bar{a} = -\sigma^{(D)}e^{(1)}.$$

Conversely, under additional conditions we can show the boundedness of $\Lambda^{(\gamma)}$ as given in the following theorem.

Theorem 4.11. *Assume that there are K, a positive definite matrix, and e, a vector, such that*

$$\begin{aligned} &\sigma^{(D)}K + \bar{A} = 0 \\ &\sigma^{(D)}e + \bar{a} = 0, \end{aligned} \tag{4.21}$$

and

$$-Q = D'K + KD + K^2$$

is negative definite. Then $\Lambda^{(\gamma)}$ is finite for each $0 < \gamma < 1$. Moreover, $\Lambda^{(\gamma)},\ 0 < \gamma < 1$ is bounded.

Proof. Let $x^u(t)$ be a process satisfying (2.11) with $|u(t)|$ bounded by r. Denote

$$W(x) = \frac{1}{2}Kx \cdot x + e \cdot x.$$

By Ito's rule,

$$\begin{aligned} dW(x^u(t)) =&(\frac{1}{2}\Delta W(x^u(t)) + b(x^u(t)) \cdot \nabla W(x^u(t)) \\ &+ \gamma \sum u_i(t)\sigma_D^{(i)} \cdot \nabla W(x^u(t)))dt + \nabla W(x^u(t)) \cdot dB(t) \\ =&(\frac{1}{2}trK + Dx^u(t) \cdot (Kx^u(t) + e) + \gamma\sigma^{(D)'}u(t) \cdot (Kx^u(t) + e))dt \\ &+ \nabla W(x^u(t)) \cdot dB(t) \\ =&(\frac{1}{2}trK - \frac{1}{2}|Kx^u(t)|^2 - \frac{1}{2}Qx^u(t) \cdot x^u(t) + Dx^u(t) \cdot e \\ &- \gamma u(t) \cdot \bar{\mu}(x^u(t)))dt + \nabla W(x^u(t)) \cdot dB(t). \end{aligned}$$

Here we use (4.21) in the last step. Then

$$
\begin{aligned}
&\gamma\int_0^T u(t)\cdot\bar\mu(x^u(t))dt\\
=&\frac{1}{2}trKT+\int_0^T(-\frac{1}{2}Qx^u(t)\cdot x^u(t)+Kx^u(t)\cdot e+\frac{1}{2}|e|^2+Dx^u(t)\cdot e)dt\\
&\quad+\int_0^T\nabla W(x^u(t))\cdot dB(t)-\frac{1}{2}\int_0^T|\nabla W(x^u(t))|^2dt-W(x^u(T))+W(x^u(0)).
\end{aligned}
$$

We have

$$
\begin{aligned}
(4.22)\quad &E_x[\exp(\int_0^T\gamma\ell^{(\gamma)}(x^u(t),u(t))dt)]\\
&=\exp(\frac{1}{2}(trK+|e|^2)T+W(x))\cdot\bar E_x[\exp(-W(x^u(T))+\int_0^T(-\frac{1}{2}\gamma(1-\gamma)|gu(t)|^2\\
&\qquad-\frac{1}{2}Qx^u(t)\cdot x^u(t)+(Dx^u(t)+Kx^u(t))\cdot e)dt)].
\end{aligned}
$$

Here $\bar E_x[\cdots]$ is the expectation with respect to the probability measure $\bar P$,

$$
\frac{d\bar P}{dP}|_{\mathcal{F}_T}=\exp(\int_0^T\nabla W(x^u(t))(x^u(t))\cdot dB(t)-\frac{1}{2}\int_0^T|\nabla W(x^u(t))(x^u(t))|^2dt).
$$

Under $\bar P$,

$$
dx^u(t)=(b(x^u(t))+\nabla W(x^u(t))+\gamma\sum u_i(t)\sigma_D^{(i)})dt+d\bar B(t),
$$

$\bar B(t)$ is a Brownian motion. Since K is positive definite, there is $c>0$ such that $-W(y)\le c$ for all y, and

$$
-\frac{1}{2}\gamma(1-\gamma)|gu|^2-\frac{1}{2}Qy\cdot y+(Dy+Ky)\cdot e\le c.
$$

From (4.22), we have

$$
E_x[\exp(\int_0^T\gamma\ell^{(\gamma)}(x^u(t),u(t))dt)]\le\exp(c+W(x))\exp((c+\frac{1}{2}(trK+|e|^2))T).
$$

This implies,

$$
\Lambda_r^{(\gamma)}\le c+\frac{1}{2}(trK+|e|^2)
$$

for all $r>0$. Therefore,

$$
\Lambda^{(\gamma)}\le c+\frac{1}{2}(trK+|e|^2)
$$

for all γ. This completes the proof.

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