Risk Sharing in Private Information Models with Asset Accumulation: Explaining the Excess Smoothness of Consumption*

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Abstract

We study testable implications for the dynamics of consumption and income of models in which first-best allocations are not achieved because of a moral hazard problem with hidden saving. We show that in this environment agents typically achieve more insurance than that obtained under self insurance with a single asset. Consumption allocations exhibit 'excess smoothness', as found and defined by Campbell and Deaton (1989). We argue that excess smoothness, in this context, is equivalent to a violation of the intertemporal budget constraint considered in a Bewley economy (with a single asset). We also show parameterizations of our model in which we can obtain a closed-form solution for the efficient insurance contract and where the excess smoothness parameter has a structural interpretation in terms of the severity of the moral hazard problem. We present tests of excess smoothness, applied to UK micro data and constructed using techniques proposed by Hansen et al. (1991) to test the intertemporal budget constraint. Our theoretical model leads us to interpret them as tests of the market structure faced by economic agents. We also construct a test based on the dynamics of the cross sectional variances of consumption and income that is, in a precise sense, complementary to that based on Hansen et al (1991) and that allows us to estimate the same structural parameter. The results we report are consistent with the implications of the model and internally coherent.

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1 Introduction

In this paper, we study the intertemporal allocation of consumption in a model with moral hazard and hidden saving. We characterize some of the empirical implications of such a model and show how it can explain some well-known puzzles in the consumption literature.

The ability to smooth out income shocks and avoid large changes in consumption is an important determinant of individual households' welfare. Resources can be moved over time and across states of the world using a variety of instruments, ranging from saving and borrowing to formal and informal insurance arrangements. The study of the intertemporal allocation of consumption is equivalent to establishing what instruments are available to individual households (that is, what are the components of the intertemporal budget constraint) and how they are used.

When thinking of the smoothing of individual shocks, two mechanisms come immediately to mind. On one hand, households can use a private technology to move resources over time, for instance saving in 'good times' and dis-saving in 'bad times'. On the other, they can enter implicit or explicit insurance arrangements against idiosyncratic shocks. There might be links between these two mechanisms: with endogenously incomplete markets, the availability of specific instruments to move resources over time is likely to affect the type of insurance arrangements that can be sustained in equilibrium. We study consumption smoothing within a unifying framework and consider explicitly the links between private (anonymous) savings and insurance in a moral hazard model. We therefore contribute to and bring together two strands of the literature: that on consumption smoothing and that on endogenously incomplete markets. In particular, we show how some empirical results in the consumption literature can be interpreted, with the help of the model we construct, as providing evidence on partial insurance and on the relevance of specific imperfections.

The permanent income/life-cycle (PILC) model has been, for a long time, the workhorse to study individual consumption smoothing and the intertemporal allocation of resources. During the 1980s and 1990s, a number of papers, starting with Campbell and Deaton (1989), pointed to the fact that consumption seems 'too smooth' to be consistent with the model's predictions, in that consumption does not react sufficiently to innovation to the permanent component of income. This evidence, derived from aggregate data, was interpreted as a failure of the PILC model.

In such a model, allocations are determined by a sequence of Euler equations and an intertemporal budget constraint, often characterized by the presence of a single asset with a fixed rate of return. We will refer to the latter as the 'risk-free IBC'. Campbell and Deaton (1989), as other papers in this literature (including West (1988) and Gali (1991)), take the risk free IBC (and in particular the instruments available to an individual to move resources over time) as given and therefore interpret the evidence of 'excess smoothness' as a failure of some of the Euler equations. Indeed, Campbell and Deaton (1989, p. 372) conclude their paper by saying:'Whatever it is that causes changes in consumption to be correlated with lagged changes in income whether [...] the marginal utility of consumption depends on other variables besides consumption, or that consumers are liquidity constrained or that consumers adjust slowly through inertia or habit formation, the same failure is responsible for the smoothness of consumption ...'. All these explanations, however, maintain a risk-free IBC in which individuals move resources over time through a single asset with a fixed interest rate. We propose a model in which excess smoothness is related to a violation of the risk free IBC which arises because the standard model neglects trades in state-contingent assets (or state-contingent transfers) that give individuals 'more insurance' than they would be able to get by self insurance and saving. In our model, the amount of risk sharing enjoyed by individuals is determined endogenously and depends on the severity of the moral hazard problem.

As we hint above, an IBC defines the structure of the market to which individuals have access. Therefore a violation of a specific IBC can be interpreted as a violation of a given market structure. Given that our approach focuses on violation of a risk-free IBC, we use a test of an IBC proposed by Hansen, Roberds, and Sargent (1991; henceforth HRS). The HRS test is particularly appropriate in our context as it can be explicitly interpreted as a test of a market structure. Moreover, because of the way it is formulated, not only does it encompass other tests (such as the Campbell and Deaton (1989) or West (1991)) but it is also robust to informational advantages of the agents relative to the econometrician. The cost of applying the HRS test is that it requires the Euler equation to hold. However, this is consistent with the model we construct and, more importantly, is an empirically testable proposition.

When framed within the context of endogenously incomplete markets, it seems intuitive that a model that predicts some additional insurance relative to the amount of (self) insurance one observes in a Bewley model would predict that consumption reacts 'less' to shocks than is predicted by the Bewley model. To make such a statement precise, however, is not a trivial exercise. In what follows, in addition to such a formal statement, we also obtain, for some versions of our model, closed-form solutions for the optimal intertemporal allocations and map the coefficient of 'excess smoothness' of the consumption literature and of the HRS test into a structural parameter of our model reflecting the severity of the moral hazard problem.

This paper, therefore, makes two main contributions. The first is of a theoretical nature: we show how a model with moral hazard and hidden saving, by providing additional insurance relative to a Bewley economy, generates what in the consumption literature has been named the excess smoothness of consumption. In addition, we construct specific examples in which we can derive a closed-form solution and relate directly the excess-smoothness coefficients to a structural parameter of our model. While the examples are surely special, we stress that, in equilibrium, they deliver an income process that coincides with the process typically used in the Permanent Income Hypothesis (PIH) literature. Moreover, the examples give an intuition that can be generalized to a larger class of models. Our theoretical result allows us to interpret in a specific fashion the empirical findings we obtain. The second contribution is empirical: we adapt the HRS test to micro data and apply it on a time-series of cross-sections from the UK. This evidence constitutes one of the first examples of an excess smoothness test performed on micro data. We also complement the HRS test with a test based on movements in the cross sectional variance of consumption. Because of the nature of our data, we are forced to use the HRS test on time-series of cross-sections and therefore focus on risk sharing across cohorts. The variance test complements the HRS one because it focuses on insurance within cohorts.

Our empirical results, obtained from UK time-series of cross-sections, are remarkably in line with the predictions of the model we construct.

The rest of the paper is organized as follows. In Section 2, we present the building blocks of our model, provide the equilibrium definition, and discuss alternative market structures. In Section 3, we characterize the equilibrium allocation for our model and present some examples that yield useful closed form solutions. In Section 4, we discuss the empirical implications of the equilibria we considered in Section 3 and present our tests and the results we obtain. Section 5 concludes. The appendices contain the proofs of the results stated in the text.

2 Model

2.1 Tastes and Technology

Consider an economy consisting of a large number of agents that are ex-ante identical. Each agent lives $T \leq \infty$ periods and is endowed with a private stochastic production technology which takes the following form (neglecting individual indices for notational ease):

$$y_t = f(\theta_t, e_t).$$

That is, the individual income $y_t \in Y \subset \Re$ is determined by the agent's effort level $e_t \in E \subset \Re$ and the shock $\theta_t \in \Theta \subset \Re$. The history of income up to period t will be denoted by $y^t = (y_1, ..., y_t)$, while the history of shocks is $\theta^t = (\theta_1, ..., \theta_t)$. Let $\Phi(\theta_{t+1} \mid \theta^t)$ be the conditional probability of θ_{t+1} conditioned on $\theta^t \in \Theta^t$. The component θ_t can be interpreted as the the agent's skill level at date t. At this stage, we do not impose any specific structure on the time-series properties of θ_t, but we assume that θ^t are iid across individuals. In each period, the effort e_t is chosen after observing θ_t. The function $f : \Theta \times E \to Y$ is assumed to be continuous and increasing in both arguments. Both the effort e and the shocks θ are assumed to be private information, giving rise to moral hazard problems, while y_t is publicly observable.

Agents are born with no wealth, have von Neumann-Morgenstern preferences, and rank deterministic sequences according to

$$\sum_{t=1}^{T} \delta^{t-1} \, u(c_t, e_t),$$

with $c_t \in C$ and $\delta \in (0,1)$. We assume u to be real valued, continuous, strictly concave, smooth, strictly increasing in c_t and strictly decreasing in e_t. Notice that, given a plan for effort levels, there is a deterministic and one-to-one mapping between histories of the private shocks θ^t and income y^t. Therefore, we can consider θ^t alone. Let μ^t be the probability measure on Θ^t and assume that the law of large numbers applies, so that $\mu^t(A)$ is also the fraction of agents with histories $\theta^t \in A$ at time t.

Since θ^t are unobservable, we make use of the revelation principle and define a reporting strategy $\sigma = \{\sigma_t\}_{t=1}^T$ as a sequence of θ^t-measurable functions such that $\sigma_t : \Theta^t \to \Theta$ and $\sigma_t(\theta^t) = \hat{\theta}_t$ for some $\hat{\theta}_t \in \Theta$. A truthful reporting strategy σ^* is such that $\sigma_t^*(\theta^t) = \theta_t$ almost surely (a.s.) for all θ_t. Let Σ be the set of all possible reporting strategies. A reporting strategy essentially generates publicly observable histories according to $h^t = \sigma(\theta^t) = (\sigma_1(\theta_1), ..., \sigma_t(\theta^t))$, with $h^t = \theta^t$ when $\sigma = \sigma^*$.

An allocation (e, c, y) consists of a triplet $\{\mathbf{e}_t, \mathbf{c}_t, \mathbf{y}_t\}_{t=1}^T$ of θ^t-measurable functions for effort, consumption, and income growths (production) such that they are 'technically' attainable:

$$\Omega = \left\{(e, c, y) : \forall\, t \geq 1, \quad \theta^t, \; \mathbf{e}_t(\theta^t) \in E, \; \mathbf{c}_t(\theta^t) \in C, \quad \text{and} \;\; \mathbf{y}_t(\theta^t) = f\left(\theta_t, \mathbf{e}_t(\theta^t)\right)\right\}.$$

The idea behind this notation is that incentive compatibility will guarantee that the agent announces truthfully his endowments (i.e. uses σ^*) so that, in equilibrium, private histories are public information.

For simplicity, we disregard aggregate shocks, and we do not allow for productive assets such as capital. In the baseline model, we assume the availability of a constant return technology that allows $q \in (0,1)$ units of consumption at time t to be transformed into one unit of consumption at time $t+1$, or vice versa. Equivalently, we are assuming a small open economy or a village economy where a money lender has access to an external market with an exogenously given interest rate r. The number $q = \frac{1}{1+r}$ can be interpreted as the time constant price of a one period bond in the credit market. In this case, absent aggregate shocks, thanks to the law of large numbers, the feasibility condition is constituted be a unique inequality:

$$\int_{\Theta^t} \left[\sum_{t=1}^{T} q^{t-1} \mathbf{c}_t(\theta^t) \right] d\mu^t \left(\theta^t\right) \leq \int_{\Theta^t} \left[\sum_{t=1}^{T} q^{t-1} \mathbf{y}_t(\theta^t) \right] d\mu^t \left(\theta^t\right). \tag{1}$$

Although we present our results with the assumption of a small open economy with a constant interest rate, in the appendices we show that the same results can be derived with a time varying interest rate and even for the case of a closed economy.[1]

2.2 Equilibrium and market arrangements

Having described agents' tastes and the technological environment they face, to characterize intertemporal allocations we need to specify the market arrangements in which they operate and define the relevant equilibrium concepts.

Our economy is characterized by private information, as we assume that effort and skill level are not observable. In addition to the moral hazard problem, consumption is also not observable (and/or contractable) and agents have hidden access to an anonymous credit market where they trade risk-free bonds at price q. The agents do not have private access to any other anonymous asset markets.[2]

The market structure we adopt is an open economy version of that proposed in Golosov and Tsyvinski (2007). In addition to the anonymous (secondary) market for assets, there is a (primary) insurance market where a continuum of identical firms offer exclusive contracts to agents. All insurance firms are owned equally by all agents. At the beginning of period 1, each firm signs a contract (e, c, y) with a continuum of agents in the economy that is binding for both parties. Firms operate in a competitive market and agents sign a contract with the firm that promises the highest ex-ante expected discounted utility. After the contract is signed, each agent chooses a reporting strategy σ, supplies effort levels so as to generate, at each node, the agreed income level $y_t\left(\sigma(\theta^t)\right)$, and receives $\mathbf{c}_t\left(\sigma(\theta^t)\right)$ units of consumption. Firms can borrow and lend at the ongoing interest rate, which they take as given.

[1]For a similar model in a small open economy, see Abraham and Pavoni (2004, 2005, and 2008). For another similar analysis in a closed economy with capital, see Golosov and Tsyvinski (2007).

[2]The assumption that in the anonymous asset market only the risk-free asset is traded is done without loss of generality in our environment as no other Arrow security would be traded (see Golosov and Tsyvinski (2007))

We define expected utility from reporting strategy $\sigma \in \Sigma$, given the allocation $(e,c,y) \in \Omega$, as

$$\mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1} u\left(c_t, e_t\right) \mid (e,c,y)\,,\ \sigma\right] := \sum_{t=1}^{T} \delta^{t-1} \int_{\Theta^t} u\left(\mathbf{c}_t\left(\sigma(\theta^t)\right), g\left(\theta_t, \mathbf{y}_t\left(\sigma(\theta^t)\right)\right)\right) d\mu^t(\theta^t),$$

where $g(y,\theta)$ represents the effort level needed to generate y when the shock is θ, i.e., g is the inverse of f with respect to e keeping θ fixed. Since y is observable, the mis-reporting agent must adjust his/her effort level so that the lie is not detected.

Let $b := \{\mathbf{b}_{t+1}\}_{t=1}^{T}$ be a plan of risk-free asset holding, where $\mathbf{b}_t$ is a θ^{t-1}-measurable function with $b_1 = 0$ and $\lim_{t\to T} q^{t-1}\mathbf{b}_{t+1}(\theta^t) \geq 0$. Given the price of the bond q, the typical firm, in equilibrium, solves the following problem:

$$\max_{b,(e,c,y)\in\Omega} \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right) \mid (e,c,y)\,,\ \sigma^*\right], \tag{2}$$

$$\text{s.t. } \mathbf{E}\left[\sum_{t=1}^{T} q^{t-1}\left(c_t - y_t\right) \mid (e,c,y)\,,\ \sigma^*\right] \leq 0, \tag{3}$$

with the incentive compatibility constraint:

$$\mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right) \mid (e,c,y)\,,\ \sigma^*\right] \geq \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(\hat{c}_t^{\sigma}, \hat{e}_t^{\sigma}\right) \mid (e,c,y)\,,\ \sigma\right] \text{ for all } \sigma \in \Sigma, \tag{4}$$

where the deviation for consumption $\hat{c}^{\sigma}$ must be such that the new path of consumption can be replicated by the use of a risk free bond. More precisely, given (e,c,y,b), for each σ, a deviation $\hat{c}^{\sigma}$ is admissible if there is a plan of bond holdings $\hat{b}^{\sigma}$ such that for all t and a.s. for all histories θ^t,

$$\hat{\mathbf{c}}_t^{\sigma}\left(\sigma_t\left(\theta^t\right)\right) + q\hat{\mathbf{b}}_{t+1}^{\sigma}(\sigma_t\left(\theta^t\right)) - \hat{\mathbf{b}}_t^{\sigma}(\sigma_{t-1}\left(\theta^{t-1}\right)) = \mathbf{c}_t\left(\sigma_t\left(\theta^t\right)\right) + q\mathbf{b}_{t+1}(\sigma_t\left(\theta^t\right)) - \mathbf{b}_t(\sigma_{t-1}\left(\theta^{t-1}\right)),$$

and $\lim_{t\to T} q^{t-1}\hat{\mathbf{b}}_{t+1}^{\sigma}(\sigma_t\left(\theta^t\right)) \geq 0$.

Definition 1 *Given q, an equilibrium for the economy is an allocation (e^*, c^*, y^*) and bond trades b^* such that*
i) firms choose (e^*, c^*, y^*, b^*) solving problem (2)-(4) (taking q as given);
ii) agents choose their reporting strategy, effort levels, consumption, and asset trades optimally as described above, given the contract (and q);
iii) the intertemporal aggregate feasibility constraint (1) holds.

It is straightforward to see that the incentive constraint (4) (considered at σ^*) implies that the equilibrium allocation must satisfy the Euler equation:

$$u_c'(c_t^*, e_t^*) = \frac{\delta}{q}\mathbf{E}_t\left[u_c'\left(c_{t+1}^*, e_{t+1}^*\right)\right], \tag{5}$$

where the marginal utilities are evaluated at the equilibrium values dictated by (e^*, c^*, y^*), and $\mathbf{E}_t[\cdot]$ is the conditional expectation operator on future histories given θ^t.[3]

2.3 Two useful extreme cases

With an eye to our empirical strategy, it is useful to compare the allocations in our imperfect information model with those that obtain in two well known alternative and extreme cases: a full information setting and a situation where intertemporal trades are exogenously restricted to the risk-free bond. Both situations can be obtained, under certain parameter settings, as special cases of our model. Moreover, both can be characterized by a different intertemporal budget constraint.

Full information. In the model where there is no private information problem, in equilibrium, the typical firm offers exclusive contracts solving problem (2) and (3) alone. That is, the firm will solve the same problem as before with the crucial exception of condition (4). Since in this case we are in a complete market setting, we can apply the well-known Welfare Theorems which imply that the equilibrium allocation solves the problem of a planner aiming to maximize the representative agent's utility subject to the feasibility constraint, condition (1).

Note that, as in our set-up, with full information the equilibrium allocation satisfies the Euler equation (5). In fact, the equality between present and future discounted marginal utility is true state by state: $u'_c(c_t, e_t) = \frac{\delta}{q} u'_c(c_{t+1}, e_{t+1})$; and hence also in expected terms.

The Bewley economy (self-insurance). We define a Bewley economy (or self-insurance) as an economy where agents can move resources over time only by participating in a simple credit market with a constant one-period bond with price q. Recall that an asset holding plan b is made of a set of θ^{t-1}-measurable function $\mathbf{b}_t$, $t = 1, ..., T$ with $b_1 = 0$. In this economy, a typical agent solves

$$\max_{b,(e,c,y)\in\Omega} \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1} u(c_t, e_t)\right]$$

subject to

$$\mathbf{c}_t(\theta^t) + q\mathbf{b}_{t+1}(\theta^t) \leq \mathbf{b}_t(\theta^{t-1}) + \mathbf{y}_t(\theta^t), \tag{6}$$

where $b_0 = 0$. As usual we rule out Ponzi games by requiring that $\lim_{t\to T} q^{t-1}\mathbf{b}_t(\theta^t) \geq 0$. Condition (6) is the budget constraint typically used in Permanent Income models when the agent has access only to a risk-free bond market. This problem can be seen as an extension of the permanent-income

[3] This condition is the first-order equivalent to the incentive constraint that ensures that the agent is not willing to deviate in assets decisions alone, while contemplating telling the truth about shock histories θ^t.

model studied by Bewley (1977), which allows for endogenous labour supply and non-stationary income.

The main implication of the self insurance model is the well-known Euler equation:

$$u'_c(c_t, e_t) = \frac{\delta}{q}\mathbf{E}_t\left[u'_c(c_{t+1}, e_{t+1})\right]. \tag{7}$$

Another important necessary condition that individual intertemporal allocations have to satisfy in this model can be derived by repeatedly applying the budget constraint (6). Starting from any node $\bar{\theta}^{t-1}$, $t \geq 1$, with asset holding level $\mathbf{b}_t(\bar{\theta}^{t-1})$, the following net present value condition (NPVC) must be satisfied

$$\sum_{n=t}^{T} q^{n-t}\left(\mathbf{c}_n(\theta^n) - \mathbf{y}_n(\theta^n)\right) \leq \mathbf{b}_t(\bar{\theta}^{t-1}), \tag{8}$$

a.s. for all histories θ^T emanating from node $\bar{\theta}^{t-1}$.

Given the income process and the price for the bond q, conditions (7) and (8) define consumption (even when a closed form solution does not exist). It is interesting to compare NPVC (8) for $t = 1$ with the corresponding equation for the full information case, equation (1). In the latter case, the agent has available a wide array of state-contingent securities that are linked in an individual budget constraint that sums over time and across histories, as all trades can be made at time 1. In the Bewley economy, instead, the agent can only trade in a single asset. This restriction on trade requires that the net present value of consumption minus income equals zero a.s. for *all* histories θ^T. Notice that there is no expectation operator involved in condition (8): the intertemporal transfer technology implied by a risk-free asset does not allow for cross-subsidizations of consumption across income histories.

The lesson we should retain from this section is that all allocations considered here satisfy the Euler equation: clearly such an equation is consistent with many stochastic processes for consumption. The key feature that distinguishes the three different equilibrium allocations we have considered is the intertemporal budget constraint, which defines the relevant market structure. For instance, the equilibrium allocation in our moral hazard model typically violates the NPVC (8) based on a *single asset* which is relevant for the permanent income model because it ignores the state-contingent payments the agents might receive from the insurance firm in our economy.

3 Characterizing the equilibrium allocation

In this section, we analyse the properties of the endogenously incomplete markets model we presented above. It is easy to show that the equilibrium allocation (e^*, c^*, y^*) and b^* can be replicated by an incentive compatible plan of lump sum transfers $\tau^* = \{\tau_t^*(\theta^t)\}_{t=1}^T$ that solves the following firm's problem:

$$\max_{\tau,(e,c,y)\in\Omega} \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right)\right]$$

s.t.

$$\mathbf{c}_t(\theta^t) = \mathbf{y}_t(\theta^t) + \tau_t(\theta^t), \tag{9}$$

the incentive constraint (4), and the intertemporal budget constraint

$$\mathbf{E}\left[\sum_{t=1}^{T} q^{t-1}\tau_t\right] \leq 0.$$

It is clear, from condition (9), that we study equilibrium allocations where agents do not trade intertemporally ($b_t^* \equiv 0$). This is done without loss of generality since the firms and the agents face the same interest rate.[4] Alternatively, τ could be chosen so that the transfer $\tau_t = \tau_t(\theta^t)$ represents the net trade on state contingent assets the agent implements at each date t and node θ^t. In this case, the lump sum transfer would solve $\mathbf{E}\left[\tau_t\right] = 0$ and all the intertemporal transfer of resources will in effect be made by the agents.

In Appendix A, we show that, under some conditions, the equilibrium allocation (e^*, c^*, y^*) and $b_t^* \equiv 0$ can be 'implemented' using a transfer scheme τ^* which is a function of income histories y^t alone.[5] This simplifies the analysis and allows us to describe the consumption allocation in terms of observables. At history y^t, an agent with asset level b_t faces the following budget constraint for all y^t

$$c_t + qb_{t+1} = y_t + \tau_t^*(y^t) + b_t.$$

3.1 Excess smoothness and the Bewley economy

Allen (1985) and Cole and Kocherlakota (2001) present asymmetric information models that give rise to intertemporal allocations that coincide with those that would prevail in the Bewley economy we described in Section 2.3. In this section, we first present a specific version of our model where the 'Allen and Cole-Kocherlakota' (ACK) result obtains. As stressed in Abraham and Pavoni (2004), to obtain the 'self-insurance' result a crucial restriction on the way effort is converted into output is needed. We then move on to relax this restriction. Within the more general case, we consider a specific parameterization of the income process that allows us to obtain a closed-form solution for the equilibrium transfers. While this example is useful because it gives very sharp predictions, some

[4] E.g., see Fudenberg et al. (1990).

[5] In particular, we assume that the optimal plan of consumption c is y^t-measurable.

of the properties of the allocations we discuss generalize to the more general case and inform our empirical specification.

Self-insurance Let us assume agents' preferences and production satisfy

$$u\left(c,e\right)=u(c-e) \qquad \text{and} \qquad f\left(\theta,e\right)=\theta+e, \tag{10}$$

with $\Theta=(\theta_{\min},\theta_{\max})$ and $E=(e_{\min},e_{\max})$. Obviously, in this environment the optimal plan of effort levels is indeterminate. As long as $0\in E$, we can hence set, without any loss of generality, $e_t^*\equiv 0$. This normalization has two advantages. First, since e_t does not change with θ^t while $f(\theta_t,e_t)$ is strictly increasing in θ_t, all variations in θ_t will induce variations in y_t, automatically guaranteeing the y^t-measurability of c (see Appendix A). Second, with constant effort we can focus on the risk sharing dimension of the equilibrium allocation. This last argument also motivates the modelling choice for our closed-form solution below.

Proposition 1 *Assume $T<\infty$ and that the utility function u and the production function f are as in (10). Then the equilibrium allocation coincides with self-insurance.*

The proof of the proposition is reported in Appendix A, where we show that incentive compatibility fully characterizes the equilibrium allocation. The main intuition of the result can be easily seen in a two period model, which we solve backwards. Consider the last period of the programme. Using the budget constraint, given any history y^{T-1} and θ_T (and last period wealth b_T) the incentive constraint (for $e_T^*=0$) is as follows

$$\begin{aligned} u\left(c_T^*-e_T^*\right) &= u\left(\theta_T+\tau_T^*\left(y^{T-1},\theta_T\right)+b_T\right)\\ &\geq u\left(\theta_T+\tau_T^*\left(y^{T-1},\theta_T+\hat{e}_T\right)+b_T\right) \text{ for all } \hat{e}_T\in E. \end{aligned}$$

Clearly, in order to be incentive compatible, the transfer scheme must be constant across y_Ts. Now consider the problem in period $T-1$. For ease of exposition, let us assume that the transfer scheme is differentiable,[6] and write the agent's first order conditions with respect to e_{T-1} (evaluated at the optimum $b_T^*=b_{T-1}^*=e_T^*=e_{T-1}^*=0$). A necessary condition for incentive compatibility is[7]

$$\frac{\partial\tau_{T-1}^*\left(y^{T-1}\right)}{\partial y_{T-1}}+\delta\frac{\partial\tau_T^*\left(y^T\right)}{\partial y_{T-1}}\mathbf{E}_{T-1}\left[\frac{u'\left(c_T^*\right)}{u'\left(c_{T-1}^*\right)}\right]=0.$$

[6] The formal proof does not assume differentiability. See Appendix A.

[7] Note that in the expression below $\frac{\partial\tau_T^*(y^T)}{\partial y_{T-1}}$ has been taken out from the expectation operator as we saw that τ_T is constant in y_T shocks.

The Euler equation - i.e., the (local) incentive constraint for bond holding - reads as $\mathbf{E}_{T-1}\left[\frac{u'(c_T^*)}{u'(c_{T-1}^*)}\right] = \frac{q}{\delta}$, which implies

$$\frac{\partial \tau_{T-1}^*(y^{T-1})}{\partial y_{T-1}} + q\frac{\partial \tau_T^*(y^T)}{\partial y_{T-1}} = 0. \tag{11}$$

Equation (11) states that the net present value of transfers must be constant across income histories. The intuition for the fact that the firm is unable to provide any consumption insurance on top of self-insurance is relatively simple in this case. First, at each t with equilibrium income $y_t^* = \theta_t$, the agent can always deviate, locally, and generate any income level such that $y_t^* - \epsilon \leq \hat{y}_t \leq y_t^* + \epsilon$. Second, the perfect substitutability between consumption and effort in the utility function on one side and between income and effort in production on the other side imply that such deviation has *zero direct cost* to the agent. Since this is true for all income levels, in a static setting, these two observations together imply that the firm will never be able to provide a transfer scheme that induces anything different from constant payments over income levels. In our model, this simple intuition extends to a general dynamic setting. Roughly speaking, the free access to the credit market implies that the agent only cares about the *present value* of transfers (i.e., he/she does not care about the exact timing of deterministic transfer payments). Hence, a simple extension of the previous argument implies that the present value of transfers must be constant across income histories, as otherwise the agent would find it profitable to perform some local deviation on effort and engage in an appropriate bond plan.

Now recall that the self insurance allocation has two defining properties: first, it must satisfy the Euler equation; second, it must satisfy the intertemporal budget constraint with one bond, i.e., the period zero net present value must be zero for all y^T. Since the Euler equation is always satisfied here, the only way of obtaining a different allocation is that the transfers scheme τ^* permits violation of the agent's period zero self insurance intertemporal budget constraint for some history y^T. The previous argument demonstrates that it cannot be the case; hence the only incentive compatible allocation coincides with self-insurance. This implies that the 'relaxed-optimal' contract obtained by using the local (first-order-condition) version of the incentive constraint corresponds to the self-insurance allocation. Since this allocation is obviously globally incentive compatible, it must be the optimal one.

The proof of Proposition 1, in Appendix A, never uses the time series properties of θ_t, which can therefore be very general. Likewise, we do not require a constant q.[8] Proposition 1 is therefore, in

[8]Both Allen (1985) and Cole and Kocherlakota (2001) assume iid shocks and constant q. We generalize Allen's result, while our model does not nest, strictly speaking, Cole and Kocherlakota's as we do not impose exogenous and binding liquidity constraints.

some dimensions, a general result. Indeed, as we prove in the Web Appendix B, Section 7.4, it holds even in a more general model that allows for two types of income shocks, with different degrees of persistence.

Proposition 1 and its proof, however, also make it clear that the coincidence between the equilibrium allocation and self-insurance is, in other dimensions, very fragile. Below, we present a full class of models with slightly more general u and f functions, where the equilibrium allocation coincides with self-insurance only for a zero measure set of parameters.[9]

Excess smoothness Consider now the following generalization of agent preferences:

$$u(c, e) = u(c - v(e)),$$

with v increasing and convex. Since consumption and *effort cost* enter the utility function in a linear fashion, we eliminate the wealth effects. This simplifies the analysis and allows for closed-form solutions. Moreover, it is crucial for the self-insurance result we derived above. While we only consider interior solutions for e, we leave the function f unspecified.

Again, the main intuition regarding the model can be gained by studying the last two periods. Let's start with the final period (T). The first order condition of the incentive constraint in this last period is

$$1 + \frac{\partial \tau_T^*\left(y^{T-1}, y_T\right)}{\partial y_T} = \frac{v'\left(e_T^*\right)}{f_e'\left(e_T^*, \theta_T\right)}. \tag{12}$$

Recall that in the ACK model we have $v'(e) = f_e'(e, \theta) = 1$ for all e, θ. Since risk sharing requires $\frac{\partial \tau_T^*\left(y^{T-1}, y_T\right)}{\partial y_T} < 0$, no insurance is possible in that environment. However, in general, $\frac{v'(e_T)}{f_e'(e_T, \theta_T)}$ might be less than 1. In fact, it is easy to see that - under some regularity conditions on f and v - for all θ_T, the optimal contract implements effort levels such that $\frac{v'(e_T)}{f_e'(e_T, \theta_T)} \leq 1$ (as long as this is technologically feasible).[10] A strict inequality is compatible with some risk sharing.

What is the intuition for this fact? If the firm's aim is to make agents share risk, the key margin for an optimal contract is to guarantee that the agent does not shirk or, equivalently, that she does not *reduce* effort. When the agent shirks so that output is reduced by one unit, the gain she gets

[9] Abraham and Pavoni (2004) also obtain similar results. They consider a more general class of models but do not derive sufficient conditions for the self-insurance result and do not obtain closed-form solutions.

[10] Assume there is at least one $\bar{e} \in E$ such that $\frac{v'(\bar{e})}{f_e'(\bar{e}, \theta)} \leq 1$. Implementing an effort level where $\frac{v'(e_T)}{f_e'(e_T, \theta_T)} > 1$ is dominated by a lower effort since by reducing marginally e_T, for such θ_T, there is a production gain. In the static case, it can be shown that if $\frac{d}{de}[\frac{f_\theta' v'}{f_e'}] \geq 0$ (i.e., the Spence-Mirrlees condition holds for the static version of our problem) the firm can provide the same ex-post welfare to the agent by saving net transfer costs as it also improves on insurance. Details on the formal derivation are available upon request.

from this reduction in effort is equivalent to $\frac{v'}{f'_e}$ units of consumption. This is the right hand side of (12). The left hand side is the net consumption loss. When the marginal tax/transfer is negative this loss will be less than one as the direct reduction of one unit of consumption is mitigated by the increase in net transfers. A small $\frac{v'}{f'_e}$ reduces shirking returns, making it easier for the firm to satisfy the incentive compatibility, and hence to provide insurance. The same intuition carries over to a multi-period setting, where the generalized version of (11) takes the form

$$\frac{\partial \tau^*_{T-1}\left(y^{T-1}\right)}{\partial y_{T-1}} + q\frac{\partial \tau^*_T\left(y^T\right)}{\partial y_{T-1}} = \frac{v'(e^*_{T-1})}{f'_e\left(\theta_{T-1}, e^*_{T-1}\right)} - 1.$$

3.2 Closed forms

We now assume that income y_t depends on exogenous shocks θ_t and effort e_t as follows:

$$y_t = f\left(\theta_t, e_t\right) = \theta_t + a\min\left\{e_t, 0\right\} + b\max\left\{e_t, 0\right\}, \tag{13}$$

with $a \geq 1 \geq b$. Notice that when $a = b = 1$, one obtains the linear specification used to obtain the self-insurance result. Preferences are as in the previous section:[11]

$$u\left(c_t, e_t\right) = u\left(c_t - e_t\right).$$

Moreover, we assume that θ_t follows an ARIMA(p) process:

$$\theta_t - \theta_{t-1} = \beta\left(L\right) v_t, \tag{14}$$

where $\beta\left(L\right)$ is a polynomial of order p in the lag operator L, invertible, and the innovation v_t is iid. Linearity of the ARIMA process helps in finding the closed form solution. Since effort will be time constant, in equilibrium, $\theta_t = y_t$, and hence the income process will display the standard representation often used in the consumption literature.[12]

In Proposition 3, stated and proved in Web Appendix B, we show that if u is exponential (CARA): $u\left(c - e\right) = -\frac{1}{\rho}\exp\left\{-\rho\left(c - e\right)\right\}$, and the shocks v_t are *normally distributed* with zero mean and variance σ_v^2, we get the following expression for consumption growth:

$$c^*_t - c^*_{t-1} = \frac{\ln(\delta/q)}{\rho} + \frac{\rho}{2}\sigma_c^2 + \frac{1}{a}\frac{(1-q)}{1-q^{T-t-1}}\left(\mathbf{E}_t - \mathbf{E}_{t-1}\right)\left[\sum_{n=0}^{T-t} q^n y^*_t\right],$$

[11] Equivalently, we could have assumed a fully linear production function, $f\left(e, \theta\right) = e + \theta$, and a kinked effort cost function, $v\left(e\right) = \frac{1}{a}\min\left\{e_t, 0\right\} + \frac{1}{b}\max\left\{e_t, 0\right\}$.

[12] See, for example, Abowd and Card (1989), Meghir and Pistaferri (2004), and Blundell et al. (2008).

where σ_c^2 indicates the variance of consumption growth. For $t \leq T - p$, the previous expression reduces to (see Web Appendix B for details)

$$\Delta c_t^* = \frac{\ln(\delta/q)}{\rho} + \frac{\rho}{2a^2}\left[\beta\left(q\right)\right]^2 \sigma_v^2 + \frac{1}{a}\beta\left(q\right)v_t. \tag{15}$$

For $a = 1$, we are back to the self-insurance case where innovations to permanent income are fully reflected in consumption changes. For $a > 1$, we get some more risk sharing over and above self-insurance, with full insurance obtainable as a limit case for $a \to \infty$. Proposition 5 in Web Appendix B, shows that a very similar (simpler) closed form can be obtained assuming quadratic u, $\delta = q$, and differentiability of the transfer scheme.[13] With CARA utility (as opposed to the model where u is quadratic), the presence of a precautionary saving motive implies that the equilibrium allocation displays increasing consumption. Notice, however, that, in our economy, $a > 1$ permits both reduction of the cross-sectional dispersion of consumption and mitigation of the precautionary saving motive, hence the steepness of consumption (i.e., 'intertemporal dispersion'). The model implies a very tight relationship between these two moments.

3.3 The Case with isoelastic utility

We now discuss the case in which agents have isoelastic preferences. Full details are to be found in Web Appendix B. We assume the following production function:

$$y_t = \theta_t e_t.$$

This model corresponds to a simple dynamic variant of the most standard version of the well-known model of Mirrlees (1971), where y is labour income, θ represents worker productivity, and e is hours of work. We assume that preferences take the CRRA (or isoelastic) form:[14]

$$u\left(c_t, e_t\right) = \frac{\left(c_t \cdot e_t^{-\frac{1}{a}}\right)^{1-\gamma}}{1-\gamma} \quad \text{for} \quad e_t \leq 1 \quad \text{and} \quad u\left(c_t, e_t\right) = \frac{\left(c_t \cdot e_t^{-\frac{1}{b}}\right)^{1-\gamma}}{1-\gamma} \quad \text{for} \quad e_t \geq 1, \quad \text{for} \quad \gamma > 1;$$

$$u\left(c_t, e_t\right) = \ln c_t - \frac{1}{a}\ln e_t \quad \text{for} \quad e_t \leq 1 \quad \text{and} \quad u\left(c_t, e_t\right) = \ln c_t - \frac{1}{b}\ln e_t \quad \text{for} \quad e_t \geq 1, \quad \text{for} \quad \gamma = 1,$$

where $a \geq 1 \geq b$. As before, therefore, we are assuming that the marginal cost of effort changes discontinuously above a threshold level, here $e_t = 1$. Moreover, we assume that

$$\ln \theta_t - \ln \theta_{t-1} = \beta\left(L\right)\ln v_t,$$

[13] In this case, no parametric assumptions are needed for the income process and consumption does not grow because a precautionary saving motive is absent. See Web Appendix B for details.

[14] Again, we could have assumed, equivalently, $u\left(c_t, e_t\right) = \frac{\left(c_t \cdot e_t^{-1}\right)^{1-\gamma}}{1-\gamma}$ and a modified Cobb-Douglas production function of the form $y_t = \theta_t e_t^a$ for $e_t \leq 1$ and $y_t = \theta_t e_t^b$ for $e_t \geq 1$, with $a \geq 1 \geq b$.

where (with a small abuse in notation) we posit that $\ln v_t$ is normally distributed with zero mean and variance σ_v^2. In this context, we can obtain a closed form for discounted net transfers that leads - for $t \leq T - p$ - to the following permanent income formulation:

$$\Delta \ln c_t^* = \frac{\ln(\delta/q)}{\gamma} + \frac{\gamma}{2}\sigma_c^2 + \frac{1}{a}\beta(q\lambda)\ln v_t, \tag{16}$$

where $\lambda = \exp\left\{\frac{\ln \delta/q}{\gamma} + \frac{1-\gamma}{2}\sigma_c^2\right\}$, and $\sigma_c^2 = \frac{1}{a^2}\left[\beta(q\lambda)\right]^2 \sigma_v^2$.[15] Note that whenever u is logarithmic ($\gamma = 1$) then $q\lambda = \delta$ (see Web Appendix B for details). When $a = 1$ and income shocks are a random walk, $\beta(q\lambda) = 1$ and one obtains the same expression as in the self-insurance model with zero wealth (e.g., Constantinides and Duffie, 1996). With non-zero wealth, the comparison with the Bewley model is more delicate. First, the coefficient that relates innovations to permanent income to consumption needs to be corrected for the wealth-income ratio, as pointed out by Banks et al. (2001) and Blundell et al. (2008). Second, equation (16) does not necessarily hold, and log-consumption is not necessarily a martingale and this might have implications for our empirical approach, which we discuss below.

'Pure' moral hazard. Finally, we can use equation (16) to draw attention to an important difference between our model and one with observable assets. It is well known that in that case, when preferences are additive separable, the inverse of the marginal utility follows a martingale and, therefore, the Euler equation is violated (see Rogerson (1985) and Ligon (1998)). In that framework, it is easy to show that with isoelastic utility, consumption growth follows[16]

$$\mathbf{E}_{t-1}\Delta \ln c_t^* = \frac{\ln(\delta/q)}{\gamma} - \frac{\gamma}{2}\sigma_c^2. \tag{17}$$

The key difference between this expression and (16) is that, according to (17), consumption growth *decreases* with an increase in the variance of consumption and therefore, income shocks. This prediction is inconsistent with the evidence in, for instance, Banks et al. (2001).

[15]More precisely, given a, γ, σ_v, and the coefficients β_j, $j = 0, 1, 2, ..., p$ defining the persistence of the income process, the conditional variance of consumption solves the system of equations

$$\lambda = \exp\left\{\frac{\ln \delta/q}{\gamma} + \frac{1-\gamma}{2}\sigma_c^2\right\} \quad \text{and} \quad \sigma_c = \frac{\sum_{j=0}^{p} \beta_j q^j \lambda^j \sigma_v}{a}.$$

[16]With CARA utility, the same expression holds true for consumption in levels.

4 Empirical implications of the model

The model we developed in the previous section has some important empirical implications. In it, while the Euler equation holds, the risk-free IBC is violated. This violation gives rise to what the literature has labelled the 'excess smoothness' of consumption: consumption does not react 'enough' to innovations in permanent income. We provide one of the first excess smoothness tests performed on micro data and we give a structural interpretation to the excess smoothness coefficient as reflecting the severity of moral hazard.[17]

In this section, we pursue these ideas by developing two different and complementary tests of the implications of our model on micro data. The first is an 'excess smoothness' test derived from a time-series model of consumption and income and based on the test of the risk-free IBC proposed by Hansen, et al. (1991) (HRS henceforth). As mentioned above, testing the validity of an IBC is equivalent, in our context, to testing a market structure. The second test is based on the dynamics of consumption and income inequality within groups, as measured by variances of log consumption. As we discuss below, the two tests complement each other.

As we study the extent to which income shocks are insured or are reflected in consumption, we need to use individual-level data. Unfortunately, however, longitudinal surveys that follow individuals over time and contain complete information on consumption are extremely rare. One of the most commonly used panels from the US, the Panel Study of Income Dynamics (PSID), for instance, only contains information on food consumption. Other data sets, which contain complete information on consumption, have a very short longitudinal dimension (such as the Consumer Expenditure Survey (CEX) in the US) or lack it completely (such as the Family Expenditure Survey (FES) in the UK). As a long time period is crucial to identify the time-series properties of the variables of interest, this lack of longitudinal data is a problem for us. To overcome this difficulty, we use synthetic cohort data or pseudo-panels, along the lines proposed by Deaton (1985) and Browning, et al. (1985).

Our main data source is the UK FES from 1974 to 2002. The FES is a time series of repeated cross-sections which is collected for the main purpose of computing the weights for the Consumer Price Index. Each survey consists of about 7,000 families contacted over two-week periods throughout the year. As households are interviewed every week throughout the year, the FES data are used to construct quarterly time series. This allows us to exploit a relatively long time-series horizon. We

[17] Nalewaik (2006) adapts the Hansen et al. (1991) test to US micro data but does not stress the relationship with excess smoothness, nor provids an excess sensitivity test.

use data on households headed by individuals born in the 1930s, 1940s, 1950s, and 1960s to form pseudo-panels for four year-of-birth cohorts. As we truncate the samples so as to have individuals aged between 25 and 60, the four cohorts form an unbalanced sample. The 1930s cohort is observed over later periods of its life cycle and exits before the end of our sample, while the opposite is true for the 1960s cohort. Apart from the year of birth, the other selection criterion we used for this study is marital status. As we want to study relatively homogeneous groups, we excluded from our sample unmarried individuals. We also excluded the self-employed.

This data set, which has been used in many studies of consumption (see, for instance, Attanasio and Weber, (1993)), probably constitutes the best quality micro data set with consumption information and that covering the longest time period. It contains detailed information on consumption, income, and various demographic and economic variables. We report results obtained using two different definitions of consumption. The first takes as 'consumption' expenditure on non-durable items and services, in real terms and divided by the number of adult equivalents in the household (where for the latter we use the McClements definition of adult equivalents). The second also includes expenditure on durables.

Within our model, the risk-free IBC is violated because it considers disposable income from labour and a single asset and ignores state-contingent transfers. The latter can be interpreted as the returns to other assets with state contingent returns or other income sources, such as government transfers. This implies that if one were to modify the definition of income to include all these state-contingent transfers, one would go back to a version of the PIH and to its implications. Therefore, one should not be able to detect 'excess smoothness' with this changed definition of income. Intuitively, this suggests changing the definition of income in our empirical exercise: if one considers income definitions that include transfers that are conceived to insure individual shocks, one should observe less 'excess smoothness' when applying our tests. This, however, is not a formal statement, as our results are silent about the decentralization of the constrained efficient allocations and, empirically, we do not observe any individual or cohort of individuals for the entire life cycle. However, in what follows we apply our tests using different definitions of income: first gross earnings, then gross earnings plus public transfers, and finally net earnings and transfers.

4.1 The HRS approach

In constructing our first test, we follow HRS. They consider an income process y_t which is one element of the information structure available to the consumer and assume that it admits the following representation:

$$(1-L)y_t := \Delta y_t = \beta(L)w_t, \tag{18}$$

where w_t is an n-dimensional vector of orthogonal covariance stationary random variables that represent the information available to the consumer. $\beta(L)$ is a $1 \times n$ vector of polynomials in the lag operator L. Notice that this specification for the income process encompasses both models considered in Section 3. Here we adopt only in part the HRS notation and adapt it to ours. In particular, without loss of generality, we start from a representation for the income process that has already been rotated so that its first component represents the innovation for the process that generates consumption. It is useful to decompose the right-hand side of equation (18) into its first component and the remaining ones:

$$\Delta y_t = \beta_1(L)w_{1t} + \beta_2(L)w_{2t}. \tag{19}$$

The first result that HRS prove in their paper is that an intertemporal budget constraint has empirical content, in that it imposes testable restrictions on the time-series behaviour of income and consumption, only if one has restrictions on the time-series property of consumption, e.g., that it follows a martingale. When this is not true, one can always find a time series representation for consumption and disposable income that satisfies an IBC. In our model an Euler equation is always satisfied because of the availability of the hidden asset, so that the HRS test has empirical content. Moreover, whether consumption satisfies an Euler equation can be checked empirically, which we do.[18]

The martingale restriction on consumption implies that it can be represented by

$$\Delta c_t = \pi w_{1t}, \tag{20}$$

where deterministic trends have been removed from consumption and π is a scalar different from zero whenever insurance markets are not complete. The coefficient π represents the extent to which

[18] A short digression is in order here. Evidence of the 'excess sensitivity' of consumption to income, as reported by Campbell and Mankiw (1989) and others, is usually interpreted as a violation of the orthogonality conditions implied by the Euler equation for consumption. Such evidence would constitute a problem for the use of the HRS test we are proposing. However, there is evidence that many of the violations reported in the literature are based on aggregate time-series data and are explained by aggregation problems and/or the failure to control for the evolution of needs over the life cycle. See Attanasio and Weber (1993 and 1995). Attanasio (2000) presents a critical discussion of these issues. Another reason that can explain income and consumption tracking each other is the non-separability of consumption and leisure in the utility function, a point made early on by Heckman (1974) and on which Attanasio and Weber (1993 and 1995) and Blundell et al. (1994) provide some evidence.

income news is reflected in consumption. Notice that equation (20) does not include lags. HRS show that, given this structure, the net present value (NPV) condition implies some restrictions on the coefficients of equations (20) and (19). In particular, the intertemporal budget constraint implies that

$$\pi = \beta_1(\widetilde{q}), \tag{21}$$

$$\beta_2(\widetilde{q}) = 0, \tag{22}$$

where $\widetilde{q} = q$ in the case in levels, while $\widetilde{q} = \lambda q$ for the case in logs.[19] HRS show that restriction (21) is testable, while restriction (22) is not, in that there exist other representations for income that are observationally equivalent to (19) for which the restriction holds by construction. The *alternative hypothesis that* $\pi < \beta_1(\widetilde{q})$ is equivalent to what Campbell and Deaton (1989) and West (1988) define as 'excess smoothness' of consumption.

The theoretical structure we have illustrated in the previous section provides a structural interpretation of the 'excess smoothness' and HRS tests. Notice the similarity of equations (14) and (15) to equations (19) and (20).[20] The null considered by HRS corresponds to the Bewley model we considered in Section 2.3, which also corresponds to a special case of our model (see Section 3.1). The moral hazard model we constructed generates a specific deviation from the null: it implies $\beta_1(\widetilde{q})/\pi = a$. In our context, the extent of 'excess smoothness' represents the severity of the incentive problem.

An important feature of the HRS approach is that the test of the NPV restriction does not require the econometrician to identify *all* information available to the consumer. Intuitively, the test uses two facts. First, under the null, the intertemporal budget constraint with a single asset must hold whatever is the information set available to the agent. Second, under the assumption that the agent has no coarser information than the econometrician, the validity of the Euler equation implies that consumption innovations reveal part of the information available to the agent. By following the HRS strategy, we test the intertemporal budget constraint along the dimensions of information identified via the Euler equation (which is not zero as long as insurance markets are not complete).

The representation in equations (20) and (19) is derived under the assumption that preferences are time separable. A more general formulation of equation (20) is the following:

$$\Delta c_t = \pi \psi(L) w_{1t}, \tag{23}$$

[19]HRS work with quadratic utility and $q = \delta$. As we saw above, similar relations can be obtained for CARA utility and for CRRA when working in logs. In these last two cases, we allow for $q \neq \delta$.

[20]Simply set $\pi = \frac{1}{a}$, $w_{1t} = v_t$, $\beta_1(L) = \beta(L)$, and $\beta_2(L) = 0$.

where c in equation (23) represents total consumption expenditure, which enters the budget constraint, while utility is defined over the consumption services which, in turn, are a function of current and, possibly, past expenditure. The polynomial in the lag operator $\psi(L)$ reflects these non-separabilities or other complications, such as *iid* taste shocks to the instantaneous utility function. The non-separabilities considered in HRS imply that $\psi(\tilde{\delta}) = 1$, where $\tilde{\delta} = \delta = q$ if utility is quadratic, while $\tilde{\delta} = 1$ with CARA and CRRA utility. The condition $\psi(\tilde{\delta}) = 1$ imposes restrictions on the way lagged shocks enter the equation for consumption growth which imply that the NPV restriction, in this case, takes the form $\pi = \beta_1(\tilde{q})$, as in equation (21).[21] We will need this extension to interpret some of our results.

We follow HRS and consider a time-series representation of consumption and income as a function of two (unobservable) factors. As in HRS, this gives rise to an MA representation of the following form:

$$\begin{aligned} \Delta c_{ht} &= \alpha^{cc}(L) v_{h1t} \\ \Delta y_{ht} &= \alpha^{yc}(L) v_{h1t} + \alpha^{yy}(L) v_{h2t}, \end{aligned} \tag{24}$$

where we have added the subscript h to denote households and stress that the application will be on micro data and assume that the two vectors v are independent of each other and over time (we allow, however, correlation between v_{hit} and $v_{kit}, i = 1, 2; h \neq k$, to take into account aggregate shocks). The system (24), which is identified by a standard triangular assumption that consumption is only affected by the first factor, can be estimated by maximum likelihood, making some assumptions about the distribution of the relevant variables. As in HRS, it is straightforward to show that, in the case of the intertemporal non-separability implied by equation (23), the intertemporal budget constraint with a single asset can still be tested as an hypothesis on the coefficients of the system (24), provided that the process for consumption satisfies some restrictions we discuss below. In particular, if the IBC holds (and therefore there is no excess smoothness), the relevant restrictions to be tested are $\alpha^{cc}(\tilde{\delta}) = \alpha^{yc}(\tilde{q})$, while excess smoothness will imply $\alpha^{cc}(\tilde{\delta}) \leq \alpha^{yc}(\tilde{q})$. And once again, $\alpha^{cc}(\tilde{\delta})/\alpha^{yc}(\tilde{q}) = 1/a$ reflects the severity of the moral hazard problem in our model.

The presence of intertemporal non-separabilities implies an equation like (23) for consumption changes, and, in particular, the presence of lagged values of w_1. Consumption is not a martingale

[21] Attanasio (2000) (section 3.6) discusses the case with preference shocks which leads to extended equations such as (23). HRS (in section 4) and Alessie and Lusardi (1997) consider habit formation. HRS only consider the case with quadratic utility, while Alessie and Lusardi, within a simpler model, allow for both quadratic and CARA utilities. The condition $\pi = \beta_1(\tilde{q})$ is immediate from their equations (3) and (9) for the quadratic and CARA utilities respectively. The CRRA utility case is similar as long as habit is multiplicative. Details are available upon request.

anymore. However, as we mention above, the relevant Euler equation still provides enough restrictions to make the IBC testable. Moreover, some of these restrictions are testable in a system like (24), as they translate into restrictions on the α coefficients. While the hypothesis that $\psi(\tilde{\delta}) = 1$ cannot be tested because the coefficients of ψ are not identified ($\alpha^{cc}(\tilde{\delta})$ is *proportional* to $\psi(\tilde{q})$), we can identify coefficients on lagged values of v_2 in the first equation of the system (24) and test the hypothesis that they are zero. The alternative that they are different from zero corresponds to the standard excess sensitivity tests, which have been applied many times in the literature, since Hall (1978). Intuitively, a coefficient on lagged values of v_2 in the consumption equation would imply that consumption is excessively sensitive to expected changes in income to be consistent with the PIH. We implement these tests on our data below. In our context, these tests are important because, as stressed by HRS, if consumption does not satisfy the restrictions implied by the Euler equation, we cannot meaningfully test the intertemporal budget constraint.

To implement the estimation of system (24) on our data, some important modifications of the standard procedure used by HRS are necessary. In particular, we need to take into account that we are using household- level data and that we do not have longitudinal data. To address these two problems, we use the approach recently developed by Attanasio and Borella (2006).

Because our data do not have a longitudinal dimension, we do not observe the quantities on the left-hand side of equations (24). As mentioned above, we overcome this problem by using synthetic cohort techniques (see Deaton (1985)). In particular, given groups of fixed membership, indexed by g, and an individual variable z^h_{gt}, we can state, without loss of generality,

$$z^h_{gt} = \overline{z}_{gt} + \eta^h_{gt},$$

where the first term on the right-hand side defines the population group mean. We do not observe $\overline{z}_{gt}$, but we can obtain a consistent estimate $\widetilde{z}_{ct}$ of it from our sample. This will differ from the population mean by an error whose variance can be consistently estimated given the within-cell variability and cell size (see Deaton (1985)). The presence of this measurement error in the levels will induce an additional MA(1) component in the time-series behaviour of the changes in the variables of interest. The variability of this component will have to be taken into account when estimating the parameters of the model. We do so by assuming that the information on within-cell variability provides an exact measurement of the variance of this component. Given the sample sizes involved, this assumption is not a very strong one. Given the known values for the variance-covariance matrix of the sampling error component, the likelihood function of the MA system in (24) can be computed using the Kalman filter (for details see Attanasio and Borella (2006)).

Given these considerations, aggregating the household-level equations (24) at the group level and assuming that the degree of the polynomials in the lag operator is 2, the system that we will be estimating can be written as

$$\begin{aligned} \Delta c_{gt} &= v_{1gt} + \alpha_1^{cc} v_{1gt-1} + \alpha_2^{cc} v_{1gt-2} \\ \Delta y_{gt} &= \alpha_0^{yc} v_{1gt} + \alpha_1^{yc} v_{1gt-1} + \alpha_2^{yc} v_{1gt-2} + v_{2gt} + \alpha_1^{yy} v_{2gt-1} + \alpha_2^{yy} v_{2gt-2}, \end{aligned} \tag{25}$$

where we have normalized the coefficient of the contemporaneous first factor in the consumption equation and of the second factor in the income equation to be 1. In addition to estimating the coefficients in the above system, we also test the hypothesis that coefficients on the lagged values of v_{2gt} do not enter the consumption equation.

4.1.1 Results in levels

In Tables 1 and 2, we report the estimates we obtain estimating the MA system (25) by maximum likelihood. As we allow the variance-covariance matrix in the system to be cohort specific, we limit the estimation to cohorts that are observed over a long time period. This means using balanced pseudo panels with two cohorts: that born in the 1940s and that born in the 1950s. We experimented with several specifications that differed in terms of the number of lags considered in the system. The most general specification included up to eight lags in both the consumption and income equation. However, no coefficient beyond lag 2 was either individually or jointly significant. In the tables, therefore, we focus on the specification with two lags.

Table 1 uses as a definition of consumption the expenditure on non-durables and services. We use three different definitions of income. The first is gross earnings, the second gross earnings plus all public transfers (the most important of which are unemployment insurance and housing benefits - benefits awarded to the needy to cover housing expenses), and the third is net earnings plus benefits. For each of the three definitions, we report two specifications: one with two lags in each of the two equations and one where the insignificant coefficients are restricted to zero.

Several interesting elements come out of the table. First, the dynamics of income are richer than those of consumption. However, and perhaps surprisingly, the coefficients on the lags of v_{2gt} are not statistically significant and are constrained to zero in columns 2, 4 and 6. In the consumption equation, the coefficient on the first lag of v_{1gt} is consistently significant and attracts a negative sign. As discussed above, this could be a sign of intertemporal non-separability of preference, maybe induced by some elements of non-durable consumption having some durability at the quarterly frequency.

To implement the HRS test of the IBC, it is necessary that the marginal utility of consumption satisfies a martingale property. In Table 1, such an assumption is imposed, in that lagged 'income shocks' do not enter the consumption equation. This hypothesis, however, can be tested. If we do not impose the restriction, the estimated coefficients are small in size and never significantly different from zero, either individually or jointly. In the table, for each specification, we report the value of the LR test that the coefficients on the first and second lag are jointly zero and its corresponding p-value. This is an important result as it corresponds to a non-violation of the excess sensitivity test.

The test of the intertemporal budget constraint, which is parametrized as $\pi\psi(\tilde{\delta}) - \beta_1(\tilde{q})$, clearly shows the presence of excess smoothness. Interestingly, such evidence is stronger for gross earnings. The value of the test does not change much when we add benefits to gross earnings (as in columns 3 and 4). However, when we consider net earnings plus benefits, the value of the test is greatly reduced in absolute value (moving from -0.49 to -0.26), although still statistically different from zero. Therefore, when we use a definition of income that includes an important smoothing mechanism, we find much less evidence of consumption 'excess smoothness' *relative to that income definition.*

Table 2 mirrors the content of Table 1, with the difference that the definition of consumption we use now includes expenditure on durables. The results we obtain are, in many ways, similar to those of Table 1. Perhaps surprisingly, the coefficient on lagged v_{1gt} in the consumption equation is smaller in absolute value than in Table 1 and for two of the three income definitions, not statistically different from zero. The test of excess sensitivity, as in Table 1, does not reject the null that the coefficients on lagged income shocks are zero in the consumption equation (although the p-value for the test corresponding to column 1 is 0.098). The most interesting piece of evidence, however, is that the coefficient that measures excess smoothness is now considerably lower in absolute value, indicating less consumption smoothing relative to the null of the Bewley model. This is suggestive of the fact that durables might be playing an important role in the absorption of shocks, as speculated, for instance, by Browning and Crossley (2009). However, when we consider different income definitions, the evidence is consistent with that reported in Table 1, in that consumption consumption exhibit much less 'excess smoothness' relative to net earnings than to gross earnings.

4.1.2 Results in logs

When re-estimating the system using the specification in logs, we try to use the same sample used in the specification in levels. However, as we aggregate the non-linear relationship (i.e., we take the group average of logs), we are forced to drop observations that have zero or negative income. Apart from this, the sample is the same. We report our estimates in Table 3. Given the evidence on the dynamics of consumption discussed above, we only report the results for total consumption, which includes expenditure on durables. Results for non-durables and services are available upon request. Our estimates of a are obtained under the assumption that $\lambda = 1$, which corresponds to log-utility and $\delta = q$. If $q > \delta$, and $\gamma > 1$, λ would be less than unity. We have explored values for λ below 1, obtaining similar results.

As with Tables 1 and 2, we test the hypothesis that lagged coefficients of the 'income' shock do not appear significantly in the consumption equation. In the case of the log specification we should stress, as mentioned in Section 3.3, that one could get a rejection either because of binding liquidity constraints or because the second moments in equation (16) might depend on lagged variables. While this is possible, empirically, this 'excess sensitivity' test does not reject the null at standard significance values.[22]

As with the results in levels, we do find evidence of excess smoothness. The drop in the size of the excess smoothness parameter when we move to definitions of income that include some smoothing mechanisms is even more dramatic than in Table 2. In the last column, corresponding to net earnings plus benefits, the excess smoothness parameter, while still negative, is not significantly different from zero. This result suggests that most of the insurance available over and above self-insurance comes from public transfers and taxation.

4.2 The evolution of cross-sectional variances

The empirical implications of the model we have stressed so far focus on the means of consumption and income. Because of the lack of longitudinal data at the individual level, we were forced to use synthetic panel data. This implies that purely idiosyncratic shocks are averaged away and the 'excess smoothness test' identifies risk sharing across cohorts. To complement this evidence, it can therefore be useful to consider the implications of the theory for the cross-sectional variances of income and consumption. The test we develop here focuses on risk sharing *within* cohorts. For this purpose, the closed-form solution derived in Web Appendix B for the model with two independent shocks, one of which is assumed to be temporary and one permanent, is particularly useful. In this case, we have (see equation (63))

$$\ln c_t^i = \frac{1}{a^p} \ln x_t^i + \frac{(1-\lambda q)}{a^T} \sum_{s=0}^{t-1} \ln \xi_{t-s}^i + t\frac{\gamma}{2}\left[\left(\frac{1}{a^p}\right)^2 \sigma_{v^p}^2 + \left(\frac{1-\lambda q}{a^T}\right)^2 \sigma_{v^T}^2\right] - t\frac{\ln \frac{q}{\delta}}{\gamma} + \mu^i + z_t. \quad (26)$$

[22] The issue of the estimation of log-linearized Euler equations like equation (16) is discussed in Attanasio and Low (2004). The results we find are consistent with the findings of their simulations.

The term μ^i allows for ex-ante heterogeneity, which could capture distributional issues, the initial level of assets of individual i, or fixed effects which can be observable to the firm but may be unobservable to the econometrician. The term z_t allows for aggregate shocks, which will be assumed to be orthogonal to individual shocks and included in the information set of all agents in the economy. If we compute the cross-sectional variance at time t of both sides of equation (26), we have

$$\begin{aligned} Var\left(\ln c_t^i\right) &= \left(\frac{1}{a^p}\right)^2 Var\left(\ln x_t^i\right) + \left(\frac{1-\lambda q}{a^T}\right)^2 Var\left(\sum_{s=0}^{t-1} \ln \xi_{t-s}^i\right) + Var\left(\mu^i\right) \\ &+2\left[\frac{Cov\left(\ln x_t^i, \mu^i\right)}{a^p} + \frac{(1-\lambda q)}{a^T}\sum_s Cov\left(\ln \xi_{t-s}^i, \mu^i\right)\right], \end{aligned} \tag{27}$$

where we have used the fact that $Cov\left(\ln x_t, \ln \xi_{t-s}\right) = 0$ for all s.[23] We start by assuming that both $Cov\left(\ln \xi_{t-s}^i, \mu^i\right)$ and $Cov\left(\ln x_t^i, \mu^i\right)$ are time invariant. The time invariance of these terms can be obtained by assuming constant μ's across agents in the same group (as in our theoretical model). If we now take the first difference of equation (27), neglecting the individual indices, we have

$$\Delta Var\left(\ln c_t\right) = \left(\frac{1}{a^p}\right)^2 \Delta Var\left(\ln x_t\right) + \left(\frac{1-\lambda q}{a^T}\right)^2 Var\left(\xi_t^i\right), \tag{28}$$

where $\Delta Var\left(\ln c_t\right) := Var\left(\ln c_t\right) - Var\left(\ln c_{t-1}\right)$ and $\Delta Var\left(\ln x_t\right)$ is similarly defined. Under our assumptions, $\Delta Var\left(\ln y_t\right) = \Delta Var\left(\ln x_t + \ln \xi_t\right) = \Delta Var\left(\ln x_t\right)$, which implies two things. First, we can replace the permanent component x_t by total income y_t in the first term of the right hand side of equation (28). Second, without independent identification of the two components of income (x_t and ξ_t), only $a^p (= a)$ can be identified.

Notice that equation (28) allows again the identification of the structural parameter a, which reflects the severity of the moral hazard problem. As noted by Deaton and Paxson (1994), under perfect risk sharing, the cross sectional variance of consumption is constant over time.[24]

Under the PIH, as pointed out by Blundell and Preston (1998), the changes in the variance of consumption reflect changes in the variance of (permanent) income. Here, we consider a specific alternative to the perfect insurance hypothesis that implies that consumption variance grows, but less than the increase in the variance of permanent income.

[23] Here we are making use of the income process with two separate shocks typically assumed in the PIH literature, which we introduce in Web Appendix B. If we were to use the process with a single shock used in Section 3, equation (26) for log consumption would be much more complicated, as reported in Web Appendix B. It is easy to see, however, that the whole analysis goes through for $Cov\left(\ln x_t, \ln \xi_{t-s}\right) \neq 0$, as long as it is time constant for all $s \geq 0$.

[24] Attanasio and Szekely (2004) propose a test of perfect risk sharing based on changes in the cross-sectional variances of marginal utilities.

The empirical strategy we follow here is quite different from Blundell et al. (2008). They study the evolution of the cross- sectional variance of consumption *growth*,[25] while we start from the specification for consumption *levels* in equation (26) to derive equation (28). We notice that, as we do not have longitudinal data, we cannot identify separately a^p and a^T, while Blundell et al. (1998) do identify the proportion of permanent and transitory shocks that can be insured.

The estimation of equation (28) also requires the identification of groups. Here the group implicitly defines the participants in a risk- sharing arrangement and the test will identify the amount of risk sharing within that group. As with the estimation of the HRS system, the lack of truly longitudinal data and the use of time series of cross-sections imply that the estimated variances (for income and consumption) will have an error component induced by the variability of the sample. This is particularly important for the changes in the variance of income on the right-hand side: the problem induced is effectively a measurement error problem which induces a bias in the estimated coefficient. However, as explained in Web Appendix C, it is easy to obtain an expression for the bias implied by an OLS estimator in finite samples and correct it.

To estimate the parameters in equation (28), we use the same sample we used for the HRS test, with the only difference being that we do not limit ourselves to the balanced pseudo-panel but use four cohorts, although the youngest and oldest are only used for part of the time period. Otherwise, the selection criteria used to form our sample are the same as above.

The results are reported in Table 4. There are four columns in the table, each reporting the slope coefficient of equation (28) and the implied a with the corresponding standard errors. The standard error of a is computed by the delta method. In the first two columns, we use expenditure on non-durables and services as our definition of consumption, while in the last two we use total expenditure. In column 2 and 4, the consumption figure is divided by the number of adult equivalents. The three panels correspond to the same three definitions of income we used for the HRS test.

The first aspect to be noted is that all the slope coefficients are positive and statistically different from zero. Moreover, consistently with the theory, they all imply a value of a greater than unity. Finally, the results are affected only minimally by the consideration of adult equivalents.

If we analyse the difference across income definitions, we find results that are consistent with the implications of the model and, by and large, with the evidence from the HRS approach. The coefficient on the changes in the variance of gross earnings is much smaller than the one on the other income definitions. This is consistent with the evidence from Tables 1 and 2, which showed more 'excess smoothness' for this definition. Unlike in Tables 1 and 2, however, the main difference

[25] This has important advantages, but it forces to use the approximation $Var(\Delta c_t) \approx \Delta Var(c_t)$.

in the size of the coefficient is between the first income definition on one side and the second and third on the other. With the HRS approach, the main difference was between the first and second on one side and the third on the other.

Finally, if we look at the differences between the definitions of consumption that include durables and those that do not, we find that the coefficients are (except for the first income definition) larger for the former than the latter. Again, this is consistent with the evidence from the HRS approach, which finds less 'excess smoothness' when one includes durables in the definition of consumption, i.e., some self-insurance mechanisms seem to be at work via durables.

The consistency of the results obtained with the variance approach and those obtained with the HRS approach is remarkable because the two tests, as stressed above, focus on different aspects of risk sharing: the latter on insurance across groups and the former on insurance within groups. It is remarkable that both yield results that are in line with our model and indicate that the observed amount of risk sharing is in between that predicted by a simple Permanent Income model and that predicted by perfect insurance markets. Comparing the magnitudes of the coefficients from the two approaches, we can obtain a measure of the different degrees of risk- sharing possibilities that are available within cohorts as opposed to those available across cohorts.

5 Conclusions

In this paper, we discuss the theoretical and empirical implications of a model where perfect risk sharing is not achieved because of information problems. A specific and important feature of our model is that in addition to the standard moral hazard problem, in the economy we study, agents have hidden access to the credit market. After characterizing the equilibrium of this model, we have shown how it can be useful to interpret individual data on consumption and income.

Developing results in Abraham and Pavoni (2004), we have shown that in a competitive equilibrium of our model, agents typically obtain more insurance than in a Bewley set-up. Moreover, we are able to construct examples in which we can get closed-form solutions for consumption. These results have more than an aesthetic value: in our empirical approach they allow us to give a structural interpretation to some of the empirical results in the literature and to those we obtain. In particular, in our model one *should* observe the so-called 'excess smoothness' of consumption. Moreover, 'excess smoothness' is distinct from the so-called 'excess sensitivity' of income to consumption. Finally, we can map the excess smoothness parameter into a structural parameter.

The presence of excess smoothness follows from the fact that, even in the presence of moral hazard and hidden assets, in general, a competitive equilibrium is able to provide some insurance over and above what individuals achieve on their own by self-insurance. This additional insurance is what generates excess smoothness in consumption, which can then be interpreted as a violation of the intertemporal budget constraint with a single asset. The equilibrium allocations generated by our model violate the IBC with a single asset because they neglect some state-contingent transfers the agents use to share risk.

Tests of the risk-free intertemporal budget constraint become tests of market structure in our framework. For this reason, we start our empirical work using a test of 'excess smoothness' that was first proposed as a test of the intertemporal budget constraint with a single asset by Hansen,et al. (1991). We extend their approach so that it can be applied to micro data.

Because longitudinal data on consumption are rarely available, we work with time series of cross-sections and synthetic cohort data. As we are forced to aggregate the consumption and income of individuals belonging to a given year-of-birth cohort, we necessarily lose some of the variability in idiosyncratic income and the possibility of studying the amount of risk sharing of these shocks. This is one of the reasons why, in addition to the extension of the HRS test, we propose an additional test, which uses the movements in the cross-sectional variances of consumption and income to identify the same structural parameters of our model.

While related to the work of Deaton and Paxson (1994), Blundell and Preston (1998) and Blundell, et al. (2008), our approach is different in that it focuses on the variance in the *level* rather than *changes* of consumption. Moreover, as is the case for the version of the HRS test we present, we can give the coefficients we estimate a structural interpretation in terms of our moral hazard model.

While many papers, starting with Campbell and Deaton (1989) have documented the 'excess smoothness' of consumption using aggregate time- series data, the evidence based on micro data is recent and very limited. Using our two different approaches and data from the UK, we forcefully reject both perfect risk sharing and the simple Bewley economy, while we do not reject the hypothesis of 'no excess sensitivity' of consumption to income. More generally, our results are consistent with the model with moral hazard and hidden assets we considered. Particularly suggestive is the fact that when we consider income definitions that include smoothing mechanisms, such as social assistance and net taxes, we find less evidence of 'excess smoothness'.

Our results have obvious policy implications, as one could, in principle, be able to quantify in terms of welfare the insurance role played by the taxation system or unemployment insurance. Such computations would be immediately feasible from our analysis. We would also be able to perform accurate counterfactuals in order to evaluate the effects of a given policy. All normative issues, however, are left for future research.

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6 Appendix A

6.1 Implementing the efficient allocation with income taxes

In this section we show that under some conditions the equilibrium allocation (e, c, y) can be 'described' using a transfer scheme τ which is function of income histories y^t alone. This will simplify the analysis and allow us to describe the consumption allocation in terms of observables.

Notice first, that through $y_t = y_t(\theta^t)$ the y component of the equilibrium allocation generates histories of income levels y^t. Let's denote by $y^t(\theta^t) = \left(\mathbf{y}_1(\theta^1), ..., \mathbf{y}_t(\theta^t)\right)$ this mapping. In general $y_t(\theta^t)$ is not invertible, as it might be the case that for a positive measure of histories of shocks θ^t we get the same history of incomes y^t. A generalization of the argument used in Kocherlakota (2005) however shows that it suffices to *assume that the optimal plan of consumption c alone is y^t-measurable.* That is, that there exists a sequence of y^t-measurable functions c^* such that for all t, θ^t we have $\mathbf{c}_t^*(y^t(\theta^t)) = \mathbf{c}_t(\theta^t)$. We now show that under fairly general conditions the implementation idea of Kocherlakota (2005) extends to the general case with hidden savings.

Now, notice that $\mathbf{y}_t$ is y^t-measurable by construction. As a consequence, from (9) is easy to see that the y^t-measurability of c implies that τ is y^t-measurable as well. From the transfer scheme τ, we can hence obtain the new y^t-measurable scheme τ^* as follows: $\tau_t^*(y^t(\theta^t)) = \tau_t(\theta^t)$. Given τ^*, let

$$\mathbf{E}\left[\sum_{t=1}^{T}\delta^{t-1}\,u(c_t^*, \hat{e}_t)\ \mid\ \hat{e}\right] := \sum_{t=1}^{T}\delta^{t-1}\int_{\Theta^t} u\left(\mathbf{c}_t^*\left(\hat{y}^t(\theta^t)\right), \hat{\mathbf{e}}_t\left(\theta^t\right)\right) d\mu^t(\theta^t)$$

where $\mathbf{c}_t^*(\hat{y}^t(\theta^t)) = \tau_t^*(\hat{y}^t(\theta^t)) + \mathbf{y}_t^*(\hat{y}^t(\theta^t))$, and the new mapping is induced by $\hat{e}$ as follows: $\hat{y}_t(\theta^t) = f(\theta_t, \hat{\mathbf{e}}_t(\theta^t))$ for all t, θ^t. For any history of shocks θ^t, a plan $\hat{e}$ not only entails different effort costs, it also generates a different distribution over income histories y^t hence on transfers and consumption. This justifies our notation for the conditional expectation.

We say that the equilibrium allocation (e, c, y) can be *described* with y^t-measurable transfers if the agent does not have incentive to deviate from c^*, e^* given τ^*. The incentive constraint in this case is as follows:

$$\mathbf{E}\left[\sum_{t=1}^{T}\delta^{t-1}\,u(c_t^*, e_t^*)\ \mid\ e^*\right] \geq \mathbf{E}\left[\sum_{t=1}^{T}\delta^{t-1}\,u(\hat{c}_t, \hat{e}_t)\ \mid\ \hat{e}\right], \tag{29}$$

where, as usual, the deviation path of consumption $\hat{c}$ must be replicated by a plan of risk free bonds $\hat{b}$. An important restriction in the deviations $\hat{e}$ contemplated in constraint (29) is that they are required to generate 'attainable' histories of $y's$, i.e. histories of $y's$ that can happen in an equilibrium allocation. The idea is that any off-the-equilibrium value for y^t will detect a deviation with certainty. One can hence set the firm's transfers to a very low value (perhaps minus infinity) in these cases, so that the agent will never have incentive to generate such off-the-equilibrium histories.

Finally, suppose the agent chooses an effort plan $\hat{e}$ so that the realized history $\hat{y}^t$ is attainable in equilibrium. This means both that there is a reporting strategy $\hat{\sigma}$ so that $\hat{y}^t = (\mathbf{y}_1(\hat{\sigma}), \mathbf{y}_2(\hat{\sigma}), ..., \mathbf{y}_t(\hat{\sigma}))$ and that given a consumption plan $\hat{c}$ the utility the agent gets is $\mathbf{E}\left[\sum_{t=1}^T \delta^{t-1} u(\hat{c}_t, e_t) \mid (e, c, y); \hat{\sigma}\right]$, where the notation is that in the main text.[26] This effectively completes the proof since the incentive constraint (4) guarantees that the agent will chose the truth-telling strategy which implies the equilibrium plans e and c as optimal for him.

6.2 Proof of Proposition 1

We now show that, for the specification of preferences and production function we stated in Proposition 1, incentive compatibility fully characterizes the efficient allocation. In fact, we will allow for a generic sequence of bond prices $\{q_t\}_{t=1}^{T-1}$ faced by both the agents and the firms in the economy.

In order to avoid the use of the taxation principle, we will consider transfers schemes that depend on the revelation plan σ. This notation would also be more directly related to the Bewley model of Section 2.3. Recall that $T < \infty$, and consider the last period of the program. Using the budget constraint, assuming the agent declared history θ^{T-1} so far, and has wealth b_T, after the realization of θ_T, his/her preferences over the report σ_T of θ_T can be represented as follows:

$$u(c_T - e_T) = u\left(\theta_T + \tau_T\left(\theta^{T-1}, \sigma_T\right) + b_T\right).$$

The key aspect to notice here is that since utility depends on the declaration σ_T only through the transfer, the agent will declare the productivity level delivering the maximal transfer whenever possible. More precisely, for any value of $e_T(\theta^{T-1}, \theta_T)$, the local incentive compatibility implies that for a sufficiently small $\epsilon > 0$ we have

$$u\left(\theta_T + \tau_T\left(\theta^{T-1}, \theta_T\right) + b_T\right)$$

[26]More in detail, notice that by definition we have $\hat{\mathbf{e}}_t(\theta^t) = g(\theta_t, \hat{\mathbf{y}}_t(\theta^t))$. Since $\hat{\mathbf{y}}_t(\theta^t)$ is 'attainable' it can be induced from y by a 'lie', i.e., there exists a $\hat{\sigma}$ such that $\hat{\mathbf{y}}_t(\theta^t) = \mathbf{y}_t(\hat{\sigma}(\theta^t))$. But then $\hat{\mathbf{e}}_t(\theta^t) = g(\theta_t, \mathbf{y}_t(\hat{\sigma}(\theta^t)))$ and from the definition of τ^* we have $\mathbf{c}_t^*(\hat{\mathbf{y}}_t(\theta^t)) = \mathbf{c}_t^*(\mathbf{y}_t(\hat{\sigma}(\theta^t))) = \mathbf{c}_t(\hat{\sigma}(\theta^t))$, which implies that

$$\mathbf{E}\left[\sum_{t=1}^T \delta^{t-1} u(c_t^*, \hat{e}_t) \mid \hat{e}\right] = \mathbf{E}\left[\sum_{t=1}^T \delta^{t-1} u(c_t, e_t) \mid (e, c, y), \hat{\sigma}\right]$$

for some $\hat{\sigma} \in \Sigma$. Finally, it is easy to see that c^* (and $b^* \equiv 0$) is incentive compatible among the deviations that solve $\hat{\mathbf{c}}_t^\sigma(\sigma(\theta^t)) + q\hat{\mathbf{b}}_{t+1}^\sigma(\sigma(\theta^t)) - \hat{\mathbf{b}}_t^\sigma(\sigma(\theta^{t-1})) = \mathbf{c}_t^*(\sigma(\theta^t))$ and $\lim_{t\to T} q^{t-1}\hat{\mathbf{b}}_{t+1}^\sigma(\sigma(\theta^t)) \geq 0$ a.s. for all histories θ^t. A final remark. One can easily show that under the same conditions, e must also be x^t-measurable. If e is not x^t-measurable it means that for at least two some θ^t, and $\bar{\theta}^t$ we have $\mathbf{e}_t(\theta^t) \neq \mathbf{e}_t(\bar{\theta}^t)$ while $f(\theta_t, \mathbf{e}_t(\theta^t)) = f(\theta_t, \mathbf{e}_t(\bar{\theta}^t))$. However, since u is decreasing in e, effort incentive compatibility (at $b_t = 0$) implies that $\tau_s(\theta^s) \neq \tau_s(\bar{\theta}^s)$ for some $s \geq t$ with τ not y^s-measurable. A contradiction to the fact that the equilibrium transfer scheme τ is y^t-measurable.

$$\geq \quad u\left(\theta_T + \tau_T\left(\theta^{T-1}, \sigma_T\right) + b_T\right) \text{ for all } \sigma_T \in [\theta - \epsilon, \theta + \epsilon].$$

This condition applied to all $\theta_T \in \Theta$ implies that a transfer payment τ_T which is invariant across θ_T's is the sole incentive compatible possibility.

Now consider the problem in period $T-1$. We now show that - given θ^{T-2} and θ_{T-1} - an incentive compatible transfers scheme must induce constant number for the present value of transfers $\tau_{T-1}\left(\theta^{T-1}\right) + q_{T-1}\tau_T\left(\theta^{T-1}\right)$ across θ_{T-1} levels. Note that in our notation we used the fact that τ is constant across θ_T. Consider a generic history θ^{T-2} and suppose that $\tau_{T-1}\left(\theta^{T-2}, \theta'_{T-1}\right) + q_{T-1}\tau_T\left(\theta^{T-2}, \theta'_{T-1}\right) < \tau_{T-1}\left(\theta^{T-2}, \theta''_{T-1}\right) + q_{T-1}\tau_T\left(\theta^{T-2}, \theta''_{T-1}\right)$ for θ' sufficiently close to θ''. This cannot be incentive compatible. The agent who receives shock θ'_{T-1} would declare θ''_{T-1} (and change effort by the difference between the two productivity levels) and adjust his/her bond plan accordingly. We just have to show that there is a bond plan so that the agent is better off by declaring θ''_{T-1}.But this is easy to see since τ_T does not depend on θ_T. More precisely, if $c_{T-1},\ c_T(\theta_T)$ is the consumption plan under θ'_{T-1} then $c_{T-1} + \frac{\kappa}{1+q_{T-1}},\ c_T(\theta_T) + \frac{\kappa}{1+q_{T-1}}$ is one of the consumption paths attainable by deviating, where $\kappa > 0$ is the difference between the two net present values of transfers. A similar argument applies to the case where the inequality is reversed. In this case, the profitable deviation would be takes by the agent who receives shock θ''_{T-1}. The discounted value of transfer must hence be constant across θ_{T-1} as well, that is, $\tau_{T-1}(\theta^{T-1}) + q_{T-1}\tau_T(\theta^T)$ is θ^{T-2}-measurable. Going backward, we get that for all t the quantity

$$NPV_t := \sum_{n=0}^{T-t} \left(\Pi_{s=1}^{n} q_{t-1+s}\right) \tau_{t+n}(\theta^{t+n})$$

is at most θ^{t-1}-measurable since it is a constant number for all continuation histories $\theta_t, \theta_{t+1}, ..., \theta_T$ following node θ^{t-1}. If we consider the initial period ($t = 1$), (local) incentive compatibility implies that $NPV_1\left(\theta^T\right) := \sum_{n=1}^{T} \left(\Pi_{s=1}^{n-1} q_s\right) \tau_n$ is one number a.s. for all histories θ^T. As we argued already in the main text while discussing equation (5), another necessary condition for incentive compatibility is the agent's Euler equation. The problem of the firm facing a relaxed (local) incentive compatibility constraint hence reduces to the choice of the unique number NPV_1 and the plan of bonds holdings for the agent. Since the agent and the firm face the same sequence of bond prices, it is easy to see that the firm is indifferent across all bond plans. This implies that the only number NPV_1 consistent with nonnegative profits and with maximizing the agent's utility is zero. Moreover, this number can be obtained by setting $\tau_t \equiv 0$ (equivalently we could assume the transfer τ is set so that the agent chooses $b_t \equiv 0$). It is now obvious that $\tau_t \equiv 0$ globally incentive compatible, implying that it must be the optimal one. We have hence shown that the optimal allocation corresponds to the bond economy allocation. **Q.E.D.**

More extensively, in all cases, the optimal contract must solve the standard Euler equation at each node and the condition

$$NPV_1\left(\theta^T\right) := \sum_{t=1}^{T} \left(\Pi_{s=1}^{t-1} q_s\right) \tau_t^*\left(\theta^t\right) = \sum_{t=1}^{T} \left(\Pi_{s=1}^{t-1} q_s\right) \left(\mathbf{c}_t^*(\theta^t) - \mathbf{y}_t^*(\theta^t)\right) = 0 \text{ a.s. for all } \theta^T.$$

But these are precisely the two defining properties for the self-insurance allocation: first, it must satisfy the Euler equation. Second, it must satisfy the intertemporal budget constraint with the risk free bond, i.e., the period zero net present value must be zero for all θ^T. Since the Euler equation (5) is always satisfied in our allocation, the only way of obtaining a different allocation is that the transfers scheme τ permitted to violate the agent's period zero self insurance intertemporal budget constraint for some history θ^T. The whole point of Proposition 1 is to demonstrate that it cannot be the case.

A final remark. Although the proof uses finite time, by adapting the proof of Proposition 7 in Cole and Kocherlakota (2001), we can show the result for $T = \infty$, at least as long as u is a bounded function.

7 Appendix B: Closed forms

The outcome of this section will be a set of closed form solutions to our model which give a structural interpretation, in terms of the marginal cost/return of effort, of the coefficient ϕ coming from a generalized permanent income equation of the form:

$$\Delta c_t = \Gamma_t + \phi \Delta y_t^p,$$

where the variable will be expressed in levels or in logs, depending on the specification, and

$$\phi = \frac{1}{a},$$

with $a \geq 1$, and where $\frac{1}{a}$ is the marginal return to shirking. Since in our model wealth effects are absent (at least in the chosen space), the equilibrium contract implements a constant effort level in all periods, which will be normalized to a given number: the first best level of effort. So the whole margin in welfare will come from risk sharing. The incentive compatibility constraint will hence dictate the degree of such insurance as a function of the marginal cost of effort. A lower effort cost/return allows the firm to insure a lot the agent without inducing him to shirk. And the firm will use the whole available margin to impose transfers and obtain consumption smoothing.

7.1 Closed form in levels: CARA utility

7.1.1 Model

Recall that we can perform a change in variable and assume $y_t = \theta_t + e_t$ and $u(c, e) = u(c - v(e))$, where

$$v(e_t) := \frac{1}{a} \min\{e_t, 0\} + \frac{1}{b} \max\{e_t, 0\}, \quad \text{with} \quad a \geq 1 \geq b, \tag{30}$$

Interestingly, as we have seen in Section (3.1) for $a = b = 1$ we are in the standard ACK case, hence there is no room for risk sharing at all (on top of self insurance) and the allocation replicates that of the Bewley model.

Finally, notice that as long as $a > 1$ (and $b < 1$) the first best effort level would be zero. However, the first best allocation would also imply a constant consumption. This allocation can only be obtained by imposing a constant tax rate such that $\tau_t' = -1$. Obviously, this allocation is not incentive feasible in a world where effort and productivity are private information of the agent.

The main steps towards the derivation of our closed form will be as follows. First, we consider a relaxed optimization problem. More precisely, we consider an auxiliary problem for the firm that imposes strictly less stringent incentive constrains and the same objective function and the same technological constraints. Then we show that the solution for the relaxed problem corresponds to our closed form. Finally, we show that our closed form satisfies the original incentive compatibility constraint. This implies that the closed form solution solves the original maximization problem of the firm.

7.1.2 The relaxed problem

We eliminate the time subscript whenever possible. Consider the following problem:

$$
\begin{aligned}
\textbf{(R)} \qquad & \max_{\tau,\mathbf{y}} \ \mathbf{E}_0 \left[\sum_{t=1}^{T} \delta^{t-1} u(y\left(\theta^t\right) + \tau(\theta^t) - v(y\left(\theta^t\right) - \theta_t)) \right] \quad \text{s.t.} \quad \text{for all } \theta^t, t \geq 1 : \\
& \max\nolimits_{b,\ \hat{\theta}_t \leq \theta_t} u\left(y\left(\theta^{t-1}, \hat{\theta}_t\right) + \tau(\theta^{t-1}, \hat{\theta}_t) - q_t b - v(y\left(\theta^{t-1}, \hat{\theta}_t\right) - \theta_t)\right) + \delta V_t\left(\left(\theta^{t-1}, \theta_t\right), \hat{\theta}_t, b\right) \\
\leq \ & u(y\left(\theta^t\right) + \tau(\theta^t) - v(y\left(\theta^t\right) - \theta_t)) + \delta U_t\left(\theta^t\right); \text{ and} \\
0 \ \geq \ & \mathbf{E}_0 \left[\sum_t \left(\Pi_{n=0}^t q_s\right) \tau(\theta^t)) \right].
\end{aligned}
$$

In the above formulation, $U_t\left(\theta^t\right)$ is the equilibrium utility, and solves

$$
\begin{aligned}
U_t\left(\theta^t\right) \quad : &= \mathbf{E}_t \left[\sum_{s=1}^{T-t} \delta^{s-1} u(y\left(\theta^{t+s}\right) + \tau(\theta^{t+s}) - v\left(y\left(\theta^{t+s}\right) - \theta_{t+s}\right)) \right] \\
&= \int_\Theta \left[u(y\left(\theta^t, \theta_{t+1}\right) + \tau(\theta^t, \theta_{t+1}) - v(y\left(\theta^{t+1}\right) - \theta_{t+1})) + \delta U_{t+1}\left(\theta^t, \theta_{t+1}\right)\right] d\Phi\left(\theta_{t+1} \mid \theta^t\right),
\end{aligned}
$$

while $V_t\left(\left(\theta^{t-1}, \theta_t\right), \hat{\theta}_t, b\right)$ represents the highest utility the agent can get by choosing freely the plan of bonds but telling the truth

$$
\begin{aligned}
V_t\left(\left(\theta^{t-1}, \theta_t\right), \hat{\theta}_t, b\right) &= \max_{\mathbf{c},\mathbf{b}} \ \mathbf{E} \left[\sum_{s=1}^{T-t} \delta^{s-1} u(c\left(\theta^{t+s}\right) - v\left(y\left(\hat{\theta}^{t+s}\right) - \theta_{t+s}\right)) \ \mid \theta^t \right] \\
s.t. \ : \quad c_{t+s}\left(\theta^{t+s}\right) &= y\left(\hat{\theta}^{t+s}\right) + \tau(\hat{\theta}^{t+s}) + q_{t+s} b_{t+s+1}\left(\theta^{t+s}\right) - b_{t+s}\left(\theta^{t+s}\right),
\end{aligned}
$$

where in the previous expression, for all $s \geq 1$, we denote $\hat{\theta}^{t+s} := \left(\theta^{t-1}, \hat{\theta}_t, \theta_{t+1}, ...\theta_{t+s}\right)$. Clearly, $U_{T+1} \equiv V_{T+1} \equiv 0$.

The maximization problem is relaxed with respect to the original problem solved by the firm in equilibrium in a number of dimensions. First, the incentive constraints are only for downward deviations. Second, the deviation is 'local' because it assumes that the agent never lies for more than one period although he is allowed to deviate for more than one period in the bond decisions after a first deviation. Finally, also bond deviations are 'local' since the agent starts with zero wealth at each node in equilibrium.

Lemma 1. *The contract solving problem **(R)** implements* $e(\theta^t) = 0$ for all θ^t.

Proof. Take any contract and suppose that for some history θ^t we have $e(\theta^t) > 0$. Then consider the contract that keeps all transfers and recommendations as the previous one, but at history θ^t where it recommends income $\tilde{y}(\theta^t) = y(\theta^t) - e(\theta^t)$ and zero effort: $\tilde{e}(\theta_{T-1}, \theta_T) = 0$, and transfers $\tilde{\tau}(\theta^t) = \tau(\theta^t) + (1 - \frac{1}{b})e(\theta^t)$. It is easy to see that - at all histories - the new contract delivers exactly the same argument of the utility function u in equilibrium. We have to show that the incentive constraints are all satisfied under the new contract. The fact that the in equilibrium for all histories the arguments of u are all the same implies that $U_s(\theta^s)$ are unchanged for all s and θ^s. Moreover, it should also be clear that future values $V_{t+s}\left(\left(\theta^{t+s-1},\theta_{t+s}\right),\hat{\theta}_{t+s},b\right)$ for all $s\geq 0$, θ^{t+s} and $\hat{\theta}_{t+s},b$ do not change either. Finally, since the modification of the contract leaves the equilibrium utilities unchanged at all nodes (including node θ^t), the values $V_{t-k}\left(\left(\theta^{t-k-1},\theta_{t-k}\right),\hat{\theta}_{t-k},b\right)$ for all $k\geq 1$ are also unaffected by the change: since the argument of the utility flow u in equilibrium is unchanged - and $V_{t-k}\left(\left(\theta^{t-k-1},\theta_{t-k}\right),\hat{\theta}_{t-k},b\right)$ does not contemplate deviations over declarations after period $t-k$ - the set of consumption plans available by deviating only in the bond are unchanged by the modification to the contract.

Consider now how the change in the contract might affect the incentive constraint in period t. Clearly it can affect the incentives for productivity levels above θ_t, call these values $\bar{\theta}_t\geq\theta_t$ (we include $\bar{\theta}_t=\theta_t$ since the agent with productivity θ_t might find profitable the bond deviation under then new contract). Since the equilibrium utilities (both flows u and values U_t) do not change, in order to verify the new transfer scheme solves period t incentive constraint, it suffices to show that for all b and $\bar{\theta}_t>\theta_t$ we have

$$\begin{aligned}&u\left(y\left(\theta^{t-1},\theta_t\right)+\tau(\theta^{t-1},\theta_t)-q_tb-v(y\left(\theta^{t-1},\theta_t\right)-\bar{\theta}_t)\right)+\delta U_t\left(\left(\theta^{t-1},\theta_t\right),\bar{\theta}_t,b\right)\\ \geq\;&\\ &u\left(\tilde{y}\left(\theta^{t-1},\theta_t\right)+\tilde{\tau}(\theta^{t-1},\theta_t)-q_tb-v(\tilde{y}\left(\theta^{t-1},\theta_t\right)-\bar{\theta}_t)\right)+\delta U_t\left(\left(\theta^{t-1},\theta_t\right),\bar{\theta}_t,b\right).\end{aligned}$$

But again, since $U_t\left(\left(\theta^{t-1},\theta_t\right),\bar{\theta}_t,b\right)$ is unaffected by the change, it suffices to show that

$$u\left(x\left(\theta^{t-1},\theta_t\right)+\tau(\theta^{t-1},\theta_t)-q_tb-v(x\left(\theta^{t-1},\theta_t\right)-\bar{\theta}_t)\right)\geq u\left(\tilde{y}\left(\theta^{t-1},\theta_t\right)+\tilde{\tau}(\theta^{t-1},\theta_t)-q_tb-v(\tilde{y}\left(\theta^{t-1},\theta_t\right)-\bar{\theta}_t)\right).$$

The last inequality is true because of the following. If $y(\theta^{t-1},\theta_t)-\bar{\theta}_t>e(\theta^{t-1},\theta_t)>0$ then the change in transfer scheme generates exactly the same utility to the deviating agent. If $y(\theta^{t-1},\theta_t)-\bar{\theta}_t<e(\theta^{t-1},\theta_t)$ then the utility from deviation decreases. We show it assuming that $y(\theta^{t-1},\theta_t)-\bar{\theta}_t<0$. The case where $e\left(\theta^t\right)>y(\theta^t)-\bar{\theta}_t>0$ is a combination of this cases and that we just analyzed. We have

$$\begin{aligned}&u(y(\theta^{t-1},\theta_t)+\tau(\theta^{t-1},\theta_t)-\frac{1}{a}(y(\theta^{t-1},\theta_t)-\bar{\theta}_t))\\ =\;&u(y(\theta^{t-1},\theta_t)-\frac{1}{a}y(\theta^{t-1},\theta_t)+\tau(\theta^{t-1},\theta_t)+\frac{1}{a}\bar{\theta}_t)\\ >\;&u(y(\theta^{t-1},\theta_t)-\frac{1}{a}y(\theta^{t-1},\theta_t)+\tau(\theta^{t-1},\theta_t)+(\frac{1}{a}-\frac{1}{b})e(\theta^{t-1},\theta_t)+\frac{1}{a}\bar{\theta}_t),\end{aligned}$$

since $(\frac{1}{a}-\frac{1}{b})e(\theta^{t-1},\theta_t)<0$. The case against $e(\theta^t)<0$ follows from a similar line of proof. This implies that the equilibrium utility of the agent is unchanged. At this point one can try to actually increase agent's utility but we only need to show that our closed form solution belongs to the set of optimal contracts. **Q.E.D.**

Lemma 1 implies that we can re-write problem **(R)** as follows

$$\textbf{(R')}\qquad\max_{\tau(\theta^t)}\ \mathbf{E}_0\left[\sum_{t=1}^{T}\delta^{t-1}u(\theta_t+\tau(\theta^t))\right]\quad\text{s.t.}\quad\text{for all }\theta^t,t\geq 1:$$

$$\begin{aligned}
&\max{}_{b,\hat{\theta}_t\le\theta_t}\, u\left(\hat{\theta}_t+\tau(\theta^{t-1},\hat{\theta}_t)-q_t b-\frac{1}{a}(\hat{\theta}_t-\theta_t)\right)+\delta V_t\left(\left(\theta^{t-1},\theta_t\right),\hat{\theta}_t,b\right)\\
\le\;& u(\theta_t+\tau(\theta^t))+\delta U_t\left(\theta^t\right);\text{ and} \qquad (31)\\
0\;\ge\;& \mathbf{E}_0\left[\sum_t q^t\tau(\theta^t))\right],
\end{aligned}$$

where

$$U_t\left(\theta^t\right):=\;\mathbf{E}_t\left[\sum_{s=1}^{T-t}\delta^{s-1}u(\theta_{t+s}+\tau(\theta^{t+s}))\right]=\int_\Theta\left[u(\theta_{t+1}+\tau(\theta^t,\theta_{t+1}))+\delta U_{t+1}\left(\theta^t,\theta_{t+1}\right)\right]d\Phi\left(\theta_{t+1}\mid\theta^t\right),$$

and

$$\begin{aligned}
V_t\left(\left(\theta^{t-1},\theta_t\right),\hat{\theta}_t,b\right)\;:&=\max_{\mathbf{c},\mathbf{b}}\;\mathbf{E}\left[\sum_{s=1}^{T-t}\delta^{s-1}u(c\left(\hat{\theta}^{t+s}\right))\;\mid\;\theta^t\right]\\
s.t.\;:&\quad c_{t+s}\left(\theta^{t+s}\right)=\theta_{t+s}+\tau(\hat{\theta}^{t+s})+q_{t+s}b_{t+s+1}\left(\theta^{t+s}\right)-b_{t+s}\left(\theta^{t+s}\right).
\end{aligned}$$

Assumption 1 *The utility function takes the exponential (CARA) form:*

$$u\left(c-v(e)\right)=-\frac{1}{\rho}\exp\left\{-\rho(c-v(e))\right\}\qquad \textit{with } \rho>0 \textit{ and the function } v \text{ is as in } \textit{(30).}$$

Proposition 2. *If preferences are CARA, for each given θ^{t-1}, the present value of transfers solving problem (**R'**) - which are defined as $PVT_t=\sum_{n=0}^{T-t}\left(\Pi_{s=0}^n q_{t+s-1}\right)\tau_{t+n}\left(\theta^{t+n}\right)$ - obeys to the following. There are θ^{t-1}-measurable functions $\{\eta_t\}_{t=1}^T$ such that for all $\theta^t, t\ge 1$:*

$$\sum_{n=0}^{T-t}\left(\Pi_{s=0}^n q_{t+s-1}\right)\tau\left(\theta^{t+n}\right)=\eta_t\left(\theta^{t-1}\right)+\sum_{n=0}^{T-t}\left(\Pi_{s=0}^n q_{t+s-1}\right)\left[\left(\frac{1}{a}-1\right)\theta_{t+n}\right] \qquad (32)$$

or, equivalently, for all $\theta^t, t\ge 1$

$$\tau_t\left(\theta^t\right)+\sum_{n=1}^{T-t}\left(\Pi_{s=1}^n q_{t+s-1}\right)\eta_{t+n}\left(\theta^{t+n}\right)=\eta_t\left(\theta^{t-1}\right)+\left(\frac{1}{a}-1\right)\theta_t.$$

In particular, $\sum_{n=0}^{T-t}\left(\Pi_{s=0}^n q_{t+s-1}\right)\tau\left(\theta^{t+n}\right)$ admits partial derivative with respect to θ_t and for each fixed past history θ^{t-1}, and fixed future $\theta_{t+1},...,\theta_T$, we have

$$\frac{\partial}{\partial\theta_t}\sum_{n=0}^{T-t}\left(\Pi_{s=0}^n q_{t+s-1}\right)\tau\left(\theta^{t+n}\right)=\left(\frac{1}{a}-1\right).$$

Proof. Keep in mind that we must show the following set of equations:

$$\begin{aligned}
\tau_T\left(\theta^T\right)\;&=\;\eta_T\left(\theta^{T-1}\right)+\left(\frac{1}{a}-1\right)\theta_T\\
\tau_{T-1}\left(\theta^{T-1}\right)+q_{T-1}\eta_T\left(\theta^{T-1}\right)\;&=\;\eta_{T-1}\left(\theta^{T-2}\right)+\left(\frac{1}{a}-1\right)\theta_{T-1}\\
&\quad\;....\\
\tau_1\left(\theta_1\right)+\sum_{n=1}^{T-1}\left(\Pi_{s=1}^n q_{t+s-1}\right)\eta_{n+1}\left(\theta^n\right)\;&=\;\eta_1+\left(\frac{1}{a}-1\right)\theta_1.
\end{aligned}$$

We will prove our proposition backwards. Let's consider our problem in the last two periods. It is easy to see from our relaxed problem that, since the agent has von Neumann-Morgenstern utility and the firm maximizes the expected discounted value of profits, the only link across states comes from the incentive constraints In the proof below we will only consider the relevant incentive constraints.

$$\begin{aligned}
&\text{s.t.} \quad \text{for all } \theta^{T-1}: \\
&u(\theta_{T-1}+\tau(\theta^{T-1}))+\delta\int_{\Theta}u(\theta_T+\tau(\theta^{T-1},\theta_T))d\Phi(\theta_T\mid\theta^{T-1})\geq \qquad (33)\\
&\max_{b,\hat{\theta}_{T-1}\leq\theta_{T-1}}u(\hat{\theta}_{T-1}+\tau(\theta^{T-2},\hat{\theta}_{T-1})-q_{T-1}b-\frac{1}{a}(\hat{\theta}_{T-1}-\theta_{T-1}))+\\
&+\delta\int_{\Theta}u(\theta_T+b+\tau(\theta^{T-2},\hat{\theta}_{T-1},\theta_T))d\Phi(\theta_T\mid\theta^{T-1});\\
&\text{and}\\
&\text{and for all } \theta_{T-1},\theta_T \text{ and } \hat{\theta}_T\leq\theta_T: \\
&u(\theta_T+\tau(\theta^{T-1},\theta_T))\geq u(\hat{\theta}_T+\tau(\theta^{T-1},\hat{\theta}_T)-\frac{1}{a}(\hat{\theta}_T-\theta_T)); \qquad (34)
\end{aligned}$$

Lemma 2 *If the utility function is CARA, the transfer scheme solving problem (**R'**) satisfies the following condition: for all θ^{T-1} we have $\tau(\theta^{T-1},\theta'_T)-\tau(\theta^{T-1},\theta''_T)=-(1-\frac{1}{a})(\theta'_T-\theta''_T)$ for all θ'_T,θ''_T. In particular, the partial derivative $\frac{\partial}{\partial\theta_T}\tau\left(\theta^T\right)$ exists and it equals $(\frac{1}{a}-1)$ for all θ^{T-1}.*

Proof. It is easy to see from (34) (by taking the inverse of u transformation to both sides and apply it to all θ) that $\tau(\theta^{T-1},\theta'_T)-\tau(\theta^{T-1},\theta''_T)\geq-(1-\frac{1}{a})(\theta'_T-\theta''_T)$ for all θ'_T,θ''_T is a necessary condition for incentive compatibility.[27] Now, suppose that for a range of productivities we have $\tau(\theta^{T-1},\theta'_T)-\tau(\theta^{T-1},\theta''_T)>-(1-\frac{1}{a})(\theta'_T-\theta''_T)$ for all $\theta'_T,\theta''_T\in[\theta^0_T-\varepsilon,\theta^0_T+\varepsilon]$. We claim that there is a modification to the contract that keeps the same utility to the agent and reduces the net present value of the transfers for the firm. The new scheme is such that $\tilde{\tau}(\theta^{T-1},\theta'_T)-\tilde{\tau}(\theta^{T-1},\theta''_T)=-(1-\frac{1}{a})(\theta'_T-\theta''_T)$ and for each node θ^{T-1} the new transfer solves $\int_{\theta-\varepsilon}^{\theta+\varepsilon}u(\theta_T+\tilde{\tau}(\theta^{T-1},\theta_T))d\Phi(\theta_T\mid\theta^{T-1})=\int_{\theta-\varepsilon}^{\theta+\varepsilon}u(\theta_T+\tau(\theta^{T-1},\theta_T))d\Phi(\theta_T\mid\theta^{T-1})$. The fact that the new scheme imposes less consumption dispersion to the agent implies that it is potentially able to deliver the same agent's expected utility with lower average transfers. We have to show that this change is incentive feasible. Let's start with condition (34). The new transfer scheme is incentive compatible in the range $[\theta^0_T-\varepsilon,\theta^0_T+\varepsilon]$ by construction. Moreover, it reduces the utility at the top extreme of the range while it increases agent's utility at the bottom of the range. Now, from the specific form of u we have

$$\begin{aligned}
\int_{\Theta}u(\theta_T+b+\tilde{\tau}(\theta^{T-1},\theta_T))d\Phi(\theta_T\mid\theta^{T-1}) &= \exp\{-\rho b\}\int_{\Theta}u(\theta_T+\tilde{\tau}(\theta^{T-1},\theta_T))d\Phi(\theta_T\mid\theta^{T-1})\\
&= \exp\{-\rho b\}\int_{\Theta}u(\theta_T+\tau(\theta^{T-1},\theta_T))d\Phi(\theta_T\mid\theta^{T-1})\\
&= \int_{\Theta}u(\theta_T+b+\tau(\theta^{T-1},\theta_T))d\Phi(\theta_T\mid\theta^{T-1})
\end{aligned}$$

for all b. The equality in the second row is true since the new scheme solves $\int_{\Theta}u(\theta_T+\tilde{\tau}(\theta^{T-1},\theta_T))d\Phi(\theta_T\mid\theta^{T-1})=$

[27] If for a $\theta_T<\infty$, the transfer scheme has slope less that $\left(1-\frac{1}{a}\right)$, we can choose $\theta'_T>\theta_T$ and obtain the violation of the incentive compatibility constraint when the agent has shock θ'_T.

$\int_\Theta u(\theta_T + \tau(\theta^{T-1}, \theta_T))d\Phi(\theta_T \mid \theta^{T-1})$. This implies that the change does not affects (33) or any other incentive constraint (31) as $V_t\left(\left(\theta^{t-1}, \theta_t\right), \hat{\theta}_t, b\right)$ are unchanged for all t, b. Note that this result implies that the transfer τ_T is partially differentiable in θ_T with derivative equal to $(1 - \frac{1}{a})$ for all θ^{T-1} and $\theta_T < \theta_{\max}$.[28] Note that when $\theta_{\max} = \infty$ the function τ_T is partially differentiable everywhere. **Q.E.D.**

To complete the induction argument we need the following.

Lemma 3. *If a transfer scheme solving **(R')** is such that for all $s > t$ $\frac{\partial}{\partial \theta_s} PVT_s(\theta^s) = \frac{1}{a} - 1$ for all θ^s and all $(\theta_{s+1}, ...\theta_T)$, then $\frac{\partial}{\partial \theta_t} PVT_t(\theta^t) = \frac{1}{a} - 1$ for all θ^t and all $(\theta_{t+1}, ...\theta_T)$.*

Proof. First, note that from the incentive constraint we have for all $\theta_t' \leq \theta_t''$ $PVT_t(\theta^{t-1}, \theta_t'') - PVT_t(\theta^{t-1}, \theta_t') \geq (1 - \frac{1}{a})(\theta_t' - \theta_t'')$. If this were not true then the agent with realization θ_t'' would declare θ_t' and improve welfare. In particular, let $\kappa = PVT_t(\theta^{t-1}, \theta_t'') - PVT_t(\theta^{t-1}, \theta_t')$ and suppose $\kappa < (1 - \frac{1}{a})(\theta_t' - \theta_t'')$. Consider an agent with productivity θ_t'' declaring θ_t'. The agent would (have to) reduce effort so that the argument of the flow utility u in case of zero bond decision would be $\theta_t' + \tau(\theta^{t-1}, \theta_t') - \frac{1}{a}(\theta_t' - \theta_t'')$ as opposed to $\theta_t^{''} + \tau(\theta^{t-1}, \theta_t^{''})$ when telling the truth. We now show that there is a plan of bonds $\hat{b}$ such that telling the truth in the future and choosing the constructed bonds plan improves agents's welfare. Namely, we will show that constraint (31) is violated at node θ^t. The bond plan $\hat{b}$ is constructed so that the deviating agent gets exactly the same argument in the flow utility u all nodes but the last period one, where in each of the last period nodes the agent consumes $\hat{c}_T\left(\theta^T\right) = c_T\left(\theta^T\right) + \frac{(1-\frac{1}{a})(\theta_t' - \theta_t'') - \kappa}{\Pi_s q_s}$. The bond plan $\hat{b}$ is constructed as follows. Let $\theta^{t+s} := \left(\theta^{t-1}, \theta_t^{''}, \theta_{t+1}, ...\theta_{t+s}\right)$ the true history of shocks. Note that the agent expectations are taken according the distribution implied by this history. Moreover, let $\hat{\theta}^{t+s} := \left(\theta^{t-1}, \theta_t', \theta_{t+1}, ...\theta_{t+s}\right)$. In order to obtain the plan of consumption $\hat{c}_{t+s}\left(\hat{\theta}^{t+s}\right) = c_{t+s}\left(\theta^{t+s}\right) = \theta_{t+s} + \tau(\theta^{t+s})$ for $s > 1$, the bond plan must solve for all $s > 1$

$$b_{t+s}\left(\theta^{t+s-1}\right) - q_{t+s} b_{t+s+1}\left(\theta^{t+s}\right) = \eta_{t+s}(\theta^{t+s-1}) - \eta_{t+s}(\hat{\theta}^{t+s-1}).$$

Moreover, in period t we have

$$q_t b_{t+1}\left(\theta^t\right) = \left[\theta_t^{''} + \tau(\theta^{t-1}, \theta_t^{''})\right] - \left[\theta_t' + \tau(\theta^{t-1}, \theta_t') - \frac{1}{a}(\theta_t' - \theta_t'')\right].$$

It is east to see - by straightforward calculations - that the plan satisfies two key properties. First, it delivers the same consumption plan to the agent at all nodes but the last as claimed. This is so because our inductive hypothesis. Second, the plan is budget feasible if $\hat{c}_T\left(\theta^T\right) = c_T\left(\theta^T\right) + \frac{(1-\frac{1}{a})(\theta_t' - \theta_t'') - \kappa}{\Pi_s q_s} > c_T\left(\theta^T\right)$ as claimed.

We now have to show that it cannot be the case that the inequality is strict. Suppose that for some range of skills $[\theta_t^0 - \varepsilon, \theta_t^0 + \varepsilon]$ and consider the modification to the contract that makes it an equality and delivers the same expected utility to the agent over this range. We now show that this change is incentive compatible. The argument is a generalization of the last part of the proof of Lemma 2. **Q.E.D.**

[28] More precisely, for each $\theta_T < \theta_{\max}$ choose $\theta_T' > \theta_T$. We have just shown that for all θ_T such that $\theta_T \leq \theta_T'$ the transfer scheme has constant slope which equals $\left(1 - \frac{1}{a}\right)$.

Assumption 2. *The stochastic process for skills follows:* $\theta_t - \theta_{t-1} = \beta(L) v_t$, *where* $\beta(\cdot)$ *is a polynomial of order* p *in the lag operator* L, *and the innovation* v_t *is white noise (serially uncorrelated) process assumed to be normally distributed with zero mean and variance* σ_v^2. *The moving average process is invertible, that is the roots of the polynomial* $\beta(L)$ *lie outside the unit circle (we normalize* $\beta_0 = 1$*).*[29] *Moreover, assume that* $q_t = q$ *for all* t.

It should be clear from the proof, that next proposition - with the appropriate adjustments in notation - can be shown with slightly more general processes for θ_t, as long as linearity in the law and the assumption of Gaussian shocks are maintained. Moreover, the assumption of constant q is done only for notational simplicity. The obtained expressions are those in the main text.

Proposition 3. *Assume A1 and A2. For all* t, T *such that* $T - t - 1 \geq p$, *the consumption process follows*

$$c_{t+1}^* - c_t^* = \frac{\ln(\delta/q)}{\rho} + \frac{\rho}{2a^2}[\beta(q)]^2 \sigma_v^2 + \frac{1}{a}\beta(q) v_{t+1}.$$

In particular, if the productivity process follows $\theta_t = \theta_{t-1} + v_t$, *we have* $c_{t+1}^* - c_t^* = \Gamma_t + \frac{1}{a}\left(y_{t+1}^* - y_t^*\right) = \frac{\ln(\delta/q_t)}{\rho} + \frac{\rho}{2a^2}\sigma_v^2 + \frac{1}{a}v_{t+1}$ *no matter what is the time horizon and the sequence of bond prices.*

Proof. First, from $c_t = y_t + \tau_t$ at all nodes,[30] we have that both

$$\begin{aligned} \mathbf{E}_t \sum_{n=0}^{T-t-1} q^n c_{t+1+n}^* &= \mathbf{E}_t \sum_{n=0}^{T-t-1} q^n \left(y_{t+1+n}^* + \tau_{t+1+n}^*\left(y^{t+1+n}\right)\right) \\ &\text{and} \\ \mathbf{E}_{t+1} \sum_{n=0}^{T-t-1} q^n c_{t+1+n}^* &= \mathbf{E}_{t+1} \sum_{n=0}^{T-t-1} q^n \left(y_{t+1+n}^* + \tau_{t+1+n}^*\left(y^{t+1+n}\right)\right). \end{aligned} \tag{35}$$

Using the Euler equation

$$\exp\{-\rho(c_t^*)\} = \left(\frac{\delta}{q}\right)^s \mathbf{E}_t\left[\exp\{-\rho(c_{t+s}^*)\}\right].$$

and the properties of the normal distribution, we have, for $s \geq 1$

$$\begin{aligned} \mathbf{E}_t c_{t+s}^* &= c_t^* + s\frac{\ln\frac{\delta}{q}}{\rho} - \frac{\rho}{2}\sum_{n=1}^{s}\sigma_{c_{t+n}}^2 \\ \mathbf{E}_{t+1} c_{t+s}^* &= c_{t+1}^* + (s-1)\frac{\ln\frac{\delta}{q}}{\rho} - \frac{\rho}{2}\sum_{n=1}^{s-1}\sigma_{c_{t+1+n}}^2, \end{aligned}$$

where $\sigma_{c_t}^2$ is the variance of consumption growth in period t. This implies

$$\sum_{s=1}^{T-t-1}\left(\mathbf{E}_{t+1} - \mathbf{E}_t\right) q^s c_{t+s}^* = \frac{1-q^{T-t}}{1-q}\left[c_{t+1}^* - c_t^* + \frac{\ln\frac{q}{\delta}}{\rho} - \frac{\rho}{2}\sigma_{c_{t+1}}^2\right],$$

[29] Obviously, we assume the following initial conditions for the process: $\theta_0 = v_0 = v_{-1} = ..v_{-p} = 0$, where p is the maximum number of lags in the MA component of the process.

[30] One would obtain the same for any process for bonds using the standard rearrangements in the permanent income literature (e.g., Deaton, 1992).

and, using (35):

$$c_{t+1}^* - c_t^* = \Gamma_t + \frac{1-q}{1-q^{T-t}} \left(\mathbf{E}_{t+1} - \mathbf{E}_t\right) \left[\sum_{n=0}^{T-t-1} q^n \left(y_{t+1+n}^* + \tau_{t+1+n}^* \left(y^{t+1+n}\right)\right)\right],$$

with $\Gamma_t = \frac{\ln(\delta/q)}{\rho} + \frac{\rho}{2}\sigma_{c_t}^2$.

Second, if we apply Lemma 3 - in particular, see equation (32) - together with $\theta_{t+n} = y_{t+n}^*$ for all t, n, since $\left(\mathbf{E}_{t+1} - \mathbf{E}_t\right) \eta_{t+1}\left(y^t\right) = 0$, we obtain

$$\left(\mathbf{E}_{t+1} - \mathbf{E}_t\right) \left[\sum_{n=0}^{T-t-1} q^n \tau_{t+1+n}^* \left(y^{t+1+n}\right)\right] = \left(\frac{1}{a} - 1\right) \left(\mathbf{E}_{t+1} - \mathbf{E}_t\right) \left[\sum_{n=0}^{T-t-1} q^n y_{t+1+n}^*\right].$$

If we now combine the two last expressions, we obtain:

$$c_{t+1}^* - c_t^* = \Gamma_t + \frac{1}{a}\frac{1-q}{1-q^{T-t}} \left(\mathbf{E}_{t+1} - \mathbf{E}_t\right) \left[\sum_{n=0}^{T-t-1} q^n y_{t+1+n}^*\right]. \tag{36}$$

hence $\Gamma_t = \frac{\ln \frac{\delta}{q}}{\rho} + \frac{\rho}{2}\sigma_{c_t}^2$.

From Assumption 2, for $T-t-1 \geq p$, equation (36) becomes[31]

$$c_{t+1}^* - c_t^* = \frac{\ln \frac{\delta}{q}}{\rho} + \frac{\rho}{2}\sigma_{c_t}^2 + \frac{1}{a}\frac{1-q}{1-q^{T-t}} \sum_{n=0}^{T-t-1} q^n \beta(q)\, v_{t+1} = \frac{\ln \frac{\delta}{q}}{\rho} + \frac{\rho}{2}\sigma_{c_t}^2 + \frac{1}{a}\beta(q)\, v_{t+1}.$$

Finally, since the above expression implies that $\sigma_{c_t}^2 := var_t\left(\Delta c_{t+1}^*\right) = \frac{[\beta(q)]^2}{a^2}\sigma_v^2$, we obtain the claimed expression for consumption growth.

Clearly, the case with purely permanent shocks corresponds to the case where $\beta_i = 0$ for $i \geq 1$, hence the result is trivial. It is also easy to show that in this case, $b_t^* \equiv 0$ is consistent with

$$\begin{aligned} \frac{\partial \tau_t\left(y^t\right)}{\partial y_t} &= \frac{1}{a} - 1 \text{ for all } t; \text{ and} \\ \frac{\partial \tau_t\left(y^t\right)}{\partial y_{t-s}} &= 0 \text{ for all } t,\ s > 0. \end{aligned}$$

[31] Recall that y_t follows

$$y_{t+1}^* = y_t^* + \beta(L)\, v_{t+1},$$

with $\beta(\cdot)$ of order p. We hence have:

$$\begin{aligned} \left(\mathbf{E}_{t+1} - \mathbf{E}_t\right) y_{t+1}^* &= v_{t+1} \\ \left(\mathbf{E}_{t+1} - \mathbf{E}_t\right) q y_{t+2}^* &= q\left(1+\beta_1\right) v_{t+1} \\ \left(\mathbf{E}_{t+1} - \mathbf{E}_t\right) q^2 y_{t+3}^* &= q^2\left(1+\beta_1+\beta_2\right) v_{t+1} \\ &\;\;.... \\ \left(\mathbf{E}_{t+1} - \mathbf{E}_t\right) q^n y_{t+1+n}^* &= q^n\left(1+\beta_1+...+\beta_p\right) v_{t+1} \text{ for } n \geq p. \end{aligned}$$

As long as $T-t-1 \geq p$, collecting terms 'vertically' the expression takes the stable form we indicate in the main text.

It is hence easy to see that

$$\Delta c_{t+1}^{*}=\frac{\ln\frac{\delta}{q}}{\rho}+\frac{\rho}{2a^{2}}\sigma_{v}^{2}+\frac{1}{a}\left(y_{t+1}^{*}-y_{t}^{*}\right), \tag{37}$$

for all $T<\infty$ *and all* $\{q_t\}_{t=1}^{T-1}$. **Q.E.D.**

We now use that fact that the tax scheme is linear to show the following Lemma that concludes the proof.

Proposition 4. *If the agent has CARA preferences, when facing the above tax, the agent's problem is concave, so the derived tax scheme is optimal.*

Proof. Note that so far we have shown that the transfer scheme is differentiable. Moreover, the agent's necessary conditions for $e_t^*\left(\theta^t\right)=0$ to be an optimal choice is

$$\frac{\partial}{\partial\theta_t}\sum_{n=0}^{T-t}q^n\tau\left(\theta^{t+n}\right)\in\left[\frac{1}{a}-1,\frac{1}{b}-1\right].$$

Since we have shown that $\frac{\partial}{\partial\theta_t}\sum_{n=0}^{T-t}q^n\tau\left(\theta^{t+n}\right)=\frac{1}{a}-1$ the condition is met.

Now note that since $e_t^*\left(\theta^t\right)\equiv 0$, at all nodes we have $y_t\left(\theta^t\right)=\theta^t$. We can hence invert the identity map and write the transfer scheme as a function of income histories y^t. We have to show that, when facing the optimal tax scheme, the agent's problem is jointly concave in $\{e_t\left(\theta^t\right)\}_{t=0}^{T}$ and $\{b_{t+1}\left(\theta^t\right)\}_{t=0}^{T}$. Consider two contingent plans e^1,b^1,c^1 and e^2,b^2,c^2. Now consider the plan $e^\alpha,b^\alpha,c^\alpha$ where for all y^t, and $\alpha\in[0,1]$ we have $e_t^\alpha\left(\theta^t\right):=\alpha e_t^1\left(\theta^t\right)+(1-\alpha)\,e_t^2\left(\theta^t\right)$, and similarly for b_t^α and c_t^α. First of all, since assets enter linearly in the agent's budget constraint and effort enters linearly in the production function, the concavity of the agent's utility in $c-v\left(e\right)$, and the additive separability over time and states imply that if we show that $c_t^\alpha-v\left(e_t^\alpha\right)\geq\alpha\left[c_t^1-v\left(e_t^1\right)\right]+(1-\alpha)\left[c_t^2-v\left(e_t^2\right)\right]$, we are done. If we set k_t the constant of integration of τ_t, an agent who chooses plan e^α of effort, at node θ^t gets

$$\begin{aligned}
c_t^\alpha-v\left(e_t^\alpha\right) &= y_t^\alpha+\sum_{i=0}^{t-1}\tau_t^{(t-i)}y_{t-i}^\alpha+k_t-v\left(e_t^\alpha\right)\\
&= \theta_t+e_t^\alpha+\sum_{i=0}^{t-1}\tau_t^{(t-i)}\left[\theta_{t-i}+e_{t-i}^\alpha\right]+k_t-v\left(e_t^\alpha\right)\\
&\geq \left[\alpha\left(\theta_t+e_t^1\right)+(1-\alpha)\left(\theta_t+e_t^2\right)\right]+\sum_{i=0}^{t-1}\tau_t^{(t-i)}\left[\theta_{t-i}+e_{t-i}^\alpha\right]\left[\alpha\left(\theta_{t-i}+e_{t-i}^1\right)+(1-\alpha)\left(\theta_{t-i}+e_{t-i}^2\right)\right]\\
&\qquad +k_t-\alpha\left[v\left(e_t^1\right)\right]+(1-\alpha)\left[v\left(e_t^2\right)\right]\\
&= \alpha\left[c_t^1-v\left(e_t^1\right)\right]+(1-\alpha)\left[c_t^2-v\left(e_t^2\right)\right],
\end{aligned}$$

where the inequality in the penultimate row comes from the concavity of v in e. The last line uses the agent's budget constraint $c_t\left(y^t\right)=y_t+\tau\left(y^t\right)$. **Q.E.D.**

A final remark. Although the proof of the closed form uses finite time, we conjecture that adapting the proof of Proposition 7 in Cole and Kocherlakota (2001), we are able to show that same close form solution for $T=\infty$ despite u is unbounded below.

7.2 Quadratic utility

We now maintain the same assumptions on the cost function v (or the production function f) as in (30). Moreover, we keep the linearity assumption for the process θ_t - $\Delta\theta_t = \beta(L)\,v_t$ - but we do not assume any parametric distribution for the iid shocks v_t (of course, we need to be able to take expectations). In fact, we now need to assume that Θ is bounded above by $\theta_{\max} < \infty$, and that agent's preferences are quadratic:

$$u(c - v(e)) := -\frac{1}{2}\left(\bar{B} - (c - v(e))\right)^2 \qquad \text{with } \bar{B} >> T\theta_{\max}. \tag{38}$$

Finally, we are able to derive the closed form only within the class of transfer schemes that admit symmetric cross partial derivative. Making assumptions on endogenous variables is of course not desirable, but note that the incentive constraint will always impose some degree of monotonicity on the transfer scheme. Since monotone function on compact sets are absolutely continuous, under few further regularity conditions, we conjecture that one would be able to show at least almost everywhere differentiability of the transfer scheme. Of course, the symmetry of the Hessian is an even stronger condition which we did not investigate how to show from primitives.

We have the following.

Proposition 5. *If the agent has preferences as in (38) and $\theta_t = \theta_{t-1} + \beta(L)\,v_t$, within the class of transfer schemes that admit symmetric cross derivatives, taxes are linear in income histories. Moreover, if $\delta = q$, the expression of marginal taxes is exactly as in the CARA case. In particular, for $T \geq t+p+1$, we have*

$$\Delta c^*_{t+1} = \frac{1}{a}\beta(q)\,v_{t+1}.$$

Proof. First of all, from Lemma 1, in equilibrium we will get $e^*_t \equiv 0$, hence the transfer scheme is invertible and we can write it in terms of income histories y^t. We now need a crucial lemma, which uses differentiability.

Lemma 4. *Within the class of within the class of transfer schemes that admit symmetric cross derivatives, the discounted value of marginal transfers $\sum_{n=0}^{T-t} q^n \frac{\partial \tau_{t+n}(y^{t+n})}{\partial y_s}$ does not depend on $(y_t, ..., y_T)$ for all s. They are hence linear functions of y_s given y^t.*

Proof. Consider the following relaxed problem the firm: Maximize expected discounted profits, choosing the transfer scheme, subject to the first order conditions of the agent, namely for all $t \geq 1$, and $t \geq s \geq 0$

$$\frac{1}{b} - 1 \geq \mathbf{E}_{t-s}\sum_{n=0}^{T-t}\delta^n\left[\frac{\partial \tau_{t+n}(y^{t+n})}{\partial y_t}\frac{u'(c_{t+n} - e_{t+n})}{u'(c_t - e_t)}\right] = \mathbf{E}_{t-s}\mathbf{E}_t\left[\sum_{n=0}^{T-t}\delta^n\frac{\partial \tau_{t+n}(y^{t+n})}{\partial y_t}\frac{u'(c_{t+n} - e_{t+n})}{u'(c_t - e_t)}\right] \geq \frac{1}{a} - 1; \tag{39}$$

and the Euler equations corresponding to u as in (38)

$$-\bar{B} + \mathbf{c}_t(y^t) = \left(\frac{\delta}{q}\right)^s \mathbf{E}_t\left[\mathbf{c}_{t+s}(y^{t+s})\right] - \left(\frac{\delta}{q}\right)^s \bar{B}. \tag{40}$$

The proof is by backwards induction. By looking at the last period of the problem, we have

$$\frac{1}{b} \geq 1 + \frac{\partial \tau_T\left(y^T\right)}{\partial y_T} \geq \frac{1}{a}.$$

Since the firm aims at insuring the agent, the relevant inequality is the second one. Moreover, given that there is not gain in efficiency in changing the implemented level of effort, and (40) is not affected as long as the average value of transfers does not change, the firm will set $\frac{\partial \tau_T\left(y^{T-1}, y_T\right)}{\partial y_T} = \frac{1}{a} - 1$ for all y^{T-1} and y_T. This implies a zero cross derivative: $\frac{\partial \tau_T\left(y^{T-1}, y_T\right)}{\partial y_T \partial y_t} = 0$ for all t. Given our assumptions on the class of transfer schemes, by symmetry, it must be that $\frac{\partial \tau_T\left(y^T\right)}{\partial y_t}$ is constant in y_T for all $t < T$.

Now consider τ_{T-1}. Since $\frac{\partial \tau_T\left(y^T\right)}{\partial y_{T-1}}$ does not depend on y_T, the effort incentive compatibility can be written as follows:

$$\frac{\partial \tau_{T-1}\left(y^{T-1}\right)}{\partial y_{T-1}} + \delta \frac{\partial \tau_T\left(y^T\right)}{\partial y_{T-1}} \mathbf{E}_{T-1}\left[\frac{u'\left(c_T\right)}{u'\left(c_{T-1}\right)}\right] = \frac{1}{a} - 1, \qquad \text{for all } y^{T-2} \text{ and } y_{T-1}.$$

Since $\mathbf{E}_{T-1}\left[\frac{u'(c_T)}{u'(c_{T-1})}\right] = \frac{q}{\delta}$, we have that $\frac{\partial \tau_{T-1}\left(y^{T-1}\right)}{\partial y_{T-1}} + q\frac{\partial \tau_T\left(y^T\right)}{\partial y_{T-1}}$ is a constant for all all y^{T-2} and y_{T-1}. Again, since the transfer scheme is assumed to have symmetric cross derivative, this property implies that $\frac{\partial \tau_{T-1}\left(y^{T-1}\right)}{\partial y_t} + q\frac{\partial \tau_T\left(y^T\right)}{\partial y_t}$ is also constant in y_{T-1} (and y_T) for all t. Going backwards, we have our result: $\sum_{n=t}^{T} q^n \frac{\partial \tau_n\left(y^n\right)}{\partial y_s}$ is constant in y_t, ..., y_T for all s. **Q.E.D.**

Given the above results we can apply the law of iterated expectations and get, for a generic δ and q

$$\begin{aligned}
& \mathbf{E}_t\left[\sum_{n=0}^{T-t} \delta^n \frac{\partial \tau_{t+n}\left(y^{t+n}\right)}{\partial y_t} \frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)}\right] \\
= \; & \mathbf{E}_t\left[\sum_{n=0}^{T-t-1} \delta^n \frac{\partial \tau_{t+n}\left(y^{t+n}\right)}{\partial y_t} \frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)} + \delta^{T-t} \mathbf{E}_{T-1} \frac{\partial \tau_T\left(y^T\right)}{\partial y_t} \frac{u'\left(c_T-e_T\right)}{u'\left(c_t-e_t\right)}\right] \\
= \; & \mathbf{E}_t\left[\sum_{n=0}^{T-t-1} \delta^n \frac{\partial \tau_{t+n}\left(y^{t+n}\right)}{\partial y_t} \frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)} + \delta^{T-t} \frac{\partial \tau_T\left(y^T\right)}{\partial y_t} \mathbf{E}_{T-1} \frac{u'\left(c_T-e_T\right)}{u'\left(c_{T-1}-e_{T-1}\right)} \frac{u'\left(c_{T-1}-e_{T-1}\right)}{u'\left(c_t-e_t\right)}\right] \\
= \; & \mathbf{E}_t\left[\sum_{n=0}^{T-t-1} \delta^n \frac{\partial \tau_{t+n}\left(y^{t+n}\right)}{\partial y_t} \frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)} + \delta^{T-t-1} q \frac{\partial \tau_T\left(y^T\right)}{\partial y_t} \frac{u'\left(c_{T-1}-e_{T-1}\right)}{u'\left(c_t-e_t\right)}\right] \\
= \; & \mathbf{E}_t\left[\sum_{n=0}^{T-t-2} \delta^n \frac{\partial \tau_{t+n}\left(y^{t+n}\right)}{\partial y_t} \frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)} + \delta^{T-t-1} \mathbf{E}_{T-2}\left(\frac{\partial \tau_{T-1}\left(y^{T-1}\right)}{\partial y_t} + q\frac{\partial \tau_T\left(y^T\right)}{\partial y_t}\right) \frac{u'\left(c_{T-1}-e_{T-1}\right)}{u'\left(c_t-e_t\right)}\right] \\
= \; & \mathbf{E}_t\left[\sum_{n=0}^{T-t-2} \delta^n \frac{\partial \tau_{t+n}\left(y^{t+n}\right)}{\partial y_t} \frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)} + \delta^{T-t-2} q\left(\frac{\partial \tau_{T-1}\left(y^{T-1}\right)}{\partial y_t} + q\frac{\partial \tau_T\left(y^T\right)}{\partial y_t}\right) \frac{u'\left(c_{T-2}-e_{T-2}\right)}{u'\left(c_t-e_t\right)}\right] \\
& \ldots \\
= \; & \mathbf{E}_t\left[\sum_{n=0}^{T-t} q^n \frac{\partial \tau_{t+n}\left(y^{t+n}\right)}{\partial y_t}\right]. \qquad (41)
\end{aligned}$$

where we repeatedly used the linearity of expectations and the Euler equation. We are hence done since - given that the obtained taxes are linear - Proposition 4 implies that this transfer scheme is optimal (now within the class of schemes we consider). Moreover, we are now able to follow the steps for the derivation of the closed form for CARA utility and obtain a very similar closed form.

Since from the incentive compatibility for effort e_t, we have $\mathbf{E}_t\left[\sum_{n=0}^{T-t} q^n \frac{\partial\tau^*_{t+n}\left(y^{t+n}\right)}{\partial y_t}\right] = \frac{1}{a} - 1$, by using the law of iterated expectations, one obtains:

$$\left(\mathbf{E}_{t+1} - \mathbf{E}_t\right)\left[\sum_{n=0}^{T-t-1} q^n \tau^*_{t+1+n}\left(y^{t+1+n}\right)\right] = \left(\frac{1}{a} - 1\right)\left(\mathbf{E}_{t+1} - \mathbf{E}_t\right)\left[\sum_{n=0}^{T-t-1} q^n y^*_{t+1+n}\right]. \tag{42}$$

The expressions for the optimal individual taxes τ_t can be obtained working backwards. In the working paper Attanasio and Pavoni (2006) we consider generic q and δ. When $q \neq \delta$ the expressions can get quite complicated even for the purely temporary shocks. When $\delta = q$ however, from the Euler equation we have $\mathbf{E}_t c^*_{t+s} = c^*_t$ for all s. So, following exactly the lines of proof of Proposition 3 above for CARA - namely using the standard re-arrangements of the permanent income literature - we obtain that

$$c^*_{t+1} - c^*_t = \frac{1-q}{1-q^{T-t}}\left(\mathbf{E}_{t+1} - \mathbf{E}_t\right)\left[\sum_{n=0}^{T-t-1} q^n\left(y^*_{t+1+n} + \tau^*_{t+1+n}\left(y^{t+1+n}\right)\right)\right] = \frac{1}{a}\frac{1-q}{1-q^{T-t}}\left(\mathbf{E}_{t+1} - \mathbf{E}_t\right)\left[\sum_{n=0}^{T-t-1} q^n y^*_{t+1+n}\right].$$

Again, for $T - t - 1 \geq p$, the expression stabilizes to the claimed one:

$$c^*_{t+1} - c^*_t = \frac{1}{a}\beta\left(q\right) v_{t+1}.$$

Q.E.D.

7.3 Isoelastic utility: A closed-form in logs

The outcome of this section will be an expression for innovation in *log* consumption of the form analogous to those we obtained in Proposition 3 and 4 above for the CARA and quadratic agent's utilities

$$\ln c^*_{t+1} - \ln c^*_t = \frac{\ln\frac{\delta}{q}}{\gamma} + \frac{\gamma}{2a^2}\left[\beta\left(\lambda q\right)\right]^2\sigma_v^2 + \frac{1}{a}\beta\left(\lambda q\right) v_{t+1},$$

where v_{t+1} is the innovation to *log* of income and $\frac{1}{\gamma}$ is the intertemporal elasticity of substitution of consumption at two consecutive dates, and $\lambda > 0$ is such that $\lambda q \leq \delta$ for $\gamma \geq 1$. We will also obtain expressions for tax rates at different dates.

7.3.1 Model and derivation of the Permanent Income equation

Assume a production function of the form:

$$\ln y_t = \ln\theta_t + \ln e_t,$$

and the following process for skills:

$$\ln\theta_t = \ln\theta_{t-1} + \beta\left(L\right) v_t.$$

As for the CARA case, an additional assumption, which will be crucial for us to get an exact closed form, is that the shocks v_t are *normally distributed* with zero mean and variance σ_v^2 (note that we slightly abuse in notation here).

Recall our specification for preferences:

$$\frac{\left(c_t \cdot e_t^{-\phi(e_t)}\right)^{1-\gamma}}{1-\gamma} = \frac{1}{1-\gamma} \exp\left\{(1-\gamma)\left(\ln c_t - \phi\left(e_t\right) \ln e_t\right)\right\},$$

where $\phi(e) = \frac{1}{a}$ for $e \leq 1$ and $\phi(e) = \frac{1}{b}$ for $e \geq 1$. Our aim is to write the problem in logs in order to exploit the analogies to the case in levels. Clearly, the objective function of the agent is concave in log decisions whenever $\gamma > 1$, and assumption consistent with empirical findings.[32] Since in equilibrium we have $e_t^* \equiv 1$, the Euler equation is the usual one

$$\mathbf{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right] = \mathbf{E}_t\left[\exp\left(-\gamma\frac{\ln c_{t+1}}{\ln c_t}\right)\right] = \exp\left(-\gamma\mu_t + \gamma^2\frac{1}{2}\sigma_t^2\right) = \frac{q}{\delta}, \tag{43}$$

where we used the fact that in equilibrium c_{t+1} will be log-normally distributed,[33] with μ_t and σ_t^2 being the conditional mean and conditional variance of $\Delta \ln c_{t+1}$ respectively.

Since we implement $b_t^* \equiv 0$, the budget constraint in equilibrium implies that $\ln c_t^*\left(y^t\right) = \ln y_t^* + \ln \tau_t^*\left(y^t\right).$ In what follows, for notational simplicity, we will abuse in notation and use y^t to denote the history of *log incomes.* Since the logarithmic function is strictly monotone (and $y_t \geq 0$) every function of y_t can be written as a function of $\ln y_t$, and vice versa. The objective function for effort plans hence becomes

$$\mathbf{E}_0 \sum_{t=0} \delta^t \frac{1}{1-\gamma} \exp\left\{(1-\gamma)\left(\ln y_t + \ln \tau_t\left(y^t\right) - v\left(\ln e_t\right)\right)\right\}.$$

Given our specification for v and u, the objective function can be all expressed in logs. It is now easy to see the strong analogy to the case in levels considered above. In particular, we will follow the main line of proof we adopted for the quadratic utility with the additional feature of log normality to obtain precise expression for (deterministic) consumption growth rates as in the CARA case. If *we assume that the transfer scheme τ is differentiable*, the first order condition for log of effort $\ln e_t$ is

$$\mathbf{E}_t \sum_{n=0}^{T-t} \delta^n \left(\frac{c_{t+n}}{c_t}\right)^{1-\gamma} \frac{\partial \ln \tau_{t+n}\left(y^{t+n}\right)}{\partial \ln y_t} = \frac{1}{a} - 1 \tag{44}$$

Once again if *the transfer scheme admits symmetric cross derivative* we can show backwards that the conditional expectations can be decomposed since $\frac{\partial \ln \tau_{t+n}\left(y^{t+n}\right)}{\partial \ln y_t}$ does not depend on (log) y_{t+n}.

The strong similarity with the model in levels has one last caveat. Since c_t is log normally distributed, we have

$$\mathbf{E}_t\left[\left(\frac{c_{t+n}}{c_t}\right)^{1-\gamma}\right] = \mathbf{E}_t\left[\exp\left((1-\gamma)\frac{\ln c_{t+1}}{\ln c_t}\right)\right] = \exp\left\{(1-\gamma)\mu_t + \frac{1}{2}(1-\gamma)^2\sigma_t^2\right\}.$$

[32]For the UK, see Attanasio and Weber (1993).

[33]For a more extensive argument on this, see the very last section in the Appendix of Attanasio and Pavoni (2006).

Moreover, from the Euler equation (43) we obtain:

$$\begin{aligned}
\exp\left\{(1-\gamma)\mu_t + \frac{1}{2}(1-\gamma)^2\sigma_t^2\right\} &= \exp\left\{-\gamma\mu_t + \gamma^2\frac{1}{2}\sigma_t^2\right\}\exp\left\{\mu_t + \frac{1}{2}(1-2\gamma)\sigma_t^2\right\} \\
&= \frac{q}{\delta}\exp\left\{\mu_t + \frac{1}{2}\sigma_t^2 - \gamma\sigma_t^2\right\} \\
&: = \frac{q}{\delta}\lambda_t. \qquad (45)
\end{aligned}$$

where $\lambda_t := \exp\left(\mu_t + \left(\frac{1}{2}-\gamma\right)\sigma_t^2\right) > 0$. In the log utility case, when $\delta = q$ then $\lambda_t = 1$.[34] Similarly, by the law of iterated expectations, assuming constant μ_t and σ_2^2, we get

$$\mathbf{E}_t\left[\left(\frac{c_{t+n}}{c_t}\right)^{-\gamma}\right] = \mathbf{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\mathbf{E}_{t+1}\left(\frac{c_{t+2}}{c_{t+1}}\right)^{-\gamma}\cdot..\cdot\mathbf{E}_{t+n-1}\left(\frac{c_{t+n}}{c_{t+n-1}}\right)^{-\gamma}\right] = \left(\frac{q}{\delta}\right)^n,$$

and

$$\begin{aligned}
\mathbf{E}_t\left[\exp\left((1-\gamma)\frac{\ln c_{t+n}}{\ln c_t}\right)\right] &= \mathbf{E}_t\left[\exp\left\{-\gamma\frac{\ln c_{t+1}}{\ln c_t}\right\}\lambda\mathbf{E}_{t+1}\exp\left\{-\gamma\frac{\ln c_{t+2}}{\ln c_{t+1}}\right\}\lambda\cdot..\cdot\mathbf{E}_{t+n-1}\exp\left\{-\gamma\frac{\ln c_{t+n}}{\ln c_{t+n-1}}\right\}\lambda\right] \\
&= \left(\frac{q\lambda}{\delta}\right)^n.
\end{aligned}$$

Again, using using the law of iterated expectations in the same way we did to derive equation (41), the incentive constraint (44) can be written as

$$\mathbf{E}_t\sum_{n=0}^{T-t}(q\lambda)^n\frac{\partial\ln\tau_{t+n}(y^{t+n})}{\partial\ln y_t} = \frac{1}{a} - 1. \qquad (46)$$

Since in the log space, taxes are linear and the agent's objective function is concave for $\gamma \geq 1$, the so derived scheme is the optimal one within the class of differentiable schemes with symmetric cross-derivative as it solves the relaxed problem (only subject to the first order conditions), while being globally incentive compatible.

We can now follow the same steps as for the model in levels to obtain the desired permanent income expressions: from $\ln c_t = \ln y_t + \ln\tau_t$ at all nodes, we get

$$\begin{aligned}
\mathbf{E}_t\sum_{n=0}^{T-t-1}(q\lambda)^n\ln c^*_{t+1+n} &= \mathbf{E}_t\sum_{n=0}^{T-t-1}(q\lambda)^n\left(\ln y^*_{t+1+n} + \ln\tau^*_{t+1+n}\left(y^{t+1+n}\right)\right) \\
\mathbf{E}_{t+1}\sum_{n=0}^{T-t-1}(q\lambda)^n\ln c^*_{t+1+n} &= \mathbf{E}_{t+1}\sum_{n=0}^{T-t-1}(q\lambda)^n\left(\ln y^*_{t+1+n} + \ln\tau^*_{t+1+n}\left(y^{t+1+n}\right)\right).
\end{aligned}$$

[34] Since $\mu_t = \frac{\ln\delta/q}{\gamma} + \frac{\gamma}{2}\sigma_t^2$, $\lambda_t = \exp\left(\frac{\ln\delta/q}{\gamma} + \frac{1-\gamma}{2}\sigma_t^2\right)$ which implies $\lambda_t \leq \frac{\delta}{q}$ since $\gamma \geq 1$. Moreover, when u is logarithmic, we have $\lambda = \frac{\delta}{q}$, which is obviously consistent with

$$\lim_{\gamma\to1^+}\mathbf{E}_t\left[\exp\left((1-\gamma)\frac{\ln c_{t+n}}{\ln c_t}\right)\right] = 1.$$

By repeatedly using Euler equation (43), together with the properties of the normal distribution, we obtain

$$\ln c_{t+1}^* - \ln c_t^* = \frac{\ln(\delta/q)}{\gamma} + \frac{\gamma}{2}\sigma^2 + \frac{1-q\delta}{1-(q\delta)^{T-t}}\left(\mathbf{E}_{t+1} - \mathbf{E}_t\right)\left[\sum_{n=0}^{T-t-1} (q\lambda)^n \left(\ln y_{t+1+n}^* + \ln \tau_{t+1+n}^*\left(y^{t+1+n}\right)\right)\right],$$

Finally, from the expression for marginal log taxes, we obtain

$$\ln c_{t+1}^* - \ln c_t^* = \frac{\ln(\delta/q)}{\gamma} + \frac{\gamma}{2}\sigma^2 + \frac{1}{a}\frac{1-q\delta}{1-(q\delta)^{T-t}}\left(\mathbf{E}_{t+1} - \mathbf{E}_t\right)\left[\sum_{n=0}^{T-t-1} (q\lambda)^n \left(\ln y_{t+1+n}^*\right)\right].$$

It is hence again easy to see that for $T \geq t+1+p$ - using the properties of the ARIMA(p) process we postulated above - the previous expression stabilizes into

$$\Delta \ln c_{t+1}^* = \frac{\ln(\delta/q)}{\gamma} + \frac{\gamma}{2}\sigma^2 + \frac{1}{a}\frac{1-q\delta}{1-(q\delta)^{T-t}}\sum_{n=0}^{T-t-1} (q\lambda)^n \beta(q\lambda) v_{t+1} = \frac{\ln(\delta/q)}{\gamma} + \frac{\gamma}{2}\sigma^2 + \frac{1}{a}\beta(q\lambda) v_{t+1}. \quad (47)$$

Since the polynomial is invertible, and $0 < q\lambda \leq \delta$, all expressions are well defined. Moreover, from (47) we obtain that that $\sigma^2 := var_t\left(\Delta \ln c_{t+1}^*\right) = \frac{\left[\Sigma_i\left((q\lambda)^i \beta_i\right)\right]^2}{a^2}\sigma_v^2 = \frac{[\beta(\lambda q)]^2}{a^2}\sigma_v^2$ as claimed above.

7.3.2 Expressions for taxes

The analysis is tedious but straightforward. Taxes will be defined by the two incentive constraints: The effort incentive constraints, and the Euler equations. Moreover, since taxes take very complicated expressions for the periods close to T, we will derive the expressions only for stable values, hence for sufficiently large T.

The whole analysis will be considerably simplified if we describe the transfer scheme in terms of the histories of the shocks v_t. In order to simplify the notation we keep all symbols as above (although this is an abuse in notation of course). Let $v^t = (v_1, ..., v_t)$ be a given history of shocks. By repeatedly applying the law for $\ln \theta_t$, we have

$$\ln \theta_t = \ln \theta_0 + \sum_{s=0}^{t-1} \beta(L) v_{t-s}. \quad (48)$$

We will normalized $\theta_0 = 1$, and that lagged terms in the MA expressions v_{-s}, $s = 0, ..p$, will be set to zero as well by the other initial conditions. It is easy to see that in the last period we have

$$\frac{\partial \ln \tau_T\left(v^T\right)}{\partial v_T} = \frac{1}{a} - 1. \quad (49)$$

this is so since, given v^{T-1}, the agent can lie over v_T exactly in the same way as he would lie over θ_t, with exactly the same marginal net costs/returns, as $\beta_0 = 1$. As before, it is easy to show that taxes are linear in v^t. Note however that a lie over v_t today affects future income not only through the transfer scheme, but also via the persistence pattern of the process for θ_t. In particular, consider the an agent lying over v_t and then telling the truth over future v_{t+s}. He/she will have to lie (implicitly) over all future θ_{t+s}, precisely by the amount of the future effect of v_t over θ_{t+s}. Of course, he/she will them forced to make income levels to appear consistently with the lie, namely $\hat{y}_{t+s} = \hat{\theta}_{t+s}$. For $t \leq T-p$ we hence have (note that we can eliminate the conditional expectation because of the linearity of taxes):

$$\sum_{s=0}^{T-t}(q\lambda)^s\frac{\partial\ln\tau_{t+s}\left(v^{t+s}\right)}{\partial v_t}=\left(\frac{1}{a}-1\right)\left[1+(q\lambda)\left(1+\beta_1\right)+(q\lambda)^2\left(1+\beta_1+\beta_2\right)+...+(q\lambda)^p\frac{\beta\left(1\right)}{1-q\lambda}\right]. \quad (50)$$

Consider now, the Euler equation between periods t and $t+1$, for $t\leq T-p$. In order for $b_t^*\equiv 0$ to be incentive compatible at each node, we have

$$\exp\left\{-\rho(\ln\theta_t+\ln\tau_t\left(v^t\right))\right\}=\frac{\delta}{q}\mathbf{E}_t\left[\exp\left\{-\rho(\ln\theta_{t+1}+\ln\tau_{t+1}\left(v^{t+1}\right))\right\}\right]. \quad (51)$$

As we saw, the incentive compatibility constraint together with the symmetric partial derivative assumption implies that there is a function $\hat{\eta}_{t+1}$ such that $\ln\tau_{t+1}\left(v^{t+1}\right)=\hat{\eta}_{t+1}\left(v^t\right)+\tau_{t+1}^{(t+1)}v_{t+1}$ (note that the functions $\hat{\eta}$ are not the same as the function η in Proposition 3 but very similar in nature, namely for all $s\geq 0$, $\frac{\partial\hat{\eta}_{t+1}\left(v^t\right)}{\partial v_{t-s}}=\frac{\partial\ln\tau_{t+1}\left(v^{t+1}\right)}{\partial v_{t-s}}$). Since θ_{t+1} is normally distributed, taking the log operator in both sides and use the properties of the normal distribution, since $\mathbf{E}_tv_{t+1}=0$, (51) becomes

$$\ln\theta_t+\ln\tau_t\left(v^t\right)=\Gamma_t^{t+1}+\mathbf{E}_t\ln\theta_{t+1}+\hat{\eta}_{t+1}\left(v^t\right)=\Gamma_t^{t+1}+\ln\theta_t+\sum_{i=1}^{p}\beta_iv_{t+1-i}+\hat{\eta}_{t+1}\left(v^t\right). \quad (52)$$

where we used the projection result: $\mathbf{E}_t\theta_{t+1}=\theta_t+\sum_{i=1}^{p}\beta_iv_{t+1-i}$. We also used the linearity of the tax on v_{t+1} together with $\mathbf{E}_tv_{t+1}=0$. More in general, for all $t,s\geq 1$, we have

$$\ln\theta_t+\ln\tau_t\left(v^t\right)=\Gamma_t^{t+s}+\ln\theta_t+\sum_{n=1}^{\min\{s,p\}}\sum_{i=n}^{p}\beta_iv_{t+n-i}+\hat{\eta}_{t+s}\left(v^t\right). \quad (53)$$

Now, in order for (52) to hold true for all v_t given v^{t-1}, it must be that

$$\frac{\partial\ln\tau_t\left(v^t\right)}{\partial v_t}=\frac{\partial\hat{\eta}_{t+1}\left(v^t\right)}{\partial v_t}+\beta_1=\frac{\partial\ln\tau_{t+1}\left(v^{t+1}\right)}{\partial v_t}+\beta_1.$$

In general, the Euler equation between period t and $t+s$, $s\geq 1$ implies

$$\frac{\partial\ln\tau_t\left(v^t\right)}{\partial v_t}=\frac{\partial\ln\tau_{t+s}\left(v^{t+s}\right)}{\partial v_t}+\sum_{i=1}^{\min\{s,p\}}\beta_i. \quad (54)$$

Hence, for $s\geq p$, marginal taxes become constant. Now, in order for both the Euler equations and the incentive constraint (50) to hold simultaneously, by repeatedly using (54) we have:

$$\begin{aligned}\sum_{s=0}^{T-t}(q\lambda)^s\frac{\partial\ln\tau_{t+s}\left(v^{t+s}\right)}{\partial v_t} &= \frac{\partial\ln\tau_t\left(v^t\right)}{\partial v_t}\left[1+q\lambda\left(1-\beta_1\right)+(q\lambda)^2\left(1-\beta_1-\beta_2\right)+...+(q\lambda)^p\frac{2-\beta\left(1\right)}{1-q\lambda}\right]\\ &= \left(\frac{1}{a}-1\right)\left[1+q\lambda\left(1+\beta_1\right)+(q\lambda)^2\left(1+\beta_1+\beta_2\right)+...+(q\lambda)^p\frac{\beta\left(1\right)}{1-q\lambda}\right].\end{aligned}$$

It is hence easy to see that

$$\frac{\partial\ln\tau_t\left(v^t\right)}{\partial v_t}=\left(\frac{1}{a}-1\right)\frac{\left[1+q\lambda\left(1+\beta_1\right)+(q\lambda)^2\left(1+\beta_1+\beta_2\right)+...+(q\lambda)^p\frac{\beta(1)}{1-q\lambda}\right]}{\left[1+q\lambda\left(1-\beta_1\right)+(q\lambda)^2\left(1-\beta_1-\beta_2\right)+...+(q\lambda)^p\frac{2-\beta(1)}{1-q\lambda}\right]}:=\left(\frac{1}{a}-1\right)\kappa,$$

and of course all other taxes can be obtained from this expression from (54). Note that $\kappa > 0$ and, for future reference, that when $\beta_i = 0$ for all $i > 0$, $\kappa = 1$ so $\frac{\partial \ln \tau_t(v^t)}{\partial v_t} = \frac{\partial \ln \tau_{t+s}(v^{t+s})}{\partial v_t} = \frac{1}{a} - 1$.

Finally, we derive the expression relating the change in the cross sectional variance of consumption with the change in cross sectional variance of income. Again, in order to have stable formulas we assume $t \geq p$ and $t \leq T - p$, so that all above expressions apply fully. We have

$$\begin{aligned} \ln c_t^*\left(\theta^t\right) &= \ln y_t^* + \ln \tau_t^*\left(v^t\right) + \ln \theta_t + \ln \tau_t^*\left(v^t\right) \\ &= \theta_t + \tau_t^{(0)} v_t + \tau_t^{(-1)} v_{t-1} + .. + \tau_t^{(-p)} v_{t-p} + \tau_t^{(-p)} v_{t-p-1} + ... + \tau_t^{(-p)} v_1 + \tau_0 + t\Gamma. \end{aligned} \tag{55}$$

where, the constant of integration τ_0 is chosen to satisfy the planner's budget constraint, and - as we have shown above - for all t: $\tau_t^{(-s)} = \left(\frac{1}{a} - 1\right)\kappa - \beta_1 - \beta_2 - ...\beta_s$. Similarly, for $t+1$, we have

$$\ln c_{t+1}^*\left(\theta^{t+1}\right) = \theta_{t+1} + \tau_{t+1}^{(0)} v_{t+1} + \tau_{t+1}^{(-1)} v_t + .. + \tau_{t+1}^{(-p)} v_{t+1-p} + \tau_t^{(-p)} v_{t-p} + ... + \tau_t^{(-p)} v_1 + \tau_0 + (t+1)\Gamma,$$

where for all n we have $\tau_{t+1}^{(-n)} = \tau_t^{(-n)}$. Recall that we are interested in computing the unconditional variance of both the above term and that in (55), and then take the difference. This stated difference in variances can be stated as follows:

$$\begin{aligned} &\Delta var\left(\ln c_{t+1}^*\right) := var\left(\ln c_{t+1}^*\right) - var\left(\ln c_t^*\right) \\ = \; &var\left(\theta_{t+1}\right) - var\left(\theta_t\right) + var\left(\tau_{t+1}^*\left(v^{t+1}\right)\right) - var\left(\tau_t^*\left(v^t\right)\right) + 2\left[cov\left(\theta_{t+1}, \tau_{t+1}^*\left(v^{t+1}\right)\right) - cov\left(\theta_t, \tau_t^*\left(v^t\right)\right)\right]. \end{aligned}$$

Now, note the following:

$$\begin{aligned} var\left(\ln \tau_t^*\left(v^t\right)\right) &= \left(\left[\tau_t^{(0)}\right]^2 + .. + (1 + t - p)\left[\tau_t^{(-p)}\right]^2\right)\sigma_v^2 \\ &\text{while} \\ var\left(\ln \tau_{t+1}^*\left(v^{t+1}\right)\right) &= \left(\left[\tau_{t+1}^{(0)}\right]^2 + .. + (2 + t - p)\left[\tau_{t+1}^{(-p)}\right]^2\right)\sigma_v^2. \end{aligned}$$

Moreover, for $t \geq p$, $\tau_{t+1}^{(-n)} = \tau_t^{(-n)}$, we have

$$var\left(\ln \tau_{t+1}^*\left(v^{t+1}\right)\right) - var\left(\tau_t^*\left(v^t\right)\right) = \left[\tau_t^{(-p)}\right]^2 \sigma_v^2.$$

and

$$\begin{aligned} cov\left(\theta_t, \tau_t^{(-s)} v_{t-s}\right) &= cov\left(\theta_{t+1}, \tau_{t+1}^{(-s)} v_{t+1-s}\right), & \text{for } s \leq p, \text{ and} \\ cov\left(\theta_t, \tau_t^{(-p)} v_{t-s}\right) &= cov\left(\theta_{t+1}, \tau_{t+1}^{(-p)} v_{t+1-s}\right) = \tau_{t+1}^{(-p)}\beta(1)\sigma_v^2, & \text{for } s \geq p. \end{aligned}$$

Given that for $t \geq p$ only the correlation with v_1 remains in the $t+1$ terms, we have

$$cov\left(\theta_{t+1}, \tau_{t+1}^*\left(v^{t+1}\right)\right) - cov\left(\theta_t, \tau_t^*\left(v^t\right)\right) = \tau_{t+1}^{(-p)}\beta(1)\sigma_v^2.$$

In summary:

$$\Delta var\left(\ln c_{t+1}^*\right) = \Delta var\left(\theta_{t+1}\right) + \left[\tau_t^{(-p)}\right]^2 \sigma_v^2 + 2\tau_{t+1}^{(-p)}\beta(1)\sigma_v^2.$$

Finally, since from the definition of θ_t in (48), for $t \geq p$ we have

$$\Delta var\left(\ln y_{t+1}^{*}\right) = \Delta var\left(\ln \theta_{t+1}\right) = \left[\beta\left(1\right)\right]^{2} \sigma_{v}^{2} > 0,$$

which implies

$$\Delta var\left(\ln c_{t+1}^{*}\right) = \left[\beta\left(1\right) + \tau_{t+1}^{(-p)}\right]^{2} \sigma_{v}^{2} = \frac{\left[\beta\left(1\right) + \tau_{t}^{(-p)}\right]^{2}}{\left[\beta\left(1\right)\right]^{2}} \Delta var\left(\ln y_{t+1}^{*}\right),$$

where, recall that

$$\tau_{t+1}^{(-p)} = \left(\frac{1}{a} - 1\right) \frac{\left[1 + q\lambda\left(1+\beta_1\right) + \left(q\lambda\right)^{2}\left(1+\beta_1+\beta_2\right) + ...\right]}{\left[1 + q\lambda\left(1-\beta_1\right) + \left(q\lambda\right)^{2}\left(1-\beta_1-\beta_2\right) + ...\right]} - \beta\left(1\right) + 1$$

hence $\beta\left(1\right) + \tau_{t}^{(-p)} = 1 + \left(\frac{1}{a} - 1\right)\kappa$ and

$$\Delta var\left(\ln c_{t+1}^{*}\right) = \left(\frac{1 + \left(\frac{1}{a} - 1\right)\kappa}{\beta\left(1\right)}\right)^{2} \Delta var\left(\ln y_{t+1}^{*}\right). \tag{56}$$

Since both $\beta\left(1\right) > 0$ and $\kappa > 0$ the parameter a is identified. Moreover, for $\beta_i = 0$ for all i, we have

$$\Delta var\left(\ln c_{t+1}^{*}\right) = \frac{1}{a^{2}} \Delta var\left(\ln y_{t+1}^{*}\right). \tag{57}$$

7.4 An extended model with two types of shocks

We now briefly present an extension of our model that allows for two types of (independent) shocks to income, with different degrees of persistence. Although we develop the model in levels, very similar expressions can be derived for the log-linear case.

Assume agents have preferences over c_t, l_t and e_t as follows: $-\frac{1}{\rho}\exp\left\{-\rho\left(c_t - e_t - l_t\right)\right\}$. Moreover, assume that individual income can be decomposed into two components: $y_t = x_t + \xi_t$, where $x_t = f\left(\theta_t^{p}, e_t\right)$, and $\xi_t = g\left(v_t^{T}, l_t\right)$. In this model, x_t represents the permanent component of income as $\theta_t^{p} = \theta_{t-1}^{p} + v_t^{p}$, with v_t^{p} iid; while ξ_t represents the temporary one, as v_t^{T} is iid. The production function f is as in (13), and a similar functional form for g is assumed:

$$\xi_t = g\left(v_t^{T}, l_t\right) = v_t^{T} + a^{T}\min\left\{l_t, 0\right\} + b^{T}\max\left\{l_t, 0\right\} \text{ with } a^{T} > 1 > b^{T}.$$

Since effort will be again time constant, in equilibrium, the income process will display the following process:[35]

$$y_t = y_{t-1} + v_t^{p} + \Delta v_t^{T}. \tag{58}$$

[35] We could easily allow the temporary shock v_t^{T} to follow a $MA\left(p\right)$ process.

We now follow a line of proof very similar to that used for the baseline model, and show that the reaction of consumption to the different shocks for $T \to \infty$ can be written as[36]

$$\Delta c_t^* = \frac{\ln(\delta/q)}{\rho} + \frac{\rho}{2}\left[\left(\frac{1}{a^p}\right)^2 \sigma_{v^p}^2 + \left(\frac{1-q}{a^T}\right)^2 \sigma_{v^T}^2\right] + \frac{1}{a^p} v_t^p + \frac{1-q}{a^T} v_t^T, \tag{59}$$

where, for consistency, we denoted by a^p the slope of f for $e_t \leq 0$.

The closed form for the version of our model with two types of shocks provides a structural interpretation of recent empirical evidence. Using the evolution of the cross sectional variance and covariance of consumption and income, Blundell et al. (2008) estimate two parameters, ϕ and ψ, representing the fraction of permanent and temporary shocks reflected into consumption. Within our model estimates of these parameters can be interpreted as the severity of informational problems for income shocks of different persistence.

7.4.1 Proof of the closed form expression (59).

The analysis is performed separately for the two type of shocks. Obviously, $e_t^* = l_t^* = 0$ at all nodes. We can hence equivalently describe the transfer scheme in terms of incomes. In presence of both permanent and temporary shocks, the firm should obviously condition its transfers on $\xi_t = g\left(v_t^T, l_t\right)$ realizations as well. Denote by $h^t = (x^t, \xi^t)$ the combined public history. In the CARA case, by following the same line of proof of Proposition 3 we can show the differentiability of the scheme and the first order conditions of the agent solving

$$\begin{aligned}
\mathbf{E}_t \sum_{n=0}^{T-t} \delta^n \left[\frac{\partial \tau_{t+n}\left(h^{t+n}\right)}{\partial \xi_t} \frac{u'\left(c_{t+n} - e_{t+n} - l_{t+n}\right)}{u'\left(c_t - e_t - l_t\right)}\right] &= \frac{1}{a^T} - 1 \\
&\text{and} \\
\mathbf{E}_t \sum_{n=0}^{T-t} \delta^n \left[\frac{\partial \tau_{t+n}\left(h^{t+n}\right)}{\partial x_t} \frac{u'\left(c_{t+n} - e_{t+n} - l_{t+n}\right)}{u'\left(c_t - e_t - l_t\right)}\right] &= \frac{1}{a^p} - 1,
\end{aligned}$$

where, for consistency, we denoted by a^p the slope of f for $e_t \leq 0$. By the same proposition, the slopes $\frac{\partial \tau_{t+n}\left(h^{t+n}\right)}{\partial \xi_t}$ do not depend on histories before or after period t, we can uses the Euler equation and apply the law of iterated expectations and get, for a generic δ and a deterministic sequence of bond prices (in the notation below ζ stays for x or ξ)

$$\mathbf{E}_t \left[\sum_{n=0}^{T-t} \delta^n \frac{\partial \tau_{t+n}\left(h^{t+n}\right)}{\partial \zeta_t} \frac{u'\left(c_{t+n} - e_{t+n} - l_{t+n}\right)}{u'\left(c_t - e_t - l_t\right)}\right] = \mathbf{E}_t \left[\sum_{n=0}^{T-t} \left(\Pi_{s=0}^n q_{t+s-1}\right) \frac{\partial \tau_{t+n}\left(h^{t+n}\right)}{\partial \zeta_t}\right].$$

Of course, in the quadratic utility case, exactly the same expression for marginal taxes can be obtained assuming the transfer scheme admits symmetric cross-derivatives in all elements of h^t. If we write the

[36] The corresponding expression for the model in logs is

$$\Delta c_{t+1}^* = \frac{\ln(\delta/q)}{\gamma} + \frac{\gamma}{2}\left[\left(\frac{1}{a^p}\right)^2 \sigma_{v^p}^2 + \left(\frac{1-\lambda q}{a^T}\right)^2 \sigma_{v^T}^2\right] + \frac{1}{a^p} v_{t+1}^p + \frac{1-\lambda q}{a^T} v_{t+1}^T.$$

expressions for a constant q, we get:

$$\mathbf{E}_t \sum_{n=0}^{T-t} q^n \frac{\partial \tau_{t+n}\left(h^{t+n}\right)}{\partial \xi_t} = \frac{1}{a^T} - 1 \quad \text{and} \quad \mathbf{E}_t \sum_{n=0}^{T-t} q^n \frac{\partial \tau_{t+n}\left(h^{t+n}\right)}{\partial x_t} = \frac{1}{a^p} - 1. \tag{60}$$

Assuming CARA (or quadratic) preferences, for permanent shocks (i.e., x_t follows an ARIMA(0)), the Euler equation implies that only contemporaneous marginal taxes are positive, and

$$\frac{\partial \ln \tau_t\left(y^t\right)}{\partial \ln x_t} = \frac{1}{a} - 1.$$

In this case, absent temporary shocks, we would have

$$\Delta \ln c_{t+1} = \frac{\ln \frac{\delta}{q}}{\gamma} + \frac{\gamma}{2}\sigma_c^2 + \frac{1}{a}v_{t+1},$$

Since, from the above expression, the variance of log consumption is $\sigma_c^2 = \frac{1}{a^2}\sigma_v^2$, we have

$$\ln \tau_t\left(y^t\right) = \left(\frac{1}{a} - 1\right)\ln y_t + t\left[\frac{\ln \frac{\delta}{q}}{\gamma} + \frac{\gamma}{2a^2}\sigma_v^2\right] + \ln \tau_0. \tag{61}$$

If we add temporary shocks. Since the analysis can be done independently, by comparing Euler equations ad different dates, one can easily show that the tax rates for the purely temporary shock are related as follows:

$$1 + \frac{\partial \tau_t\left(h^t\right)}{\partial \xi_t} = \frac{\partial \tau_{t+s}\left(h^{t+s}\right)}{\partial \xi_t} \geq 0, \text{ for all } \text{t}, s > 0. \tag{62}$$

It is hence easy to see by direct inspection of (62) and (60) that, as $T \to \infty$, the expressions for transfers become:

$$1 + \tau_x = \frac{1}{a^p} \quad \text{and} \quad 1 + \tau_\xi = \frac{1-q}{a^T},$$

where $1+\tau_x = 1+\frac{\partial \tau_t\left(h^t\right)}{\partial x_t}$ and $1+\tau_\xi = 1+\frac{\partial \tau_t\left(h^t\right)}{\partial \xi_t} = \frac{\partial \tau_{t+k}\left(h^{t+k}\right)}{\partial \xi_t}$ for $k > 0$. Hence tax rates are time-invariant, and the agent's consumption reaction to income shocks is given by:[37]

$$\Delta c_{t+1} = \Gamma + \frac{1}{a^p}\Delta x_{t+1} + \frac{1-q}{a^T}\Delta \xi_{t+1} = \Gamma + \frac{1}{a^p}v_{t+1}^p + \frac{1-q}{a^T}v_{t+1}^T,$$

where $\Gamma \geq 0$ and $\Gamma = 0$ when u is quadratic and $\delta = q$.

As explained in the proof of Proposition 4, all the above expressions constitute optimal transfer schemes since the agent's problem is concave as all taxes are linear in all arguments.

[37] As it should be clear from the analysis for the isoelastic model, the corresponding equation for the model in logs is

$$\Delta \ln c_{t+1} = \Gamma + \frac{1}{a^p}\Delta \ln x_{t+1} + \frac{1-\lambda q}{a^T}\Delta \ln \xi_{t+1} = \Gamma + \frac{1}{a^p}v_{t+1}^p + \frac{1-\lambda q}{a^T}v_{t+1}^T,$$

where $\lambda = \exp\left\{\frac{\ln \delta/q}{\gamma} + \frac{1-\gamma}{2}\sigma_c^2\right\}$.

Finally, given the expressions for marginal taxes, we have

$$
\begin{aligned}
c_t^*\left(h^t\right) &= y_t^* + \tau_t^*\left(h^t\right) = x_t^* + \xi_t^* + \left(\frac{1}{a^p} - 1\right) x_t^* + \left(\frac{1-q}{a^T} - 1\right) \xi_t^* + \sum_{s=1}^{t-1} \frac{1-q}{a^T} \xi_{t-s}^* \\
&= \frac{1}{a^p} x_t^* + \frac{1-q}{a^T} \sum_{s=0}^{t-1} \xi_{t-s}^* + t\Gamma + \tau_0. \qquad (63)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\Delta var\left(c_t^*\left(h^t\right)\right) &= \left[var\left(\frac{1}{a^p} x_t^*\right) - var\left(\frac{1}{a^p} x_{t-1}^*\right)\right] + var\left(\frac{1-q}{a^T} \sum_{s=0}^{t-1} \xi_{t-s}^*\right) - var\left(\frac{1-q}{a^T} \sum_{s=0}^{t-2} \xi_{t-s}^*\right) \\
&= \left(\frac{1}{a^p}\right)^2 \sigma_{v^p}^2 + \left(\frac{1-q}{a^T}\right)^2 \sigma_{v^T}^2 \qquad (64) \\
&= \left(\frac{1}{a^p}\right)^2 \Delta var\left(y_t^*\right) + \psi,
\end{aligned}
$$

where $\psi := \left(\frac{1-q}{a^T}\right)^2 \sigma_{v^T}^2$ is a constant in the regression, and the last lines uses the fact that $var\left(y_t^*\right) = var\left(x_t^*\right) + var\left(\xi_t^*\right) + 2cov\left(x_t^*, \xi_t^*\right) = var\left(x_t^*\right) + \sigma_{v^T}^2$ hence $\Delta var\left(y_t^*\right) = \Delta var\left(x_t^*\right)$.

8 Appendix C: Bias correction for the variance based test

The observable version of equation (28) is:

$$\Delta\overline{Var\,(c_{gt})} = \frac{1}{a^2}\Delta\overline{Var\,(y_{gt})} + \frac{1}{a^2}\Delta\varepsilon^y_{gt} - \Delta\varepsilon^c_{gt}, \tag{65}$$

where $\varepsilon^y_{gt} = Var(y_{gt}) - \overline{Var\,(y_{gt})}$, and $\varepsilon^c_{gt} = Var(c_{gt}) - \overline{Var\,(c_{gt})}$. The variance of the residuals ε will go to zero as the size of the cells in each time period increases. Moreover, information on the within cell variability can be used to correct OLS estimates of the coefficients in equation (65). In particular, a bias correct estimator will be given by the following expression:

$$\widehat{\theta} = A^{-1}[\widetilde{\theta} - B] \tag{66}$$

where $\widetilde{\theta} = (Z'Z)^{-1}Z'w$ is the OLS estimator, $B = (Z'Z)^{-1}\left\{\frac{1}{T-1}\sum_{t=2}^{T}\frac{\sigma_{cygt}}{N_{gt}} + \frac{\sigma_{cygt-1}}{N_{gt-1}}\right\}$allows for the possibility of correlation between the ε^y_t and ε^c_t and A is given by

$$A = \left[I - (Z'Z)^{-1}\frac{1}{T-1}\sum_{t=2}^{T}\left(\frac{\sigma^2_{ygt}}{N_{gt}} + \frac{\sigma^2_{ygt-1}}{N_{gt-1}}\right)\right],$$

In computing the variance covariance matrix of this estimator it will be necessary to take into account the MA structure of the residuals as well as the possibility that observations for different groups observed at the same time will be correlated.

Table 1
Non Durable Consumption

	gross earnings	gross earnings	gross earnings + benefits	gross earnings + benefits	net earnings + benefits	net earnings + benefits
income equation						
Own shock						
ayy(t)	1	1	1	1	1	1
	-	-	-	-	-	-
ayy(t-1)	0.505	-	0.333	-	0.376	-
	(20.735)	-	*(25.683)*	-	*(18.240)*	-
ayy(t-2)	-0.672	-	-0.748	-	-0.507	-
	(12.252)	-	*(18.183)*	-	*(10.247)*	-
Income/Consumption Shock						
ayc(t)	1.159	1.171	1.073	1.064	0.771	0.793
	(0.346)	*(0.317)*	*(0.329)*	*(0.317)*	*(0.253)*	*(0.243)*
ayc(t-1)	-1.140	-1.148	-0.887	-0.894	-0.602	-0.587
	(0.454)	*(0.415)*	*(0.492)*	*(0.459)*	*(0.342)*	*(0.323)*
ayc(t-2)	0.992	0.995	0.827	0.844	0.619	0.568
	(0.370)	*(0.331)*	*(0.356)*	*(0.311)*	*(0.257)*	*(0.227)*
consumption equation						
Income/Consumption Shock						
acc(t)	1	1	1	1	1	1
	-	-	-	-	-	-
acc(t-1)	-0.577	-0.493	-0.604	-0.491	-0.612	-0.499
	(0.196)	*(0.114)*	*(0.193)*	*(0.116)*	*(0.191)*	*(0.111)*
acc(t-2)	0.084	-	0.116	-	0.118	-
	(0.201)	-	*(0.191)*	-	*(0.193)*	-
Log L	-773.3	-773.9	-759.5	-760.6	-685.0	-685.5
excess smoothness	-0.491	-0.499	-0.489	-0.493	-0.273	-0.263
se	*(0.171)*	*(0.165)*	*(0.160)*	*(0.153)*	*(0.132)*	*(0.128)*
excess sensitivity						
Log L unrestricted model	-771.87	-773.4	-759.45	-758.9	-684.64	-684.4
LR	2.9	0.94	0.1	3.38	0.68	2.22
P-value	*0.235*	*0.625*	*0.951*	*0.185*	*0.712*	*0.330*
Comparison with 4 lags model						
Log L 4lags model	-771.5	-773.6	-756.3	-760.4	-681.7	-685.3
LR	3.7	0.54	6.34	0.42	6.52	0.34
P-value	*0.717*	*0.763*	*0.386*	*0.811*	*0.368*	*0.844*

NOTES:

- all data are in (first diff of) levels
- SE in parentheses
- excess smoothness test computed as sum(acc(t-L))-sum(ayy(t-L))=0, with L=0,…,4
- interest rate =0.01
- *Income/consumption shock* is the shock that enters both the income and the consumption equation
- excess sensitivity test computed as LR test (restrictions: acy(t-1)=acy(t-2)=0)

Table 2
Total Consumption Expenditure

	gross earnings	gross earnings	gross earnings + benefits	gross earnings + benefits	net earnings + benefits	net earnings + benefits
income equation						
Own shock						
ayy(t)	1	1	1	1	1	1
	-	-	-	-	-	-
ayy(t-1)	51.615	-	0.400	-	0.321	-
	(1535.290)	-	*(9.124)*	-	*(28.027)*	-
ayy(t-2)	-36.884	-	-0.968	-	-0.765	-
	(1072.450)	-	*(7.222)*	-	*(20.287)*	-
Income/Consumption Shock						
ayc(t)	0.596	0.586	0.730	0.497	0.683	0.662
	(0.427)	*(0.351)*	*(0.383)*	*(0.346)*	*(0.236)*	*(0.205)*
ayc(t-1)	-0.341	-0.673	-0.472	-0.533	-0.277	-0.273
	(0.638)	*(0.473)*	*(0.572)*	*(0.477)*	*(0.358)*	*(0.304)*
ayc(t-2)	0.572	0.981	0.561	0.910	0.330	0.363
	(0.451)	*(0.467)*	*(0.430)*	*(0.451)*	*(0.280)*	*(0.238)*
consumption equation						
Income/Consumption Shock						
acc(t)	1	1	1	1	1	1
	-	-	-	-	-	-
acc(t-1)	-0.349	-0.346	-0.386	-0.345	-0.372	-0.395
	(0.406)	*(0.208)*	*(0.366)*	*(0.202)*	*(0.367)*	*(0.172)*
acc(t-2)	-0.069	-	-0.022	-	-0.031	-
	(0.358)	-	*(0.330)*	-	*(0.319)*	-
Log L	-885.2	-886.0	-869.4	-870.5	-788.0	-788.9
excess smoothness	-0.233	-0.224	-0.217	-0.203	-0.131	-0.174
se	*(0.128)*	*(0.144)*	*(0.116)*	*(0.113)*	*(0.099)*	*(0.128)*
excess sensitivity						
Log L unrestricted model	-882.9	-885.3	-869.25	-868.7	-786.8	-788.8
LR	4.64	1.36	0.36	3.64	2.4	0.1
P-value	*0.098*	*0.507*	*0.835*	*0.162*	*0.301*	*0.951*
Comparison with 4 lags model						
Log L 4lags model	-882.3	-885.4	-867.8	-869.6	-786.2	-788.3
LR	5.78	1.24	3.26	1.82	3.54	1.1
	0.448	*0.538*	*0.776*	*0.403*	*0.739*	*0.577*

NOTES:

all data are in (first diff of) levels
SE in parentheses
excess smoothness test computed as sum(axx(t-L))-sum(ayx(t-L))=0, with L=0,…,4
interest rate =0.01
excess sensitivity test computed as LR test (restrictions: acy(t-1)=acy(t-2)=0)

Table 3

Total Consumption Expenditure: log specification

	gross earnings	**gross earnings**	**gross earnings + benefits**	**gross earnings + benefits**	**net earnings + benefits**	**net earnings + benefits**
income equation						
Own shock						
ayy(t)	1	1	1	1	1	1
	-	-	-	-	-	-
ayy(t-1)	0.332	-	0.181	-	0.159	-
	(14.452)	-	*(4.936)*	-	*(1.982)*	-
ayy(t-2)	-0.748	-	-0.779	-	-0.595	-
	(10.305)	-	*(3.957)*	-	*(1.426)*	-
Income/Consumption Shock						
ayc(t)	0.889	0.816	0.717	0.748	0.799	0.570
	(0.200)	*(0.155)*	*(0.162)*	*(0.146)*	*(0.191)*	*(0.187)*
ayc(t-1)	-1.102	-1.250	-0.685	-0.836	-0.523	-0.222
	(0.249)	*(0.206)*	*(0.216)*	*(0.177)*	*(0.253)*	*(0.211)*
ayc(t-2)	0.393	0.660	0.094	0.223	-0.120	0.086
	(0.229)	*(0.199)*	*(0.191)*	*(0.139)*	*(0.214)*	*(0.152)*
consumption equation						
Income/Consumption Shock						
acc(t)	1	1	1	1	1	1
	-	-	-	-	-	-
acc(t-1)	-0.601	-1.011	-0.647	-1.011	-0.578	-0.619
	(0.243)	*(0.008)*	*(0.279)*	*(0.008)*	*(0.324)*	*(0.125)*
acc(t-2)	-0.411	-	-0.368	-	-0.444	-
	(0.247)	-	*(0.282)*	-	*(0.330)*	-
Log L	100.9	97.2	153.3	148.6	173.8	168.2
LR		7.4		*9.38*		*11.16*
P-Value		*0.060*		*0.025*		*0.011*
excess smoothness	-0.181	-0.226	-0.133	-0.141	-0.170	-0.048
se	*(0.100)*	*(0.079)*	*(0.050)*	*(0.053)*	*(0.066)*	*(0.106)*
excess sensitivity						
Log L unrestricted mo	101.9	97.9	153.5	149.9	174.4	168.6
LR	2.0	1.4	0.4	2.58	1.24	0.8
P-value	*0.368*	*0.497*	*0.819*	*0.275*	*0.538*	*0.670*

NOTES:

all data are in (first diff of) levels
SE in parentheses
excess smoothness test computed as sum(axx(t-L))-sum(ayx(t-L))=0, with L=0,…,4
interest rate =0.01
excess sensitivity test computed as LR test (restrictions: acy(t-1)=acy(t-2)=0)

Table 4
Variance Based Test

Ind. Var.	**non durable consumption**	**non durable consumption per ad.eq.**	**total consunmption**	**total consumption per ad.eq.**
Gross earnings	0.0709	0.0376	0.0765	0.0547
	0.0133	0.0154	0.0177	0.0196
implied a	3.7556	5.1571	3.6144	4.2746
	0.0484	0.0900	0.0610	0.0865
Gross earnings+ benefits	0.2357	0.1476	0.3019	0.2495
	0.0302	0.0355	0.0401	0.0448
implied a	2.0596	2.6032	1.8200	2.0021
	0.0447	0.0747	0.0492	0.0634
Net earnings + benefits	0.2601	0.1466	0.3478	0.2733
	0.0351	0.0413	0.0463	0.0519
implied a	1.9608	2.6121	1.6957	1.9129
	0.0482	0.0871	0.0511	0.0686
Number of observ	505	505	505	505