Risk Sensitive Portfolio Management With Cox-Ingersoll-Ross Interest Rates: the HJB Equation *

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Abstract

This paper presents an application of risk sensitive control theory in financial decision making. The investor has an infinite horizon objective that can be interpreted as maximizing the portfolio's risk adjusted exponential growth rate. There are two assets, a stock and a bank account, and two underlying Brownian motions, so this model is incomplete. The novel feature here is that the interest rate for the bank account is governed by Cox-Ingersoll-Ross dynamics. This is significant for risk sensitive portfolio management because the factor process, unlike in the Gaussian and all other cases treated in the literature, cannot be negative.

Keywords: risk sensitive control, optimal portfolios, CIR interest rates, incomplete model

AMS Subject Classification: 60J20, 90A09, 90C40, 93E20

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1 Introduction

Beginning with the pioneering work by Merton [20], [21], [22] and continuing through the recent books by Karatzas and Shreve [17] and Korn [18], some very sophisticated stochastic control methods have been developed for portfolio management. Virtually all of these studies make use of an expected utility criterion. But recently a new criterion has emerged from the control theory literature. Called the *risk sensitive* criterion, this was originally used (see, for example, Whittle [25]) for a decision maker seeking to maximize some (random) cash reward (or minimize some cash payment) while simultaneously being concerned about the risk or uncertainty in the size of the reward. Essentially, this criterion equals the expected value of the reward minus a penalty term that is proportional to the variance of the reward. The constant of proportionality is a parameter whose value can be chosen in order to achieve for the decision maker an appropriate trade-off between the expectation of the reward and its variance.

Recognizing its relevance to portfolio management, Bielecki and Pliska [5] applied the risk sensitive idea to a version of Merton's [21] intertemporal capital asset pricing model. The result was an infinite horizon criterion that they called the *risk adjusted growth rate* and viewed as being analogous to the classical Markowitz single-period approach except that instead of trading off single-period criteria the investor is trading off the portfolio's long run growth rate versus its average volatility (see Bielecki and Pliska [9] for a detailed study of various economic and mathematical properties of this criterion). Bielecki and Pliska also showed in [5] and subsequent work (see [2], [3], [4], [6], [7], [8], and [10]) that the resulting models usually have the virtue of being more tractable than corresponding models which use traditional expected utility criteria. Other studies of the risk sensitive criterion for portfolio management include Bagchi and Kumar [1], Fleming and Sheu [13], [14], [15], Kuroda and Nagai [19], Nagai [23], and Nagai and Peng [24]. Kaise and Sheu [16] discuss the solution of a general equation (in R^n) that is related to the HJB equation in this paper.

Throughout all this work on risk sensitive portfolio management the underlying factor process, if any, was taken to be Gaussian or, at least (see Nagai [23]), a process whose domain is all of some Euclidean space. The aim of this paper is to provide some initial results on risk sensitive portfolio management for a case where this kind of condition does not hold. Since interest rate processes are commonly taken as factor processes and since the so-called Cox-Ingersoll-Ross [11] interest rate process (a popular one in finance literature) cannot be negative, this model of the factor process was chosen for our object of study.

The result is a risk sensitive portfolio optimization model having a factor process whose domain is the non-negative portion of the real line. Since this is a model of interest rates, it is more realistic than, say, Gaussian models, but it comes with a price: the resulting analysis is exceptionally lengthy, complex, and technical. This is true even though our model is rather simple, having just this scalar-valued factor process, two assets (the usual bank account and a risky stock), and two underlying Brownian motions. Consequently, this paper will study only the associated Hamilton-Jacobi-Bellman equation, saving the verification of optimality and related issues for a separate paper.

After formulation of our model in Section 2, the main results are presented in Section 3. Chief among these is Theorem 3.1, which asserts the HJB equation has a unique solution. Needed for its proof and of separate interest are some results pertaining to a related, "truncated" problem: for some fixed number M the investor is required to keep all of his or her money in the bank account whenever the interest rate exceeds M. Existence of a unique solution to the HJB equation for this truncated problem is established by Theorem 3.2. Intuitively, one should expect the solution of the truncated HJB equation to converge to the solution of the original one as $M \to \infty$; this is indeed the case, as stated in Theorem 3.3. The rest of the paper is devoted to the proofs of these three theorems. Theorem 3.2 is proved in Section 4, whereas the other two are proved in Section 5.

2 Formulation of the Optimal Risk Sensitive Asset Management Problem

In this section we formulate an optimal dynamic asset management problem featuring a risk sensitive optimality criterion. Let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, \mathbf{P})$ be the underlying probability space. The securities market involves a single factor, namely, an interest rate r that is subject to the so-called Cox-Ingersoll-Ross [11] dynamics

$$dr_t = -c(r_t - \bar{r})dt + \lambda\sqrt{r_t}dW_t, \tag{1}$$

where $c, \bar{r}$, and λ are three specified positive scalar parameters. In order to ensure that the interest rate process is always strictly positive, we make the following (see Feller [12])

Assumption: $2c\bar{r} > \lambda^2$.

There are two assets. One is the customary bank account:

$$\frac{dS_0(t)}{S_0(t)} = r_t dt; \tag{2}$$

here $S_0(t)$ represents the time t amount of money in the bank account assuming none is added or withdrawn after time 0. The other asset is a stock (or stock index) whose price process satisfies

$$\frac{dS_1(t)}{S_1(t)} = \mu(r_t)dt + \sigma dW_t + \rho d\bar{W}_t. \tag{3}$$

Here W_t and $\bar{W}_t$ are two independent Brownian motions, σ and ρ are two specified scalar parameters, and

$$\mu(r) := \mu_1 + \mu_2 r, \tag{4}$$

where μ_1 and μ_2 are two specified scalar parameters. Note that with $\mu_2 \neq 0$ we can allow the level of interest rates to affect the return properties of the stock, and with $\sigma \neq 0$ the residuals of the interest rate process will be correlated with the residuals of the stock's return process. For instance, with suitable values of σ and ρ this correlation is negative.

Trading strategies will be adapted real-valued stochastic processes that are denoted h. We shall interpret h_t as the proportion of the investor's time-t wealth that is invested in the stock. In general, for each time-t we allow h_t to be any real number, that is, we do not impose any short selling restrictions, etc. Additional assumptions about admissible trading strategies will be provided below.

The investor's time-t wealth will be denoted V_t. Under the trading strategy h, the corresponding wealth process V will satisfy

$$\frac{dV_t}{V_t} = [(1-h_t)r_t + h_t\mu(r_t)]dt + h_t(\sigma dW_t + \rho d\bar{W}_t). \tag{5}$$

By standard results, there exists a unique, strong, and almost surely positive solution to this equation; it is given by

$$V_t = V_0 exp\Big(\int_0^t h_t\sigma dW_t + \int_0^t h_t\rho d\bar{W}_t + \int_0^t [-\frac{1}{2}(\sigma^2+\rho^2)h_t^2 + (1-h_t)r_t + h_t\mu(r_t)]dt\Big). \tag{6}$$

In this paper we consider the following family of risk sensitized optimal investment problems, labeled as $\mathcal{P}_\theta$:

for $\theta \in (0,\infty)$, *maximize the risk sensitized expected growth rate*

$$J_\theta(v,r;h) := \liminf_{t\to\infty}(-2/\theta)t^{-1}ln\ \mathbf{E}^h\ [e^{-(\theta/2)lnV_t}|V_0 = v, r_0 = r] \tag{7}$$

over the class of all admissible investment processes h,

where $\mathbf{E}^h$ is the expectation with respect to $\mathbf{P}$. The notation $\mathbf{E}^h$ emphasizes that the expectation is evaluated for the wealth process V corresponding to the investment strategy h.

The parameter θ here is interpreted as the measure of the investor's attitude toward risk; the bigger the value of θ, the more risk averse the investor. This is because the criterion can be interpreted, at least approximately, as the portfolio's exponential growth rate minus a penalty term which equals $\theta/4$ times the portfolio's asymptotic variance. A comprehensive interpretation of this risk sensitive objective for portfolio management can be found in Bielecki and Pliska [9].

We note that the techniques used in this paper can also be used to study problems $\mathcal{P}_\theta$ for negative values of θ, corresponding to risk seeking investors. The risk null case, for $\theta = 0$, can be studied independently or as the limit of the risk averse situation when the risk-sensitivity parameter θ goes to zero. However, we shall not consider the cases where $\theta \leq 0$ in this paper.

For much of what follows we find it convenient to introduce the scalar parameter

$$\gamma := -\theta/2. \tag{8}$$

Since θ is always strictly positive, the parameter γ should always be regarded as strictly negative. Moreover, the reader should keep in mind that corresponding to any appearance of the parameter γ is $\theta = -2\gamma$.

3 Analysis of the Hamilton-Jacobi-Bellman Equation

In this section we formulate our model and present our main results concerning the Hamilton-Jacobi-Bellman equation corresponding to the investor's portfolio optimization problem $\mathcal{P}_\theta$. We not only establish existence and uniqueness of a solution, we also establish some important properties of this solution. This analysis is rather involved, and so the balance of this paper is devoted to the proof of the results in this section.

In view of our risk sensitive objective, we are interested in computing the expectation of quantities like V_t^γ for some $\gamma < 0$. Since by equation (6)

$$V_t^\gamma = V_0^\gamma exp\Big(\gamma \int_0^t h_t \sigma dW_t + \gamma \int_0^t h_t \rho d\bar{W}_t + \int_0^t \gamma[-\frac{1}{2}(\sigma^2+\rho^2)h_t^2 + (1-h_t)r_t + h_t\mu(r_t)]dt\Big), \quad (9)$$

we recognize that it is convenient to make a Girsanov-type change of probability measure. In particular, it is straightforward to show for each trading strategy h and $T > 0$ that

$$E[V_T^\gamma] = \tilde{E}\Big[V_0^\gamma exp\Big(\gamma \int_0^T L(r_t, h_t)dt\Big)\Big], \quad (10)$$

where we have introduced the notation $\tilde{E}$ for expectation under the new probability measure and the additional functions

$$L(r,u) := -\frac{1}{2}(1-\gamma)(\sigma^2+\rho^2)u^2 + \bar{\mu}(r)u + r, \quad (11)$$

and

$$\bar{\mu}(r) := \mu(r) - r. \quad (12)$$

Moreover, under this new probability measure the dynamics for the interest rate process r are given by

$$dr_t = (-c(r_t - \bar{r}) + \gamma\sigma\lambda\sqrt{r_t}h_t)dt + \lambda\sqrt{r_t}d\tilde{W}_t, \quad (13)$$

where $\tilde{W}$ denotes a (scalar valued) Brownian motion under this new probability measure.

Using standard methods of risk sensitive control theory (see, for example, [8], [14], and [19]), it is now straightforward to specify the Hamilton-Jacobi-Bellman dynamic programming equation. This is

$$\Lambda = \frac{1}{2}\lambda^2 r\frac{d^2\Phi}{dr^2} - c(r-\bar{r})\frac{d\Phi}{dr} + \frac{1}{2}\lambda^2 r\Big(\frac{d\Phi}{dr}\Big)^2 + \inf_{\{u\in\mathbf{R}\}}\Big[\gamma\sigma\lambda\sqrt{r}u\frac{d\Phi}{dr} + \gamma L(r,u)\Big]. \quad (14)$$

We seek a solution in terms of the scalar Λ and the *bias function* Φ such that Λ is the optimal risk adjusted growth rate in problem $\mathcal{P}_\theta$ and such that the minimal selector identifies an optimal (or, at least, an ϵ-optimal) trading strategy.

It is convenient to transform this equation into a simpler form. Since the stock proportion h_t is unrestricted, we see that the minimizing value of u in the HJB equation must satisfy the first order condition $\gamma\sigma\lambda\sqrt{r}\Phi' + \gamma[-(1-\gamma)(\sigma^2+\rho^2)u + \bar{\mu}(r)] = 0$. In other words, our candidate h^* for the optimal trading strategy will satisfy the expression $h_t^* = u^*(r_t)$, where

$$u^*(r) := \frac{1}{1-\gamma}\frac{1}{\sigma^2+\rho^2}\Big(\bar{\mu}(r) + \sigma\lambda\sqrt{r}\frac{d\Phi}{dr}\Big). \tag{15}$$

Substituting this value of u in the HJB equation, introducing the function

$$g := \frac{d\Phi}{dr},$$

and doing a little algebra enables one to see that the original HJB equation is equivalent to

$$\Lambda = \frac{1}{2}\lambda^2 r\frac{dg}{dr} + \frac{1}{2}\lambda^2 r\Big(1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}\Big)g^2 + b(r)g + d(r), \tag{16}$$

where we have introduced for convenience the functions

$$b(r) := -c(r-\bar{r}) + \frac{\gamma}{1-\gamma}\frac{\sigma\lambda}{\sigma^2+\rho^2}\sqrt{r}\bar{\mu}(r) \tag{17}$$

and

$$d(r) := \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{1}{\sigma^2+\rho^2}[\bar{\mu}(r)]^2 + \gamma r. \tag{18}$$

Here is our main result about the HJB equation:

Theorem 3.1 *The HJB equation (16) has a unique solution (Λ^*, g^*) satisfying the following two properties:*

$$\lim_{r\to 0} g^*(r) = \frac{1}{c\bar{r}}\Big[\Lambda^* - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{\mu_1^2}{\sigma^2+\rho^2}\Big] \tag{19}$$

and either

$$\lim_{r\to\infty}\frac{g^*(r)}{\sqrt{r}} = -\Big(1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}\Big)^{-1}\Big[\frac{|\mu_2-1|}{\lambda}\sqrt{\frac{-\gamma}{1-\gamma}\frac{1}{\sigma^2+\rho^2}} + \frac{\gamma}{1-\gamma}\frac{\sigma}{\sigma^2+\rho^2}\frac{\mu_2-1}{\lambda}\Big] \tag{20}$$

for $\mu_2 \neq 1$ or

$$\lim_{r\to\infty} g^*(r) = -\Big(1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}\Big)^{-1}\Big(\frac{c}{\lambda^2} - \Big(\frac{c^2}{\lambda^4} - 2\frac{\gamma}{\lambda^2}\Big(1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}\Big)\Big)^{\frac{1}{2}}\Big) \tag{21}$$

for $\mu_2 = 1$. Moreover, Λ^* is characterized as the smallest Λ such that the HJB equation has a solution defined for all r.

In order to study problem $\mathcal{P}_\theta$, as well as to investigate a related problem of separate interest, consider exactly the same problem except that now, for some arbitrary positive number M, we impose the trading strategy constraint that $h_t = 0$ if $r_t > M$. Analogous to the unconstrained problem, the dynamic programming equation for this constrained, truncated problem is

$$\Lambda_M = \frac{1}{2}\lambda^2 r\frac{dg}{dr} + \frac{1}{2}\lambda^2 r\Big(1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}\Big)g^2 + b(r)g + d(r), \quad r \le M, \tag{22}$$

$$\Lambda_M = \frac{1}{2}\lambda^2 r \frac{dg}{dr} + \frac{1}{2}\lambda^2 r g^2 - c(r-\bar{r})g + \gamma r, \quad r > M. \tag{23}$$

In addition, our candidate for the optimal trading strategy is now given by

$$u_M^*(r) := \frac{1}{1-\gamma}\frac{1}{\sigma^2+\rho^2}\Big(\bar{\mu}(r) + \sigma\lambda\sqrt{r}g(r)\Big), \qquad r \leq M, \tag{24}$$

$$u_M^*(r) := 0, \qquad r > M.$$

Moreover, for this constrained problem we have the following important result:

Theorem 3.2 *The HJB equation (22), (23) for the constrained problem has a unique solution* (Λ_M^*, g_M^*) *satisfying the following two properties:*

$$\lim_{r\to 0} g_M^*(r) = \frac{1}{c\bar{r}}\left[\Lambda_M^* - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{\mu_1^2}{\sigma^2+\rho^2}\right] \tag{25}$$

and

$$\lim_{r\to\infty} g_M^*(r) = \frac{c}{\lambda^2} - \sqrt{\frac{c^2}{\lambda^4} - \frac{2\gamma}{\lambda^2}}. \tag{26}$$

As our next main result indicates, the solutions of the two kinds of investment problems are related in an intuitive manner.

Theorem 3.3 *The following hold:*

$$\lim_{M\to\infty} g_M^*(r) = g^*(r) \tag{27}$$

and

$$\lim_{M\to\infty} \Lambda_M^* = \Lambda^*. \tag{28}$$

Remark. For the equation (16), there is a smallest Λ such that (16) has a smooth solution W. This follows from the argument in [16]. We can show that Λ^* in Theorem 3.1 is the smallest Λ mentioned above. See 5.3 . The argument in [16] is applicable to the equations in multidimensional spaces, and therefore, can be applied to a model with several assets and multiple factor processes. However, there is difficulty to obtain W^* and to understand its behavior.

These three theorems are proved in the following two sections. We now conclude this section by suggesting a procedure for computing the solution (Λ^*, g^*) of (16). Suppose for any number Λ we can solve for the function g satisfying (16) and (19). It turns out that Λ^* is characterized as the smallest Λ such that this solution g is finite for all $r > 0$. Therefore, if some value Λ gives a finite solution then $\Lambda^* \leq \Lambda$. On the other hand, if some value Λ does not correspond to a finite g, then $\Lambda < \Lambda^*$. Hence a suitable iterative procedure should converge to (Λ^*, g^*).

4 Proof of Theorem 3.2

We begin by making a transformation of the constrained HJB equation (22). Denoting

$$A := 1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2},$$

$$\bar{\Lambda} := A\Lambda,$$

$$\bar{d}(r) := Ad(r),$$

and

$$\bar{g} := Ag, \tag{29}$$

we see by simple substitution that (22) is equivalent to

$$\bar{\Lambda} = \frac{1}{2}\lambda^2 r\frac{d\bar{g}}{dr} + \frac{1}{2}\lambda^2 r\bar{g}^2 + b(r)\bar{g} + \bar{d}(r), \quad r \leq M. \tag{30}$$

We would like to know that this equation has a (possibly unique) solution $\bar{g}$ for an arbitrary $\bar{\Lambda}$, but establishing this is not so easy because the second term on the right hand side is nonlinear and the coefficient of the first derivative term is degenerate at $r = 0$. Our approach will be to address these issues by studying the function

$$\tilde{g}(r) := \bar{g}(r)e(r)$$

where

$$e(r) := r^{\frac{2c\bar{r}}{\lambda^2}} \exp\Big(-\frac{2c}{\lambda^2}r + \frac{2\gamma\sigma}{(1-\gamma)\lambda(\sigma^2+\rho^2)}\int_0^r \frac{\bar{\mu}(s)}{\sqrt{s}}ds\Big).$$

This is because $\bar{g}$ satisfies equation (30) if and only if $\tilde{g}$ satisfies

$$\frac{d\tilde{g}}{dr} + \frac{1}{e(r)}\tilde{g}^2 = \frac{2}{\lambda^2 r}e(r)[\bar{\Lambda} - \bar{d}(r)], \tag{31}$$

and this latter differential equation will be easier to analyse.

Lemma 4.1 *If the differential equation (31) has a solution $\tilde{g}$ on $(0, r_0]$ for some $r_0 > 0$, then $\tilde{g}(r) \to 0$ as $r \to 0$.*

Proof. We first prove that for any $c_1 > 0$ there are $r_n, n = 1, 2, \cdots$, which tend to 0 as n tends to infinity and which satisfy $\tilde{g}(r_n) > -c_1$. If not, there is $r_1 > 0$ such that $\tilde{g}(r) \leq -c_1$ for all $0 < r \leq r_1$. Since

$$\frac{\frac{d\tilde{g}}{dr}}{\tilde{g}^2} + \frac{1}{e(r)} = \frac{2}{\lambda^2 r}e(r)[\bar{\Lambda} - \bar{d}(r)]\frac{1}{\tilde{g}^2(r)}$$

we have

$$\frac{1}{\tilde{g}(r)} - \frac{1}{\tilde{g}(r_1)} = -\int_r^{r_1} \frac{1}{e(s)}ds + \int_r^{r_1} \frac{2}{\lambda^2 s}e(s)[\bar{\Lambda} - \bar{d}(s)]\frac{1}{\tilde{g}^2(s)}ds. \tag{32}$$

For small $r > 0$, the first term on the right-hand side is bounded above by $-c_2 r^{-\frac{2c\bar{r}}{\lambda^2}+1}$ for some $c_2 > 0$ and the second term is bounded above by c_3/c_1^2 for some $c_3 > 0$. From this, the right hand side converges to $-\infty$ as $r \to 0$, in which case $\tilde{g}(r) \to 0$ as $r \to 0$, a contradiction.

By this result we can take a sufficiently small $r_1 > 0$ such that $\tilde{g}(r_1) > -c_1$. Next we show that

$$\tilde{g}(r) > -c_1, \ r \leq r_1. \tag{33}$$

To see this, suppose this is not true. Then there is some $r_2 < r_1$ such that $\tilde{g}(r_2) = -c_1$ and

$$\tilde{g}(r) > -c_1, \quad r_2 < r < r_1.$$

By (31)

$$\frac{d\tilde{g}}{dr}(r_2) = -\frac{1}{e(r_2)}c_1^2 + \frac{2}{\lambda^2 r_2}e(r_2)[\bar{\Lambda} - \bar{d}(r_2)] < 0.$$

This contradicts the specified property of c_1.

Finally, it suffices to show that for any $c_1 > 0$ there is some $r_3 > 0$ such that

$$\tilde{g}(r) \le c_1, \ r \le r_3, \tag{34}$$

because it is easy to see that our lemma follows from (33) and (34). To prove (34), suppose there is $r_4 > 0$ such that $\tilde{g}(r_4) > c_1$. Then

$$\frac{d\tilde{g}}{dr}(r_4) \le -\frac{1}{e(r_4)}c_1^2 + \frac{2}{\lambda^2 r_4}e(r_4)[\bar{\Lambda} - \bar{d}(r_4)] < 0.$$

Therefore, $\tilde{g}(r)$ is decreasing at r_4. This argument also shows that $\tilde{g}$ is decreasing on the set $\{\tilde{g}(r) > c_1\}$. Then we must have $\tilde{g} > c_1$ on $(0, r_4]$. This leads to a contradiction since using (32) with $r_1 = r_4$ and small r we have the right hand side tending to $-\infty$ while the left hand side is bounded. This completes the proof of the Lemma. □

Theorem 4.1 *If $\bar{g}$ is a solution of (30) defined on $(0, r_0]$ for some $r_0 > 0$, then either*

$$\lim_{r\to 0} r\bar{g}(r) = -\frac{2c\bar{r}}{\lambda^2} + 1 \tag{35}$$

or

$$\lim_{r\to 0} \bar{g}(r) = \frac{1}{c\bar{r}}\Big(\bar{\Lambda} - \frac{1}{2}A\frac{\gamma}{1-\gamma}\frac{\mu_1^2}{\sigma^2+\rho^2}\Big). \tag{36}$$

Remark. Note that (36) is equivalent to (25).

Proof. By (31) we have

$$\frac{d\tilde{g}}{dr} \le \frac{2}{\lambda^2 r}e(r)[\bar{\Lambda} - \bar{d}(r)],$$

so by Lemma 4.1 we have

$$\tilde{g}(r) \le \int_0^r \frac{2}{\lambda^2 s}e(s)[\bar{\Lambda} - \bar{d}(s)]ds.$$

Substituting for $e(s)$ and so forth, it is apparent this implies for some number $c_1 > 0$ that

$$\tilde{g}(r) \le c_1 r^{\frac{2c\bar{r}}{\lambda^2}}, \quad 0 < r < r_0. \tag{37}$$

The next main step is to show for some number $c_2 > 0$ that

$$\tilde{g}(r) > -c_2 r^{\frac{2c\bar{r}}{\lambda^2}-1}, \quad 0 < r < r_0. \tag{38}$$

We consider two cases, depending upon whether or not

$$\bar{\Lambda} \ge \frac{A}{2}\frac{\gamma}{1-\gamma}\frac{\mu_1^2}{\sigma^2+\rho^2}. \tag{39}$$

First we suppose inequality (39) is true, in which case $\bar{\Lambda} - \bar{d}(r) > 0$ for all positive r in some neighborhood of zero. We have one of the following two possibilities: there are infinitely many

$r_n > 0, n = 1, 2, \cdots$, which tend to 0 as n tends to infinity and are such that $\tilde{g}(r_n) < 0$; or there is $r_1 > 0$ such that $\tilde{g}(r) \geq 0$ for $0 < r \leq r_1$.

We consider the first possibility. Then there is sufficiently small r_1 such that $\tilde{g}(r_1) < 0$. It is easy to see by (31) that $\tilde{g}(r) < 0$ for all $r \leq r_1$. The conditions (39) and (31) imply

$$\frac{\frac{d\tilde{g}}{dr}}{\tilde{g}^2} \geq -\frac{1}{e(r)},$$

in which case

$$\frac{1}{\tilde{g}(r)} - \frac{1}{\tilde{g}(r_1)} \geq -\int_r^{r_1} \frac{1}{e(s)} ds,$$

that is,

$$-\frac{1}{\tilde{g}(r)} \leq -\frac{1}{\tilde{g}(r_1)} + \int_r^{r_1} \frac{1}{e(s)} ds.$$

It follows that for some constants $c_1, \bar{c}_1$, we have

$$-\tilde{g}(r) > \bar{c}_1 r^{\frac{2c\bar{r}}{\lambda^2}-1}, \tag{40}$$

since

$$\int_r^{r_1} \frac{1}{e(s)} ds \leq c_1 r^{-\frac{2c\bar{r}}{\lambda^2}+1}.$$

By (31) again we have

$$\frac{1}{\tilde{g}(r)} - \frac{1}{\tilde{g}(r_1)} = -\int_r^{r_1} \frac{1}{e(s)} ds + \int_r^{r_1} \frac{2}{\lambda^2 s} e(s)[\bar{\Lambda} - \bar{d}(s)] \frac{1}{\tilde{g}^2(s)} ds. \tag{41}$$

We use this to study the limiting behavior of $\tilde{g}(r)$. For the first term on the right hand side we have by L'Hospital's rule

$$\lim_{r \to 0} r^{\frac{2c\bar{r}}{\lambda^2}-1} \int_r^{r_0} \frac{1}{e(s)} ds = \frac{1}{\frac{2c\bar{r}}{\lambda^2} - 1}.$$

For the second term on the right hand side of (41) we have

$$\int_r^{r_0} \frac{2}{\lambda^2 s} e(s)[\bar{\Lambda} - \bar{d}(s)] \frac{1}{\tilde{g}^2(s)} ds \leq c \int_r^{r_0} s^{-1+\frac{2c\bar{r}}{\lambda^2}-2\frac{2c\bar{r}}{\lambda^2}+2} ds \leq c r^{-\frac{2c\bar{r}}{\lambda^2}+2}.$$

Here we use (40). So by (41) we have

$$\lim_{r \to 0} \frac{\tilde{g}(r)}{r^{\frac{2c\bar{r}}{\lambda^2}-1}} = -\frac{2c\bar{r}}{\lambda^2} + 1. \tag{42}$$

Hence for the first possibility (i.e, $\tilde{g}(r_1) < 0$ for some small $r_1 > 0$) and when (39) holds, we have proved (38) and (35).

Now we consider the second possibility: there is $r_1 > 0$ such that $\tilde{g}(r) \geq 0$ for all $r \leq r_1$. Since

$$\tilde{g}(r) = -\int_0^r \frac{1}{e(s)} \tilde{g}^2(s) ds + \int_0^r \frac{2}{\lambda^2 s} e(s)[\bar{\Lambda} - \bar{d}(s)] ds, \tag{43}$$

then it follows by L'Hospital's rule and (37) that

$$\lim_{r \to 0} \frac{\tilde{g}(r)}{r^{\frac{2c\bar{r}}{\lambda^2}}} = \frac{1}{c\bar{r}} \Big(\bar{\Lambda} - \frac{1}{2} A \frac{\gamma}{1-\gamma} \frac{\mu_1^2}{\sigma^2 + \rho^2} \Big). \tag{44}$$

This with the relation $\tilde{g}(r) = \bar{g}(r)e(r)$ and the definition of $e(\cdot)$ imply (36).

We summarize what we have shown. Assuming the condition (39), we have (42) or (44). They are equivalent to (35) and (36), respectively. They also imply (38).

For the remainder of this proof we consider the opposite case, namely, where inequality (39) does not hold. We choose $r_1 > 0$ such that $\bar{\Lambda} - \bar{d}(r) < 0$ for $0 < r \le r_1$. Now (41) and the definition of $e(\cdot)$ imply

$$\frac{1}{\tilde{g}(r)} - \frac{1}{\tilde{g}(r_1)} \le -\int_r^{r_1} \frac{1}{e(s)} ds \le -c_3\big(r^{-\frac{2c\bar{r}}{\lambda^2}+1}\big)$$

for some number $c_3 > 0$. The second inequality holds for $r > 0$ small. By (43), $\tilde{g}(r) < 0$ for $0 < r < r_1$ if r_1 is small. In particular, $\tilde{g}(r_1) < 0$. So by the above inequality we can prove (38).

We now assert that

$$-\tilde{g}(r_1) > r_1^{\frac{2c\bar{r}}{\lambda^2}-\delta-\frac{1}{2}}$$

for some small $r_1 > 0$ and some positive δ with $\delta < \frac{1}{2}$ implies

$$-\tilde{g}(r) > r^{\frac{2c\bar{r}}{\lambda^2}-\delta-\frac{1}{2}}, \quad r \le r_1.$$

If not, we can find $0 < r_2 < r_1$ such that

$$f(r) < 0, \ r_2 < r \le r_1$$

and $f(r_2) = 0$, where

$$f(r) := \tilde{g}(r) + r^{\frac{2c\bar{r}}{\lambda^2}-\delta-\frac{1}{2}}.$$

Since

$$\frac{df}{dr} = \frac{d\tilde{g}}{dr} + \Big(\frac{2c\bar{r}}{\lambda^2} - \delta - \frac{1}{2}\Big) r^{\frac{2c\bar{r}}{\lambda^2}-\delta-\frac{3}{2}} = -\frac{\tilde{g}^2(r)}{e(r)} + \frac{2}{\lambda^2 r}[\bar{\Lambda} - \bar{d}(r)]e(r) + \Big(\frac{2c\bar{r}}{\lambda^2} - \delta - \frac{1}{2}\Big) r^{\frac{2c\bar{r}}{\lambda^2}-\delta-\frac{3}{2}},$$

it is easy to see this is strictly positive at $r = r_2$ by using $f(r_2) = 0$. This contradicts the property of r_2, and so the assertion is established.

Let $0 < \delta < 1/2$ and $r_1 > 0$ be small, and consider two situations, depending upon whether

$$-\tilde{g}(r) > r^{\frac{2c\bar{r}}{\lambda^2}+\delta-1}, \ 0 < r \le r_1. \tag{45}$$

For the first situation, assume (45) does not hold for infinitely many r_1 which tend to 0. Then by the preceding assertion there is $r_2 > 0$ such that

$$-\tilde{g}(r) \le r^{\frac{2c\bar{r}}{\lambda^2}+\delta-1}, \quad r \le r_2.$$

Note that we have already established the property $\tilde{g}(r) < 0$ for r small. From these, by (31) we have

$$\tilde{g}(r) = -\int_0^r \frac{\tilde{g}^2(s)}{e(s)} ds + \int_0^r \frac{2e(s)}{\lambda^2 s}[\bar{\Lambda} - \bar{d}(s)]ds \ge -c_1 r^{\frac{2c\bar{r}}{\lambda^2}-1+2\delta} - c_2 r^{\frac{2c\bar{r}}{\lambda^2}} \ge -c r^{\frac{2c\bar{r}}{\lambda^2}-1+2\delta},$$

since $\delta < 1/2$. That is,

$$-\tilde{g}(r) \le c r^{\frac{2c\bar{r}}{\lambda^2}-1+2\delta}.$$

Continuing in an iterative fashion one obtains

$$-\tilde{g}(r) \leq c_1 r^{\frac{2c\bar{r}}{\lambda^2}-1+2^m\delta},$$

if m is such that $2^m\delta < 1$, where c_1 may depend on m and δ. Especially, this holds for $m = m_0$, $2^{m_0}\delta < 1 \leq 2^{m_0+1}\delta$. Apply this same procedure once more to obtain

$$\tilde{g}(r) \geq -c_1 r^{\frac{2c\bar{r}}{\lambda^2}-1+2\bar{\delta}} - c_2 r^{\frac{2c\bar{r}}{\lambda^2}} \geq -cr^{\frac{2c\bar{r}}{\lambda^2}}, \tag{46}$$

where $\bar{\delta} = 2^{m_0}\delta$. The last step is due to $2\bar{\delta} > 1$. We shall now show (44) by the following calculation. In view of (37) and (38) we have

$$\lim_{r\to 0} r^{-\frac{2c\bar{r}}{\lambda^2}} \int_0^r \frac{2e(s)}{\lambda^2 s}[\bar{\Lambda} - \bar{d}(s)]ds = \frac{1}{c\bar{r}}\Big(\bar{\Lambda} - \frac{1}{2}A\frac{\gamma}{1-\gamma}\frac{\mu_1^2}{\sigma^2+\rho^2}\Big). \tag{47}$$

In addtion, by (37) and (46) we have

$$\lim_{r\to 0} r^{-\frac{2c\bar{r}}{\lambda^2}} \int_0^r \frac{1}{e(s)}\tilde{g}(s)^2 ds = 0.$$

This with (47) and (43) imply (44), which is equivalent to (36).

Now we consider the opposite situation, namely, there is $0 < \delta < 1/2$ such that (45) does hold for some r_1. Then

$$\frac{1}{\tilde{g}(r)} - \frac{1}{\tilde{g}(r_1)} = -\int_r^{r_1} \frac{1}{e(s)}ds + \int_r^{r_1} \frac{2}{\lambda^2 s}e(s)[\bar{\Lambda} - \bar{d}(s)]\frac{1}{\tilde{g}^2(s)}ds.$$

Corresponding to the two terms on the right hand side we have

$$\lim_{r\to 0} r^{\frac{2c\bar{r}}{\lambda^2}-1} \int_r^{r_1} \frac{1}{e(s)}ds = \frac{1}{\frac{2c\bar{r}}{\lambda^2}-1}$$

and

$$\lim_{r\to 0} r^{\frac{2c\bar{r}}{\lambda^2}-1} \int_r^{r_1} \frac{2}{\lambda^2 s}e(s)[\bar{\Lambda} - \bar{d}(s)]\frac{1}{\tilde{g}^2(s)}ds = 0.$$

Hence

$$\lim_{r\to 0} \frac{\tilde{g}(r)}{r^{\frac{2c\bar{r}}{\lambda^2}-1}} = -\frac{2c\bar{r}}{\lambda^s} + 1, \tag{48}$$

so by the definition of $e(\cdot)$ and the relationship between $\bar{g}$ and $\tilde{g}$ we see that (35) holds. This completes the proof of this theorem. □

From now on we shall focus on solutions of (30) that satisfy (36) rather than (35). The reason will become apparent below. In particular, see Corollary 4.1 which gives special properties of the solution satisfying (36). In the proof of Theorem 3.1 it will be seen that the smallest $\bar{\Lambda}$ such that (30) has a finite solution for all $r < M$ corresponds to a $\bar{g}$ which satisfies (36). See 5.3.

Suppose a solution $\bar{g}$ of (30) and (36) exists, and consider the corresponding solution $\tilde{g}$ of (31), so that $\tilde{g}$ also satisfies (43). Denote

$$\tilde{g}^{(0)}(r) := \int_0^r \frac{2}{\lambda^2 s}e(s)[\bar{\Lambda} - \bar{d}(s)]ds$$

and

$$\tilde{g}_0(r) := \tilde{g}(r) - \tilde{g}^{(0)}(r).$$

Then

$$\tilde{g}_0(r) = -\int_0^r \tilde{g}^2(s)\frac{1}{e(s)}ds.$$

Since (36) implies

$$|\tilde{g}(r)| \le cr^{\frac{2c\bar{r}}{\lambda^2}}$$

for r small, we have,

$$|\tilde{g}_0(r)| \le cr^{\frac{2c\bar{r}}{\lambda^2}+1}.$$

Moreover,

$$\tilde{g}(r) = \tilde{g}^{(0)}(r) - \int_0^r [\tilde{g}^{(0)}(s) + \tilde{g}_0(s)]^2\frac{1}{e(s)}ds$$

$$= \tilde{g}^{(0)}(r) - \int_0^r \tilde{g}^{(0)}(s)^2\frac{1}{e(s)}ds - 2\int_0^r \tilde{g}^{(0)}(s)\tilde{g}_0(s)\frac{1}{e(s)}ds - \int_0^r \tilde{g}_0(s)^2\frac{1}{e(s)}ds.$$

Continuing this procedure, we may define $\tilde{g}^{(n)}(r)$ recursively by

$$\tilde{g}^{(n+1)}(r) = \tilde{g}^{(0)}(r) - \int_0^t \frac{1}{e(s)}\tilde{g}^{(n)}(s)^2ds.$$

We can show by induction that the following holds :

$$\tilde{g}(r) = \tilde{g}^{(n)}(r) + O\big(r^{\frac{2c\bar{r}}{\lambda^2}+n+1}\big). \tag{49}$$

This gives an asymptotic expansion of $\tilde{g}(r)$ for small $r > 0$.

Lemma 4.2 *Fix $\bar{\Lambda}$. For small enough $r_0 > 0$ there exists a unique $\tilde{g}$ satisfying for all $r \in (0, r_0]$ both (31) and (43). It also satisfies*

$$|\tilde{g}(r)| \le c_1 r^{\frac{2c\bar{r}}{\lambda^2}}, \qquad r \le r_0, \tag{50}$$

for a positive number c_1, as well as (44)(this is equivalent to (36) if we take $\bar{g}(r) = \tilde{g}(r)/e(r)$). In addition, we have (49) and

$$\tilde{g}(r) \le \int_0^r \frac{2}{\lambda^2 s}e(s)[\bar{\Lambda} - \bar{d}(s)]ds, \qquad r \le r_0. \tag{51}$$

Remark. Since there is a one to one correspondence between solutions of (22) satisfying (25) and solutions of (31) satisfying (36) (see the beginning of this section), it follows from Lemma 4.2 that there exists a solution g of (22) satisfying (25), at least a solution in some neighborhood of $r = 0$.

Proof. For some suitable positive numbers r_0 and c_1 (to be decided later), consider the operator T defined for $f \in \mathbf{F}_{c_1}$, where

$$Tf(r) := -\int_0^r f^2(s)\frac{1}{e(s)}ds + \int_0^r \frac{2}{\lambda^2 s}e(s)[\bar{\Lambda} - \bar{d}(s)]ds$$

and

$$\mathbf{F}_{c_1} := \{f : |f(r)| \le c_1 r^{\frac{2c\bar{r}}{\lambda^2}}, 0 \le r \le r_0\}.$$

In order to show that $Tf \in \mathbf{F}_{c_1}$ we need to estimate $|Tf(r)|$. The first term in the definition of T is bounded by

$$c_1^2 \bar{c}_2 \int_0^r s^\delta ds = c_1^2 c_2 r^{\delta+1},$$

where $\delta := \frac{2c\bar{r}}{\lambda^2}$ and $1/e(s) \le \bar{c}_2 s^{-\delta}$, $c_2 = \bar{c}_2/(1+\delta)$. The second term is bounded by

$$c_1(\bar{\Lambda})\bar{c}_3 \int_0^r s^{\delta-1} ds = c_1(\bar{\Lambda}) c_3 r^\delta,$$

where $c_3 = \bar{c}_3/\delta$ and $e(r) \le \bar{c}_3 r^\delta$ for r small, and where

$$c_1(\bar{\Lambda}) := \max_{0<r\le 1} |\frac{2}{\lambda^2}[\bar{\Lambda} - \bar{d}(r)]|.$$

Therefore

$$|Tf(r)| \le [c_1^2 c_2 r + c_1(\bar{\Lambda})c_3] r^\delta \le [c_1^2 c_2 r_0 + c_1(\bar{\Lambda})c_3] r^\delta$$

if $r \le r_0$. It now follows by taking $c_1 = 2c_1(\bar{\Lambda})c_3$ and $r_0 = 1/[4c_2 c_3 c_1(\bar{\Lambda})]$ that

$$|Tf(r)| \le c_1 r^{\frac{2c\bar{r}}{\lambda^2}},$$

and so $T : \mathbf{F}_{c_1} \to \mathbf{F}_{c_1}$.

On the other hand, for $r \le r_0$

$$|Tf_1(r) - Tf_2(r)| \le \int_0^r |f_1(s) + f_2(s)||f_1(s) - f_2(s)| \frac{1}{e(s)} ds$$

$$\le ||f_1 - f_2|| 2c_1 \int_0^{r_0} s^{\frac{2c\bar{r}}{\lambda^2}} \frac{1}{e(s)} ds \le 2c_1 \bar{c}_2 r_0 ||f_1 - f_2||, \quad r \le r_0,$$

where $||\cdot||$ denotes the supnorm on $[0, r_0]$ and

$$\bar{c}_2 := \max_{r\le 1} \frac{1}{e(r)} r^{\frac{2c\bar{r}}{\lambda^2}}.$$

Hence by taking r_0 small enough so that $2c_1\bar{c}_2 r_0 < 1$ we see that T will be a contraction mapping from $\mathbf{F}_{c_1}$ into $\mathbf{F}_{c_1}$. Hence T has a unique fixed point, say $\tilde{g}$, which means that $\tilde{g}$ satisfies (43) and thus (31).

If we define $\bar{g}(r) = \tilde{g}(r)/e(r)$, then $\bar{g}$ is a solution of (30) defined on $(0, r_0]$. Therefore, in view of Theorem 4.1, either one of (35) or (36) holds. Since $\tilde{g}$ is in $\mathbf{F}_{c_1}$, we have (50). Then (35) cannot be true. Since (36) is equivalent to (44), (44) holds. Finally, (51) is a consequence of (43). This completes the proof of the Lemma. □

Lemma 4.3 *Let $\tilde{g}_1$ and $\tilde{g}_2$ be the two solutions of (31) (equivalently, (43)) corresponding to $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$, respectively, as defined in Lemma 4.2. If $\bar{\Lambda}_1 < \bar{\Lambda}_2$ then $\tilde{g}_1 < \tilde{g}_2$.*

Proof. Since $\tilde{g}_1$ and $\tilde{g}_2$ both satisfy equation (31) (with their respective values of $\bar{\Lambda}$) we can subtract one equation from the other to obtain

$$\frac{d}{dr}(\tilde{g}_1 - \tilde{g}_2) + \frac{1}{e(r)}[\tilde{g}_1 + \tilde{g}_2][\tilde{g}_1 - \tilde{g}_2] = \frac{2}{\lambda^2 r} e(r)[\bar{\Lambda}_1 - \bar{\Lambda}_2].$$

We thus have

$$\tilde{g}_1(r) - \tilde{g}_2(r) = \int_0^r \frac{2e(s)}{\lambda^2 s}[\bar{\Lambda}_1 - \bar{\Lambda}_2] \exp\Big(-\int_s^r \frac{\tilde{g}_1(u) + \tilde{g}_2(u)}{e(u)} du\Big) ds.$$

This is strictly negative if $\bar{\Lambda}_1 < \bar{\Lambda}_2$, so Lemma 4.3 is established. □

Corollary 4.1 *For each $\bar{\Lambda}$, (31) has only one solution satisfying (36). For fixed $\bar{\Lambda}$, assume $\bar{g}_0$ defined on $(0, r_0]$ is a solution of (31) satisfying (36). Let $y < \bar{g}_0(r_0)$, and suppose $\bar{g}$ is the solution of (31) such that $\bar{g}(r_0) = y$. Then $\bar{g}(r)$ exists for $r \in (0, r_0]$ and $\bar{g}$ satisfies (35).*

Remark. While uniqueness of this solution is true for general r_0, existence of a solution has only been established for small enough $r_0 > 0$.

We note that if $\tilde{g}(r_0)$ is finite, then $\tilde{g}$ is well defined for $r > r_0$, up to a (possibly infinite) point denoted $r(\bar{\Lambda})$ where $\lim_{r \to r(\bar{\Lambda})} \tilde{g}(r) = -\infty$ (If $\tilde{g}$ explodes, then by (51) it explodes in the negative direction). For each $M > 0$ we now define

$$\bar{\Lambda}_*(M) := \inf\{\bar{\Lambda} : \text{the corresponding solution } \tilde{g} \text{ of (31) satisfying (36) is finite for all } r \leq M\},$$

and note solutions of (31) satisfying (36) are given by (43). The preceding results now imply the following:

Corollary 4.2 *Fix arbitrary $0 < M < \infty$. Then $\bar{\Lambda}_*(M) < \infty$ and, for each $\bar{\Lambda} > \bar{\Lambda}_*(M)$, the corresponding solution $\tilde{g}$ of (43) is finite for all $r \leq M$. Moreover, $\tilde{g}(M) \to \infty$ as $\bar{\Lambda} \to \infty$ and $\tilde{g}(M) \to -\infty$ as $\bar{\Lambda} \to -\infty$.*

Remark. Recall that a solution $\bar{g}$ of (30) is well defined if and only if a solution $\tilde{g}$ of (43) is well defined. Since $\bar{g} = Ag$, the preceding corollaries tell us when the constrained dynamical programming equation (22) has a unique solution for $r \leq M$. In particular, since $g(r) = \bar{g}(r)/A$ and $\bar{\Lambda} = A\Lambda$, we see that (36) implies (25). Also, note that in Lemma 5.2 below we prove that $\bar{\Lambda}_*(M) > -\infty$.

Proof. We prove $\bar{\Lambda}_*(M) < \infty$. The rest is a consequence of either Lemma 4.3 or a similar argument.

We first take a r_0 small enough and a finite $\bar{\Lambda}_0$, and

$$\tilde{g}_0(r) = \int_0^r \frac{2}{\lambda^2 s} e(s)(\bar{\Lambda}_0 - \bar{d}(s)) ds - \int_0^r \frac{1}{e(s)} \tilde{g}_0^2(s) ds, \ r \leq r_0.$$

We know that $\tilde{g}_0(r)$ is finite for $r \in [0, r_0]$. Now for $\theta > 0$ we consider

$$\tilde{g}_\theta(r) = \int_0^r \frac{2}{\lambda^2 s} e(s)(\bar{\Lambda}_0 + \theta - \bar{d}(s)) ds - \int_0^r \frac{1}{e(s)} \tilde{g}_\theta^2(s) ds, \ r \leq r_0.$$

This has solution $\tilde{g}_\theta(\cdot)$ in a neighborhood of 0 as given in Lemma 4.2 with $\bar{\Lambda} = \bar{\Lambda}_0 + \theta$. We know that $\tilde{g}_\theta(r)$ is finite for $r \in [0, r_0]$ and that

$$\tilde{g}_\theta(r) \geq \tilde{g}_0(r), \ r \leq r_0.$$

Let us fix $0 < r_1 < r_0$, a large $K > 0, K > \|\tilde{g}_0\|_{[r_1,r_0]}$, the maximum of $|\tilde{g}_0(r)|, r \in [r_1, r_0]$. There is a θ_0 such that for $\theta > \theta_0$, $\tilde{g}_\theta(\cdot)$ is increasing for $r_1 \leq r \leq r_0$, if $|\tilde{g}_\theta(r)| \leq K$. This is due to the following calculation:

$$\frac{d}{dr}\tilde{g}_\theta(r) = \frac{2}{\lambda^2 r} e(r)(\bar{\Lambda}_0 + \theta - \bar{d}(r)) - \frac{1}{e(r)}\tilde{g}_\theta^2(r)$$
$$\geq \frac{2}{\lambda^2 r_0} \inf_{[r_1,r_0]} \{e(r)(\bar{\Lambda}_0 + \theta_0 - \|\bar{d}\|_{[r_1,r_0]})\} - \|\frac{1}{e(\cdot)}\|_{[r_1,r_0]} K^2 > 0,$$

where the last inequality holds if θ_0 is large enough.

From this, for a fixed K, we must have $\tilde{g}_\theta(r_0) > K$ if θ is large enough. Suppose not. Then using the fact that $\tilde{g}_\theta(r) > \tilde{g}_0(r), \ r_1 \leq r \leq r_0$ and the above monotonicity result we can conclude that

$$|\tilde{g}_\theta(r)| < K, \ r_1 \leq r \leq r_0.$$

Thus

$$\frac{d}{dr}\tilde{g}_\theta(r) \geq (\frac{2}{\lambda^2 r_0} \inf_{[r_1,r_0]} \{e(r)(\bar{\Lambda}_0 + \theta - \|\bar{d}\|_{[r_1,r_0]})\} - \|\frac{1}{e(r)}\|_{[r_1,r_0]} K^2)$$

is larger than a given number (say L) for $r_1 \leq r \leq r_0$ if θ is large enough. For such θ,

$$\tilde{g}_\theta(r_0) = \tilde{g}_\theta(r_1) + \int_{r_1}^{r_0} \frac{d}{dr}\tilde{g}_\theta(r)dr \geq \tilde{g}_0(r_1) + L(r_0 - r_1),$$

and this is larger than K if L is large enough. This gives a contradiction.

Next, for a fixed $K > 0$ if θ is large enough, then $\tilde{g}_\theta(r_0) > K$ implies $\tilde{g}_\theta(r) > K, \ r_0 \leq r \leq M$. This follows by using the properties that

$$\inf_{r_0 \leq r \leq M} \frac{1}{r} e(r) > 0, \ \sup_{r_0 \leq r \leq M} \frac{1}{e(r)} < \infty$$

and the estimate

$$\frac{d}{dr}\tilde{g}_\theta(r) \geq \frac{2}{\lambda^2} \inf_{r_0 \leq r \leq M} \{\frac{1}{r} e(r)(\bar{\Lambda}_0 + \theta_0 - \|\bar{d}\|_{[r_0,M]})\} - \|\frac{1}{e(r)}\|_{[r_0,M]} K^2 > 0$$

for a $r_0 \leq r \leq M$ satisfying $\tilde{g}_\theta(r) = K$ if θ is large enough.

We conclude from the above analysis that for a $K > \|\tilde{g}_0\|_{[r_1,r_0]}$, there is a θ sufficiently large such that $\tilde{g}_\theta(M) > K$. This implies $\bar{\Lambda}_*(M) \leq \bar{\Lambda}_0 + \theta < \infty$. As a consequence of this argument, we also have that $\tilde{g}_\theta(M)$ tends to ∞ as θ tends to ∞. This ends the proof. □

We now turn to the study of the solution of the constrained dynamical programming equation for $r > M$, that is, equation (23). But the solution of this differential equation must satisfy the boundary condition at $r = M$ that has $g(M)$ taking the value that comes from the solution of (22) and (25) for $r \leq M$.

Lemma 4.4 *Given a specified value of $g(M)$, equation (23) has a unique solution g on $[M, r_1)$, where $r_1 := \sup\{r > M : g(r) > -\infty\}$. Also, there exists some $K < \infty$, which does not depend on r_1, such that $g(r) \leq K$ on $[M, r_1)$ (but may depend on $g(M)$).*

Proof. It is sufficient to prove the existence of K. The existence of r_1 follows from the theory of ordinary differential equations.

First note that (23) can be rewritten as

$$\frac{dg}{dr} - \frac{2c}{\lambda^2}[1 - \frac{\bar{r}}{r}]g + g^2 = \frac{2}{\lambda^2 r}[\Lambda_M - \gamma r], \quad r \geq M. \tag{52}$$

It suffices to show that if a solution is such that $g(r_0) < c_2$, then $g(r) < c_2$ for all $r > r_0$, where r_0 and c_2 here are large. Suppose, on the contrary, there is some $r_1 > r_0$ such that $g(r) < c_2$ for $r_0 \leq r < r_1$ and $g(r_1) = c_2$. Then by differential equation (52) we must have $\frac{dg}{dr}(r_1) < 0$. But this is a contradiction, so Lemma 4.4 is established. □

For fixed M and any $\Lambda > \bar{\Lambda}_*(M)/A$ we know by Corollary 4.2 that (22) with $\Lambda_M = \Lambda$ has a solution g on $(0, M]$ with $g(M)$ finite. So corresponding to each such Λ we can, as in the following lemma, consider the solution g of (23) on $[M, r_1)$ that takes this corresponding value of $g(M)$ at $r = M$. In other words, for each $\Lambda > \bar{\Lambda}_*(M)/A$ we have a solution of (22) and (23) that is continuous on $[0, r_1)$ for some $r_1 > M$.

Lemma 4.5 *Let g_1 and g_2 be two solutions of (22) and (23) corresponding to Λ_1 and Λ_2, respectively, where $\Lambda_1, \Lambda_2 > \bar{\Lambda}_*(M)/A$. Then $\Lambda_1 < \Lambda_2$ implies $g_1(r) < g_2(r)$ if g_1 is defined at r.*

Proof. First consider two differential equations (52), one satisfied by $(g_1, \Lambda_M = \Lambda_1)$ and the other by $(g_2, \Lambda_M = \Lambda_2)$. Subtracting one from the other gives

$$\frac{d}{dr}(g_2 - g_1) + \left[-\frac{2c}{\lambda^2}(1 - \bar{r}/r) + g_2 + g_1\right](g_2 - g_1) = \frac{2}{\lambda^2 r}[\Lambda_2 - \Lambda_1],$$

in which case

$$g_2(r) - g_1(r) = \exp\Big(-\int_M^r \big(-\frac{2c}{\lambda^2}(1 - \bar{r}/s) + g_1(s) + g_2(s)\big)ds\Big)[g_2(M) - g_1(M)]$$

$$+ \int_M^r \frac{2}{\lambda^2 s}(\Lambda_2 - \Lambda_1)\exp\Big(-\int_s^r \big(-\frac{2c}{\lambda^2}(1 - \bar{r}/u) + g_1(u) + g_2(u)\big)du\Big)ds.$$

Since $g_2(M) - g_1(M) > 0$, Lemma 4.5 follows from this. □

For each $M > 0$, we now define

$$\Lambda_M^* := \inf\{\Lambda_M : \text{the corresponding solution } g \text{ of (22), (23) satisfying (25) is finite for all } r \geq 0\},$$

and we observe that $\Lambda_M^* \geq \bar{\Lambda}_*(M)/A$. The preceding results imply the following:

Corollary 4.3 *For each fixed number $M < \infty$ we have $\Lambda_M^* < \infty$.*

Proof. We need to prove the existence of a Λ_M such that $g(r)$ is finite for all $r > 0$, where g is the solution for (22), (23) and (25). By (52), if $g(M) > 0$ and $\Lambda_M > 0$, then $g(r) > 0$ for all $r > M$. We can show by the argument in the proof of Lemma 4.5 that $g(M) > 0$ if Λ_M is sufficiently large. This completes the proof. □

By Corollary 4.3 we know for $\Lambda_M \geq \Lambda_M^*$ that there exists a solution of (22), (23) and (25). We now investigate the limiting behavior of this solution g as $r \to \infty$.

Theorem 4.2 *Fix $M < \infty$ and arbitrary $\Lambda_M \geq \Lambda_M^*$, and consider the solution $g = g_M$ of (22), (23) satisfying (25). Then exactly one of the following two conditions will hold, that is, either*

$$\lim_{r\to\infty} g(r) = \frac{c}{\lambda^2} - \sqrt{\frac{c^2}{\lambda^4} - \frac{2\gamma}{\lambda^2}} \tag{53}$$

or

$$\lim_{r\to\infty} g(r) = \frac{c}{\lambda^2} + \sqrt{\frac{c^2}{\lambda^4} - \frac{2\gamma}{\lambda^2}}. \tag{54}$$

Proof. Denote

$$\alpha := \frac{c}{\lambda^2} - \sqrt{\frac{c^2}{\lambda^4} - \frac{2\gamma}{\lambda^2}},$$

and note that α is negative and satisfies

$$\alpha^2 - \frac{2c}{\lambda^2}\alpha + \frac{2\gamma}{\lambda^2} = 0.$$

Next, define[1]

$$\bar{g}(r) = g(r) - \alpha,$$

and note that, in view of (23), we must have

$$\frac{d\bar{g}}{dr} + \Big(-2\sqrt{\frac{c^2}{\lambda^4} - \frac{2\gamma}{\lambda^2}} + \frac{2c\bar{r}}{\lambda^2}\frac{1}{r}\Big)\bar{g} + \bar{g}^2 = \Big(\frac{2\Lambda_M}{\lambda^2} - \frac{2c\bar{r}}{\lambda^2}\alpha\Big)\frac{1}{r}, \quad r > M. \tag{55}$$

We now claim there is c_1 large enough such that

$$\bar{g}(r) \geq -c_1/r, \quad r \geq M. \tag{56}$$

If not, then not only is there some $r_0 > M$ such that $\bar{g}(r_0) < -c_1/r_0$, but there is some $r_0 > M$ such that

$$\bar{g}(r) < -c_1/r, \quad r \geq r_0. \tag{57}$$

To see this, suppose $\bar{g}(r_0) < -c_1/r_0$ but (57) is false. Then there is some r_1 such that $\bar{g}(r_1) = -c_1/r_1$ and $\bar{g}(r) < -c_1/r$ for $r_0 \leq r < r_1$. It then follows from (55) that

$$\frac{d\bar{g}}{dr}(r_1) \leq \Big[-2c_1\sqrt{\frac{c^2}{\lambda^4} - \frac{2\gamma}{\lambda^2}} + \Big(\frac{2\Lambda_M}{\lambda^2} - \frac{2c\bar{r}}{\lambda^2}\alpha\Big)\Big]\frac{1}{r_1}.$$

This implies $\frac{df}{dr}(r_1) < 0$, where $f(r) := \bar{g}(r) + c_1/r$. But this is a contradiction, so we see that if (56) is false, then there exists some $r_0 > M$ such that (57) is true.

[1] The $\bar{g}$ used in this proof must not be confused with the $\bar{g}$ that is the solution of differential equation (30).

Using (55) and (57) one can show that

$$\frac{d\bar{g}}{dr} + \frac{1}{2}\bar{g}^2 < 0, \quad r \geq r_0.$$

This, in turn, implies

$$-\frac{1}{\bar{g}(r)} + \frac{1}{\bar{g}(r_0)} + \frac{1}{2}(r - r_0) < 0, \quad r \geq r_0.$$

But this cannot be true for all $r \geq r_0$, so (56) must be true.

We now consider two cases, depending upon whether or not

$$\frac{2\Lambda_M}{\lambda^2} - \frac{2c\bar{r}}{\lambda^2}\alpha > 0. \tag{58}$$

If (58) is true and $\bar{g}(r_0) > 0$ for some $r_0 > M$, then $\bar{g}(r) > 0$ for all $r \geq r_0$. From this, we conclude that one of the following two possibilities holds: either there is $r_0 > M$ such that

$$\bar{g}(r) > 0, \; r \geq r_0, \tag{59}$$

or there is $r_0 > M$ such that

$$\bar{g}(r) < 0, \; r \geq r_0. \tag{60}$$

We have the same conclusion if the opposite of (58) holds, so it suffices to consider (59) and (60) separately. First we assume (60). This together with (56) implies

$$\lim_{r\to\infty} \bar{g}(r) = 0.$$

In other words,

$$\lim_{r\to\infty} g(r) = \alpha, \tag{61}$$

which is equivalent to (53) in this case.

For the rest of this proof we shall assume (59) and show that for small $c_1 > 0$ and some $r_1 > M$ then either

$$\bar{g}(r) + \Big(-2\sqrt{\frac{c^2}{\lambda^4} - \frac{2\gamma}{\lambda^2}} + \frac{2c\bar{r}}{\lambda^2}\frac{1}{r}\Big) > -c_1, \quad r \geq r_1, \tag{62}$$

or

$$\bar{g}(r) + \Big(-2\sqrt{\frac{c^2}{\lambda^4} - \frac{2\gamma}{\lambda^2}} + \frac{2c\bar{r}}{\lambda^2}\frac{1}{r}\Big) < -c_1, \quad r \geq r_1. \tag{63}$$

Indeed, it is easy to see that in order to prove this assertion it suffices to show that if

$$\bar{g}(r) + \Big(-2\sqrt{\frac{c^2}{\lambda^4} - \frac{2\gamma}{\lambda^2}} + \frac{2c\bar{r}}{\lambda^2}\frac{1}{r}\Big) > -c_1 \tag{64}$$

holds for some $c_1 > 0$ and for $r = r_0 > 0$, then (64) in fact holds for all $r \geq r_0$.

To prove this last statement, assume the contrary: there is some $r_1 > r_0$ such that (64) holds for $r_0 \leq r < r_1$ and equality holds in (64) for $r = r_1$. In this case

$$\frac{df}{dr}(r_1) = -c_1\Big(c_1 - 2\sqrt{\frac{c^2}{\lambda^4} - \frac{2\gamma}{\lambda^2}} + \frac{2c\bar{r}}{\lambda^2}\frac{1}{r_1}\Big) + \Big(\frac{2\Lambda_M}{\lambda^2} - \frac{2c\bar{r}}{\lambda^2}\alpha\Big)\frac{1}{r_1} - \frac{2c\bar{r}}{\lambda^2}\frac{1}{r_1^2} > 0,$$

where we have defined

$$f(r) := \bar{g}(r) + \Big(-2\sqrt{\frac{c^2}{\lambda^4} - \frac{2\gamma}{\lambda^2}} + \frac{2c\bar{r}}{\lambda^2}\frac{1}{r}\Big).$$

But this is a contradiction, so (64) must be true for all $r \geq r_0$.

Our next step is to show (63) cannot hold. This is because it and (55) would imply

$$\frac{d\bar{g}}{dr} > c_1\bar{g} + \Big(\frac{2\Lambda_M}{\lambda^2} - \frac{2c\bar{r}}{\lambda^2}\alpha\Big)\frac{1}{r}, \quad r \geq r_1,$$

since $\bar{g}(r) > 0$, in which case

$$\bar{g}(r) \geq \exp\{c_1(r - r_0)\}\bar{g}(r_0) + \int_{r_0}^{r}\Big(\frac{2\Lambda_M}{\lambda^2} - \frac{2c\bar{r}}{\lambda^2}\alpha\Big)\frac{1}{s}\exp\{c_1(r - s)\}ds.$$

But this contradicts Lemma 4.4, so we conclude that if $\bar{g}(r) > 0$ for all $r \geq r_0$ (i.e., if (59) holds), then (62) holds for any $c_1 > 0$ provided r_1 is large enough.

Having established (62), we now prove that for a fixed large r_1 there is a c_2 large enough such that

$$\bar{g}(r) + \Big(-2\sqrt{\frac{c^2}{\lambda^4} - \frac{2\gamma}{\lambda^2}} + \frac{2c\bar{r}}{\lambda^2}\frac{1}{r}\Big) < \frac{c_2}{r}, \ r \geq r_1. \tag{65}$$

To prove this, we consider the function

$$f(r) := \bar{g}(r) + \Big(-2\sqrt{\frac{c^2}{\lambda^4} - \frac{2\gamma}{\lambda^2}} + \frac{2c\bar{r}}{\lambda^2}\frac{1}{r}\Big).$$

We choose c_2 such that

$$f(r) < c_2/r \tag{66}$$

for $r = r_0$. Our next objective in this proof is to show that this implies (66) is true for all $r \geq r_0$. Otherwise, there is some $r_1 > r_0$ such that (66) is true for $r_0 \leq r < r_1$ and equality holds in (66) for $r = r_1$. By (55) again it is easy to see that

$$\frac{df}{dr}(r_1) + c_2/r_1^2 = \bar{g}(r_1)(-c_2/r_1) + \Big(\frac{2\Lambda_M}{\lambda^2} - \frac{2c\bar{r}}{\lambda^2}\alpha\Big)\frac{1}{r_1} + c_2\frac{1}{r_1^2} < 0$$

if c_2 is large enough. But this is another contradiction, so (66) must be true for all $r \geq r_0$.

Now (62) and (65) imply

$$\lim_{r\to\infty} f(r) = 0.$$

This is equivalent to (54). This together with (61) completes the proof of Theorem 4.2. □

It turns out that the limit (53) is the one we want; (54) will now be ignored. See Lemma 4.7. Here again we are interested in the smallest Λ_M such that (22), (23), and (25) have a solution for all r. The following lemma says there is at most one value of Λ_M giving a solution of (22), (23), and (25) that also satisfies (53).

Lemma 4.6 *Suppose g_1 and g_2 are two solutions of (22), (23), and (25) corresponding to $\Lambda_M = \Lambda_1, \Lambda_2$, respectively. If both g_1 and g_2 satisfy (53), then $g_1 = g_2$ and $\Lambda_1 = \Lambda_2$.*

Proof. Subtracting the equation for g_2 from the equation for g_1 gives

$$\frac{d}{dr}(g_2 - g_1) + \Big(-\frac{2c}{\lambda^2} + \frac{2c\bar{r}}{\lambda^2}\frac{1}{r} + g_1 + g_2\Big)(g_2 - g_1) = \frac{2}{\lambda^2 r}(\Lambda_2 - \Lambda_1).$$

By (53) we then have

$$-(g_2 - g_1)(r)\bar{e}(r) = \int_r^\infty \frac{2}{\lambda^2 s}(\Lambda_2 - \Lambda_1)\bar{e}(s)ds, \tag{67}$$

where we have introduced the function

$$\bar{e}(r) := \exp\Big\{\int_{r_0}^r \Big(-\frac{2c}{\lambda^2} + \frac{2c\bar{r}}{\lambda^2}\frac{1}{s} + g_1(s) + g_2(s)\Big)ds\Big\}.$$

Here $r_0 > M$ is fixed. The integral on the right hand side of (67) is finite by (53). Moreover, (67) implies $g_2 - g_1 > 0$ if $\Lambda_2 - \Lambda_1 < 0$. But we also have $g_2 - g_1 < 0$ if $\Lambda_2 - \Lambda_1 < 0$. Therefore $\Lambda_1 = \Lambda_2$ and $g_1 = g_2$, that is, the proof of Lemma 4.6 is completed. □

It remains to prove the solution g^* corresponding to $\Lambda_M = \Lambda_M^*$ satisfies (53). In other words, with Lemma 4.6 establishing uniqueness, it remains to establish existence. This is a consequence of the following lemma, because if for $\Lambda_M = \bar{\Lambda}$ the corresponding limit is (54), then for all $\Lambda_M < \bar{\Lambda}$ in some neighborhood of $\bar{\Lambda}$ the corresponding limits also satisfy (54). Hence if there exists a solution for $\Lambda_M = \Lambda_M^*$ (the infimum of Λ_M for which there exists a solution; Λ_M^* is finite by Corollary 4.2 above and Lemma 5.2 below, and the infimum is attained by the same kind of argument used below in the proof of Theorem 5.1), this solution must satisfy the other limit, namely (53).

Lemma 4.7 *If $\Lambda_M = \hat{\Lambda}$ is such that the corresponding solution of (22), (23) satisfies (25) and (54), then there exists some $\delta > 0$ such that for any $\Lambda_M > \hat{\Lambda} - \delta$ the solution of (22), (23) exists for all r.*

Proof. Let $\hat{g}$ be the solution corresponding to $\hat{\Lambda}$, so

$$\frac{d\hat{g}}{dr} - \frac{2c}{\lambda^2}(1 - \frac{\bar{r}}{r})\hat{g} + \hat{g}^2 = \frac{2}{\lambda^2 r}(\hat{\Lambda} - \gamma r), \quad r > M.$$

With g being the solution of (22) and (23) corresponding to Λ, write

$$\bar{g} := g - \hat{g}$$

so

$$\frac{d\bar{g}}{dr} + \frac{d\hat{g}}{dr} - \frac{2c}{\lambda^2}(1 - \frac{\bar{r}}{r})(\bar{g} + \hat{g}) + (\bar{g} + \hat{g})^2 = \frac{2}{\lambda^2 r}(\Lambda - \gamma r).$$

This implies

$$\frac{d\bar{g}}{dr} + \big(\frac{2c}{\lambda^2}(1 - \frac{\bar{r}}{r}) + 2\hat{g}\big)\bar{g} + \bar{g}^2 = \frac{2}{\lambda^2 r}(\Lambda - \hat{\Lambda}). \tag{68}$$

We now seek the solution of (68) such that $||\bar{g}|| \le \delta_1$, where

$$||\bar{g}|| := \sup_{r \ge M} |\bar{g}(r)|.$$

Here δ_1 will be chosen later in a manner which depends on δ, where $|\Lambda - \hat{\Lambda}| < \delta$. Note that (68) can be rewritten as

$$\bar{g}(r) = \bar{g}(M)\frac{1}{\bar{e}(r)} - \int_M^r \frac{\bar{e}(s)}{\bar{e}(r)}\bar{g}^2(s)ds + \frac{2}{\lambda^2}(\Lambda - \hat{\Lambda})\int_M^r \frac{\bar{e}(s)}{\bar{e}(r)}\frac{1}{s}ds,$$

where we introduced the function

$$\bar{e}(r) := \exp\Big\{\int_M^r \Big(-\frac{2c}{\lambda^2}(1-\bar{r}/s) + 2\hat{g}(s)\Big)ds\Big\}.$$

We use again the fixed-point argument to get a solution g. Denote

$$\mathbf{F} := \{f : [M,\infty) \to \mathbf{R}, ||f|| \le \delta_1, f(M) = \bar{g}(M)\},$$

where $\bar{g}(M) = g(M) - \hat{g}(M)$, and where g is the solution of (22) corresponding to $\Lambda_M = \Lambda$. We denote for $f \in \mathbf{F}$

$$Tf(r) := \bar{g}(M)\frac{1}{\bar{e}(r)} - \int_M^r \frac{\bar{e}(s)}{\bar{e}(r)}f^2(s)ds + \frac{2}{\lambda^2}(\Lambda - \hat{\Lambda})\int_M^r \frac{\bar{e}(s)}{\bar{e}(r)}\frac{1}{s}ds.$$

We know $\bar{g}(M) \to 0$ if $\Lambda \to \hat{\Lambda}$. We consider

$$|\Lambda - \hat{\Lambda}| < \delta, \;\; \bar{\delta} = |\bar{g}(M)| \max_{r \le M} \frac{1}{\bar{e}(r)},$$

where δ is small. Then take $\delta_1 > 0$ satisfying

$$\delta_1^2 \sup_{r \ge M} \int_M^r \frac{\bar{e}(s)}{\bar{e}(r)}ds + \bar{\delta} + \frac{2\delta}{\lambda^2} \sup_{r \ge M}\{\int_M^r \frac{\bar{e}(s)}{\bar{e}(r)}\frac{1}{s}ds\} < \delta_1.$$

Note for δ small enough we can take

$$\delta_1 = 2(\bar{\delta} + \frac{2\delta}{\lambda^2}\sup_{r \ge M}\{\int_M^r \frac{\bar{e}(s)}{\bar{e}(r)}\frac{1}{s}ds\}).$$

Then it is not difficult to show that the operator $T : \mathbf{F} \to \mathbf{F}$. Moreover, for arbitrary $f_1, f_2 \in \mathbf{F}$ we have

$$||Tf_1 - Tf_2|| \le 2\delta_1 \sup_{r \ge M} \frac{1}{\bar{e}(r)} \int_M^r \bar{e}(s)ds||f_1 - f_2|| = K||f_1 - f_2||.$$

By taking δ_1 small enough one has the number $K < 1$. Then T is a contraction with a unique fixed point in $\mathbf{F}$, which is the unique solution of (23). This completes the proof of Lemma 4.7. □

5 Proofs of Theorems 3.1 and 3.3

Our first result shows that the solutions of Theorem 3.2 converge as $M \to \infty$ to a solution of the HJB equation (16) that also satisfies conditions (19) and (20)(if $\mu_2 \neq 1$, (21) if $\mu_2 = 1$). Later in this section we will show uniqueness, thereby completing the proofs of both Theorems 3.1 and Theorem 3.3.

Theorem 5.1 *Let Λ_M^* and $g_M^*(r)$ be as in Theorem 3.2. Then $\Lambda_M^* \to \Lambda$ and $g_M^*(r) \to g(r)$ as $M \to \infty$, where Λ and $g(r)$ satisfy (16) and $g(r)$ also satisfies (19) and either (20) (in the case $\mu_2 \neq 1$) or (21) (in the case $\mu_2 = 1$).*

To prove Theorem 5.1 we need the following four lemmas. The first two of these are based upon the following equation:

$$\Lambda = \frac{1}{2}\lambda^2 r\frac{dg}{dr} + \frac{1}{2}\lambda^2 r(1+\frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2})g^2 + b(r)g + d(r),\ r \leq R_0+1. \tag{69}$$

Here $R_0 > 0$, and note this equation is essentially the same as (22), which is part of the dynamical programming equation in Theorem 3.2.

Lemma 5.1 *Let $r_0 < R_0$ be arbitrary. Then there is $K > 0$, depending on r_0, R_0, and Λ, such that if (69) has a solution g, then*

$$|g(r)| \leq K,\ r_0 \leq r \leq R_0.$$

Moreover, K can be chosen to be increasing in Λ. Therefore, for r_0, R_0, Λ fixed, the set $\{|g(r)| : r_0 \leq r \leq R_0; g \ \ \text{satisfies (69)}\}$ is bounded.

Remark. If the value of the ODE solution g is specified at $r_0 < R_0$, say, then the values of g for all r will be determined. The most interesting part of this lemma is the conclusion that regardless of the initial value we choose for g at r_0, if $g(r)$ is finite in $(0, R_0+1]$, then $|g(r)| \leq K$ for all $r_0 \leq r \leq R_0$, where K is independent of g, although it may depend on R_0. Consequently, in order to have a solution of (68) we cannot arbitrarily assign a value of g at r_0. Regarding the dependence of K on Λ, an expression for K is provided after (71) below. This dependence will be used in the proof of Theorem 5.1.

Proof. Let g satisfy (69). Take $\phi : [0,\infty) \to [0,1]$ smooth such that

$$\begin{aligned} \phi(r) = &\ 1,\ r_0 \leq r \leq R_0, \\ = &\ 0,\ 0 \leq r \leq \tfrac{r_0}{2},\ R_0+1 < r. \end{aligned} \tag{70}$$

Without loss of generality, we can take ϕ such that it satisfies the following property:

$$|\frac{1}{\sqrt{\phi(r)}}\frac{d\phi(r)}{dr}| \leq K_1.$$

Here K_1 is some number that may depend on r_0 and R_0. To see this, we denote $h(r)$ by

$$h(r) := \frac{1}{\sqrt{\phi(r)}}\frac{d\phi(r)}{dr},\ r \geq R_0,$$

so

$$\sqrt{\phi(r)} = 1 + 2\int_{R_0}^{r} h(u)du,\ r > R_0.$$

We then choose $h(\cdot)$ such that $h(\cdot)$ is bounded and

$$2\int_{R_0}^{R_0+1} h(u)du = -1.$$

Moreover, we choose $h(r) = 0, r \geq R_0+1$. The derivatives of h of any order at R_0 and R_0+1 are 0. Thus ϕ satisfies the required property on $[R_0,\infty)$. We can apply a similar argument for $r \in (0, r_0]$.

Consider

$$f(r) = \frac{1}{2}\phi(r)g^2(r).$$

Then $f(r)$ takes a maximum at some r_1 satisfying $r_0/2 \le r_1 \le R_0 + 1$. Denote

$$X^2 = 2f(r_1) = 2\max f(r).$$

Then

$$\frac{df}{dr}(r_1) = 0,$$

that is,

$$\phi(r_1)g(r_1)\frac{dg}{dr}(r_1) = -\frac{1}{2}g^2(r_1)\frac{d\phi}{dr}(r_1). \tag{71}$$

We multiply (69) at $r = r_1$ by $\phi(r_1)g(r_1)$ and use (71) to get

$$\begin{aligned}\Lambda\phi(r_1)g(r_1) = \quad & -\tfrac{1}{4}\lambda^2 r_1 g^2(r_1)\tfrac{d\phi}{dr}(r_1) + \tfrac{1}{2}\lambda^2 r_1(1 + \tfrac{\gamma}{1-\gamma}\tfrac{\sigma^2}{\sigma^2+\rho^2})\phi(r_1)g^3(r_1) \\ & +b(r_1)\phi(r_1)g^2(r_1) + d(r_1)\phi(r_1)g(r_1).\end{aligned}$$

Assume $g(r_1) \neq 0$. Then divide the above relation by $g(r_1)$ to obtain

$$\begin{aligned}\Lambda\phi(r_1) = \quad & -\tfrac{1}{4}\lambda^2 r_1 g(r_1)\tfrac{d\phi}{dr}(r_1) + \tfrac{1}{2}\lambda^2 r_1(1 + \tfrac{\gamma}{1-\gamma}\tfrac{\sigma^2}{\sigma^2+\rho^2})\phi(r_1)g^2(r_1) \\ & +b(r_1)\phi(r_1)g(r_1) + d(r_1)\phi(r_1).\end{aligned}$$

Then

$$X^2 + 2\alpha X = \beta, \tag{72}$$

where

$$\alpha = \alpha(r_1) = \frac{2}{\lambda^2 r_1(1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2})}(b(r_1)\sqrt{\phi(r_1)} - \frac{1}{4}\lambda^2 r_1\frac{1}{\sqrt{\phi(r_1)}}\frac{d\phi}{dr}(r_1)),$$

$$\beta = \beta(r_1) = \frac{2}{\lambda^2 r_1(1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2})}(-d(r_1)\phi(r_1) + \Lambda\phi(r_1)).$$

From (72),

$$X = -\alpha \pm \sqrt{\alpha^2 + \beta},$$

in which case

$$|X| \le |\alpha| + \sqrt{\alpha^2 + \beta}.$$

We see $|X| \le K$, where

$$K = \max\{|\alpha(r)| + \sqrt{\alpha(r)^2 + \beta(r)}, \frac{r_0}{2} \le r \le R_0 + 1\}$$

so that K depends on r_0, R_0, and Λ. Since

$$\max_{r_0 \le r \le R_0} |g(r)|^2 \le X^2,$$

the result follows. □

The next lemma says that the set of all Λ such that (69) has a solution is bounded below.

Lemma 5.2 *For a fixed R_0, there is a $\Lambda(R_0)$ such that if (69) has a solution g, then $\Lambda \ge \Lambda(R_0)$.*

Proof. We take $0 < r_0 < R_0$ and a smooth function ϕ as in (70). We can define $\hat{b}(r)$ for all $r > 0$ such that

$$\hat{b}(r) = b(r),\ r \leq R_0$$

and such that the diffusion process defined by

$$d\hat{r}(t) = \hat{b}(\hat{r}(t))dt + \lambda\sqrt{\hat{r}(t)}dB(t)$$

has an invariant density that we denote by $\hat{p}(r)$.

Denote

$$\Phi(r) = \exp((1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2})W(r)),$$

where

$$W(r) = \int_{r_0}^{r} g(u)du.$$

Then

$$\hat{L}\Phi(r) = \hat{\Lambda}\Phi(r) - \hat{d}(r)\Phi(r),\ r \leq R_0,$$

where

$$\hat{\Lambda} = (1 + \tfrac{\gamma}{1-\gamma}\tfrac{\sigma^2}{\sigma^2+\rho^2})\Lambda,$$

$$\hat{d}(r) = (1 + \tfrac{\gamma}{1-\gamma}\tfrac{\sigma^2}{\sigma^2+\rho^2})d(r),$$

$$\hat{L}f(r) = \tfrac{1}{2}\lambda^2 r\tfrac{d^2 f}{dr^2}(r) + \hat{b}(r)\tfrac{df}{dr}(r).$$

We have

$$\int \hat{L}\Phi(r)\phi(r)\hat{p}(r)dr = \int \hat{\Lambda}\Phi(r)\phi(r)\hat{p}(r)dr - \int \hat{d}(r)\Phi(r)\phi(r)\hat{p}(r)dr.$$

The equation for the invariant density $\hat{p}$ is:

$$\frac{1}{2}\frac{d^2}{dr^2}(\lambda^2 r\hat{p}(r)) - \frac{d}{dr}(\hat{b}(r)\hat{p}(r)) = 0.$$

In one dimension, we have

$$\frac{1}{2}\frac{d}{dr}(\lambda^2 r\hat{p}(r)) - \hat{b}(r)\hat{p}(r) = 0.$$

From this and the integration by parts formula we then have

$$\int \hat{L}\Phi(r)\phi(r)\hat{p}(r)dr = -\frac{1}{2}\int \lambda^2 r\frac{d\Phi}{dr}(r)\frac{d\phi}{dr}(r)\hat{p}(r)dr.$$

Consequently,

$$\hat{\Lambda}\int \Phi(r)\phi(r)\hat{p}(r)dr = \int \hat{d}(r)\Phi(r)\phi(r)\hat{p}(r)dr - \frac{1}{2}\int \lambda^2 r\frac{d\Phi}{dr}(r)\frac{d\phi}{dr}(r)\hat{p}(r)dr. \tag{73}$$

By Lemma 5.1 there is some number K, which depends on Λ, such that

$$\tfrac{1}{K} \leq \Phi(r) \leq K,$$

$$|\tfrac{d\Phi}{dr}(r)| \leq K$$

for $r_0 \le r \le R_0$. From this and (73), we have

$$|\hat{\Lambda}| \int \Phi(r)\phi(r)\hat{p}(r)dr \le K(\|\hat{d}\| + \lambda^2 R_0 \|\frac{d\phi}{dr}\|).$$

The left hand side is larger than

$$|\hat{\Lambda}| \frac{1}{K} \int \phi(r)\hat{p}(r)dr.$$

From these, $|\hat{\Lambda}|$ has an upper bound depending only on R_0. This completes the proof. □

For the following lemma and subsequent use we shall make us of a quantity that was defined in Section 4, namely,

$$\Lambda_M^* := \inf\{\Lambda_M : \quad (22) \text{ and } (23) \text{ has a solution satisfying } (25)\}.$$

Lemma 5.3 *For each $M_0 > 0$, the set $\{\Lambda_M^*; M \ge M_0\}$ is bounded above.*

Proof. It is enough to show that there is Λ such that (22)-(23) has a solution $g = g_M$ satisfying (25) such that $g(r) > 0$ for all r, for $\Lambda_M = \Lambda$ with $M \ge M_0$, since this will imply $\Lambda_M^* \le \Lambda$ by the definition of Λ_M^*.

We take Λ large enough such that (22) has a solution g satisfying

$$\lim_{r \to 0} g(r) = \frac{1}{c\bar{r}}(\Lambda - \frac{\gamma}{2(1-\gamma)} \frac{\mu_1^2}{\sigma^2 + \rho^2}) > 0.$$

By (22), it is easy to see that $g(r) > 0$ for $0 < r \le M$, since in $0 < r \le M$, g is increasing at the zeros of g. This argument also applies to $M \le r$. That is, $g(r)$ cannot be $-\infty$ for finite r. Therefore, we get a unique solution of (22)-(23) satisfying (25). □

Lemma 5.4 *Let $g = g_M$ be a solution of (22) and (23) satisfying (25) with $\Lambda = \Lambda_M \ge \Lambda_M^*$. Then there is some $M_0 > 0$ such that for $M \ge M_0$,*

$$g(r) < 0, \;\; r > M_0.$$

Proof. By (53), $g(r)$ will be negative if r is large enough. From equation (23) and the fact that Λ is bounded below (see Lemma 5.2), it is easy to see that $g(r) < 0$ for $r > M$ if M is large enough, since $g(r)$ for $r > M$ is increasing at zeros of g. This argument also applies to $M_0 \le r \le M$. Therefore, $g(r) < 0$ for $r > M_0$ if M is large enough. This completes the proof. □

Armed with these lemmas, we can now prove Theorem 5.1.

Proof of Theorem 5.1

By Lemmas 5.2 and 5.3, for a fixed $M_0 > 0$, $\{\Lambda_M^*, M \ge M_0\}$ is bounded above and below. We can take a sequence $M_n \to \infty$ as $n \to \infty$ such that $\Lambda_{M_n}^*$ converges to some Λ. Boundedness of $\{\Lambda_{M_n}^*\}$ also implies the uniform boundedness of $\{|g_{M_n}^*(r)|\}$ on compact sets by Lemma 5.1. This further implies the uniform boundedness of $\{|\frac{dg_{M_n}^*}{dr}(r)|\}$ on compact sets, by using (22) and (23).

Therefore, we can take a subsequence of $\{M_n\}$ (still denoted by $\{M_n\}$), such that $g^*_{M_n}(r)$ converges to $g(r)$ uniformly on compact sets.

We know Λ, g satisfy (16) and g satisfies (19). In fact, we only need to rule out the possibility that g satisfies (35). But since the $g^*_{M_n}(r)$ satisfy (50) for c_1 and r_0 independent of n (see the proof of Lemma 4.2), it follows that (35) cannot hold for g.

It remains to prove that (20) or (21) (depending on the case) holds for g, because then $(\Lambda^*, g^*) = (\Lambda, g)$ satisfies the properties in Theorem 3.1 (see also 5.3 below). From this it follows that the limit of (Λ^*_M, g^*_M) as $M \to \infty$ is unique, and so Theorem 5.1 will be proved.

We now prove that (20) holds for g when $\mu_2 \neq 1$. By Lemma 5.4, there is M_0 such that

$$g(r) < 0, \; r \geq M_0. \tag{74}$$

We need to know the behaviors of the solutions of (16) as $r \to \infty$. This will be given in Theorem 5.2. Now g given above is a solution of (16). Define $\bar{g} = Ag$, $A = 1 + \gamma\sigma^2/(1-\gamma)(\sigma^2+\rho^2)$. According to this theorem, either (75) or (76) holds. From (80), we can conclude the following. If (75) holds, then $g(r) < 0$ for r large. If (76) holds, then $g(r) > 0$ for r large. Since (74) holds, we must have (75). This in turn implies (20) by a simple calculation. The case $\mu_2 = 1$ is treated in a similar manner. This completes the proof. □

Theorem 5.2 *Let (Λ, g) be a solution of (16) for $0 < r < \infty$. Then exactly one of the following relations holds:*

either

$$\lim_{r\to\infty} r(\bar{g}(r) - \bar{g}_0(r)) = -\frac{1}{8}\left(1 - \frac{\lambda}{|\lambda|}\left(\frac{-\gamma}{1-\gamma}\right)^{\frac{1}{2}} \frac{\sigma}{\sqrt{\sigma^2+\rho^2}}\right) \tag{75}$$

or

$$\lim_{r\to\infty} \frac{1}{\sqrt{r}}(\bar{g}(r) - \bar{g}_0(r)) = 2\left(-\frac{\gamma}{1-\gamma}\frac{1}{\sigma^2+\rho^2}\right)^{\frac{1}{2}} \frac{|\mu_2 - 1|}{|\lambda|}, \; \mu_2 \neq 1, \tag{76}$$

$$\lim_{r\to\infty} (\bar{g}(r) - \bar{g}_0(r)) = 2\left(\frac{c^2}{\lambda^4} - 2\frac{\gamma}{\lambda^2}\left(1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}\right)\right)^{\frac{1}{2}}, \; \mu_2 = 1. \tag{77}$$

Here

$$\bar{g}_0(r) := -\frac{b(r)}{\lambda^2 r} - \left(-\frac{2(1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2})d(r)}{\lambda^2 r} + \frac{b(r)^2}{\lambda^4 r^2}\right)^{\frac{1}{2}},$$

while $b(\cdot), d(\cdot)$, and $\bar{g}(\cdot)$ are defined by (17), (18), and (29), respectively.

In order to prove Theorem 5.2 we need three more lemmas. For these we consider a function $g(r)$ that is finite for all r and satisfies (16) and (19). Using this and $\bar{g}_0(r)$ as specified in Theorem 5.2, we then define

$$\hat{g} := \bar{g} - \bar{g}_0.$$

Since $\bar{g}_0$ satisfies

$$\frac{1}{2}\lambda^2 r \bar{g}_0(r)^2 + b(r)\bar{g}_0(r) + \bar{d}(r) = 0,$$

it follows that

$$\frac{1}{2}\lambda^2 r\frac{d}{dr}\hat{g}(r)+\frac{1}{2}\lambda^2 r\hat{g}(r)^2+\tilde{b}(r)\hat{g}(r)=L(r),$$

where

$$L(r):=\Lambda-\frac{1}{2}\lambda^2 r\frac{d}{dr}\bar{g}_0(r)$$

and

$$\tilde{b}(r):=b(r)+\lambda^2 r\bar{g}_0(r)=-\lambda^2 r\big(-\frac{2\bar{d}(r)}{\lambda^2 r}+\frac{b(r)^2}{\lambda^4 r^2}\big)^{\frac{1}{2}}=-\big(-2\lambda^2 r\bar{d}(r)+b(r)^2\big)^{\frac{1}{2}}.$$

Notice this equation can be rewritten as

$$\frac{d\hat{g}}{dr}+\hat{g}^2+\frac{2\tilde{b}(r)}{\lambda^2 r}\hat{g}=\frac{2L(r)}{\lambda^2 r}. \tag{78}$$

In order to investigate the asymptotic properties of (78) we calculate

$$\begin{aligned}
-\frac{2\bar{d}(r)}{\lambda^2 r}+\frac{b(r)^2}{\lambda^4 r^2}&=-\frac{2\gamma}{\lambda^2 r}\Big(1+\frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}\Big)\Big(\frac{1}{2}\frac{1}{1-\gamma}\frac{1}{\sigma^2+\rho^2}\bar{\mu}(r)^2+r\Big)\\
&\quad+\frac{1}{\lambda^4 r^2}\Big(-c(r-\bar{r})+\frac{\gamma}{1-\gamma}\frac{\sigma\lambda}{\sigma^2+\rho^2}\sqrt{r}\bar{\mu}(r)\Big)^2\\
&=\bar{\mu}(r)^2\bigg(-\frac{\gamma}{1-\gamma}\frac{1}{\sigma^2+\rho^2}\Big(1+\frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}\Big)\frac{1}{\lambda^2 r}+\Big(\frac{\gamma}{1-\gamma}\Big)^2\frac{\sigma^2}{(\sigma^2+\rho^2)^2}\frac{1}{\lambda^2 r}\bigg)\\
&\quad-2\frac{c\sigma\lambda}{\lambda^4}\frac{1}{\sigma^2+\rho^2}\frac{r-\bar{r}}{r}\frac{1}{\sqrt{r}}\bar{\mu}(r)+\frac{c^2}{\lambda^4}\frac{(r-\bar{r})^2}{r^2}-2\frac{\gamma}{\lambda^2}\Big(1+\frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}\Big)\\
&=-\frac{\gamma}{1-\gamma}\frac{1}{\sigma^2+\rho^2}\frac{1}{\lambda^2 r}\bar{\mu}(r)^2-2\frac{c\sigma\lambda}{\lambda^4}\frac{1}{\sigma^2+\rho^2}\frac{r-\bar{r}}{r}\frac{1}{\sqrt{r}}\bar{\mu}(r)+\frac{c^2}{\lambda^4}\frac{(r-\bar{r})^2}{r^2}-2\frac{\gamma}{\lambda^2}\Big(1+\frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}\Big).
\end{aligned}$$

Since

$$\frac{2\tilde{b}(r)}{\lambda^2 r}=-2\Big(-\frac{2\bar{d}(r)}{\lambda^2 r}+\frac{b(r)^2}{\lambda^4 r^2}\Big)^{\frac{1}{2}},$$

it follows when $\mu_2\neq 1$ that

$$\frac{2\tilde{b}(r)}{\lambda^2 r}\cong -2\Big(-\frac{\gamma}{1-\gamma}\frac{1}{\sigma^2+\rho^2}\Big)^{\frac{1}{2}}\frac{|\mu_2-1|}{|\lambda|}\sqrt{r}+O(1)\quad as\quad r\to\infty. \tag{79}$$

Moreover,

$$\bar{g}_0(r)\cong -\frac{1}{\lambda}\frac{\gamma}{1-\gamma}\frac{\sigma}{\sigma^2+\rho^2}(\mu_2-1)\sqrt{r}-\Big(\frac{-\gamma}{1-\gamma}\frac{1}{\sigma^2+\rho^2}\frac{1}{\lambda^2}\Big)^{\frac{1}{2}}|\mu_2-1|\sqrt{r}+O(1)\quad as\quad r\to\infty. \tag{80}$$

On the other hand, if $\mu_2=1$ then

$$\frac{2\tilde{b}(r)}{\lambda^2 r}\cong -2\bigg(\frac{c^2}{\lambda^4}-2\frac{\gamma}{\lambda^2}\Big(1+\frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}\Big)\bigg)^{\frac{1}{2}}+O\Big(\frac{1}{\sqrt{r}}\Big)\quad as\quad r\to\infty \tag{81}$$

and

$$\bar{g}_0(r)\cong \frac{c}{\lambda^2}-\bigg(\frac{c^2}{\lambda^4}-2\frac{\gamma}{\lambda^2}\Big(1+\frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2}\Big)\bigg)^{\frac{1}{2}}+O\Big(\frac{1}{\sqrt{r}}\Big)\quad as\quad r\to\infty.$$

From this we see that if $\mu_2 \neq 1$ then

$$L(r) \cong \frac{|\lambda|}{4}\Big(\Big(\frac{-\gamma}{1-\gamma}\frac{1}{\sigma^2+\rho^2}\Big)^{\frac{1}{2}} + \frac{\lambda}{|\lambda|}\frac{\gamma}{1-\gamma}\frac{\sigma}{\sigma^2+\rho^2}\Big)|\mu_2 - 1|\sqrt{r} + O\Big(\frac{1}{\sqrt{r}}\Big) \quad as \quad r \to \infty, \tag{82}$$

whereas if $\mu_2 = 1$ then

$$L(r) \cong \Lambda + O\Big(\frac{1}{\sqrt{r}}\Big) \quad as \quad r \to \infty.$$

We are now ready for the first of the three lemmas that will be used in the proof of Theorem 5.2.

Lemma 5.5 *There exist positive numbers c_1 and r_1 such that*

$$\hat{g}(r) > -\frac{c_1}{r}, \quad all \quad r \geq r_1. \tag{83}$$

Proof. This proof is by contradiction. Suppose it is false. Then for any $c_1 > 0$ and $r_2 > 0$ there exists some $r_0 > r_2$ such that

$$\hat{g}(r_0) \leq -c_1/r_0.$$

From this we shall prove that

$$\hat{g}(r) \leq -\frac{c_1}{r}, \quad all \quad r \geq r_0. \tag{84}$$

But if this is not true, then without loss of generality there is some $r_1 > r_0$ such that $\hat{g}(r_1) = -c_1/r_1$ and

$$\hat{g}(r) < -c_1/r, \quad r_0 < r < r_1.$$

Denoting $f(r) := \hat{g}(r) + c_1/r$, we then see that

$$\frac{df(r_1)}{dr} = -\hat{g}(r_1)^2 + \frac{L(r_1)}{\lambda^2 r_1} - \frac{2\tilde{b}(r_1)}{\lambda^2 r_1}\hat{g}(r_1) - c_1\frac{1}{r_1^2} < 0$$

if we take c_1 large enough. This is a contradiction; (84) must be true if this lemma is false.

By (84) and (78), we have

$$\frac{d\hat{g}}{dr} + \hat{g}^2 < 0.$$

Then

$$\frac{\frac{d\hat{g}}{dr}}{\hat{g}^2} + 1 < 0,$$

which implies

$$\frac{1}{\hat{g}(r_0)} - \frac{1}{\hat{g}(r)} + (r - r_0) < 0$$

for all $r > r_0$. This cannot be true for all $r \geq r_0$, so (84) leads to a contradiction. The proof is complete. □

Lemma 5.6 *Suppose for some large $r_0 > 0$ that with $r = r_0$ we have*

$$\frac{c_2}{r} < \hat{g}(r) < -\frac{2\tilde{b}(r)}{\lambda^2 r} + \frac{c_2}{r}.$$

If c_2 is large, then this inequality also holds for all $r \geq r_0$.

Proof. By (78) and (82), $\hat{g} - c_2/r$ is increasing at $r \geq r_0$ such that $\hat{g} - c_2/r = 0$. Therefore,

$$\frac{c_2}{r} < \hat{g}(r),\ r \geq r_0.$$

Denote

$$r_1 := \inf\left\{ r > r_0 : \hat{g}(r) \geq -\frac{2\tilde{b}(r)}{\lambda^2 r} + \frac{c_2}{r} \right\}.$$

We have shown $\hat{g}(r) > c_2/r$ for $r_0 < r < r_1$, so it suffices to show we have $r_1 = \infty$.

Assume not. Then $\hat{g}(r_1) = -2\tilde{b}(r_1)/(\lambda^2 r_1) + c_2/r_1$ and

$$\hat{g}(r) < -\frac{2\tilde{b}(r)}{\lambda^2 r} + \frac{c_2}{r}, \quad r_0 \leq r < r_1.$$

We now consider

$$f(r) := \hat{g}(r) + \frac{2\tilde{b}(r)}{\lambda^2 r} - \frac{c_2}{r}$$

and show that $\frac{d}{dr} f(r_1) < 0$, which leads to a contradiction. We have

$$\begin{aligned} \tfrac{d}{dr} f(r_1) = &\ \tfrac{d}{dr}\hat{g}(r_1) + \tfrac{d}{dr}\left(\tfrac{2\tilde{b}(r)}{\lambda^2 r}\right)(r_1) - \tfrac{c_2}{r_1^2} \\ = &\ \tfrac{2L(r_1)}{\lambda^2 r_1} - \tfrac{c_2}{r_1}\left(-\tfrac{2\tilde{b}(r)}{\lambda^2 r} + \tfrac{c_2}{r_1}\right) + \tfrac{d}{dr}\left(\tfrac{2\tilde{b}(r)}{\lambda^2 r}\right)(r_1) - \tfrac{c_2}{r_1^2}. \end{aligned}$$

From this and (79),(82) we can show $\frac{d}{dr} f(r_1) < 0$. This completes the proof. □

Lemma 5.7 *Let $c_1 > 0$ be small. With r_0, c_2 as in the preceding lemma such that c_2 is large enough, there exists some $r_1 > r_0$ such that*

$$\hat{g}(r) + \frac{2\tilde{b}(r)}{\lambda^2 r} \geq -c_1$$

for all $r \geq r_1$.

Proof. We first show that there is some $r_1 > r_0$ satisfying

$$\hat{g}(r_1) + \frac{2\tilde{b}(r_1)}{\lambda^2 r_1} \geq -c_1.$$

Otherwise,

$$\hat{g}(r) + \frac{2\tilde{b}(r)}{\lambda^2 r} < -c_1, \quad all \quad r > r_0. \tag{85}$$

By (78), we have

$$\frac{d\hat{g}}{dr} \geq c_1\hat{g} + \frac{2L(r)}{\lambda^2 r} \geq c_1\hat{g} - \bar{c}\frac{1}{r} \geq \frac{c_1}{2}\hat{g}$$

if c_2 is large enough. Then

$$\hat{g}(r) \geq \exp\{\frac{c_1}{2}(r - r_0)\}\hat{g}(r_0).$$

But this contradicts (85).

With r_1 as above, we now show that

$$\hat{g}(r) + \frac{2\tilde{b}(r)}{\lambda^2 r} \geq -c_1, \quad all \quad r > r_1.$$

Otherwise, there is some $r_2 > r_1$ such that $\hat{g}(r_2) + \frac{2\tilde{b}(r_2)}{\lambda^2 r_2} = -c_1$ and

$$\hat{g}(r) + \frac{2\tilde{b}(r)}{\lambda^2 r} > -c_1, \quad all \quad r_1 < r < r_2.$$

In this case we consider

$$f(r) := \hat{g}(r) + \frac{2\tilde{b}(r)}{\lambda^2 r}$$

and thus

$$\frac{d}{dr}f(r_2) = \frac{d}{dr}\hat{g}(r_2) + \frac{d}{dr}\Big(\frac{2\tilde{b}(r)}{\lambda^2 r}\Big)(r_2) = c_1\Big(-c_1 - \frac{2\tilde{b}(r_2)}{\lambda^2 r_2}\Big) + \frac{2L(r_2)}{\lambda^2 r_2} + \frac{d}{dr}\Big(\frac{2\tilde{b}(r)}{\lambda^2 r}\Big)(r_2).$$

By (79) and (82), it is easy to see that the right hand side is positive. But this is another contradiction, so this proof is complete. □

We are now ready for the proof of Theorem 5.2. Recall that $\hat{g} := \bar{g} - \bar{g}_0$.

Proof of Theorem 5.2

By Lemmas 5.5-5.7, for positive numbers r_0 and c_1, c_2 (c_1 small, c_2 large) either

$$-c_2/r < -\hat{g}(r) < c_2/r, \quad r \geq r_0 \tag{86}$$

or

$$-c_1 < \hat{g}(r) + \frac{2\tilde{b}(r)}{\lambda^2 r} < c_2/r, \quad r \geq r_0. \tag{87}$$

We first suppose that (86) holds. Denote

$$e(r) := \exp\Big(\int_{r_0}^{r} \frac{2\tilde{b}(s)}{\lambda^2 s} ds\Big),$$

so that we have for $r \geq r_0$

$$\hat{g}(r) = -\int_r^\infty \frac{L(s)}{\lambda^2 s}\frac{e(s)}{e(r)}ds + \int_r^\infty \hat{g}(s)^2\frac{e(s)}{e(r)}ds.$$

By (79) and L'Hospital's Rule we then have

$$\lim_{r\to\infty}\frac{r}{e(r)}\int_r^\infty \frac{L(s)}{\lambda^2 s}e(s)ds = \frac{1}{8}\Big(1 - \frac{\lambda\sigma}{|\lambda|}\Big(\frac{-\gamma}{1-\gamma}\Big)^{\frac{1}{2}}\Big(\frac{1}{\sigma^2+\rho^2}\Big)^{\frac{1}{2}}\Big)$$

and

$$\lim_{r\to\infty}\frac{r}{e(r)}\int_r^\infty \hat{g}(s)^2 e(s)ds = 0.$$

This implies (75).

On the other hand, suppose (87) holds. Our next step is to show that

$$-c_2\frac{1}{r} < \hat{g}(r) + \frac{2\tilde{b}(r)}{\lambda^2 r} < c_2/r, \quad r \geq r_0. \tag{88}$$

We do this by first showing that there is some number $r_1 > r_0$ satisfying (88) for $r = r_1$. This is true, for if not then

$$\hat{g}(r) + \frac{2\tilde{b}(r)}{\lambda^2 r} \leq -c_2 \frac{1}{r}, \quad r \geq r_0.$$

Then (78) implies

$$\frac{d\hat{g}}{dr} \geq c_2 \frac{1}{r}\hat{g} + \frac{2L(r)}{\lambda^2 r} \geq \frac{c_2}{2}\frac{1}{r}\hat{g},$$

where c_1 is given in (87). Integrating this we obtain

$$\hat{g}(r) \geq \hat{g}(r_0)\exp(\frac{c_2}{2}\ln r/r_0) = \hat{g}(r_0)(\frac{r}{r_0})^{c_2/2}.$$

But this contradicts the assertion that $\hat{g}(r) + \frac{2\tilde{b}(r)}{\lambda^2 r} < -c_2/r$ for all $r \geq r_0$, so we know (88) holds for some $r_1 > r_0$.

For the final step, we use an argument by contradiction, as at the beginning of this proof, to show that the inequalities in (88) hold for all $r \geq r_1$. This also implies (88) by choosing a large c_2, if necessary. Finally, (76) follows directly from (88), so this proof is completed. □

We now have fully established Theorem 5.1. Thus to complete the proofs of Theorems 3.1 and 3.3 it only remains to establish uniqueness of the solution of the HJB equation. This is accomplished by the following lemma.

Lemma 5.8 *Let g_1 and g_2 be solutions of (16) satisfying (19) corresponding to Λ_1 and Λ_2, respectively. Let $\hat{g}_1 = \bar{g}_1 - \bar{g}_0$ and $\hat{g}_2 = \bar{g}_2 - \bar{g}_0$ with $\bar{g}_0$ defined as in Theorem 5.2, and suppose $\hat{g}_1$ and $\hat{g}_2$ both satisfy limit (75). Then $g_1 = g_2$ and $\Lambda_1 = \Lambda_2$.*

Proof. Denote

$$\bar{\Lambda}_1 = (1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2})\Lambda_1, \ \bar{\Lambda}_2 = (1 + \frac{\gamma}{1-\gamma}\frac{\sigma^2}{\sigma^2+\rho^2})\Lambda_2.$$

We subtract the equation for $\hat{g}_2$ from the equation for $\hat{g}_1$, thereby obtaining

$$\frac{d}{dr}(\hat{g}_2 - \hat{g}_1) + \Big(\frac{2\tilde{b}(r)}{\lambda^2 r} + \hat{g}_1 + \hat{g}_2\Big)(\hat{g}_2 - \hat{g}_1) = \frac{\bar{\Lambda}_2 - \bar{\Lambda}_1}{\lambda^2 r}.$$

Denote

$$\tilde{e}(r) := \exp\Bigg(\int_{r_0}^{r}\Big(\frac{2\tilde{b}(s)}{\lambda^2 s} + \hat{g}_1(s) + \hat{g}_2(s)\Big)ds\Bigg).$$

Then

$$\frac{d}{dr}\Big(\big(\hat{g}_2(r) - \hat{g}_1(r)\big)\tilde{e}(r)\Big) = \frac{\bar{\Lambda}_2 - \bar{\Lambda}_1}{\lambda^2 r}\tilde{e}(r),$$

and so

$$\big(\big(\hat{g}_2(r) - \hat{g}_1(r)\big)\tilde{e}(r) = -\int_r^\infty \frac{\Lambda_2 - \Lambda_1}{\lambda^2 s}\tilde{e}(s)ds.$$

Without loss of generality, suppose $\Lambda_2 - \Lambda_1 \geq 0$, in which case $\hat{g}_2(r) - \hat{g}_1(r) \leq 0$. But Lemma 4.3 implies $\hat{g}_2(r) - \hat{g}_1(r) \geq 0$. Therefore, $\hat{g}_2(r) = \hat{g}_1(r)$, $\Lambda_2 = \Lambda_1$, and this proof is completed. □

Ramark. The following result, not crucial for the proofs of Theorems 3.1 or 3.3, says that Λ^* is the smallest number such that (16) has a solution defined on $[0, \infty)$. For $\Lambda = \Lambda^*$, (16) has a unique solution. A more general result of this kind is given in the paper by Kaise and Sheu [16].

Theorem 5.3 *Let* Λ^* *be given in Theorem 3.1. Then there is only one solution for (16) on* $[0, \infty)$ *with* $\Lambda = \Lambda^*$.

If (16) has a solution on $[0, \infty)$, *then* $\Lambda \geq \Lambda^*$.

Proof. We consider only $\mu_2 \neq 1$. The argument for the case $\mu_2 = 1$ is similar. Assume $\Lambda = \Lambda^*$ and g is a solution of (16) on $[0, \infty)$. Assume $g \neq g^*$ given in Theorem 3.1. Then (35) holds for g. Since g^* satisfies (36), a simple comparison argument for ODE shows that $g(r) < g^*(r)$ for all r. But g satisfies either one of (75) or (76). Since g^* satisfies (75), therefore $g < g^*$ implies that g also satisfies (75). Now by Lemma 5.8, we conclude $g = g^*$, a contradiction.

We now consider Λ such that (16) has a solution g_0 defined on $[0, \infty)$. Then (16) must have a solution g defined on $[0, \infty)$ satisfying (36). If g_0 also satisfies (36), then $g = g_0$. See Corollary 4.1. If g_0 satisfies (35), then we have $g(r) > g_0(r)$ for small $r > 0$, and hence for all r. This implies g is also defined for all $r > 0$. Now if $\Lambda < \Lambda^*$, then $g(r) < g^*(r)$ for all r by Lemma 4.3. By Theorem 5.2, g either satisfies (75) or (76). We know g^* satisfies (75). Together with $g < g^*$, we conclude that g satisfies (75). By Lemma 5.8, we have $g = g^*, \Lambda = \Lambda^*$, a contradiction. This completes the proof. □

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