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Risk-Taking Behavior with Limited Liability and Risk Aversion

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Risk-Taking Behavior with Limited Liability and Risk Aversion [1]

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Abstract: We consider in this paper the problem of a risk-averse firm with limited liability. The firm has to select the size of its investment in a risky project. We show that the optimal exposure to risk of the limited liability firm is always larger than under full liability. Moreover, there exists a positive lower bound on the value of the firm below which the firm will "bet for resurrection," i.e. it will invest the largest positive amount in the risky project. We also consider the standard portfolio problem with more than one risky asset. We show that limited liability may induce the firm to specialize in no Mean-Variance efficient assets.

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1 Introduction

This article aims at formalizing the consequences of limited liability on the risk-taking behavior of a rational decision maker (D M). By limited liability we mean that this DM is explicitly or implicitly guaranteed a minimum wealth, even if his/her activity generates considerable losses. This can be applied for instance to two different, but equally important contexts.

The first context is that of the damages (both physical or material) that an individual can cause to others, either in the course of his/her profession (medecine, surgery, house-building) or because of other activities (e.g. driving a car). These activities are in general covered by compulsory liability insurance, which means that even if our DM inflicts important damages to others, he/she will only be liable for a limited amount.

The second context, on which we will concentrate in this paper is the general case of a limited liability firm, the owners of which are not responsible for debts that could exceed the amount of their stake. Of course, if debtors can monitor in real time the activity of the firm, they can condition the interest rate they demand on the riskiness of this activity, and the limited liability problem disappears. Most of the time, this real time monitoring is impossible and a moral hazard problem appears. The most striking example is that of a financial intermediary (bank, saving and loan, security broker, insurance company) who has to select risky investments, which are financed in a large proportion by outsiders' funds.

If these outsiders cannot monitor the firm's investments in real time, the limited liability clause gives the DM the equivalent of a free put option (Stiglitz-Weiss (1981)). Consequently, if the DM is risk-neutral, or if the owners of the firm are perfectly diversified (so that they agree on the objective function of the firm, i.e. the market value of its profit, including the option) the DM will seek to maximize the expectation of a *convex* function of the firm's profit. As a result, the DM will systematically exhibit a *risk-loving* behavior, and adopt a very risky attitude.

Our objective in this paper is to rationalize a more contrasted behavior, exemplified for instance by the Saving and Loans crisis in the USA. It is true that well documented examples abound of "zombie" S. and Ls, which adopted such very risky attitudes, in an attempt of "betting for resurrection". However this was not the systematic attitude of all Saving and Loans, even before strict capital requirements were introduced: well capitalized S. and L. persisted in a sound investment strategy. Still, it can be shown (cf sections 2 and 6) that if security markets are competitive, a risk neutral limited liability portfolio manager will always select an extremely risky and specialized portfolio. Another element has to be introduced, namely risk-aversion of the DM.

This is what we do in this paper, by assuming that the DM maximizes the expectation of a concave VNM utility function. This is perfectly justified in the case of an individual (entrepreneur), but more controversial in the case of a firm. In the latter case, we only see it as a proxy for taking into account the imperfection of capital markets, and more specifically the fact that the firm's stockholders cannot perfectly diversify their own portfolios. This is consistent with the very existence of the financial intermediaries that we want to study.

The paper is organized as follows. After presenting the model and its applications (section 2) we characterize the optimal policy (section 3) of a risk averse DM under limited liability: risk exposure is always higher than under full liability (proposition 2) and is often maximal (propositions 1 and 3). More specifically, proposition 4 shows the existence of a critical level of initial wealth under which the DM systematically chooses maximal risk exposure. In section 4 we explore the consequences of a possible remedy, namely forced recapitalization. The results depend on the monotonicity properties of the risk aversion indexes. In section 5 we present a geometrical analysis of risk taking behavior under limited liability. This analysis is then applied (in section 6 to an extension of the model to the case of multiple risks. We show in particular that for low levels of initial wealth, the DM may make portfolio choices which are mean-variance inefficient.

2 The model and its applications

As in Rochet (1992) and Posey (1992), our model is static with two dates : at time t = 0, the decision-maker (DM) selects his/her exposure to risk. At time t = 1, uncertainty is resolved. As in Gollier (1995), we consider a linear payoff function

$$\tilde{k} = k_0 + \alpha\tilde{x}.$$

$\tilde{k}$ is the net wealth at the end-date 1, whereas k_0 is the initial wealth at date 0. $\tilde{x}$ is any (discrete, continuous or mixed) random variable with cumulative probability distribution F. Variable α is the decision variable measuring the scale of exposure to risk. It is constrained to be nonnegative and not to exceed an upper bound $\bar{\alpha}$ which is exogenous. In section 6, we extend this model to allow for more than one source of risk.

There is no intermediary consumption. Due to limited liability, the final consumption of the DM is

$$\tilde{z} = \max(0, \tilde{k}) = \max(0, k_0 + \alpha\tilde{x}).$$

The DM is endowed with a von Neumann-Morgenstern utility function $u(z)$ of his final consumption that is twice continuously differentiable, increasing and concave. Without loss of generality, we assume that $u(0)$ = O.

The problem of the DM is to select the exposure to risk that maximizes his/her expected utility:[1]

$$\alpha^* \in \arg\max_{0\leq\alpha\leq\bar{\alpha}} \quad H(\alpha\,;k_0) = Eu(\tilde{z}) = \int_{-\frac{k_0}{\alpha}}^{+\infty} u(k_0+\alpha x)dF(x) \tag{1}$$

If the DM faced full liability, his problem would rather be as follows :

$$\hat{\alpha} \in \arg\max_{0\leq\alpha\leq\bar{\alpha}} \quad \hat{H}(\alpha\,;k_0) = Eu(\tilde{k}) = \int_{-\infty}^{+\infty} u(k_0+\alpha x)dF(x) \tag{2}$$

The standard application of the linear payoff model is the portfolio problem: an investor with initial wealth k_0 may invest in either a riskfree asset with zero interest rate, and a risky asset whose return is distributed as $\tilde{x}$. α is the demand for the risky asset. Under unlimited liability, this problem is equivalent to the one analyzed by Pratt (1964) and Arrow (1965).2 This model is also generic for the coinsurance problem and for the problem of the firm under output-price uncertainty, as stated in Dionne, Eeckhoudt, Gollier (1993). Our aim is to examine the impact of limited liability on the basic properties of the Arrow-Pratt portfolio problem. As a benchmark, remember that under unlimited liability, the objective function $\hat{H}$ is concave, and that $\hat{\alpha}$ is positive as long as $E\tilde{x}$ is positive.

In this paper, we are interested in the policy implications of limited liability for the regulation of the banking system. The use of an expected utility approach for the objective of the bank is motivated by the incompleteness of markets. Two different interpretations of the above model can be given. In both cases, a "bank" has equity capital K and deposits D at date O. It is assumed that there is a deposit insurance system, financed by premia which only depend on the volume of deposits. The cost of deposits is the sum of the interest paid to the depositors and the insurance premium. The unit cost of deposits is denoted R. In a competitive market, R is exogenous for the bank. It does not depend upon the probability of failure, because of the misprizing of deposit insurance. The above linear model has two possible interpretations:

1. *Portfolio management:* The liability side of the balance sheet is exogenous, i.e. D is not a decision variable. On the asset side, the bank may invest in two securities : a riskfree asset with interest rate R_f and a risky asset with a rate of return $\tilde{R}$. If the bank invests α in the risky asset at date O, its equity capital at date 1 is

$$\tilde{K} = K(1+R_f) + D(R_f - R) + \alpha(\tilde{R} - R_f). \tag{3}$$

[1] H is formally defined as in (1) only for positive α. For $\alpha = 0$, it is defined as $u(\max(0,k_0))$.

[2] Notice that if u is linear, H is convex and $\hat{H}$ is linear. As a result both α^* and $\hat{\alpha}$ are always extreme points, that is either 0 or $\bar{\alpha}$.

$\bar{\alpha} = K + D$ represents the maximum amount that may be invested in the risky asset.

2. *Optimal leverage :* The asset side of the balance sheet is exogenous and the rate of return of assets is denoted $\tilde{R}$. The bank selects the volume of deposits D that maximizes its expected utility. Its final equity capital is written as

$$\tilde{K} = KR + (K + D)(\tilde{R} - R) \tag{4}$$

If β is the maximum leverage D/K allowed, the upper bound $\bar{\alpha}$ on $\alpha = K + D$ is $K(1 + \beta)$.

Without entering into details, a simple rewording of these applications suggests that our model could also be applied to the study of solvency regulations for insurance companies.

3 Characterization of the optimal policy

If the cumulative distribution function F has a derivative $f = F'$, one can compute the first two derivatives of the objective function H with respect to α :

$$\frac{\partial H}{\partial \alpha} = \int_{-\frac{k_0}{\alpha}}^{+\infty} x u'(k_0 + \alpha x) dF(x) \tag{5}$$

$$\frac{\partial^2 H}{\partial \alpha^2} = \int_{-\frac{k_0}{\alpha}}^{+\infty} x^2 u''(k_0 + \alpha x) dF(x) + u'(0) f\left(-\frac{k_0}{\alpha}\right) \frac{k_0^2}{\alpha^3} \tag{6}$$

The first-order condition for α^* is written as

$$\left.\frac{\partial H}{\partial \alpha}\right|_{\alpha=\alpha^*} \begin{cases} = 0 & \text{if} \quad 0 < \alpha^* < \bar{\alpha}\ ; \\ \geq 0 & \text{if} \quad \alpha^* = \bar{\alpha}, \\ \leq 0 & \text{if} \quad \alpha^* = 0 \end{cases} \tag{7}$$

whereas the second-order condition is

$$\left.\frac{\partial^2 H}{\partial \alpha^2}\right|_{\alpha=\alpha^*} \leq 0, \tag{8}$$

if $0 < \alpha^* < \bar{\alpha}$.

Contrary to the-well-known unlimited liability model represented by the objective function H, the objective function H under limited liability needs not to be a concave function of the decision variable. In the case of a continuous random variable, $\frac{\partial^2 H}{\partial \alpha^2}$ is the sum of a negative term and a positive one for $\alpha \geq 0$, as shown in equation (6). Its sign is thus ambiguous. Moreover, if there is an atom at z $= x_0$, H has a kink at $\alpha = -k_0/x_0$. If $\tilde{x}$ is a discrete random variable, $\frac{\partial^2 H}{\partial \alpha^2}$ is negative whenever it exists but it has upward jumps, so that H is never (globally) concave.

An important consequence of the non-concavity of the objective function is that a continuous change in a parameter (for example, a change in k_0) can generate a discontinuous change in the optimal exposure to risk. As a numerical example, consider the concave utility function $u(z) = \min(z, \omega)$ and the following distribution for $\tilde{x} \equiv (-2, 1/3\ ; 1, 1/3\ ; 4, 1/3)$. One can verify that $\alpha^*(k_0)$ is discontinuous at $k_0 = k = 5\omega/9$. Indeed, $\alpha^*(k_-) = 4\omega/9$ and $\alpha^*(k_+) = \omega/9$. In this example, there is a 75% downwards jump in the optimal scale of the risk for a marginal increase in initial wealth.

For the sake of simplicity, we hereafter assume that x is a continuous random variable. All results can be extended to the discrete case.

Proposition 1 *If* $k_0 \leq 0$, *then* $\alpha^* = \bar{\alpha}$.

Proof: Since $-\frac{k_0}{\alpha}$ is non negative, $\frac{\partial H}{\partial \alpha}$ is positive in interval $[0, \bar{\alpha}]$.

■

This proposition is a strong version of "betting for resurrection": when initial wealth is zero or negative, it is always optimal to accept the greatest risk available. This proposition allows us to hereafter assume that initial wealth is positive: $k_0 > 0$.

Proposition 2 $\alpha^* \geq \hat{\alpha}$.

Proof: This follows from the fact that

$$\frac{\partial \hat{H}}{\partial \alpha}(\alpha^*\ ; k_0) = \int_{-\infty}^{-\frac{k_0}{\alpha^*}} x u'(k_0 + \alpha^* x) dF(x) + \frac{\partial H}{\partial \alpha}(\alpha^*\ ; k_0)$$

Assume first that $\alpha^* \notin \{0, \bar{\alpha}\}$. The first term in the RHS of this equation is negative, whereas the second is zero. By concavity of H, we get the result. The same argument holds in the case $\alpha^* = 0$ since the second term in the RHS is also negative. When $\alpha^* = \bar{\alpha}$, the result is obvious.

■

Thus; introducing limited liability increases the optimal exposure to risk of risk-averse agents. This result is an extension of the well-known Arrow-Pratt result that less risk aversion raises the demand for risky assets. Indeed, $v(z) \equiv \max(0, u(z))$ is a convex transformation of u. It is however not a strict corollary of the Arrow-Pratt result since these authors assumed that v was still concave, which is not the case here. Even in the case of an actuarially unfair risk ($E\tilde{x} < 0$), a risk-averse DM may purchase a positive amount of the risk. Under unlimited liability, we would have $\hat{\alpha} = 0$. This fact clearly shows the distortion that is introduced by limited liability.

Under unlimited liability, the optimal scale of the risk can be either finite or infinite. It is indeed straightforward to prove that the following condition

$$\frac{\inf_{k\in R} u'(k)}{\sup_{k\in R} u'(k)} < \frac{\int_{-\infty}^{0} -x dF(x)}{\int_{0}^{+\infty} x dF(x)}$$

is necessary and sufficient to have a bounded solution for $\hat{\alpha}$ (assuming that $E\tilde{x}$ is positive). It is only when the utility function satisfies the Inada conditions (marginal utility becomes infinite at zero wealth, and zero at infinite wealth) that we can guarantee that the optimal exposure to risk is finite, whatever the distribution of $\tilde{x}$. This result strikingly differs from the following property in the limited liability case:

Proposition 3 *Suppose that there is no constraint on* α^* : $\bar{\alpha} = +\infty$. *If* $Proba[\tilde{x} > 0] > 0$ *and if* u *is unbounded above, then* α^* *is infinite.*

Proof: By assumption, there exists a positive scalar ϵ such that $P[\tilde{x} \geq \epsilon]$ is positive. We obtain the following sequence of inequalities:

$$\begin{aligned} H(\alpha\,; k_0) &= \int_{-\frac{k_0}{\alpha}}^{+\infty} u(k_0 + \alpha x) dF(x) \\ &\geq \int_{\epsilon}^{+\infty} u(k_0 + \alpha x) dF(x) \\ &\geq u(k_0 + \alpha\epsilon) Proba[\tilde{x} \geq \epsilon], \end{aligned}$$

for any $\alpha \geq 0$. Since $u(k_0 + \alpha\epsilon)$ tends to infinity with α, so does $H(\alpha)$.

■

When the utility function is unbounded above, it is always optimal to accept risks without limit, under limited liability. It is only when there is no chance to get a positive gain that it will be optimal not to accept the bet[3]. Of course, this is a rather negative result. It reinforces the intuition that utility functions should be. bounded above, following the well-known argument based on super-St. Petersburg games. Still, the same phenomenon may occur with bounded utility functions, for DMs that are close to bankruptcy.

Proposition 4 *Suppose that* $\bar{\alpha} < +\infty$, *and* $P[\tilde{x} > 0] > 0$. *There exists a threshold* $\bar{k}_0$ *such that* $\alpha^* = \bar{\alpha}$ *for any* $k_0 \leq \bar{k}_0$.

Proof: Denote $k_0^*(\alpha) \equiv \{k_0 \geq 0; \frac{\partial H}{\partial \alpha}(\alpha\,; k_0) = 0\} \cup \{+\infty\}$. We would be done if we proved that $\bar{k}_0 = \inf_{0 \leq \alpha \leq \bar{\alpha}} k_0^*(\alpha)$ is positive. We first prove that $\frac{\partial H}{\partial \alpha}(\alpha\,; 0)$ is positive for any $\alpha \geq 0$. This is obvious for positive α. For $\alpha = 0$, we get that

[3]If $\tilde{x}$ represents the net payoff of a lottery, i.e. the gross payoff minus the price of the lottery ticket, the demand for the lottery would be zero as long as the price exceeds the largest possible gross payoff. But as soon as the price goes below this bound, the demand for lottery tickets goes to infinity. In this example, it is assumed that the DM has no borrowing constraint, clearly an unrealistic assumption.

$$\begin{aligned}\frac{\partial H}{\partial \alpha}(0\,;0) &= \lim_{\alpha\to 0}\frac{H(\alpha\,;0)-H(0\,;0)}{\alpha}\\ &= \lim_{\alpha\to 0}\frac{\int_0^{+\infty}u(\alpha x)dF(x)}{\alpha}\\ &= \int_0^{+\infty} xu'(0)dF(x)\\ &= u'(0)E[\max(0,\tilde{x})],\end{aligned}$$

which is also positive. This implies that

$$\inf_{0\le\alpha\le\bar{\alpha}}\frac{\partial H}{\partial\alpha}(\alpha\,;0)$$

is positive. By continuity of $\frac{\partial H}{\partial\alpha}$ with respect to k_0, this implies in turn that $k_0^*(\alpha)$ is positive for all α. Therefore, $\bar{k}_0 \geq 0$. If it were zero, there would exist $(\alpha_n) \subset [0,\bar{\alpha}]$ such that $\alpha_n \to \alpha \in [0,\bar{\alpha}]$ and $k_0^*(\alpha_n) \to 0$. We would thus have that $\inf_{0<\alpha<\bar{\alpha}} \frac{\partial H}{\partial\alpha}(\alpha\,;0) = 0$, a contradiction.

n

This result is a more general version of "betting for resurrection" than the one presented in Proposition 1. This is not only for bankrupt initial positions, but also for initial positions that are "close enough" to bankruptcy, that the DM will adopt a highly risk-loving attitude. A particular case of this proposition is in Shaven (1986) in the context of insurance demand. Shaven considered two-point distribution functions for $\tilde{x}$.

4 The consequences of recapitalization

A classical remedy to the financial distress of corporations is forced recapitalization. In this section, we examine the consequences of such a policy on the risk-taking behavior of our decision maker, i.e. we examine the properties of the mapping $k_0 \to \alpha^*(k_0)$. In the "classical (full-liability) case, this behavior is completely determined by that of the absolute risk aversion index : $k \to r(k) \stackrel{def}{=} -\frac{u''(k)}{u'(k)}$. More specifically, Arrow (1965) and Pratt (1964) have proved the following result:

Proposition (Arrow (1965), Pratt (1964)):
If $k \to r(k)$ is increasing (resp. decreasing) then $k_0 \to \hat{\alpha}(k_0)$ is decreasing (resp. increasing).

When limited liability is introduced, we have already noticed that recapitalization tended to moderate the risk exposure chosen by the DM, because it decreased the option value of limited liability. When absolute risk aversion is increasing, the wealth effect also goes in that direction. Therefore it is not surprising that the first part of the Arrow-Pratt result extends to the case of limited liability:

Proposition 5 *If $k \longrightarrow r(k)$ is increasing, then $k_0 \longrightarrow \alpha^*(k_0)$ is decreasing. As a result, recapitalizat ion induces a decrease in risk exposure.*

Proof: Differentiating (5) with respect to k_0 yields

$$\frac{\partial^2 H}{\partial\alpha\partial k_0} = \int_{-\frac{k_0}{\alpha}}^{+\infty} xu''(k_0+\alpha x)dF(x) - u'(0)\frac{k_0}{\alpha^2}f\left(-\frac{k_0}{\alpha}\right).$$

The second term in the RHS is positive. Using the definition of r, we obtain that

$$\frac{\partial^2 H}{\partial\alpha\partial k_0} \leq \int_{-\frac{k_0}{\alpha}}^{+\infty} -xr(k_0+\alpha x)u'(k_0+\alpha x)dF(x).$$

Since r is increasing, $-xr(k_0+\alpha x)$ is strictly less than $-xr(k_0)$, for all x. It yields

$$\frac{\partial^2 H}{\partial\alpha\partial k_0} < -r(k_0)\int_{-\frac{k_0}{\alpha}}^{+\infty} xu'(k_0+\alpha x)dF(x) = -r(k_0)\frac{\partial H}{\partial\alpha} \tag{9}$$

If $\alpha^*(\cdot)$ is continuous and differentiable at k_0 and if $\alpha^*(k_0)$ is interior (the case of a corner solution is trivial), differentiating the first order condition yields:

$$\frac{d\alpha^*}{dk_0} = -\frac{\frac{\partial^2 H}{\partial\alpha\partial k_0}}{\frac{\partial^2 H}{\partial\alpha^2}}(\alpha^*(k_0), k_0).$$

Since $\frac{\partial^2 H}{\partial k_0}(\alpha^*, k_0)$ is necessarily negative by the second order condition, $\frac{d\alpha^*}{dk_0}$ has the same sign as $\frac{\partial^2 H}{\partial\alpha\partial k_0}(\alpha^*, k_0)$. Applying inequality (9) to (α^*, k_0) proves that this sign is negative.

If $\alpha^*(\cdot)$ has a jump in k_0, we have to prove that it is necessary a downward jump. Such a discontinuity can occur only when $H(\cdot, k_0)$ has two maxima α_1 and α_2 (with $\alpha_1 < \alpha_2$). Therefore we have:

$$H(\alpha_1, k_0) = H(\alpha_2, k_0) \quad \text{and} \quad \frac{\partial H}{\partial\alpha}(\alpha_1, k_0) = \frac{\partial H}{\partial\alpha}(\alpha_2, k_0) = 0$$

Assuming that second order derivatives are non singular, we can apply the implicit function theorem in the neighborhood of both (α_1, k_0) and (α_2, k_0). We obtain two mappings $k \longrightarrow \alpha_1(k)$ and $k \longrightarrow \alpha_2(k)$ such that

$$\alpha_i(k_0) = \alpha_i \quad \text{and} \quad \frac{\partial H}{\partial\alpha}(\alpha_i(k), k) \equiv 0 \quad i = 1, 2.$$

By continuity $\alpha^*(k)$ is necessarily equal, in a neighborhood of k_0, either to $\alpha_1(k)$ or to $\alpha_2(k)$. We are going to prove that in fact (locally):

$$\begin{cases} \alpha^*(k) &= \alpha_2(k) & \text{for} & k < k_0 \\ \alpha^*(k) &= \alpha_1(k) & \text{for} & k > k_0 \end{cases}$$

which will establish our result, since $\alpha_1 < \alpha_2$.

Let $\varphi(k) \stackrel{def}{=} H(\alpha_2(k), k) - H(\alpha_1(k), k)$, we have

$$\begin{aligned}\varphi'(k_0) &= \tfrac{\partial H}{\partial k}(\alpha_2, k_0) - \tfrac{\partial H}{\partial k}(\alpha_1, k_0) \\ &= \int_{\alpha_1}^{\alpha_2} \tfrac{\partial^2 H}{\partial \alpha \partial k}(\alpha, k_0) d\alpha.\end{aligned}$$

By relation (9) we have:

$$\varphi'(k_0) < -r(k_0) \int_{\alpha_1}^{\alpha_2} \frac{\partial H}{\partial \alpha}(\alpha, k_0) d\alpha,$$

or:

$$\varphi'(k_0) < -r(k_0)\varphi(k_0) = 0.$$

Thus, locally:

$$\begin{cases} \varphi(k) > 0 & \text{for} \quad k < k_0 \\ \varphi(k) < 0 & \text{for} \quad k > k_0 \end{cases}$$

and the proof is complete. n

Of course, increasing absolute risk aversion is not a very reasonable assumption. If we assume instead that u is DARA (decreasing absolute risk aversion) the wealth effect and the option effect go in opposite directions. In that case the graph of $\alpha^*(\cdot)$ is typically U-shaped (first decreasing then increasing).

A natural question to ask is whether recapitalization induces a decrease in the probability of failure, i.e. $P\left[\tilde{x} \leq -\frac{k_0}{\alpha^*(k_0)}\right]$. This is the case if and only if the mapping $k_0 \rightarrow \frac{\alpha^*(k_0)}{k_0}$ is decreasing. Again, an important benchmark is the full liability case, for which Arrow (1965) and Pratt (1964) have proven that $k_0 \rightarrow \frac{\bar{\alpha}(k_0)}{k_0}$ was decreasing (resp. increasing) when the relative index of risk aversion $\gamma(k) \stackrel{def}{=} -\frac{ku''(k)}{u'(k)}$ was increasing (resp. decreasing). In the case where $\gamma(\cdot)$ is increasing, this result extends to the limited liability context:

Proposition 6 *If $k \rightarrow \gamma(k)$ is increasing then $k_0 \rightarrow \frac{\alpha^*(k_0)}{k_0}$ is decreasing. As a result, recapitalization induces a decrease in the probability of failure.*

Proof: It is exactly analogous to that of proposition 5, after the following change of variable :

$$\tau = \frac{\alpha}{k_0} \quad \text{and} \quad L(\tau, k_0) = H(\tau k_0, k_0).$$

Increasing relative risk aversion implies indeed that

$$\frac{\partial^2 L}{\partial \tau \partial k_0}(\tau, k_0) \leq -\gamma(k_0) \frac{\partial L}{\partial k_0}(\tau, k_0)$$

5 The geometry of risk taking behavior under limited liability

In the next section we extend our analysis to the case of several sources of risk, which applies for instance to the problem of portfolio selection with several risky assets. As a useful preliminary, we study in this section the geometrical representation of the preferences of a limited liability DM in the (mean, standard deviation) plane. As a side product, this will also provide simple, graphical, intuitions for our previous results, derived analytically in sections 2 to 4.

More specifically let us define two functions

$$U(\mu,\sigma) = \int_{-\mu/\sigma}^{+\infty} u(\mu + t\sigma)dG(t), \tag{10}$$

$$\bar{U}(\mu,\sigma) = \int_{-\infty} u(\mu + t\sigma)dG(t), \tag{11}$$

interpreted respectively as the expected utility of a limited liability and a full-liability decision maker confronted with a random variable

$$\tilde{z} = \mu + \tilde{t}\sigma,$$

where $\tilde{t}$ has zero mean, unit variance and density function g (with associated cumulative G). This specification will be justified in the next section. For technical reasons we also assume that $\lim_{x \to -\infty} x^3 g(x)$ = O. Notice for the moment that if G is conveniently chosen (namely : $G(t) = F(E\tilde{x} + t\sqrt{\text{var}\tilde{x}})$), U and $\bar{U}$ are simple transformations of the functions H and $\hat{H}$ studied previously:

$$H(\alpha, k_0) = U(k_0 + \alpha E\tilde{x}, \alpha\sqrt{\text{var}\tilde{x}}), \tag{12}$$

$$\hat{H}(\alpha, k_0) = \bar{U}(k_0 + \alpha E\tilde{x}, \alpha\sqrt{\text{var}\tilde{x}}). \tag{13}$$

The properties of $\bar{U}$ are well known: it is concave, increasing in μ and decreasing in a. Moreover:

$$\frac{\partial \bar{U}}{\partial \sigma}(\mu, 0) = \int tu'(\mu)dG(t) = 0$$

which means that indifference curves have a horizontal tangent when they cross the vertical axis ($\sigma =$ O). In fact the typical shape of indifference curves is the following:

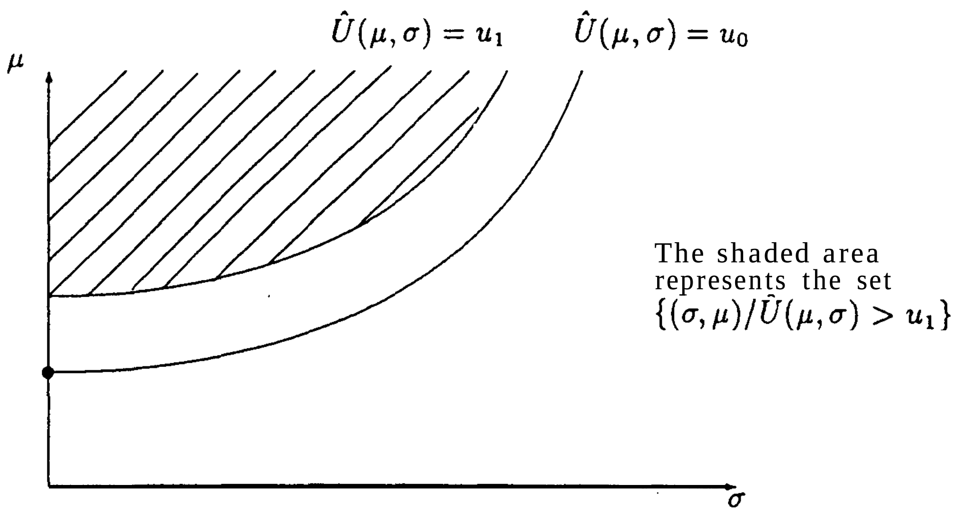

Figure 1 : The indifference curves of a full liability DM in the (standard deviation, mean) plane

The properties of U are less straightforward. It is easy to see that U is still increasing in μ but it is not everywhere decreasing in σ, and not quasi-concave in general. To see this, it is enough to study the behavior of U along the two axes:

$$U(\mu,0) = \hat{U}(\mu,0) = u(\mu),$$

$$U(0,\sigma) = \int u(t\sigma)dG(t).$$

Thus $\sigma \to U(0,\sigma)$ *increases* from 0 to $U(0,+\infty)$ when σ increases from 0 to $+\infty$! Moreover

$$U(0,+\infty) = u(+\infty) \cdot \text{Proba}[\tilde{t} > 0]$$

As a result, when u is unbounded above ($u(+\infty) = +\infty$) there exists for all m in $\boldsymbol{R}_+$ a unique $\sigma(m)$ such that:

$$U(0,\sigma(m)) = U(m,0). \tag{14}$$

Therefore the typical shape of indifference curves is the following:

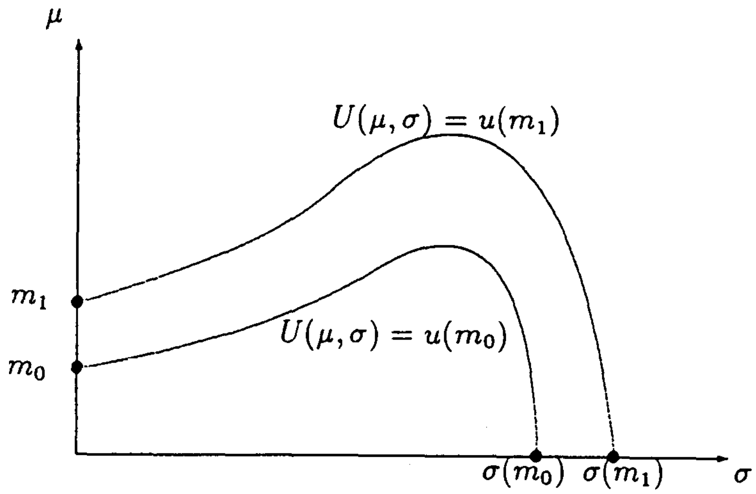

Figure 2: The indifference curves of a limited liability DM in the (standard deviation, mean) plane

The expressions of first and second order partiaL derivatives of U are easily computed:

$$\frac{\partial U}{\partial \mu}(\mu,\sigma) = \int_{-\mu/\sigma}^{+\infty} u'(\mu + t\sigma)dG(t),$$

$$\frac{\partial U}{\partial \sigma}(\mu,\sigma) = \int_{-\mu/\sigma}^{+\infty} tu'(\mu + t\sigma)dG(t),$$

$$\frac{\partial^2 U}{\partial \mu^2}(\mu,\sigma) = \int_{-\mu/\sigma}^{+\infty} u''(\mu + t\sigma)dG(t) + \frac{1}{\sigma}u'(0)g\left(-\frac{\mu}{\sigma}\right),$$

$$\frac{\partial^2 U}{\partial \mu \partial \sigma}(\mu,\sigma) = \int_{-\mu/\sigma}^{+\infty} tu''(\mu + t\sigma)dG(t) - \frac{\mu}{\sigma^2}u'(0)g\left(-\frac{\mu}{\sigma}\right),$$

$$\frac{\partial^2 U}{\partial \sigma^2}(\mu,\sigma) = \int_{-\mu/\sigma}^{+\infty} t^2u''(\mu + t\sigma)dG(t) + \frac{\mu^2}{\sigma^3}u'(0)g\left(-\frac{\mu}{\sigma}\right).$$

It is easy to see that indifference curves have still a horizontal tangent when they cross the vertical axis (like for U) :

$$\frac{\partial U}{\partial \sigma}(\mu,0) = \int_{-\infty}^{+\infty} tu'(\mu)dG(t) = 0$$

However now they also cross the horizontal axis, at least when u is unbounded above. It is also worth noticing that U is not monotonic with respect to σ : as already remarked $\frac{\partial U}{\partial \sigma}(0,\sigma)$ is positive, whereas for σ small enough, $\frac{\partial U}{\partial \sigma}(\mu,\sigma)$ has the more usual (negative) sign. To see this, remember that $\frac{\partial U}{\partial \sigma}(\mu,0)$ equals zero, whereas

$$\lim_{\sigma \to 0} \frac{\partial^2 U}{\partial \sigma^2}(\mu,\sigma) = \int_{-\infty}^{+\infty} t^2u''(\mu)g(t)dt - \frac{u'(0)}{\mu}[\lim_{x \to -\infty} x^3g(x)]$$

Since $\lim_{x \to -\infty} x^3 g(x)$ is nil, we obtain:

$$\frac{\partial^2 U}{\partial \sigma^2}(\mu, 0) = u''(\mu) E(\tilde{t}^2) < 0.$$

Thus $\frac{\partial U}{\partial \sigma}(\mu, \sigma)$ is negative for σ small enough. Therefore, if we move along an indifference curve by increasing σ, we start by a horizontal tangent for $\sigma = 0$, then μ increases (i.e. $\frac{\partial U}{\partial \sigma} < 0$) and reaches a maximum after which μ decreases (i.e. $\frac{\partial U}{\partial \sigma} > 0$), and the indifference curve eventually touches the horizontal axis, with a negative slope. We can therefore partition the plane in two regions : one in which risk aversion dominates and $\frac{\partial U}{\partial \sigma} \leq 0$, and its complement in which the convexity effect of the option value dominates and $\frac{\partial U}{\partial \sigma} > 0$.

The surprising properties obtained in sections 2 to 4 occur when U is maximum in that region. To see this it is enough to remark that, because of (13), the problem that we solved in the previous sections corresponds geometrically to finding the maximum of U on the set:

$$\Delta = \{\mu = k_0 + \alpha(E\tilde{x}), \sigma = \alpha\sqrt{var\tilde{x}}, 0 \leq \alpha \leq \bar{\alpha}\}$$

Consider for instance proposition 3, which asserts that when u is unbounded above and $\bar{\alpha}$ infinite, there is no such maximum. The geometric intuition of this result is clear : when indifference curves have the shape indicated in figure 2, then Δ (which is unbounded when $\bar{\alpha} = +\infty$) always crosses each of them.

We are going to see in the next section how this geometric representation is also useful for studying the case of multiple risks.

6 Extension to multiple risks

We consider now the case where our DM is confronted with several sources of risk. Whereas motivations for such a problem could easily be found in an insurance context, we have chosen to illustrate it here by the more classical portfolio selection problem à la Tobin-Markovitz.

More specifically, we assume that there are n risky assets, so that the decision variable to be chosen by our DM is now a vector $\alpha \in \mathbb{R}^n$ such that:

$$\tilde{x} = <\alpha, \tilde{\rho}>,$$

where $\tilde{\rho}$ represents the (random) excess returns vector. We assume that short selling is prohibited:

$$\alpha_i \geq 0 \quad i = 1, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i \leq 1.$$

Using the same method as in part 3, it is possible to extend Proposition 4 in that case. Namely, beyond a small initial wealth's level, the DM will invest nothing in the riskless asset. However, it is more delicate to determine the composition of the risky portfolio.

Without any further assumption on the distribution of returns, the analysis would be overwhelmingly complicated. A standard simplification is obtained by assuming that $\tilde{\rho}$ is elliptically distributed.[4] In such a case there exists a univariate random variable $\tilde{t}$ such that

$$\tilde{x} = \mu(\alpha) + \sigma(\alpha)\tilde{t}$$

with:

$$\mu(\alpha) = <\alpha, E\tilde{\rho}>, \sigma^2(\alpha) = var(\tilde{x}) = <\alpha, var(\tilde{\rho})\alpha>,$$

and $\tilde{t}$ is distributed according to the distribution function G, with $E\tilde{t} = 0$ and $var(\tilde{t}) = 1$. We assume that the covariance matrix of $\tilde{\rho}$, denoted $var(\tilde{\rho})$, is non-singular.

This is the justification for the specification (10) of the utility function studied in section (5), since our problem can be formalized as[5]:

$$(\mathcal{P}) \begin{cases} \max U(k_0 + \mu(\alpha), \sigma(\alpha)) \\ \alpha \in \mathbb{R}^n_+, \sum_{i=1}^n \alpha_i \leq 1 \end{cases}$$

or

$$\begin{cases} \max U(\mu, \sigma) \\ (\mu, \sigma) \in S(k_0) \end{cases}$$

where

$$S(k_0) = \left\{ \begin{array}{lll} (\mu, \sigma), & \mu = k_0 + <\alpha, E\tilde{\rho}>, & \alpha \in \mathbb{R}^n_+ \\ & \sigma^2 = <\alpha, var(\tilde{\rho})\alpha>, & \sum_{i=1}^n \alpha_i \leq 1 \end{array} \right\}$$

Since U is continuous and $S(k_0)$ is compact, problem $(\mathcal{P})$ has always a solution. It would not be the case in general without short sales constraints. For instance, consider the benchmark case of risk-neutral decision maker (linear *u*). In that case it is easy to see that the objective function of problem $(\mathcal{P})$ is convex with respect to α, and that the solution of $\mathcal{P}$ is always an extreme point of the feasible set ($\alpha_i = 1$ for some *i*). In other words there is a complete specialization of the optimal portfolio of the DM. We are going to show that this result (completely opposed to the usual diversification strategy of risk-averse portfolio managers) can also occur with a concave utility function, provided that the initial net wealth of the DM is small enough.

[4]For a detailed analysis of elliptic distributions see Ingersoll (1987, p 105)

[5]Notice that U is considered as the truncated expected utility of an elliptic random variable. The converse (i.e. the expected utility of a truncated elliptic random variable) would not work since the truncation of an elliptic variable is not elliptic.

For that purpose, we first need a representation of the set $S(0)$ (since $S(k_0)$ is deduced from $S(0)$ by a simple translation). For the sake of simplicity we will restrict to the case of two risky assets ($n = 2$). In that case $S(0)$ is a homeomorphic transformation of the triangle $\{\alpha \in \mathbb{R}^2_+, \alpha_1 + \alpha_2 \leq 1\}$. Its shape is thus determined by the image of the frontier of this triangle, which is easily obtained.

In order to have an idea of the possible phenomena, let us go back to the risk neutral case.

We denote $E(\tilde{\rho}_i) = \mu_i$ and $Var(\tilde{\rho}_i) = \sigma_i$ with, by convention, $\sigma_1 > \sigma_2$. Then:

Proposition 7 *a) If $\mu_1 > \mu_2$, then for all $k_0 \geq 0$, the risk-neutral DM invests everything in the first risky asset.*

b) If $\mu_1 < \mu_2$ and $\int_{-\mu_1/\sigma_1}^{+\infty}(\mu_1 + \sigma_1\tilde{t})dG(t) > \int_{-\mu_2/\sigma_2}^{+\infty}(\mu_2 + \sigma_2\tilde{t})dG(t)$
Then $\exists\ k_0^c > 0$ such that:
if $k_0 \in [0, k_0^c[$, the risk-neutral DM invests everything in the first risky asset.
if $k_0 \in]k_0^c, +\infty[$, the risk-neutral DM invests everything in the second risky asset.

c) If $\mu_1 < \mu_2$ and
$\int_{-\mu_1/\sigma_1}^{+\infty}(\mu_1 + \sigma_1\tilde{t})dG(t) < \int_{-\mu_2/\sigma_2}^{+\infty}(\mu_2 + \sigma_2\tilde{t})dG(t))$,
for all $k_0 \geq 0$, the risk-neutral DM invests everything in the second risky asset.

Proof: g is the density function of $\tilde{t}$. Let us define:

$$f(x) = \int_{-x}^{+\infty}(x + t)g(t)dt,$$

and:

$$\psi(k_0) = \sigma_1 f\left(\frac{k_0 + \mu_1}{\sigma_1}\right) - \sigma_2 f\left(\frac{k_0 + \mu_2}{\sigma_2}\right).$$

As, for a risk-neutral DM, $U(\mu, \sigma) = \sigma f(\frac{\mu}{\sigma}), \psi(k_0)$ is strictly positive (resp. negative) if and only if the DM chooses the first (resp. the second) risky asset. Moreover,

$$\psi'(k_0) = f'\left(\frac{k_0 + \mu_i}{\sigma_1}\right) - f'\left(\frac{k_0 + \mu_2}{\sigma_2}\right)$$

with$f'(x) = \int_{-x}^{+\infty} g(t)dt$; $f''(x) = g(-x)$. f' is a positive increasing function and $\psi'(k_0) \geq 0$ if and only if $\frac{k_0+\mu_1}{\sigma_1} \geq \frac{k_0+\mu_2}{\sigma_2}$.

If we define $A_{12} = \frac{\mu_1\sigma_2-\mu_2\sigma_1}{\sigma_1-\sigma_2}$, we obtain then:

$$\begin{aligned}
\psi'(k_0) &\geq 0 \Leftrightarrow k_0 \leq A_{12} \\
\psi'(k_0) &\leq 0 \Leftrightarrow k_0 \geq A_{12} \\
\psi(A_{12}) &= \sigma_1\ f\left(\frac{A_{12}+\mu_1}{\sigma_1}\right) - \sigma_2\ f\left(\frac{A_{12}+\mu_2}{\sigma_2}\right) \\
&= \sigma_1\ f\left(\frac{\mu_1-\mu_2}{\sigma_1-\sigma_2}\right) - \sigma_2\ f\left(\frac{\mu_1-\mu_2}{\sigma_1-\sigma_2}\right) \\
&= (\sigma_1-\sigma_2)\ f\left(\frac{\mu_1-\mu_2}{\sigma_1-\sigma_2}\right) > 0.
\end{aligned}$$

Let us examine now the behavior of $\psi(x)$ when x tends to $\pm\infty$.

We will first prove that $f(x)$ tends to zero when x tends to $-\infty$.

Indeed, $f(x) = \int_{-x}^{+\infty}(x+t)g(t)dt = \int_{-x}^{+\infty} xg(t)dt + \int_{-x}^{+\infty} tg(t)dt$.

When $x \to -\infty$, the second term tends to zero because $\int_{-\infty}^{+\infty} tg(t)dt = E(\tilde{t})$ exists. The first term is equal to $xp(\tilde{t} \geq -x)$. If it does not tend to zero when $-x \to +\infty$, then $\exists\epsilon > 0,\ \ \exists(x'_n)_{n\in\mathbb{N}} \to +\infty$ such that

$$\forall\, n\ \ x'_n(1-G(x'_n)) > \epsilon.$$

It implies $\int_{x'_n}^{+\infty} tg(t)dt \geq x'_n \int_{x'_n}^{+\infty} g(t)dt > \epsilon$; there is a contradiction with the existence of $E(\tilde{t})$.

The first point gives immediately that when x tends to $-\infty$, $\psi(x)$ tends to zero, as $\frac{x+\mu_i}{\sigma_i} \underset{x\to-\infty}{\longrightarrow} -\infty$.

When $x \to +\infty$, things are not so easy, as $f(+\infty)$ does not exist anymore. So we work directly on $\psi(x)$:

$$\begin{aligned}
\psi(x) &= \sigma_1 \int_{-\frac{x+\mu_1}{\sigma_1}}^{+\infty} \left(\frac{x+\mu_1}{\sigma_1}+t\right) g(t)dt \\
&\quad -\sigma_2 \int_{-\frac{x+\mu_2}{\sigma_2}}^{+\infty} \left(\frac{x+\mu_2}{\sigma_2}+t\right) g(t)dt \\
&= \left[\sigma_1 \int_{-\frac{x+\mu_1}{\sigma_1}}^{+\infty} tg(t)dt - \sigma_2 \int_{-\frac{x+\mu_2}{\sigma_2}}^{+\infty} tg(t)dt\right] \\
&\quad + \left[\int_{-\frac{x+\mu_1}{\sigma_1}}^{+\infty} (\mu_1-\mu_2)g(t)dt\right] + \left[\int_{-\frac{x+\mu_1}{\sigma_1}}^{+\infty} xg(t)dt - \int_{-\frac{x+\mu_2}{\sigma_2}}^{+\infty} xg(t)dt\right]
\end{aligned}$$

The first term tends to $(\sigma_1-\sigma_2)E(\tilde{t})$ and the second one to $\mu_1-\mu_2$ when x tends to $+\infty$. The third term is equal to:

$$\int_{-\frac{x+\mu_1}{\sigma_1}}^{-\frac{x+\mu_2}{\sigma_2}} dG(t).$$

As $\int_{-\infty}^{+\infty} dG(t) = 1$, $\int_{-\frac{x+\mu_1}{\sigma_1}}^{-\frac{x+\mu_2}{\sigma_2}} dG(t) \underset{x\to+\infty}{\longrightarrow} 0$

Finally:

$$\exists\ \psi(+\infty) = \mu_1-\mu_2+(\sigma_1-\sigma_2)E(\tilde{t}) = \mu_1-\mu_2$$

The proposition is now easy to establish, observing that

$$\psi(0) = \int_{-\mu_1/\sigma_1}^{+\infty}(\mu_1 + \sigma_1 t)dG(t) - \int_{-\mu_2/\sigma_2}^{+\infty}(\mu_2 + \sigma_2 t)dG(t).$$

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Proposition 7 shows us that, in some cases, the risk neutral DM will choose no mean-variance efficient assets (first part of point b of the proposition).

This effect if intrinsically due to the limited liability.

However, it can be ruled out for some utility functions and $\tilde{t}$'s distributions, by constraining the initial wealth k_0. More precisely:

Proposition 8 *If* $u(x) = 1 - e^{-ax}$, $a > 0$, $\tilde{t} \rightsquigarrow \mathcal{N}(0,1)$ *and* $k_0 + \inf_i \mu_i \geq \frac{1}{a}$, *then* $\frac{\partial U}{\partial \sigma} < 0$ *on the* (μ, σ) *feasible set.*

Proof:

$$U(\mu,\sigma) = \int_{-\mu/\sigma}^{+\infty}(1 - e^{-a(\mu+\sigma t)})\frac{e^{-t^2/2}}{\sqrt{2\Pi}}dt$$

$$= N(\mu/\sigma) - e^{\frac{a^2\sigma^2}{2}-a\mu}N(\mu/\sigma - a\sigma)$$

where N denotes the cumulative of the standard gaussian.

$$\begin{aligned}\sqrt{2\Pi}\frac{\partial U}{\partial \sigma} &= e^{-\mu^2/2\sigma^2}N\left(-\frac{\mu}{\sigma^2}\right) - \sqrt{2\Pi}a^2\sigma e^{\frac{a^2\sigma^2}{2}-a\mu}N\left(\frac{\mu}{\sigma} - a\sigma\right) \\ &+ e^{\frac{a^2\sigma^2}{2}-a\mu}e^{-\frac{1}{2}(\mu/\sigma - a\sigma)^2}\left(\frac{\mu}{\sigma^2} + a\right) \\ &= -\sqrt{2\Pi}a^2\sigma e^{\frac{a^2\sigma^2}{2}-a\mu}N\left(\frac{\mu}{\sigma} - a\sigma\right) + ae^{\frac{a^2\sigma^2}{2}-a\mu}e^{-\frac{1}{2}\left(\frac{\mu}{\sigma}-a\sigma\right)^2}\end{aligned}$$

Let us define:

$$\phi(\sigma) \stackrel{def}{=} \frac{\sqrt{2\Pi}}{a\sigma}e^{a\mu - \frac{a^2\sigma^2}{2}}\frac{\partial U}{\partial \sigma} = \frac{1}{\sigma}e^{-\frac{1}{2}\left(\frac{\mu}{\sigma}-a\sigma\right)^2} - a\int_{-\infty}^{\frac{\mu}{\sigma}-a\sigma} e^{-u^2/2}du$$

$$\phi(0^+) = -a\sqrt{2\Pi}; \quad \phi(+\infty) = 0.$$

$$\phi'(\sigma) = \frac{1}{\sigma^2}\left[\frac{\mu^2}{\sigma^2} + a\mu - 1\right]e^{-\frac{1}{2}(\frac{\mu}{\sigma}-a\sigma)^2}.$$

Hence, if $\mu \geq 1/a$, $\phi'(\sigma)$ is positive and $\frac{\sqrt{2\Pi}}{a\sigma}e^{a\mu-\frac{a^2\sigma^2}{2}}\frac{\partial U}{\partial \sigma}$ is always negative on the feasible set.

And if $k_0 + \inf_i(\mu_i) \geq 1/a$, then for all the points (σ, μ) of the feasible set, $\mu \geq 1/a$.

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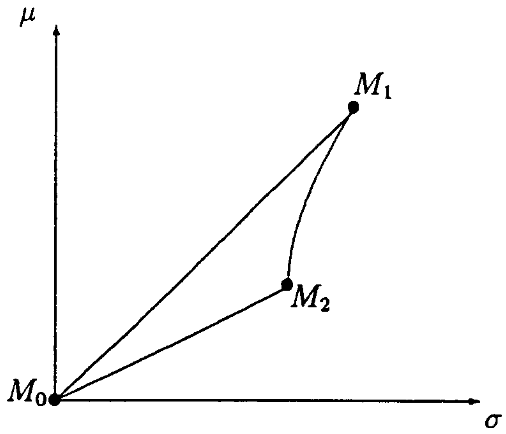

Figure 3

It follows from Proposition 8 and from the property $\frac{\partial U}{\partial \mu} > 0$ that the risk-averse DM will choose only mean-variance efficient porfolios.

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