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Robust adaptive finite-time parameter estimation and control for robotic systems

Jing Na¹*, Muhammad Nasiruddin Mahyuddin^{2*}, Guido Herrmann^{2*}, Xuemei Ren³ and Phil Barber⁴

1 Faculty of Mechanical & Electrical Engineering, Kunming University of Science & Technology, Kunming 650500, China

2 Department of Mechanical Engineering, University of Bristol, Bristol, BS8 1TR, UK

3 School of Automation, Beijing Institute of Technology, Beijing 100081, China

4 Jaguar & Land Rover Cars Limited, Abbey Road, Whitley, Coventry, CV3 4LF, UK

SUMMARY

This paper studies adaptive parameter estimation and control for nonlinear robotic systems based on parameter estimation errors. A framework to obtain an expression of the parameter estimation error is proposed first by introducing a set of auxiliary filtered variables. Then three novel adaptive laws driven by the estimation error are presented, where exponential error convergence is proved under the conventional persistent excitation (PE) condition; the direct measurement of the time derivatives of the system states are avoided. The adaptive laws are modified via a sliding mode technique to achieve finite-time (FT) convergence, and an online verification of the alternative PE condition is introduced. Leakage terms, functions of the estimation error, are incorporated into the adaptation laws to avoid windup of the adaptation algorithms. The adaptive algorithm applied to robotic systems permits that tracking control and exact parameter estimation are achieved simultaneously in finite time using a terminal sliding mode (TSM) control law. In this case, the PE condition can be replaced with a sufficient richness (SR) requirement of the command signals, and thus is verifiable a priori. The potential singularity problem encountered in TSM controls is remedied by introducing a two-phase control procedure. The robustness of the proposed methods against disturbances is investigated. Simulations based on the ‘Bristol-Elumotion-Robotic-Torso II’ (BERT II) are provided to validate the efficacy of the introduced methods.

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KEY WORDS: adaptive control; parameter estimation, robotic systems; terminal sliding mode control; finite time convergence

1. INTRODUCTION

Adaptive control [1, 2] for control systems with unknown (or immeasurable) parameters and dynamics has been widely studied to achieve output tracking, where the unknown system parameters are online updated/estimated by using control errors. With this framework, asymptotic tracking error convergence and the boundedness of the parameter estimation can be proved. However, it may be questionable to claim that the parameter estimates converge to their true

* E-mail: najing25@163.com (Jing Na), nasiruddin@usm.my (Muhammad Nasiruddin Mahyuddin), g.herrmann@bris.ac.uk (Guido Herrmann)

values due to the lack of an *online* verification of the persistent excitation (PE) condition [1]. To further improve the control performance as well as the parameter estimation, a composite adaptive law has been proposed in [3, 4], where the parameter adaptation is driven by both the tracking error and the prediction error, i.e. an extra predictor has to be designed. In [5], a new composite adaptive control was studied based on the robust integral-of-the sign-of-the-error (RISE) technique for Euler-Lagrange systems with additive uncertainties, where semi-global asymptotic stability is proved. Apart from the aforementioned results that incorporate the parameter estimation into the controller synthesis [1, 3, 6], there are also other methods that take the parameter estimation as a part of an observer design. In [7], the estimation of time-varying parameters was studied via a neural network observer. An adaptive observer was provided in [8] to guarantee arbitrarily fast exponential convergence of both the estimated parameter and states to their actual values. In [9], an identification approach based on variable structure systems (VSS) was developed for multi-input multi-output nonlinear systems. However, the observer design may impose some specific assumptions, e.g. matching condition [8], and it is not always easy to characterize the convergence rate, e.g. finite-time convergence.

It has been well-recognized that adaptive laws should preferably include some information on the parameter estimation error to improve the estimation error convergence [5]. However, this is not a trivial issue since the parameter estimation error is generally immeasurable or unknown. Recently, the desire to include parameter error information into adaptive control designs has resulted in several developments. In [10], a novel parameter estimation scheme was proposed that allows for exact reconstruction of unknown parameters in finite-time (FT). The salient feature of this method lies in that the true parameter can be recovered at any time instant as long as the PE condition is satisfied. In the subsequent work [11], the idea was incorporated into the control design, where the adaptation is driven by combining the tracking error with the parameter error to achieve exponential convergence. It was noted that the algorithm proposed in [10] needs to test online the invertibility of a regressor matrix and to compute the matrix inverse when it is appropriate. Moreover, the introduced auxiliary matrix and vector in [10, 11] have an unstable integrator and thus may increase (even to infinite values), which will result in instability phenomena in the adaptive system. Our recent work [12, 13] proposed novel filter operations to remedy the aforementioned issues (e.g. infinite growth) and developed several adaptive parameter estimation schemes, for which exponential and/or finite-time error convergence are proved without using the derivative of the system states. In particular, the developed adaptations were incorporated into the model reference adaptive control (MRAC) in [12] for a class of nonlinear systems, such that finite-time convergence of both the tracking and parameter estimation can be achieved rather than exponential convergence as in [11].

As a specific kind of nonlinear multi-input-multi-output (MIMO) systems, robotic manipulators [14, 15] have been widely used. Since the 1980s when a linear parameterization of nonlinear robot dynamics was introduced [3], adaptive control of robotic systems has been of long interest, e.g. [3, 14, 16]. Computed-torque based adaptive control [3, 14] has been widely adopted, where global error convergence can be guaranteed. For those robotic systems with unknown nonlinearities or actuator dynamics, function approximators (e.g. neural networks [17-19] or fuzzy systems[20]) have also been utilized. However, what limits the practical applicability of such adaptive controllers is that only ultimate boundedness of the parameter estimation error can be proved. Moreover, some of these control algorithms use the robot joint acceleration measurements (that are susceptible to noises) [21]. To achieve FT error convergence, the principle of terminal sliding mode (TSM) control [22] was extended to robotic systems [23, 24], where the nominal robotic model

needs to be known and the inversion of the inertia matrix needs to be online calculated. To relax the requirement of system model knowledge, neural networks were incorporated into the TSM control design in [25, 26]; however, a potential singularity problem may be encountered in the reaching phase. To avoid the singularity problem in TSM control, a modified TSM manifold was adopted [25, 26], and a two-phase control scheme was introduced in [27] for n^{th} -order SISO systems. However, it is noted that the (finite-time) parameter estimation was not addressed in the aforementioned TSM schemes.

To address these motivating questions, we revisit the adaptive parameter estimation and control design for a class of nonlinear robotic systems with unknown parameters. The parameter estimation is first studied by extending our recent work [12] for a class of general nonlinear systems, and three novel adaptive laws will be presented, which are solely driven by the parameter estimation errors and thus independent of any predictor or observer design. For this purpose, a set of auxiliary system variables are obtained by introducing stable filter operations on the system states, the regressor vector and the input. The parameter error information is thus obtained explicitly and used for the parameter estimation in constructing an adaptive law, where exponential error convergence is guaranteed, provided a filtered (integrated) regressor matrix is positive definite (This can be fulfilled under the conventional PE condition). Furthermore, by applying the sliding mode technique [28] for the adaptation, two improved adaptive laws are proposed with attractive finite-time convergence properties. Another advantage is that the developed estimation approaches avoid the *online* test for the invertibility of the regressor matrix and the direct computation of a matrix inverse in comparison to [10]. In particular, the infinite growth and possible instability of the filtered integral regressor matrix are successfully avoided, which are not necessarily prevented in [10, 11]. Moreover, we can prove that the parameter error converges to zero in finite time. A simplified online verification of a convergence condition is also suggested, which is a relaxed alternative for the usual PE-condition. These parameter estimation schemes are then applied to nonlinear robotic systems, where the torque filtering method proposed in [14, 16] is further improved so that the robot joint acceleration are not required.

The proposed parameter estimation methods for the adaptive control of nonlinear robotic systems can achieve finite-time tracking and parameter estimation simultaneously. In this case, the required PE condition can be transformed into an *a priori* verifiable sufficient richness (SR) requirement on the control reference signals [29], i.e. the rows of the regressor vector are linearly independent along a desired trajectory. In particular, a two-phase control procedure is investigated to avoid the potential singularity problem in the adaptive TSM control design: the first phase is to force the system states to enter a prescribed region in which the singularity does not occur by using a sliding mode control with linear sliding plane and exponential error convergence; the second phase is to switch to a TSM control that realizes finite-time error convergence. Consequently, finite-time convergence of the tracking error and the estimation error can be guaranteed simultaneously. Finally, the robustness of the parameter estimation and the control designs against external disturbances is studied. Simulations based on a humanoid Bristol-Elumotion-Robotic-Torso II (BERT II) arm [16] are conducted to validate the efficacy of the proposed methods. It is shown that the parameter error based adaptation can improve the performance upon the existing adaptive methods, and the proposed control can avoid the singularity problem.

The main contribution of this paper can be summarized as:

- 1) A general continuous-time framework for parameter estimation is proposed to obtain the explicit parameter error

information by using the available system dynamics and the estimated parameters. This can be achieved by introducing stable filter operations on the states, the regressor vector and the input signals.

2) Novel parameter error based adaptive parameter estimation algorithms are investigated for robotic systems; knowledge of the robot joint acceleration is avoided and finite-time convergence can be proved. Moreover, the *online* test of the invertibility of a regressor matrix and the computation of the matrix inverse are not required. A simple online verification of a convergence condition, a relaxed alternative to the usual PE-condition, is provided.

3) A two-phase TSM control is developed to avoid the potential singularity problem in TSM control of robotic systems; finite-time parameter estimation and tracking control are achieved. In this case, the required PE condition can be represented as an *a priori* verifiable SR requirement on the control command signals. The robustness of the proposed methods with disturbances is investigated.

The paper is organized as follows: three adaptive parameter estimation algorithms based on the parameter estimation error are discussed in Section 2. Section 3 is devoted to study the parameter estimation of nonlinear robotic systems; Section 4 proposes a singularity free adaptive TSM control for robotic systems with guaranteed parameter estimation. Simulation results are provided in Section 5 and conclusions are outlined in Section 6.

2. ADAPTIVE PARAMETER ESTIMATION

We first study the parameter estimation for the following nonlinear system

$$\dot{x} = \varphi(x, u) + \Phi(x, u)\theta \quad (1)$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, $\theta \in \mathbb{R}^N$ is the unknown parameter vector to be estimated. $\varphi(x, u) \in \mathbb{R}^n$ is a known function vector and $\Phi(x, u) \in \mathbb{R}^{n \times N}$ is the known regressor matrix. The functions $\varphi(x, u)$ and $\Phi(x, u)$ are bounded for bounded x and u .

To facilitate the parameter estimation, the following assumption is provided:

Assumption 2.1

The system state x and the control input u of system (1) are bounded and accessible for measurement.

Assumption 2.1 has been widely used in the estimation literature [8, 10]. This can be achieved via a control law $u = v(\bullet)$ for (1). Moreover, the state x and input u are required, but the derivative \dot{x} is not used in our estimation schemes. Essential to this paper are the concepts of persistent excitation in combination with finite time convergence. Both concepts are here introduced separately:

Definition 2.1 [2]

A vector or matrix function ϕ is persistently excited (PE) if there exist $T > 0, \varepsilon > 0$ such that $\int_t^{t+T} \phi(r)\phi^T(r)dr \geq \varepsilon I, \forall t \geq 0$. \diamond

Lemma 2.1[30]

For a continuous system $\dot{x} = \phi(x, t)$, $\phi(0, t) = 0, x \in \mathbb{R}^n$, there is a continuously differentiable positive-definite function $V(x, t)$ and real numbers $c_1 > 0, 0 < c_2 < 1$ such that $\dot{V}(x, t) \leq -c_1 V^{c_2}(x, t)$ holds, then $V(x, t)$ converges to zero in finite time with the settling time $T \leq \frac{1}{c_1(1-c_2)} V^{1-c_2}(x(t_0), t_0)$ for any given initial condition $x(t_0)$. \diamond

Unlike available results, e.g. [7, 8, 10], which are dedicated to estimate unknown parameters θ of (1) by means of observers or predictors, the parameter estimation methods to be presented are independent of any observer and

predictor design. In order to derive the parameter estimation, we define the filtered variables x_f, Φ_f, φ_f of $x, \varphi(x)$ and $\Phi(x, u)$ as

$$\begin{cases} k\dot{x}_f + x_f = x, & x_f(0) = 0 \\ k\dot{\Phi}_f + \Phi_f = \Phi, & \Phi_f(0) = 0 \\ k\dot{\varphi}_f + \varphi_f = \varphi, & \varphi_f(0) = 0 \end{cases} \quad (2)$$

where $k > 0$ is a filter parameter. Then it can be obtained from (1) and (2) that

$$\dot{x}_f = \frac{x - x_f}{k} = \varphi_f + \Phi_f \theta \quad (3)$$

Define an auxiliary filtered and ‘integrated’ regressor matrix P and vector Q as

$$\begin{cases} \dot{P} = -\ell P + \Phi_f^T \Phi_f, & P(0) = 0 \\ \dot{Q} = -\ell Q + \Phi_f^T [(x - x_f)/k - \varphi_f], & Q(0) = 0 \end{cases} \quad (4)$$

where $\ell > 0$ is another design parameter. Note that the authors [10] used a formulation similar to (4) but without the terms ℓP and ℓQ , which creates for P and Q unbounded integration operations. This is circumvented in this paper.

The solution of (4) is derived as

$$\begin{cases} P(t) = \int_0^t e^{-\ell(t-r)} \Phi_f^T(r) \Phi_f(r) dr \\ Q(t) = \int_0^t e^{-\ell(t-r)} \Phi_f^T(r) [(x(r) - x_f(r))/k - \varphi_f(r)] dr \end{cases} \quad (5)$$

We now define another auxiliary vector $W \in \mathbb{R}^N$ that can be calculated from P, Q as

$$W = P\hat{\theta} - Q \quad (6)$$

where $\hat{\theta}$ is an estimation for the unknown parameter θ . Different adaptation laws for $\hat{\theta}$ will be given in the following developments.

From (3) and (5), one can verify that $Q = P\theta$ holds such that

$$W = P\hat{\theta} - Q = P\hat{\theta} - P\theta = -P\tilde{\theta} \quad (7)$$

where $\tilde{\theta} = \theta - \hat{\theta}$ is the parameter estimation error.

The positive definite property of matrix P (i.e. $\lambda_{\min}(P(t)) > \sigma > 0$)¹ is important for parameter estimation. We will prove that this condition can be fulfilled provided the original regressor vector $\Phi(x, u)$ in (1) is PE.

Lemma 2.2 [12]

The matrix P is positive definite satisfying $\lambda_{\min}(P(t)) > \sigma$ for $t > T$ and some $\sigma > 0, T > 0$, provided the regressor matrix $\Phi(x, u)$ is persistently excited. \diamond

Proof

It is shown that the transfer function $1/(ks+1)$ in (2) is stable, minimum phase and strictly proper [2], then Φ_f defined in (2) is PE once Φ defined in (1) is PE because Φ_f is the filtered version of Φ . Moreover, based on Definition 2.1, if Φ_f is PE, there exist $T > 0$ and $\varepsilon > 0$ so that the inequality $\int_t^{t+T} \Phi_f^T(r) \Phi_f(r) dr > \varepsilon I$ holds for all $t > 0$. Since the inequality $\int_0^t e^{-\ell(t-r)} \Phi_f^T(r) \Phi_f(r) dr > e^{-\ell T} \int_{t-T}^t \Phi_f^T(r) \Phi_f(r) dr > e^{-\ell T} \varepsilon I$ holds for $t > T$, then $\lambda_{\min}(P(t)) > \sigma > 0$ holds with $\sigma = e^{-\ell T} \varepsilon$, i.e. P is positive definite. \square

According to Lemma 2.2, the conventional PE condition of the regressor vector Φ is sufficient to guarantee that

¹ Throughout this paper, $\lambda_{\max}(\cdot), \lambda_{\min}(\cdot)$ are defined as the maximum and minimum eigenvalues of the corresponding matrices.

P is positive definite. Thus, similar to other system identification and parameter estimation literature (e.g. [8, 10]), P can be retained positive definite by imposing a suitable control u and/or a dither PE signal on system (1). However, the *direct online* validation of the PE condition is difficult in particular for a nonlinear system. Thus, it will be shown that testing for the positive definiteness of P permits to numerically verify conditions of convergence of the adaptive algorithms suggested in this paper. This replaces tests for PE in the case of the suggested adaptation algorithms. Another alternative convergence condition will be provided in a later part of the paper.

From the fact $Q = P\theta$ and the assumption that P is positive definite follows easily the following:

Corollary 2.1

Assume system (1) with P, Q in (5) satisfies $\lambda_{\min}(P(t)) > \sigma > 0$ for all $t > 0$, then it follows $\theta = P^{-1}Q$. \diamond

Although Corollary 2.1 provides a possible parameter estimation, where the inverse matrix P^{-1} needs to be *online* calculated as in [10, 11], which may be difficult since it may lead to numerical problems. In the following development, we will present three online parameter estimation schemes based on the parameter error information W .

2.1 ADAPTIVE PARAMETER ESTIMATION

We first propose an adaptive law to estimate θ with exponential error convergence. The adaptive estimation algorithm for $\hat{\theta}$ is provided by

$$\dot{\hat{\theta}} = -\Gamma W \quad (8)$$

with $\Gamma > 0$ being a constant diagonal gain matrix.

Then the following result holds:

Theorem 2.1

Consider system (1) with the parameter adaptive law (8), if the filtered regressor matrix P satisfies $\lambda_{\min}(P) > \sigma > 0$, then the parameter error $\tilde{\theta}$ exponentially converges to zero with the convergence rate $\mu_1 = 2\sigma / \lambda_{\max}(\Gamma^{-1})$. \diamond

Proof

Consider the Lyapunov function as $V_1 = \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$, the derivative \dot{V}_1 along (8) is obtained as

$$\dot{V}_1 = \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} = \tilde{\theta}^T W = -\tilde{\theta}^T P \tilde{\theta} \leq -\mu_1 V_1 \quad (9)$$

where $\mu_1 = 2\sigma / \lambda_{\max}(\Gamma^{-1})$ is positive for all $t > 0$. Based on Lyapunov's Theorem [31] and (9), the error $\tilde{\theta}$ converges exponentially to zero with the rate μ_1 . \square

Remark 2.1

It is known that the inclusion of an appropriate parameter error in the parameter adaptations can improve the parameter estimation performance [1]. In this paper, W derived in (7) contains the parameter error information $P\tilde{\theta}$, which is used explicitly in the parameter adaptation; no observer or predictor design is introduced in comparison to available results [3, 7, 8, 10]. Moreover, unlike [10, 11], the matrix P and vector Q are all bounded by introducing a forgetting factor $\ell > 0$.

2.2 FINITE-TIME PARAMETER ESTIMATION

We further improve the adaptive law to achieve finite-time error convergence. The variables P , Q and W are designed as in (4)~(6), then the parameter $\hat{\theta}$ is updated by

$$\dot{\hat{\theta}} = -\Gamma \frac{P^T W}{\|W\|} \quad (10)$$

where $\Gamma > 0$ is a constant diagonal gain matrix. This leads to the following result:

Theorem 2.2

For system (1) with parameter adaptation (10) and $\lambda_{\min}(P) > \sigma > 0$, the parameter estimation error $\tilde{\theta}$ converges to zero in finite time t_a satisfying $t_a \leq \|\tilde{\theta}(0)\| \lambda_{\max}(\Gamma^{-1}) / \sigma$. \diamond

Proof

Consider the Lyapunov function as $V_2 = \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$, then it can be verified using $\lambda_{\min}(P) > \sigma > 0$ and $W = -P\tilde{\theta}$ that the derivative of V_2 can be derived as

$$\dot{V}_2 = \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} = -\tilde{\theta}^T \frac{P^T W}{\|P\hat{\theta} - Q\|} = -\frac{\tilde{\theta}^T P^T P \tilde{\theta}}{\|P\tilde{\theta}\|} \leq -\|P\tilde{\theta}\| \leq -\mu_2 \sqrt{V_2} \quad (11)$$

where $\mu_2 = \sigma \sqrt{2 / \lambda_{\max}(\Gamma^{-1})}$ is a positive constant. According to (11) and [28], finite-time convergence of the parameter estimation error $\tilde{\theta}$ can be guaranteed, and the time to achieve $\tilde{\theta} = 0$ is estimated by $t_a \leq 2\sqrt{V_2(0)} / \mu_2$. Clearly, the convergence time t_a depends on the excitation level σ and the learning gain Γ , i.e. a higher excitation level σ and larger learning gain Γ will lead to a smaller time t_a . \square

2.3 ONLINE LEARNING OF THE MATRIX INVERSE OF P

As shown above, a possible parameter estimation can be given by $P^{-1}Q = \theta$, where the inverse matrix P^{-1} needs to be calculated. In this subsection, we will propose an alternative adaptive law to learn the inverse of P *online*. Define an auxiliary matrix K as

$$\dot{K} = \ell K - K \Phi_f^T \Phi_f K, \quad K^{-1}(0) = K_0 > 0 \quad (12)$$

where $\ell > 0$ is the positive constant used in (4), and Φ_f is the filtered regressor given in (2). Consider the matrix equality $\frac{d}{dt} K K^{-1} = \dot{K} K^{-1} + K \frac{d}{dt} K^{-1} = 0$, we derive the solution of (12) as

$$K(t) = [e^{-\ell t} K_0 + \int_0^t e^{-\ell(t-r)} \Phi_f^T(r) \Phi_f(r) dr]^{-1} = [e^{-\ell t} K_0 + P(t)]^{-1} \quad (13)$$

where $K_0 = K_0^T = K^{-1}(0) > 0$ is the initial condition chosen as $K_0 = \eta I$ with $\eta > 0$ being a constant.

Furthermore, one can employ the singular value decomposition (SVD) for matrix P as

$$P = \int_0^t e^{-\ell(t-r)} \Phi_f^T(r) \Phi_f(r) dr = U S V^T \quad (14)$$

where $S = \text{diag}(s_1, \dots, s_n)$ is a matrix with s_i being the singular values of matrix P , and U, V are unitary matrices. Then based on the fact that $K_0 = \eta I$ is a diagonal matrix and (13), it follows

$$K = [e^{-\ell t} K_0 + P]^{-1} = [U(S + e^{-\ell t} \eta I) V^T]^{-1} = V(S + e^{-\ell t} \eta I)^{-1} U^T \quad (15)$$

Consequently, the matrix KP is derived as

$$KP = V(S + e^{-\ell t} \eta I)^{-1} S V^T = V \text{diag}\left(\frac{s_1}{s_1 + e^{-\ell t} \eta}, \dots, \frac{s_n}{s_n + e^{-\ell t} \eta}\right) V^T \quad (16)$$

Since $\lim_{t \rightarrow \infty} \frac{s_i}{s_i + e^{-\ell t} \eta} = 1$ if $\lambda_{\min}(P) > \sigma > 0$ and $\ell, \eta > 0$, and $V V^T = I$, the matrix KP is represented as

$$K(t)P(t) = I - E(t) \quad (17)$$

where $E(t) = V \text{diag}\left(\frac{e^{-\ell t} \eta}{s_1 + e^{-\ell t} \eta}, \dots, \frac{e^{-\ell t} \eta}{s_n + e^{-\ell t} \eta}\right) V^T$ converges to zero as $t \rightarrow \infty$, i.e. $\lim_{t \rightarrow \infty} E(t) = 0$. Then we have

$$\tilde{\theta} = \theta - \hat{\theta} = [KP + E]\theta - \hat{\theta} \quad (18)$$

An alternative parameter adaptive law can be given as

$$\dot{\hat{\theta}} = -\Gamma \frac{\hat{\theta} - KQ}{\|\hat{\theta} - KQ\|}. \quad (19)$$

Theorem 2.3

For system (1) with the adaptive parameter estimation $\hat{\theta}$ given by (19) (based on Q in (4) and K in (12) for $K^{-1}(0) = \eta I > 0$) and $\lambda_{\min}(P) > \sigma > 0$, then the estimation error $\tilde{\theta}$ converges to a residual set $\|\tilde{\theta}\| \leq c$ in finite time, where $c > 0$ is a positive constant. \diamond

Proof

Select a Lyapunov function as $V_3 = \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$, then according to (19) and the definition of P and Q in (4), we can calculate the derivative of V_3 as

$$\dot{V}_3 = -(\hat{\theta} - KP\theta - E\theta)^T \frac{\hat{\theta} - KQ}{\|\hat{\theta} - KQ\|} \leq -\|E\theta - \tilde{\theta}\| + \|E\theta\| \leq -\mu_3 \sqrt{V_3} + \rho_3 \quad (20)$$

where $\mu_3 = \sqrt{2/\lambda_{\max}(\Gamma^{-1})}$ is a positive constant, $\rho_3 = 2\|E\|\|\theta\|$ denotes the effects of the initial condition K_0 . From $\lambda_{\min}(P) > \sigma > 0$ and the discussion of equation (17), it follows $\lim_{t \rightarrow \infty} E(t) = 0$. Although the fact $\lim_{t \rightarrow \infty} E(t) = 0$ and thus $\lim_{t \rightarrow \infty} \rho_3(t) = 0$ holds and θ is bounded, ρ_3 is usually not zero in finite time since the initial condition cannot be set as zero [4], i.e. $\eta \neq 0$. Consequently, unlike Theorem 2.2, it cannot be claimed that $\tilde{\theta}$ converges to zero in finite time. However, ρ_3 is bounded for all $t \geq 0$ since E and θ are bounded, i.e. there exists $c > 0$ so that $|\rho_3| < \gamma c$ holds with $0 < \gamma < 1$. Certainly, for sufficiently large time $t > 0$, the constant $c > 0$ will be smaller. Define a compact set as $\Omega := \{\tilde{\theta} \mid V_3(\tilde{\theta}) \leq (c/\mu_3)^2\}$, then for all $\tilde{\theta}(t) \notin \Omega$, we have $\frac{1}{2} \lambda_{\max}(\Gamma^{-1}) \|\tilde{\theta}\|^2 \geq V_3(\tilde{\theta}) > (c/\mu_3)^2$ and thus $\|\tilde{\theta}\| > c$. In this case, it follows

$$\dot{V}_3 \leq -\mu_3 \sqrt{V_3} + \rho_3 \leq -c + \rho_3 < -(1-\gamma)c < 0 \quad (21)$$

Consequently, the parameter error $\tilde{\theta}$ will ultimately enter the compact set Ω within finite time. The size of the compact set can be adjusted to be arbitrarily small by tuning c small, e.g. reduction of the initial condition η of the matrix $K^{-1}(0)$. The convergence rate can also be improved by increasing the learning matrix gain Γ . \square

A clearly simple approach for parameter estimation is given by the following Corollary:

Corollary 2.2

For system (1) with Q defined in (4) and K defined in (12), it follows $\lim_{t \rightarrow \infty} KQ = \theta$ if the regressor Φ is PE. \diamond

The proof follows from the fact that $P^{-1}Q = \theta$ and $KP \rightarrow I$ as $t \rightarrow +\infty$.

According to (17), it is shown that K converges to P^{-1} due to $\lim_{t \rightarrow \infty} E(t) = 0$. This is clearly different to the results presented in [10], and thus the online check for the invertibility and the computation of the inverse of matrix P (when it is appropriate) can be avoided. However, the estimation error in this case only converges in finite time to a small bounded region around zero due to the initial condition $K^{-1}(0) = \eta I \neq 0$. In fact, with the help of the auxiliary matrix K , a practical test for the convergence condition of the algorithm of (19) or of Corollary 2.2 is carried out by verifying $KP \approx I$, implying non-singularity of P . This can be conducted *online* as shown in Section 5 for a simulation example.

Remark 2.2

From equation (17), the condition of $\lim_{t \rightarrow \infty} K(t)P(t) = I$ and $\lim_{t \rightarrow \infty} E(t) = 0$ are also satisfied if there are some constants

$\sigma_1 > 0$ and $\sigma_2 > 0$ so that $\lambda_{\min}(P) > \sigma_1 e^{-(\ell - \sigma_2)t} > 0$. In particular for $\ell > \sigma_2 > 0$, this permits an exponentially decaying value of $\lambda_{\min}(P)$, implying also a different concept of excitation of $\Phi(x, u)$. All of the adaptation laws of (8), (10) and (19) hold an inherent forgetting mechanism, which avoids the windup of the adaptation mechanism [1]. This is achieved as W is an expression of the parameter error $\tilde{\theta}$.

3. PARAMETER ESTIMATION OF ROBOTIC SYSTEM

In this section, we will apply the proposed adaptive parameter estimation algorithms for a class of robotic systems. For this purpose, appropriate filter operations will be introduced so that the measurements of the robot joint accelerations are avoided.

We consider a n -degrees of freedom (DOF) nonlinear robotic manipulator modeled by

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau \quad (22)$$

where $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$ are the robot joint position, velocity and acceleration, respectively; $\tau \in \mathbb{R}^n$ is the control torque/force vector; $M(q) \in \mathbb{R}^{n \times n}$ is the inertia matrix, $C(q, \dot{q}) \in \mathbb{R}^{n \times n}$ is the Coriolis/centripetal torque, viscous and nonlinear damping, and $G(q) \in \mathbb{R}^n$ represents the gravity torque.

The following properties are given [3, 14] for robotic system (22):

Property 3.1

The matrix $\dot{M}(q) - 2C(q, \dot{q})$ is skew-symmetric, such that

$$x^T [\dot{M}(q) - 2C(q, \dot{q})] x = 0, \quad \forall x, q, \dot{q} \in \mathbb{R}^n \quad (23)$$

Property 3.2

The left-hand dynamics of the robotic system (22) can be represented in a linearly parameterized form

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \phi(q, \dot{q}, \ddot{q})\theta \quad (24)$$

where $\theta \in \mathbb{R}^N$ is a constant parameter vector to be estimated, and $\phi(q, \dot{q}, \ddot{q}) \in \mathbb{R}^{n \times N}$ is the known regressor matrix.

Property 3.3

The matrix $M(q)$ is positive definite and satisfies $\|M(q)\| \leq \lambda_M$ for positive constant $\lambda_M > 0$.²

3.1 ADAPTIVE PARAMETER ESTIMATION

The regression matrix $\phi(q, \dot{q}, \ddot{q})$ in (24) is a function of the robot joint acceleration measurement \ddot{q} , which is sensitive to measurement noises. In this paper, as inspired by [14, 16], this regressor matrix will be reformulated, where auxiliary filtered variables are introduced to eradicate the need for the joint acceleration. Define alternative vectors as $F(q, \dot{q}) = M(q)\dot{q}$ and $H(q, \dot{q}) = -\dot{M}(q)\dot{q} + C(q, \dot{q})\dot{q} + G(q)$, then based on Property 3.2, one can derive

$$\begin{cases} F(q, \dot{q}) = M(q)\dot{q} = \phi_1(q, \dot{q})\theta \\ H(q, \dot{q}) = -\dot{M}(q)\dot{q} + C(q, \dot{q})\dot{q} + G(q) = \phi_2(q, \dot{q})\theta \end{cases} \quad (25)$$

where $\phi_1(q, \dot{q}), \phi_2(q, \dot{q}) \in \mathbb{R}^{n \times N}$ are new regressor matrices without the joint acceleration \ddot{q} .

Consequently, system (22) can be represented as

$$\dot{F}(q, \dot{q}) + H(q, \dot{q}) = \tau \quad (26)$$

² This slightly limits the class of rigid body robotic systems. In general, any robot with rotational degrees of freedom satisfies this constraint. However, in cases where an arm length can increase due to a translational motion, there is the potential for increase in inertia, should this arm also be used for rotational motion. In such cases, we may limit ourselves to local or semi-global stability concepts.

where $\dot{F}(q, \dot{q}) = \frac{d}{dt}[M(q)\dot{q}] = \dot{\phi}_1(q, \dot{q})\theta$.

To obtain $\dot{F}(q, \dot{q})$ (and thus $\dot{\phi}_1(q, \dot{q})$) without using the joint acceleration \ddot{q} , as [12, 16], we introduce stable, linear filter operations $(\bullet)_f \equiv \frac{1}{ks+1}(\bullet), k > 0$ on both sides of (26), such that

$$\dot{F}_f(q, \dot{q}) + H_f(q, \dot{q}) = [\dot{\phi}_{1f}(q, \dot{q}) + \phi_{2f}(q, \dot{q})]\theta = \tau_f \quad (27)$$

where $\phi_{1f}(q, \dot{q}) \in \mathbb{R}^{n \times N}$, $\phi_{2f}(q, \dot{q}) \in \mathbb{R}^{n \times N}$ and $\tau_f \in \mathbb{R}^n$ are the filtered version of $\phi_1(q, \dot{q})$, $\phi_2(q, \dot{q})$ and τ , respectively:

$$\begin{cases} k\dot{\phi}_{1f}(q, \dot{q}) + \phi_{1f}(q, \dot{q}) = \phi_1(q, \dot{q}), & \phi_{1f}(q, \dot{q})|_{t=0} = 0 \\ k\dot{\phi}_{2f}(q, \dot{q}) + \phi_{2f}(q, \dot{q}) = \phi_2(q, \dot{q}), & \phi_{2f}(q, \dot{q})|_{t=0} = 0 \\ k\dot{\tau}_f + \tau_f = \tau, & \tau_f|_{t=0} = 0 \end{cases} \quad (28)$$

Then substituting the first equation of (28) into (27), one can obtain

$$\left[\frac{\phi_1(q, \dot{q}) - \phi_{1f}(q, \dot{q})}{k} + \phi_{2f}(q, \dot{q}) \right] \theta = \phi_f(q, \dot{q})\theta = \tau_f \quad (29)$$

where $\phi_f(q, \dot{q}) = \frac{\phi_1(q, \dot{q}) - \phi_{1f}(q, \dot{q})}{k} + \phi_{2f}(q, \dot{q}) \in \mathbb{R}^{n \times N}$ is the new regressor matrix, which will be used for the parameter estimation. Note that $\phi_f(q, \dot{q})$ can be obtained based on (28) and (29), which is now the function of q, \dot{q} but not \ddot{q} . Thus, the accelerations required in the original regressor matrix $\phi(q, \dot{q}, \ddot{q})$ in (24) are successfully avoided.

To accommodate the parameter estimation, we define matrix $P_1 \in \mathbb{R}^{N \times N}$ and vector $Q_1 \in \mathbb{R}^N$ as

$$\begin{cases} \dot{P}_1 = -\ell P_1 + \phi_f^T \phi_f, & P_1(0) = 0 \\ \dot{Q}_1 = -\ell Q_1 + \phi_f^T \tau_f, & Q_1(0) = 0 \end{cases} \quad (30)$$

where $\ell > 0$ is a design parameter.

Define an auxiliary vector $W_1 \in \mathbb{R}^N$ that can be computed from P_1, Q_1 in (30) as

$$W_1 = P_1 \hat{\theta} - Q_1 \quad (31)$$

Then similar to Section 2, from (29) and (30), one can verify that $Q_1 = P_1 \theta$ holds such that

$$W_1 = P_1 \hat{\theta} - P_1 \theta = -P_1 \tilde{\theta} \quad (32)$$

In this section, the following assumption is also used:

Assumption 3.1

The control torque/force τ is chosen so that the regressor matrix $\phi(q, \dot{q}, \ddot{q})$ is PE and the tuple (q, \dot{q}, \ddot{q}) is bounded.

Remark 3.1

According to (29), the new regressor matrix $\phi_f(q, \dot{q})$ can be taken as the filtered version of the original regressor matrix $\phi(q, \dot{q}, \ddot{q})$ [16]. Then similar to Lemma 2.2, the PE condition of $\phi(q, \dot{q}, \ddot{q})$ (e.g. Assumption 3.1) is sufficient to guarantee that P_1 is positive definite and $\lambda_{\min}(P_1) > \sigma > 0$. This Assumption can be obtained by imposing a dither PE signal [16] on the control torque τ of (22). In Section 3, this condition will be further reduced to an *a priori* verifiable SR requirement on the reference demand [29], when the parameter estimation is incorporated into a control design.

The first adaptive law for $\hat{\theta}$ in system (22) is provided by

$$\dot{\hat{\theta}} = -\Gamma W_1 \quad (33)$$

where $\Gamma > 0$ is a constant learning gain matrix. The respective analysis of this adaptation law is summarized below:

Corollary 3.1

Consider the robotic system (22) with adaptive law (33) and Assumption 3.1, then the estimation error $\tilde{\theta}$ exponentially converges to zero with the convergence rate $\mu_1 = 2\sigma / \lambda_{\max}(\Gamma^{-1})$. \diamond

The second adaptive law for $\hat{\theta}$ in system (22) is given by

$$\dot{\hat{\theta}} = -\Gamma \frac{P_1^T W_1}{\|W_1\|} \quad (34)$$

where $\Gamma > 0$ is a constant learning gain matrix. Thus, it follows for this finite-time adaptation algorithm:

Corollary 3.2

Consider the robotic system (22) with parameter estimation (34) and Assumption 3.1, then the estimation error $\tilde{\theta}$ converges to zero in finite time t_a , satisfying $t_a \leq \|\tilde{\theta}(0)\| \lambda_{\max}(\Gamma^{-1}) / \sigma$. \diamond

The third parameter estimation algorithm for $\hat{\theta}$ can be presented as

$$\dot{\hat{\theta}} = -\Gamma \frac{\hat{\theta} - K_1 Q_1}{\|\hat{\theta} - K_1 Q_1\|} \quad (35)$$

where the auxiliary matrix K_1 is given by

$$\dot{K}_1 = \ell K_1 - K_1 \phi_f^T \phi_f K_1, \quad \text{with } K_1^{-1}(0) = \eta I > 0 \quad (36)$$

with $\ell > 0$ being the positive constant used in (30) and ϕ_f is the filtered regressor given in (29). Hence,

Corollary 3.3

Consider the robotic system (22) with parameter estimation (35) (Q_1 in (30) and K_1 in (36)) and Assumption 3.1, then the estimation error $\tilde{\theta}$ converges to a residual set $\|\tilde{\theta}\| \leq c$ in finite time for a small constant $c > 0$. \diamond

The proofs of above Corollaries are similar to those of Theorem 2.1, Theorem 2.2 and Theorem 2.3. Moreover, in the aforementioned parameter estimation schemes for the robotic system (22), we introduce alternative vectors $F(q, \dot{q})$, $H(q, \dot{q})$, and the associated filter operation (27)~(29). Consequently, the robotic joint acceleration measurements are avoided in the new regressor matrix formulations, which is practically useful. Nevertheless, similar to Corollary 2.2, we can also estimate the parameters by using the fact $\lim_{t \rightarrow \infty} K_1 Q_1 = \theta$.

3.2 ADAPTIVE PARAMETER ESTIMATION WITH DISTURBANCE

In this section, the robustness of the proposed estimation algorithms against disturbances or noises is studied. In this case, the studied robotic system is presented as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau + \xi \quad (37)$$

where $\xi \in \mathbb{R}^n$ is a bounded disturbance vector, i.e. $\|\xi\| \leq \varepsilon_\xi$, $\varepsilon_\xi > 0$. It is assumed that Assumption 3.1 holds again.

The filter operations (28)~(29) and auxiliary variables P_1 , Q_1 and W_1 in (30) ~ (31) can be redefined for system (37), while the following virtual variable ξ_f (only used for analysis) is introduced

$$k\dot{\xi}_f + \xi_f = \xi, \quad \xi_f(0) = 0 \quad (38)$$

Then according to (28) and (38), one can obtain

$$\phi_f(q, \dot{q})\theta = \tau_f + \xi_f \quad (39)$$

In this case, the parameter error information W_1 is rewritten as

$$W_1 = P_1 \hat{\theta} - Q_1 = -P_1 \tilde{\theta} + \psi \quad (40)$$

where $\psi = -\int_0^t e^{-\ell(t-r)} \phi_f^T(r) \xi_f(r) dr$ is bounded since (q, \dot{q}) is assumed to be bounded, implying ϕ and ϕ_f to be bounded. Thus, $\|\psi\| \leq \varepsilon_\psi$ for a constant $\varepsilon_\psi > 0$. These definitions now allow to prove the robustness of the introduced adaptation laws:

Lemma 3.1

For system (37) with parameter estimation (33) and Assumption 3.1, then $\tilde{\theta}$ is uniformly ultimately bounded (UUB). \diamond
Proof

Consider the Lyapunov function as $V_4 = \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$, then from (40), the derivative \dot{V}_4 is computed as

$$\dot{V}_4 = \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} = -\tilde{\theta}^T P_1 \tilde{\theta} + \tilde{\theta}^T \psi \leq -\|\tilde{\theta}\|(\sigma \|\tilde{\theta}\| - \|\psi\|) \leq -\|\tilde{\theta}\|(\mu_4 \sqrt{V_4} - \varepsilon_\psi) \quad (41)$$

where $\mu_4 = \sigma \sqrt{2 / \lambda_{\max}(\Gamma^{-1})}$ is a positive constant. Then according to the definition of UUB and the extended Lyapunov's Theorem [31], the estimation error $\tilde{\theta}$ ultimately converges to a compact set $\Omega := \{\tilde{\theta} \mid \sqrt{V_4} \leq \varepsilon_\psi / \mu_4\}$, of which the size depends on the bound of the disturbance ξ and the excitation level. \square

Lemma 3.2

For system (37) with parameter estimation (34) and Assumption 3.1, then

- i) the parameter estimation $\hat{\theta}$ is bounded;
- ii) the parameter error $\tilde{\theta}$ converges to a compact set in finite time satisfying $\lim_{t \rightarrow \infty} P_1 \tilde{\theta} = \psi$. \diamond

Proof

- i) The derivative of $P_1^{-1} W_1$ with respect to time is investigated. Consider the fact $P_1^{-1} W_1 = -\tilde{\theta} + P_1^{-1} \psi$, it follows

$$\frac{\partial P_1^{-1} W_1}{\partial t} = -\dot{\tilde{\theta}} + \frac{\partial P_1^{-1}}{\partial t} \psi + P_1^{-1} \dot{\psi} = \dot{\tilde{\theta}} - P_1^{-1} \dot{P}_1 P_1^{-1} \psi + P_1^{-1} \dot{\psi} = \dot{\tilde{\theta}} + \psi' \quad (42)$$

where ψ' is $\psi' = -P_1^{-1} \dot{P}_1 P_1^{-1} \psi + P_1^{-1} \dot{\psi}$.

Select the Lyapunov function as $V_5 = \frac{1}{2} W_1^T P_1^{-1} P_1^{-1} W_1$, then

$$\dot{V}_5 = W_1^T P_1^{-1} \frac{\partial P_1^{-1} W_1}{\partial t} = -W_1^T P_1^{-1} \Gamma \frac{P_1^T W_1}{\|W_1\|} + W_1^T P_1^{-1} \psi' \leq -(\lambda_{\min}(\Gamma) - \|P_1^{-1} \psi'\|) \|W_1\| \quad (43)$$

We now analyze the particular term $\|P_1^{-1} \psi'\|$. Consider $\psi = -\int_0^t e^{-\ell(t-r)} \phi_f^T(r) \xi_f(r) dr$, it can be verified that ψ and $\dot{\psi}$ are bounded as long as ξ and $\phi_f(q, \dot{q})$ are bounded. The matrices P_1 and \dot{P}_1 are also bounded for bounded $\phi_f(q, \dot{q})$. Moreover, the PE condition $\lambda_{\min}(P_1) > \sigma > 0$ implies that P_1^{-1} is bounded in magnitude. Thus, assuming a bounded disturbance ξ , the term $\|P_1^{-1} \psi'\|$ is bounded. Then, for large enough $\Gamma > 0$ (i.e. $\lambda_{\min}(\Gamma) > \|P_1^{-1} \psi'\|$), the boundedness of W_1 readily follows from (43), which implies $\tilde{\theta}$ and $\hat{\theta}$ are bounded.

- ii) To further analyze the error bound, we rewrite (43) as $\dot{V}_5 \leq -\mu_5 \sqrt{V_5}$, where $\mu_5 = (\lambda_{\min}(\Gamma) - \|P_1^{-1} \psi'\|) \sigma \sqrt{2}$ is a positive scalar, chosen larger than a pre-specified constant. Then according to [28], it follows that $\lim_{t \rightarrow \infty} V_5 = 0$ and thus $\lim_{t \rightarrow \infty} W_1 = 0$ holds in finite time $t_a \leq 2\sqrt{V_5(0)} / \mu_5$. This guarantees the error converges to $\lim_{t \rightarrow \infty} P_1 \tilde{\theta} = \psi$. \square

Lemma 3.3

For system (37) with adaptation (35) and Assumption 3.1, then $\tilde{\theta}$ converges to a set around zero in finite time. \diamond

Proof

Consider the Lyapunov function as $V_6 = \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$, then the derivative \dot{V}_6 along (35) and (40) is obtained as

$$\begin{aligned} \dot{V}_6 &= -\left(\hat{\theta} - K_1(Q_1 + \psi) - E\theta\right)^T \frac{\hat{\theta} - K_1 Q_1}{\|\hat{\theta} - K_1 Q_1\|} \leq -\|E\theta - \tilde{\theta} + K_1 \psi\| + \|E\theta + K_1 \psi\| \\ &\leq -\|\tilde{\theta}\| + 2\|E\theta + K_1 \psi\| \leq -\mu_6 \sqrt{V_6} + \rho_6 \end{aligned} \quad (44)$$

where $\mu_6 = \sqrt{2/\lambda_{\max}(\Gamma^{-1})}$ is a positive constant, and $\rho_6 = 2\|E\theta + K_1 \psi\|$ denotes the effects of the initial condition K_0 and the disturbance ξ , which is bounded. Then following the argument in the proof of Theorem 2.3, we can show that the parameter error $\tilde{\theta}$ will converge to a compact set around zero in finite time. The size of the compact set depends on the initial condition K_0 , the learning gain Γ and the amplitude of the disturbance ξ . \square

As stated in the above Lemmas, the proposed estimation methods (33)~(35) for robotic systems are robust against bounded disturbances, i.e. the estimated parameter converges to a small compact set around its true value. In particular, the adaptive law (34) allows the computation of explicit bounded limit values $\lim_{t \rightarrow \infty} P_1 \tilde{\theta} = \psi$.

4. ADAPTIVE FINITE-TIME CONTROL FOR ROBOTIC SYSTEM WITH GUARANTEED PARAMETER ESTIMATION

In this section, we will incorporate the estimation algorithm (10) into the control design for the robotic system (22) to achieve FT tracking control and parameter estimation. In this case, the PE condition required in Section 3 is further reduced to an *a priori* verifiable SR requirement [2, 29] on the closed-loop demand signal, which can be given as:

Assumption 4.1

The demand reference q_d for robotic system (22) is bounded, continuously differentiable and sufficiently rich [2] with respect to $\Phi_1(q_d, \dot{q}_d)$ defined in (56), i.e. for finite interval $[t, t+T]$ with $T > 0$, i.e. there exist at least N time instances t_i so that

$$\Psi_1(q_d(\bullet)) = \left[\Phi_1^T(q_d(t_1)), \Phi_1^T(q_d(t_2)), \dots, \Phi_1^T(q_d(t_N)) \right]^T \in \mathbb{R}^{nN \times N}$$

is of rank N and there is a finite constant $\delta > 0$ so that the following linear matrix inequality holds:

$$\Psi_1^T(q_d(\bullet)) \Psi_1(q_d(\bullet)) \geq \delta I \quad (45)$$

This further implies that $\Phi_1(q_d, \dot{q}_d)$ in (56) satisfies for the demand reference q_d a PE-type condition, i.e. for some $\sigma_1 > 0$

$$\int_t^{t+T} \Phi_1^T(q_d(r), \dot{q}_d(r)) \Phi_1(q_d(r), \dot{q}_d(r)) dr > \sigma_1 I \quad (46)$$

4.1 ADAPTIVE FINITE-TIME CONTROL AND ESTIMATION

To facilitate the control design, we define the error as $e = q_d - q = [e_1, \dots, e_n]^T$ and a terminal sliding mode surface $S_1 \in \mathbb{R}^n$ as

$$S_1 = \dot{e} + \lambda e^p = \dot{q}_{r1} - \dot{q} \quad (47)$$

where $e^p = [|e_1|^p \text{sign}(e_1), \dots, |e_n|^p \text{sign}(e_n)]^T \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^{n \times n}$ is a positive definite and diagonal parameter matrix, $p = p_1 / p_2$ is a positive constant with p_1, p_2 being positive odd integers satisfying $p_2 > p_1$, and the auxiliary variables \dot{q}_{r1} and its derivative \ddot{q}_{r1} can be presented as

$$\begin{aligned}\dot{q}_{r1} &= \dot{q}_d + \lambda e^p \\ \ddot{q}_{r1} &= \ddot{q}_d + p\lambda \text{diag}(|e|^{p-1})\dot{e}\end{aligned}\quad (48)$$

where the differentiation of \dot{q}_{r1} in (48) implies for the definition of $\text{diag}(|e|^{p-1})$ that $\text{diag}(|e|^{p-1}) = \text{diag}(|e_1|^{p-1}, \dots, |e_n|^{p-1})$. Note that \dot{q}_{r1} , \ddot{q}_{r1} can be obtained based on the joint position q , velocity \dot{q} and the command reference $q_d, \dot{q}_d, \ddot{q}_d$. Moreover, \ddot{q}_{r1} and $\dot{S}_1 = \ddot{q}_{r1} - \ddot{q}$ can exhibit a potential singularity for $e = 0$. According to (48), on the sliding mode surface $S_1 = 0$, we know $\ddot{q}_{r1} = \ddot{q}_d + p\lambda \text{diag}(|e|^{p-1})\dot{e} = \ddot{q}_d - \lambda^2 p e^{2p-1}$, so that the singularity problem encountered in $e = 0$ can be avoided by choosing $p > 1/2$. However, in case $S_1 \neq 0$, there may be a potential singularity problem [23, 24, 27] when $e = 0, \dot{e} \neq 0$. Several researchers [25, 26, 32] have raised this issue; they suggested the use of some smoothening technique³. However, an alternative, two-phase control strategy is suggested here in order to overcome the singularity in the reaching phase of S_1 . For this purpose, we define a function as

$$f(e, \dot{e}) = \dot{e}^T M(q) \dot{e} - \lambda_M (\lambda e^p)^T \lambda e^p \quad (49)$$

where λ_M is the upper bound of $M(q)$ defined in Property 3.3, and λ is the positive constant defined in (47). The function $f(e, \dot{e})$ is inherited from S_1 in (47). For instance, $f(e, \dot{e}) \geq 0$ implies $\dot{e}^T \dot{e} \geq (\lambda e^p)^T \lambda e^p$. The phase plane plot in Fig. 1 shows the principal two phase control idea. In particular, for $f(e, \dot{e}) \geq 0$ the linear sliding surface will be employed, while for $f(e, \dot{e}) < 0$ a finite time control law is used as explained below.

Thus, the overall control scheme can be summarized as:

- 1) A control law using a linear sliding mode surface (with no singularity), i.e. $S_1 = \dot{e} + \lambda e^p$ with $p = 1$, is used to ensure that the states enter the region $f(e, \dot{e}) \leq 0$ of the state space.
- 2) The TSM surface $S_1 = \dot{e} + \lambda e^p$ with $1/2 < p < 1$ and variables $\ddot{q}_{r1}, \dot{S}_1 = \ddot{q}_{r1} - \ddot{q}$ are only used for a control law where they do not exhibit singularity. Thus, analysis is limited to a well-defined region, $f(e, \dot{e}) \leq 0$, of the state-space (i.e. for a specific region of $e \neq 0$).

[Insert Fig.1 here]

Fig.1 The phase plane plot of the control system.

Considering Phase 2, the error equation can be derived from the robot dynamics (22) and (47)~(48) as

$$M(q)\dot{S}_1 + C(q, \dot{q})S_1 = -\tau + R_1(q, \dot{q}) \quad (50)$$

where $R_1(q, \dot{q})$ can be rewritten in a linear-in-parameter form according to Property 3.2 as

$$R_1(q, \dot{q}) = M(q)\ddot{q}_{r1} + C(q, \dot{q})\dot{q}_{r1} + G(q) = \Phi_{r1}(q, \dot{q})\theta \quad (51)$$

so that $\theta \in \mathbb{R}^N$ is the vector holding the unknown parameters, and $\Phi_{r1}(q, \dot{q}) \in \mathbb{R}^{n \times N}$ is the known regressor matrix.

We can design an adaptive control as

$$\tau = \Phi_{r1}(q, \dot{q})\hat{\theta} + K_{11}S_1 + u_r \quad (52)$$

with a robust term u_r as

$$u_r = \begin{cases} K_{12}S_1 / \|S_1\|, & S_1 \neq 0 \\ 0, & S_1 = 0 \end{cases} + \begin{cases} B(e, \dot{e}) \frac{\dot{e}}{\|\dot{e}\|}, & \text{for all } f(e, \dot{e}) > 0 \\ 0, & \text{anywhere else} \end{cases} \quad (53)$$

³ These smoothening techniques may not necessarily provide finite-time tracking convergence as the sliding plane was essentially modified.

where $K_{11}, K_{12} > 0$ are positive gain matrices, $\hat{\theta}$ is the estimation of the unknown parameters θ . The adaptive law for $\hat{\theta}$ will be specified later in (60). The scalar, non-negative function $B(e, \dot{e})$ will be introduced later in (68), where the principal idea of the second switching term will be clarified. In particular, $B(e, \dot{e})$ is well defined and finite for $\dot{e} \neq 0$.

Substituting (52) into (50) yields the following closed-loop error equation

$$M(q)\dot{S}_1 + C(q, \dot{q})S_1 = -K_{11}S_1 - u_r + \Phi_{R1}(q, \dot{q})\tilde{\theta} \quad (54)$$

In the following, we will propose the adaptive law for obtaining $\hat{\theta}$. To avoid using the robotic joint acceleration \ddot{q} of $\dot{S}_1 = \ddot{q}_{r1} - \ddot{q}$ in the adaptation, we define functions $F_1(q, \dot{q}) = M(q)S_1$ and $H_1(q, \dot{q}) = -\dot{M}(q)S_1 + C(q, \dot{q})S_1$ so that

$$\begin{cases} F_1(q, \dot{q}) = M(q)S_1 = \Phi_{F1}(q, \dot{q})\theta \\ H_1(q, \dot{q}) = -\dot{M}(q)S_1 + C(q, \dot{q})S_1 = \Phi_{H1}(q, \dot{q})\theta \end{cases} \quad (55)$$

Then system (50) can be represented as

$$\dot{F}_1(q, \dot{q}) + H_1(q, \dot{q}) - R_1(q, \dot{q}) = [\dot{\Phi}_{F1}(q, \dot{q}) + \Phi_{H1}(q, \dot{q}) - \Phi_{R1}(q, \dot{q})]\theta = \Phi_1(q, \dot{q})\theta = -\tau \quad (56)$$

where $\dot{F}_1(q, \dot{q}) = \frac{d}{dt}[M(q)S_1] = \dot{\Phi}_{F1}(q, \dot{q})\theta$, and $\Phi_1(q, \dot{q}) = [\dot{\Phi}_{F1}(q, \dot{q}) + \Phi_{H1}(q, \dot{q}) - \Phi_{R1}(q, \dot{q})]$ is the regressor matrix. Again, to obtain $\dot{\Phi}_{F1}(q, \dot{q})$ without using the joint acceleration \ddot{q} , similar to Section 3, we introduce stable filter operations on $\Phi_{R1}, \Phi_{F1}, \Phi_{H1}$ and τ as

$$\begin{cases} k\dot{\Phi}_{R1f} + \Phi_{R1f} = \Phi_{R1}, & \Phi_{R1f}|_{t=0} = 0 \\ k\dot{\Phi}_{F1f} + \Phi_{F1f} = \Phi_{F1}, & \Phi_{F1f}|_{t=0} = 0 \\ k\dot{\Phi}_{H1f} + \Phi_{H1f} = \Phi_{H1}, & \Phi_{H1f}|_{t=0} = 0 \\ k\dot{\tau}_f + \tau_f = \tau, & \tau_f|_{t=0} = 0 \end{cases} \quad (57)$$

where $\Phi_{R1f}, \Phi_{F1f}, \Phi_{H1f} \in \mathbb{R}^{n \times N}$ and $\tau_f \in \mathbb{R}^n$ are the filtered version of $\Phi_{R1}, \Phi_{F1}, \Phi_{H1}$ and τ , respectively.

Then according to (56) and (57), one can obtain

$$\left[\frac{\Phi_{F1}(q, \dot{q}) - \Phi_{F1f}(q, \dot{q})}{k} + \Phi_{H1f}(q, \dot{q}) - \Phi_{R1f}(q, \dot{q}) \right] \theta = \Phi_{1f}(q, \dot{q})\theta = -\tau_f \quad (58)$$

where $\Phi_{1f}(q, \dot{q}) = \frac{\Phi_{F1}(q, \dot{q}) - \Phi_{F1f}(q, \dot{q})}{k} + \Phi_{H1f}(q, \dot{q}) - \Phi_{R1f}(q, \dot{q}) \in \mathbb{R}^{n \times N}$ is the newly introduced regressor matrix. It

is clearly shown that the use of the derivative \dot{S}_1 and thus the acceleration measurements \ddot{q} are avoided by introducing the filter operations (55)~(58). To accommodate the parameter estimation, we define the auxiliary matrix $P_2 \in \mathbb{R}^{N \times N}$, vectors $Q_2 \in \mathbb{R}^N$ and $W_2 \in \mathbb{R}^N$ as

$$\begin{cases} \dot{P}_2 = -\ell P_2 + \Phi_{1f}^T \Phi_{1f}, & P_2(0) = 0 \\ \dot{Q}_2 = -\ell Q_2 - \Phi_{1f}^T \tau_f, & Q_2(0) = 0 \\ W_2 = P_2 \hat{\theta} - Q_2 \end{cases} \quad (59)$$

Then the fact $Q_2 = P_2 \theta$ holds so that $W_2 = P_2 \hat{\theta} - P_2 \theta = -P_2 \tilde{\theta}$ is also true. Now we will incorporate the parameter error W_2 into the adaptive parameter estimation for $\hat{\theta}$ as

$$\dot{\hat{\theta}} = \Gamma \left(\Phi_{R1}^T(q, \dot{q})S_1 - \kappa_1 \frac{P_2^T W_2}{\|W_2\|} \right) \quad (60)$$

where $\Gamma > 0$ and $\kappa_1 > 0$ are the adaptation gain matrices.

We have the following Lemma for control phase 2 to establish the reachability of S_1 :

Lemma 4.1

Consider system (22) with control (52) and adaptive law (60) under Assumptions 4.1, then for $e \neq 0$ the Lyapunov function

$$V_7 = \frac{1}{2} S_1^T M S_1 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \quad (61)$$

satisfies $\dot{V}_7 \leq -S_1^T K_{11} S_1$. \diamond

Proof

Consider the Lyapunov function as

$$V_7 = \frac{1}{2} S_1^T M S_1 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \quad (62)$$

The derivative \dot{V}_7 along (54) and (60) can be derived as

$$\begin{aligned} \dot{V}_7 &= S_1^T \left[-C(q, \dot{q}) S_1 - K_{11} S_1 - K_{12} S_1 / \|S_1\| + \Phi_{R1}(q, \dot{q}) \tilde{\theta} \right] + \frac{1}{2} S_1^T \dot{M}(q) S_1 + \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} - B(e, \dot{e}) S_1^T \begin{cases} \dot{e} / \|\dot{e}\| & \text{for all } f(e, \dot{e}) > 0 \\ 0, & \text{anywhere else} \end{cases} \\ &= -S_1^T K_{11} S_1 - K_{12} \|S_1\| + \frac{1}{2} S_1^T \left[\dot{M}(q) - 2C(q, \dot{q}) \right] S_1 - B(e, \dot{e}) S_1^T \begin{cases} \dot{e} / \|\dot{e}\| & \text{for all } f(e, \dot{e}) > 0 \\ 0, & \text{anywhere else} \end{cases} \\ &\quad + S_1^T \Phi_{R1}(q, \dot{q}) \tilde{\theta} - \tilde{\theta}^T \Phi_{R1}^T(q, \dot{q}) S_1 - \kappa_1 \frac{\tilde{\theta}^T P_2^T P_2 \tilde{\theta}}{\|P_2 \tilde{\theta}\|} \\ &\leq -S_1^T K_{11} S_1 \end{aligned} \quad (63)$$

Note that it can be derived that for all $f(e, \dot{e}) > 0$, the relation $\lambda_M \dot{e}^T \dot{e} \geq \dot{e}^T M(q) \dot{e} > \lambda_M (\lambda e^p)^T \lambda e^p$ is always true, so that $\dot{e}^T \dot{e} > (\lambda e^p)^T \lambda e^p$ holds. In this case

$$-2\dot{e}^T S_1 = -2\dot{e}^T (\dot{e} + \lambda e^p) \leq -2\dot{e}^T \dot{e} + \dot{e}^T \dot{e} + (\lambda e^p)^T \lambda e^p \leq 0 \quad (64)$$

Thus, it follows in particular $-S_1^T \begin{cases} \dot{e} / \|\dot{e}\| & \text{for all } f(e, \dot{e}) > 0 \\ 0, & \text{anywhere else} \end{cases} \leq 0$. The result of (63) will be vital for reachability of

S_1 in a final stability argument. \square

We will prove that the set $f(e, \dot{e}) \leq 0$ is a region of attraction for the control system of (50) and (52), i.e. any system state will remain within this region once it has reached the region.

Lemma 4.2

For control system (50) with (52) and the initial state satisfying $f(e(0), \dot{e}(0)) \leq 0$, the system states remain within the region $f(e, \dot{e}) \leq 0$ at all times in a semi-global sense. \diamond

Proof

We consider an arbitrary but fixed compact set Ξ in the tuple $(e, \dot{e}, \tilde{\theta})$ containing the origin in its interior. To prove Lemma 4.2, we may assume the opposite considering the following two cases. The first case deals with the possibility that $f(e(t), \dot{e}(t)) > 0$ and $e(t) \neq 0$ for some $t > 0$, where the trajectory remains within a large enough set Ξ :

Case 1: Under the assumption for this case, there exist two time instances, t_1 and t_2 , so that $f(e(t_1), \dot{e}(t_1)) = 0$, $f(e(t_2), \dot{e}(t_2)) > 0$ and $f(e(t), \dot{e}(t)) > 0$, $e(t) \neq 0$ for all $t \in (t_1, t_2]$. In particular, function $f(e(t), \dot{e}(t))$ must have increased during the interval $t \in [t_1, t_2]$. Thus, we may analyze the derivative for $f(e(t), \dot{e}(t)) > 0$, $e(t) \neq 0$. (Note that

$e(t) \neq 0$ and $f(e(t), \dot{e}(t)) > 0$ implies $\dot{e}(t) \neq 0$.) From (47) and (54), one may obtain that $M(q)\dot{S}_1 = M(q)(\ddot{e} + p\lambda \text{diag}(|e|^{p-1})\dot{e})$ so that

$$M(q)\ddot{e} = -pM(q)\lambda \text{diag}(|e|^{p-1})\dot{e} - C(q, \dot{q})(\dot{e} + \lambda e^p) - K_{11}S_1 - u_r + \Phi_{R1}(q, \dot{q})\tilde{\theta} \quad (65)$$

According to (49), the derivative of $f(e, \dot{e})$ can be computed as

$$\begin{aligned} \dot{f}(e, \dot{e}) &= 2\dot{e}^T M(q)\ddot{e} + \dot{e}^T \dot{M}(q)\dot{e} - 2\lambda_M p \dot{e}^T \lambda \text{diag}(|e|^{p-1})\lambda e^p \\ &= 2\dot{e}^T \left[-pM(q)\lambda \text{diag}(|e|^{p-1})\dot{e} + \left(\frac{\dot{M}(q)}{2} - C(q, \dot{q}) \right) \dot{e} - C(q, \dot{q})\lambda e^p - 2\lambda_M p \lambda \text{diag}(|e|^{p-1})\lambda e^p - K_{11}S_1 - u_r + \Phi_{R1}(q, \dot{q})\tilde{\theta} \right] \end{aligned} \quad (66)$$

Consequently, from Property 3.1, i.e. $\dot{M}(q) - 2C(q, \dot{q})$ is skew-symmetric, and the discussion on (64), we have

$$\dot{f}(e, \dot{e}) \leq 2\dot{e}^T \left[-pM(q)\lambda \text{diag}(|e|^{p-1})\dot{e} - C(q, \dot{q})\lambda e^p - 2\lambda_M p \lambda \text{diag}(|e|^{p-1})\lambda e^p - u_r + \Phi_{R1}(q, \dot{q})\tilde{\theta} \right] \quad (67)$$

In this case, if we choose the function $B(\bullet, \bullet)$ of the robust term u_r in (53) as

$$B(e, \dot{e}) \geq \begin{cases} \| pM(q)\lambda \text{diag}(|e|^{p-1})\dot{e} + C(q, \dot{q})\lambda e^p + 2\lambda_M p \lambda \text{diag}(|e|^{p-1})\lambda e^p + \Phi_{R1}(q, \dot{q})\tilde{\theta} \| & \text{for } e \neq 0 \\ 0 & \text{elsewhere} \end{cases} \quad (68)$$

i.e. $B(e, \dot{e})$ is a sufficiently large non-negative function in Ξ . Note that $B(e, \dot{e})$ will be only used in an infinitesimally small region of $f(e, \dot{e}) = 0$. As $B(e, \dot{e})$, in particular $\text{diag}(|e|^{p-1})\dot{e}$, is finite for $f(e, \dot{e}) = 0$ and $p > 1/2$ (see for instance discussion below (48)) and the function $B(e, \dot{e})$ acting as control gain is only used in a small vicinity of $f(e, \dot{e}) = 0$, the function $B(e, \dot{e})$ remains finite. Thus, it follows from (67) and (68) that $\dot{f}(e, \dot{e}) < 0$ is true for $t \in [t_1, t_2]$ as $f(e(t), \dot{e}(t)) > 0$. This implies that $f(e, \dot{e})$ is decreasing. This contradicts the initial assumption of Case 1, i.e. this case is rejected.

Case 2: The second case deals with the possibility that $f(e(t), \dot{e}(t)) > 0$ and $e(t) = 0$. This is only possible if there exist two time instances, t_1 and t_2 ($0 < t_1 < t_2$), so that $(e(t_1), \dot{e}(t_1)) = (0, 0)$, while $e(t_2) = 0$ and $\dot{e}(t_2) \neq 0$ for which at least for a finite time $e(t) = 0$ and $\dot{e}(t) \neq 0$ must hold. This is certainly impossible, as $e(t) = 0$ for $t \in [t_1, t_2]$ implies $\dot{e}(t_2) = 0$, which is a contradiction. Thus, Case 2 does not apply. Thus, the system state will remain within the region $f(e, \dot{e}) \leq 0$ once it is within this region. \square

This result permits now to return to Lemma 4.1 to establish a finite-time stability result for the introduced control scheme of (50) and (52) under the assumption of $f(e(0), \dot{e}(0)) \leq 0$:

Lemma 4.3

For system (22) with control (52) and (60) with Assumptions 4.1, if the initial state satisfies $f(e(0), \dot{e}(0)) \leq 0$, then

- (a) All signals in the closed-loop system are bounded;
- (b) There is a time instant $\tilde{T}_a > 0$ and a constant $\sigma > 0$ so that for $t \geq \tilde{T}_a$, the condition $\lambda_{\min}(P_2(t)) > \sigma > 0$ holds;
- (c) The sliding manifold variable S_1 and the parameter error $\tilde{\theta}$ converge to zero in finite time;
- (d) The tracking error e converges to zero in finite time. \diamond

Proof

(a) Note that it follows from $f(e(0), \dot{e}(0)) \leq 0$ that $e(t) = 0$ only if $(e(t), \dot{e}(t)) = (0, 0)$. Moreover, $(e(t), \dot{e}(t)) = (0, 0)$ implies $S_1 = 0$, for which the control singularity is avoided by choosing $p > 1/2$ (see discussion below (48)). From

Lemma 4.1, we have $\dot{V}_7 \leq -S_1^T K_{11} S_1$. This implies S_1 and $\tilde{\theta}$ are bounded, i.e. $S_1 \in L_2 \cap L_\infty$, $\tilde{\theta} \in L_\infty$, and thus $\hat{\theta}$, q, \dot{q} and $\Phi_1(q, \dot{q})$, $\Phi_{R1}(q, \dot{q})$ and the control torque τ are also bounded. The error dynamics from (54) further guarantee $\dot{S}_1 \in L_\infty$, which together with the fact $S_1 \in L_2 \cap L_\infty$ and Barbalat's Lemma [31] implies that $S_1 \rightarrow 0$ as $t \rightarrow 0$, i.e. $q \rightarrow q_d$.

(b) Recalling that $\Phi_1(q, \dot{q})$ is Lipschitz continuous with respect to q, \dot{q} , there exists a time instant T_a so that for all $t \geq T_a$, suitably chosen constant $\delta_a, \delta_b > 0$ and arbitrary constant $\varepsilon_a > 0$, the following inequalities holds

$$\|q - q_d\| \leq \delta_a, \quad \|\dot{q} - \dot{q}_d\| \leq \delta_b, \quad \|\Phi_1(q, \dot{q}) - \Phi_1(q_d, \dot{q}_d)\| \leq \varepsilon_a \quad (69)$$

i.e. the absolute continuity of $\Phi_1(q, \dot{q})$ implies the existence of T_a for arbitrarily small $\varepsilon_a > 0$ so that $\|\Phi_1(q, \dot{q}) - \Phi_1(q_d, \dot{q}_d)\| \leq \varepsilon_a$. Hence, as $t \rightarrow 0$, $\Phi_1(q, \dot{q}) \rightarrow \Phi_1(q_d, \dot{q}_d)$ is true.

According to Assumption 4.1 and the fact that the auxiliary demands q_d, \dot{q}_d are bounded, there exists a constant $\delta_1 > 0$ for small enough $\varepsilon_a > 0$ such that

$$\int_t^{t+T} \Phi_1^T(q(r)) \Phi_1(q(r)) dr > \delta_1 I \quad (70)$$

holds for all $t \geq T_a$. This means that $\Phi_1(q, \dot{q})$ is PE as long as $\Phi_1(q_d, \dot{q}_d)$ is SR [2] for system (22) with control (52).

Moreover, since $\Phi_{1f}(q, \dot{q})$ defined in (58) is the filtered version of $\Phi_1(q, \dot{q})$, one may follow the proof of Lemma 2.2 and then conclude that there must be a sufficiently large time instant \tilde{T}_a and a constant $\sigma > 0$ so that from $t \geq \tilde{T}_a$, the argument $\lambda_{\min}(P_2(t)) > \sigma > 0$ holds.

(c) To further prove FT convergence of the parameter error $\tilde{\theta}$ and terminal sliding-mode manifold S_1 , we apply the fact $\lambda_{\min}(P_2(t)) > \sigma > 0$ for (63), then it follows

$$\dot{V}_7 \leq -K_{12} \|S_1\| - \kappa_1 \|P_2 \tilde{\theta}\| \leq -\lambda_{\min}(K_{12}) \|S_1\| - \kappa_1 \sigma \|\tilde{\theta}\| \leq -\mu_7 \sqrt{V_7} \quad (71)$$

where $\mu_7 = \min \left\{ \lambda_{\min}(K_{12}) \sqrt{2/\lambda_{\max}(M)}, \kappa_1 \sigma \sqrt{2/\lambda_{\max}(\Gamma^{-1})} \right\}$ is a positive constant. Then according to [28], inequality (71) implies that $\tilde{\theta}$ and S_1 converge to zero in finite-time with the convergence time $T_r \leq 2\sqrt{V_7(0)}/\mu_7$.

(d) Finally, when the terminal sliding manifold S_1 converges to zero, i.e. $S_1 = 0$, the tracking error e in (47) can be rewritten as

$$\dot{e} + \lambda e^p = 0 \quad (72)$$

with $p < 1$. Then according to Lemma 2.1 and [30], it follows that e converges to zero in finite-time, i.e. $e_i = 0$ is the terminal attractor of system (72) with the convergence time $T_{ci} = e_i(0)^{1-p}/\lambda_i(1-p)$. \square

From Lemmas 4.2 and 4.3, we first drive the tracking error into the region $f(e, \dot{e}) \leq 0$ as shown in Fig.1 by using a singularity-free control (e.g. sliding mode control with a linear sliding plane as discussed below). Thus, the potential singularity problem with $e = 0, \dot{e} \neq 0$ can be successfully avoided, because the point $e = 0, \dot{e} \neq 0$ is outside the region $f(e, \dot{e}) \leq 0$. Then, we can switch the control to the TSM scheme (52) with adaptation (60).

To finalize the whole control procedure, we propose a sliding mode control with a linear sliding plane for the first control phase. For this purpose, we set $p = 1$ in (47), so that the error variables (i.e. S_1, q_{r1}) can be modified as

$$S_2 = \dot{e} + \lambda e = \dot{q}_{r2} - \dot{q} \quad (73)$$

Note that in this case, the error variables can be represented as $\dot{q}_{r2} = \dot{q}_d + \lambda e$ and $\ddot{q}_{r2} = \ddot{q}_d + \lambda \dot{e}$, where there is no negative power term in \ddot{q}_{r2} , so that the subsequent control is singularity-free. Thus, we redefine auxiliary variables

$R_1(q, \dot{q})$, $F_1(q, \dot{q})$, $H_1(q, \dot{q})$, $\Phi_{R1}(q, \dot{q})$, Φ_{R1f} , Φ_{F1f} , Φ_{H1f} , τ_f and P_2 , Q_2 , W_2 as (55)~(59) by using the linear sliding mode error S_2 and $\dot{q}_{r2}, \ddot{q}_{r2}$ in (73), then we design the adaptive control as

$$\tau = \Phi_{R1}(q, \dot{q})\hat{\theta} + K_{11}S_2 + \begin{cases} K_{12}S_2 / \|S_2\|, & S_2 \neq 0 \\ 0, & S_2 = 0 \end{cases} \quad (74)$$

with parameter estimation adaptation

$$\dot{\hat{\theta}} = \Gamma \left(\Phi_{R1}^T(q, \dot{q})S_2 - \kappa_1 W_2 \right) \quad (75)$$

where $\Gamma > 0$ and $\kappa_1 > 0$ are the adaptation gain matrices, and W_2 is the parameter error calculated from (59) in terms of Φ_{1f}, P_2, Q_2 .

Then the error convergence for the first control phase with (74) and (75) can be summarized as

Lemma 4.4

Consider system (22) with control (74) and adaptive law (75) under Assumptions 4.1, then the parameter error $\tilde{\theta}$ and the tracking error S_2 converge to zero. The control law enters within finite time any non-empty compact set of the combined error $(e, \dot{e}, \tilde{\theta})$ containing the origin as an interior point and converges to the hyperplane $S_2 = 0$ within finite time, so that $f(e, \dot{e}) < 0$ in finite time. \diamond

Proof

Substituting (74) into (50), we get the closed-loop dynamic equation as

$$M(q)\dot{S}_2 + C(q, \dot{q})S_2 = -K_{11}S_2 + \Phi_{R1}(q, \dot{q})\tilde{\theta} - \begin{cases} K_{12}S_2 / \|S_2\|, & S_2 \neq 0 \\ 0, & S_2 = 0 \end{cases}.$$

Similar to the analysis for TSM control, we can verify that $W_2 = -P_2\tilde{\theta}$ holds. Then consider the Lyapunov function as

$$V_8 = \frac{1}{2}S_2^T M S_2 + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \quad (76)$$

Then the derivative of V_8 can be calculated as

$$\begin{aligned} \dot{V}_8 &= S_2^T \left[-C(q, \dot{q})S_2 - K_{11}S_2 - K_{12}S_2 / \|S_2\| + \Phi_{R1}\tilde{\theta} \right] + \frac{1}{2}S_2^T \dot{M}(q)S_2 + \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} \\ &= -S_2^T K_{11}S_2 - K_{12}\|S_2\| - \tilde{\theta}^T P_2 \tilde{\theta} - \kappa_1 \sigma \|\tilde{\theta}\| \\ &\leq -S_2^T K_{11}S_2 \end{aligned} \quad (77)$$

Consequently, we know S_2 and $\tilde{\theta}$ are bounded, i.e. $S_2 \in L_2 \cap L_\infty$, $\tilde{\theta} \in L_\infty$, and $\dot{S}_2 \in L_\infty$, then according to Barbalat's Lemma, one may conclude that $S_2 \rightarrow 0$ as $t \rightarrow \infty$, i.e. $q \rightarrow q_d$. Similar to the arguments of Lemma 4.3, $\lambda_{\min}(P_2) > \sigma > 0$ is true based on Assumption 4.1, and then it follows

$$\dot{V}_8 = -S_2^T K_{11}S_2 - \kappa_1 \tilde{\theta}^T P_2 \tilde{\theta} \leq -\mu_8 V_8 \quad (78)$$

where $\mu_8 = \min \{ 2\lambda_{\min}(K_{11}) / \lambda_{\max}(M), 2\kappa_1 \sigma / \lambda_{\max}(\Gamma^{-1}) \}$ is a positive constant. According to the Lyapunov's Theorem,

$\tilde{\theta}$ and S_2 all exponentially converge to zero with the convergence rate μ_8 . We may now analyze $V_9 = \frac{1}{2}S_2^T M S_2$. It follows

$$\dot{V}_9 = S_2^T \left[-K_{11}S_2 - K_{12}S_2 / \|S_2\| + \Phi_{R1}\tilde{\theta} \right] \leq -S_2^T K_{11}S_2 - (K_{12} - \|\Phi_{R1}\tilde{\theta}\|)\|S_2\|$$

We know that $\lim_{t \rightarrow \infty} \|\Phi_{R1}\tilde{\theta}\| = 0$, i.e. at some point $K_{12} > \|\Phi_{R1}\tilde{\theta}\|$ so that V_9 converges to zero in finite time. The finite-time reaching of $S_2 = 0$ implies again exponential convergence of $(e, \dot{e}, \tilde{\theta})$ to zero. From that, it is evident that any compact set in $(e, \dot{e}, \tilde{\theta})$ containing the origin as interior point is entered within finite time. \square

Combining the linear sliding mode control (74) and (75) with the TSM control (52) and (60) as indicated in Fig.1, we will end up with a nonsingular control strategy stated as follows:

Theorem 4.1

The following control for system (22) implies FT convergence of the combined state tracking and parameter error:

- i) For system (22) with any initial state $(e(t_0), \dot{e}(t_0))$, first use control (74) and (75) to guarantee that in the time instant T_f , the trajectory will reach a point $(e_f, \dot{e}_f) \in \{(e, \dot{e}) \mid f(e, \dot{e}) \leq 0\}$.
- ii) Once $(e_f, \dot{e}_f) \in \{(e, \dot{e}) \mid f(e, \dot{e}) \leq 0\}$ is reached, the control is switched to the TSM control (52) and (60). \diamond

4.2 ADAPTIVE FINITE-TIME CONTROL AND ESTIMATION WITH DISTURBANCE

In this subsection, we will study the control and parameter estimation of the robotic system (37) with a bounded disturbance ξ . By defining the sliding mode error as (47), one can modify the error equation (54) as

$$M(q)\dot{S}_1 + C(q, \dot{q})S_1 = -K_{11}S_1 - K_{12}S_1 / \|S_1\| + \Phi_{R1}(q, \dot{q})\tilde{\theta} + \xi \quad (79)$$

where $\xi \in \mathbb{R}^n$ is a bounded disturbance vector, i.e. $\|\xi\| \leq \varepsilon_\xi, \varepsilon_\xi > 0$.

By introducing the filter operations as (55)~(58), then the system dynamics can be represented as

$$\Phi_{1f}(q, \dot{q})\theta = -\tau_f + \xi_f \quad (80)$$

where ξ_f is the filtered disturbance as defined in (38).

With the auxiliary matrix P_2 , vector Q_2 and vector W_2 defined (59), we can verify that

$$W_2 = -P_2\tilde{\theta} + \psi \quad (81)$$

where $\psi = -\int_0^t e^{-\ell(t-r)} \Phi_{2f}^T(r) \xi_f(r) dr$ is bounded for bounded Φ_2 and ξ , i.e. $\|\psi\| \leq \varepsilon_\psi$ for some $\varepsilon_\psi > 0$.

For the ease of analysis in this case, we have to restate Assumption 4.1 as Assumption 4.2:

Assumption 4.2

System (37) is sufficiently excited so that over any finite interval $[t, t+T]$ with $T > 0$ for $\Phi_{1f}(q, \dot{q})$ with respect to trajectory q , there exist N time instances t_i so that $\Psi_1^T(q(\bullet)) = [\Phi_1^T(q(t_1)), \Phi_1^T(q(t_2)), \dots, \Phi_1^T(q(t_N))]^T \in \mathbb{R}^{nN \times N}$ is of rank N . Consequently, the condition $\lambda_{\min}(P_2) > \sigma$ holds.

Then the following result holds:

Theorem 4.2

Consider system (22) with the switched control law as in Theorem 4.1, i.e. using control (74) and control (52) under Assumptions 4.2, then the closed-loop system is semi-globally stable where the tracking error $(e(t), \dot{e}(t))$ converges to zero within finite time and the parameter error $\tilde{\theta}$ remains bounded and satisfies $\lim_{t \rightarrow \infty} P_2 \tilde{\theta} = \psi$. \diamond

Proof

Much of the proof of Theorem 4.2 is based on the well-known robustness features of sliding mode control. The proof of Theorem 4.2 should follow the same structure as the proof of Theorem 4.1. Thus, the step-wise process of Lemma 4.1 to 4.2 has to be taken: The first step would be the proof showing that $f(e, \dot{e}) \leq 0$ is strictly guaranteed for control law (52), then followed by the proof of finite-time stability of the control law (52), while a linear control sliding surface from the control law (74) avoids any singularity issues. However, to keep things short, the analysis of finite-time stability of the control law under the assumption of $f(e, \dot{e}) \leq 0$ is carried out only. All other steps follow in the same manner.

For the analysis of this case, the following Lyapunov function is used:

$$V_{10} = \frac{1}{2} S_1^T M S_1 + \frac{1}{2} W_2^T P_2^{-1} \Gamma^{-1} P_2^{-1} W_2 \quad (82)$$

Then similar to the proof of Lemma 3.2, the derivative of $P_2^{-1} W_2 = -\tilde{\theta} + P_2^{-1} \psi'$ with respect to time can be given as $\frac{\partial P_2^{-1} W_2}{\partial t} = \dot{\tilde{\theta}} + \psi'$, where ψ' is defined as $\psi' = -P_2^{-1} \dot{P}_2 P_2^{-1} \psi + P_2^{-1} \dot{\psi}$. Then it follows

$$\begin{aligned} \dot{V}_{10} &= S_1^T \left[-C(q, \dot{q}) S_1 - K_{11} S_1 - K_{12} S_1 / \|S_1\| + \Phi_{r1}(q, \dot{q}) \tilde{\theta} + \xi \right] + \frac{1}{2} S_1^T \dot{M}(q) S_1 \\ &\quad + W_2^T P_2^{-1} \Gamma^{-1} \left(\dot{\tilde{\theta}} + \psi' \right) - B(e, \dot{e}) S_1^T \begin{cases} \dot{e} / \|\dot{e}\| & \text{for all } f(e, \dot{e}) > 0 \\ 0, & \text{anywhere else} \end{cases} \\ &\leq -S_1^T K_{11} S_1 - (K_{12} - \varepsilon_\xi) \|S_1\| + \psi^T P_2^{-1} \Phi_{r1}^T(q, \dot{q}) S_1 - \kappa_1 \frac{W_2^T P_2^{-1} P_2 W_2}{\|W_2\|} + W_2^T P_2^{-1} \Gamma^{-1} \psi' \\ &\leq -S_1^T K_{11} S_1 - (K_{12} - \varepsilon_\xi - \|\psi^T P_2^{-1} \Phi_{r1}^T\|) \|S_1\| - (\kappa_1 - \|P_2^{-1} \Gamma^{-1} \psi'\|) \|W_2\| \end{aligned} \quad (83)$$

We now analyze the particular terms $\|\psi^T P_2^{-1} \Phi_{r1}^T\|$ and $\|P_2^{-1} \Gamma^{-1} \psi'\|$. It is shown that $\psi = -\int_0^t e^{-\ell(t-r)} \Phi_{1f}^T(r) \xi_f(r) dr$, which implies that ψ and $\dot{\psi}$ are bounded as long as ξ and Φ_1 are bounded. This is indeed true for an initial finite time interval. In particular, the matrix P_2 and \dot{P}_2 are also bounded. Moreover, Assumption 4.2 guarantees that $\lambda_{\min}(P_2) > \sigma$ holds so that P_2^{-1} is bounded in magnitude. Thus, assuming all closed loop system parameters are suitably chosen, the terms $\|\psi^T P_2^{-1} \Phi_{r1}^T\|$ and $\|P_2^{-1} \Gamma^{-1} \psi'\|$ exist and are bounded. Thus, for large enough $K_{12} > 0$ and $\kappa_1 > 0$, semiglobal stability of (83) follows such that

$$\dot{V}_{10} \leq -S_1^T K_{11} S_1 \leq 0 \quad (84)$$

which further implies $\lim_{t \rightarrow \infty} S_1 = 0$. Consequently, the control error S_1 converges to zero and all other signals in the closed-loop are bounded. To further prove finite-time convergence, we know that (83) can be represented as

$$\begin{aligned} \dot{V}_{10} &\leq -(K_{12} - \varepsilon_\xi - \|\psi^T P_2^{-1} \Phi_{r1}^T\|) \|S_1\| - (\kappa_1 - \|P_2^{-1} \Gamma^{-1} \psi'\|) \|W_2\| \\ &\leq -\mu_{10} \sqrt{V_{10}} \end{aligned} \quad (85)$$

with $\mu_{10} = \min \left\{ (K_{12} - \varepsilon_\xi - \|\psi^T P_2^{-1} \Phi_{r1}^T\|) \sqrt{2/\lambda_{\max}(M)}, (\kappa_1 - \|P_2^{-1} \Gamma^{-1} \psi'\|) \sigma \sqrt{2/\lambda_{\max}(\Gamma^{-1})} \right\}$ being a positive scalar chosen larger than some positive constant. In this case, finite-time convergence of the tracking error S_1 and the parameter error W_2 to zero is guaranteed. This implies $\lim_{t \rightarrow \infty} P_2 \tilde{\theta} = \psi$ is true in finite time. \square

5. SIMULATIONS

To illustrate the effectiveness of the proposed estimation and control schemes, the model of the humanoid Bristol Elumotion Robotic Torso II (developed by Elumotion Ltd) [16] is utilized for simulation. In this paper, 2 joints of the robot arm are simulated, i.e. shoulder flexion and elbow flexion. Then the robotic arm dynamics are given by

$$\begin{bmatrix} M_{11}(q) & M_{12}(q) \\ M_{21}(q) & M_{22}(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} C_{11}(q, \dot{q}) & C_{12}(q, \dot{q}) \\ C_{21}(q, \dot{q}) & C_{22}(q, \dot{q}) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} G_1(q) \\ G_2(q) \end{bmatrix} = \tau \quad (86)$$

with

$$M_{11}(q) = \frac{1}{3} m_2 l_2^2 + \frac{1}{4} m_2 r_2^2 + \frac{1}{4} m_1 l_1^2 + m_2 l_1^2 + m_2 l_1 l_2 \cos(q_2) + \frac{1}{12} m_1 (3r_1^2 + l_1^2)$$

$$M_{12}(q) = M_{21}(q) = \frac{1}{12} m_2 (6l_1 l_2 \cos(q_2) + 4l_2^2 + 3r_2^2)$$

$$M_{22}(q) = \frac{1}{12} m_2 (4l_2^2 + 3r_2^2)$$

$$\begin{aligned}
C_{11}(q, \dot{q}) &= -m_2 l_1 l_2 \dot{q}_1 \dot{q}_2 \sin(q_2) - \frac{1}{2} m_2 l_1 l_2 \dot{q}_2^2 \sin(q_2) \\
C_{21}(q, \dot{q}) &= \frac{1}{2} m_2 l_1 l_2 \dot{q}_1^2 \sin(q_2) \\
C_{12}(q, \dot{q}) &= C_{22}(q, \dot{q}) = 0 \\
G_1(q) &= \frac{1}{2} m_2 g l_2 g \sin(q_1) \cos(q_2) + \frac{1}{2} m_2 g l_2 \sin(q_2) \cos(q_1) + m_2 g l_1 \sin(q_1) + \frac{1}{2} m_1 g l_1 \sin(q_1) \\
G_2(q) &= \frac{1}{2} m_2 g l_2 \sin(q_1) \cos(q_2) + \frac{1}{2} m_2 g l_2 \sin(q_2) \cos(q_1)
\end{aligned}$$

where m_1, m_2 are the mass of robot arm, l_1, l_2 and r_1, r_2 are the length and the radius for each link, and $g = 9.18$ is the gravity constant.

5.1 ADAPTIVE PARAMETER ESTIMATION

We assume that in system (86), the unknown parameters to be estimated are $\theta = [m_1, m_2]^T = [2.35, 3]^T$, then the regressor matrices $\phi_1(q, \dot{q}), \phi_2(q, \dot{q})$ of (86) can be derived as

$$\begin{aligned}
F(q, \dot{q}) &= \begin{bmatrix} M_{11}(q) & M_{12}(q) \\ M_{21}(q) & M_{22}(q) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} M_{11}(q) \dot{q}_1 + M_{12}(q) \dot{q}_2 \\ M_{21}(q) \dot{q}_1 + M_{22}(q) \dot{q}_2 \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} \frac{1}{4} l_1^2 \dot{q}_1 + \frac{1}{12} (3r_1^2 + l_1^2) \dot{q}_1 & (\frac{1}{3} l_2^2 + \frac{1}{4} r_2^2 + l_1^2 + l_1 l_2 \cos(q_2)) \dot{q}_1 + \frac{1}{12} (6l_1 l_2 \cos(q_2) + 4l_2^2 + 3r_2^2) \dot{q}_2 \\ 0 & \frac{1}{12} (6l_1 l_2 \cos(q_2) + 4l_2^2 + 3r_2^2) \dot{q}_1 + \frac{1}{12} (4l_2^2 + 3r_2^2) \dot{q}_2 \end{bmatrix}}_{\phi_1(q, \dot{q})} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \quad (87)
\end{aligned}$$

$$\begin{aligned}
H(q, \dot{q}) &= - \begin{bmatrix} \dot{M}_{11}(q) \dot{q}_1 + \dot{M}_{12}(q) \dot{q}_2 \\ \dot{M}_{21}(q) \dot{q}_1 + \dot{M}_{22}(q) \dot{q}_2 \end{bmatrix} + \begin{bmatrix} C_{11}(q, \dot{q}) \dot{q}_1 \\ C_{21}(q, \dot{q}) \dot{q}_1 \end{bmatrix} + \begin{bmatrix} G_1(q) \\ G_2(q) \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} \frac{1}{2} g l_1 \sin(q_1) & l_1 l_2 \sin(q_2) \dot{q}_1 + \frac{1}{2} l_1 l_2 \sin(q_2) \dot{q}_2 - (l_1 l_2 \dot{q}_1 \dot{q}_2 \sin(q_2) + \frac{1}{2} l_1 l_2 \dot{q}_2^2 \sin(q_2)) \dot{q}_1 \\ & + \frac{1}{2} g l_2 \sin(q_1) \cos(q_2) + \frac{1}{2} g l_2 \sin(q_2) \cos(q_1) + g l_1 \sin(q_1) \\ 0 & \frac{1}{2} l_1 l_2 \sin(q_2) \dot{q}_1 + \frac{1}{2} l_1 l_2 \dot{q}_1^2 \sin(q_2) \dot{q}_1 + \frac{1}{2} g l_2 \sin(q_1) \cos(q_2) + \frac{1}{2} g l_2 \sin(q_2) \cos(q_1) \end{bmatrix}}_{\phi_2(q, \dot{q})} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \quad (88)
\end{aligned}$$

In this part of simulation, to guarantee the PE condition (Assumption 3.1), an adaptive control proposed in [16] is applied in system (88) to track a sinusoidal reference $q_d = 20 \sin(0.3t)$. Other parameters used in (28) and (30) are $\ell = 1, k = 0.001$, and the adaptive learning gain is $\Gamma = 20I$. For comparison, the gradient parameter estimation proposed in [14] is also simulated.

The simulation results are depicted in Fig.2, in which the top subplots show the parameter estimation profiles of m_1 , while the bottom subfigures are the estimation performance of m_2 . As it can be seen, all adaptation laws (33), (34) and (35) can achieve accurate estimation of the unknown parameters and faster error convergence speed compared to the gradient-based method, which may be due to the employment of the derived parameter error information W_1 for the adaptive laws. Among the three proposed estimation schemes, the adaptation laws (34) and (35) with finite-time convergence perform superior over (33) in terms of convergence speed. In particular for (35), the best convergence performance is derived. This is reasonable because the error information used in (35) is $\hat{\theta} - K_1 Q_1 \rightarrow \tilde{\theta}$ (see Corollary 2.1), while the error information used in (33) and (34) is $P_1 \tilde{\theta}$. Hence, for (33) and (34), the parameter error $\tilde{\theta}$ is additionally conditioned by the dynamically changing matrix P_1 .

Moreover, Fig.3 provides the parameter error information W_1 used in the adaptation (33) to validate Corollary 3.1, i.e. W_1 converges to zero, and also depicts the profile of the matrix $K_1 P_1$ in (35), which converges to an identity matrix allowing for the online testing of the condition of convergence of our adaptive algorithm (See Remark 2.2).

[Insert Fig.2 here]

[Insert Fig.3 here]

Fig.2 Parameter estimation with adaptation (33), (34) and (35). Fig.3 Parameter estimation error W_1 and matrix $K_1 P_1$.

5.2 ADAPTIVE FINITE-TIME CONTROL AND ESTIMATION

In this case, the regressor matrix (51) used for control is derived as

$$R_1(q, \dot{q}) = M(q)\ddot{q}_{r1} + C(q, \dot{q})\dot{q}_{r1} + G(q)$$

$$= \begin{bmatrix} (\frac{1}{4}r_1^2 + \frac{1}{3}l_1^2)\ddot{q}_{r1} + \frac{1}{2}l_1 g \sin(q_1) & (l_1^2 + \frac{1}{4}r_2^2 + \frac{1}{3}l_2^2 + l_1 l_2 \cos(q_2))\ddot{q}_{r1} + (\frac{1}{2}l_1 l_2 \cos(q_2) + \frac{1}{3}l_2^2 + \frac{1}{4}r_2^2)\ddot{q}_{r2} + l_1 g \sin(q_1) \\ -(\frac{1}{2}l_1 l_2 \sin(q_2)\dot{q}_2^2 + l_1 l_2 \dot{q}_1 \sin(q_2)\dot{q}_2)\dot{q}_{r1} + \frac{1}{2}g l_2 \sin(q_1) \cos(q_2) + \frac{1}{2}g l_2 \sin(q_2) \cos(q_1) & \\ 0 & (\frac{1}{2}l_1 l_2 \cos(q_2) + \frac{1}{3}l_2^2 + \frac{1}{4}r_2^2)\ddot{q}_{r1} + (\frac{1}{3}l_2^2 + \frac{1}{4}r_2^2)\ddot{q}_{r2} + \frac{1}{2}l_1 l_2 \dot{q}_1^2 \sin(q_2)\dot{q}_{r1} \\ & + \frac{1}{2}g l_2 \sin(q_1) \cos(q_2) + \frac{1}{2}g l_2 \sin(q_2) \cos(q_1) \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$$

$\Phi_{R1}(q, \dot{q}, \ddot{q}_{r1}, \ddot{q}_{r2})$

(89)

where the parameters used to derive $S_1 = [S_{11}, S_{12}]^T$ and auxiliary variables $\dot{q}_{r1}, \ddot{q}_{r1}$ are set as $\lambda = \text{diag}([5, 5])$ and $p = p_1 / p_2 = 9 / 11$.

The functions $F_1(q, \dot{q}) = M(q)S_1$ and $H_1(q, \dot{q}) = -\dot{M}(q)S_1 + C(q, \dot{q})S_1$ can be rewritten as

$$F_1(q, \dot{q}) = M(q)S_1 = \begin{bmatrix} (\frac{1}{4}r_1^2 + \frac{1}{3}l_1^2)S_{11} & (l_1^2 + \frac{1}{4}r_2^2 + \frac{1}{3}l_2^2 + l_1 l_2 \cos(q_2))S_{11} + (\frac{1}{2}l_1 l_2 \cos(q_2) + \frac{1}{3}l_2^2 + \frac{1}{4}r_2^2)S_{12} \\ 0 & (\frac{1}{2}l_1 l_2 \cos(q_2) + \frac{1}{3}l_2^2 + \frac{1}{4}r_2^2)S_{11} + (\frac{1}{3}l_2^2 + \frac{1}{4}r_2^2)S_{12} \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$$

$\Phi_{F1}(q, \dot{q})$

(90)

$$H_1(q, \dot{q}) = -\dot{M}(q)S_1 + C(q, \dot{q})S_1 = \begin{bmatrix} 0 & l_1 l_2 \sin(q_2)S_{11} + \frac{1}{2}l_1 l_2 \sin(q_2)S_{12} - (l_1 l_2 \dot{q}_1 \dot{q}_2 \sin(q_2) + \frac{1}{2}l_1 l_2 \dot{q}_2^2 \sin(q_2))S_{11} \\ 0 & \frac{1}{2}l_1 l_2 \sin(q_2)S_{11} + \frac{1}{2}l_1 l_2 \dot{q}_1^2 \sin(q_2)S_{11} \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$$

$\Phi_{H1}(q, \dot{q})$

(91)

The reference is chosen as $q_d = 20 \sin(0.3t)$ to guarantee the SR condition (Assumption 4.1). The control parameters of (52) are specified as $K_{11} = \text{diag}([10, 10])$, $K_{12} = \text{diag}([0.0001, 0.0001])$, and the parameters used for adaptive law (60) are $\ell = 1, k = 0.001, \kappa_1 = 1$ and $\Gamma = 20I$. Simulation results are illustrated in Fig.4, where it is shown in Fig.4 (a) and Fig.4 (b) that fast convergence speed of tracking error and parameter estimation error can be achieved with control (52) and estimation (60), i.e. finite-time convergence as guaranteed by Theorem 4.1. To further validate the robustness of the proposed control and estimation schemes, an extra disturbance $\xi = 0.2 \sin(t)$ is applied into the system as indicated in (37), and the simulation results are provided in Fig.4(c). It is shown that the tracking error converges to zero and the parameter error converges to a small residual set around zero, as claimed in Theorem 4.2.

[Insert Fig.4 (a) here]

(a) Output tracking performance with control (52).

[Insert Fig.4 (b) here]

(b) Parameter estimation with adaptation (60).

[Insert Fig.4 (c) here]

(c) Tracking error and estimation error with disturbance.
Fig.4 Adaptive finite-time control and parameter estimation.

6. CONCLUSIONS

This paper presents some novel adaptive parameter estimation and control methods for nonlinear systems, in particular robotic systems with unknown parameters. Convergence guarantees for the parameter estimation algorithm can be verified by two alternative tests, relaxing common PE-assumptions. The adaptive laws are derived by introducing filter operations on the system dynamics, so that for instance the measurements of the robot joint accelerations are all avoided. The proposed adaptive laws for parameter estimation are driven by appropriate parameter estimation error information or can lead to an explicit computation of the identified parameters through an online learning of the inverse of a filtered regressor matrix. Moreover, the novel adaptive law with the proposed parameter error based leakage terms are incorporated into the adaptive terminal sliding mode control design. In this case, the conventional PE condition can be replaced by a sufficient richness requirement of the command signals, and thus is verifiable *a priori*. The potential singularity problem of the terminal sliding mode control was also studied by introducing a new two-phase control procedure. Theoretical results show that the proposed estimation and control algorithms are robust to bounded disturbances. Simulation studies confirm this theoretical analysis, in addition to practical evidence that the novel finite time estimator provides accurate estimation results which are faster than the usual gradient algorithm.

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Figures used in the manuscript

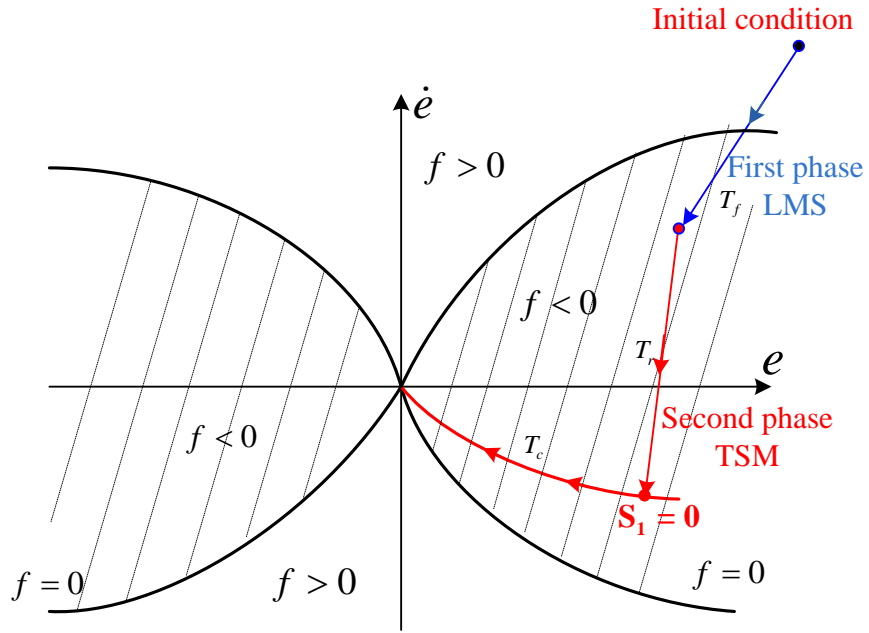


Fig.1 The phase plane plot of the control system.

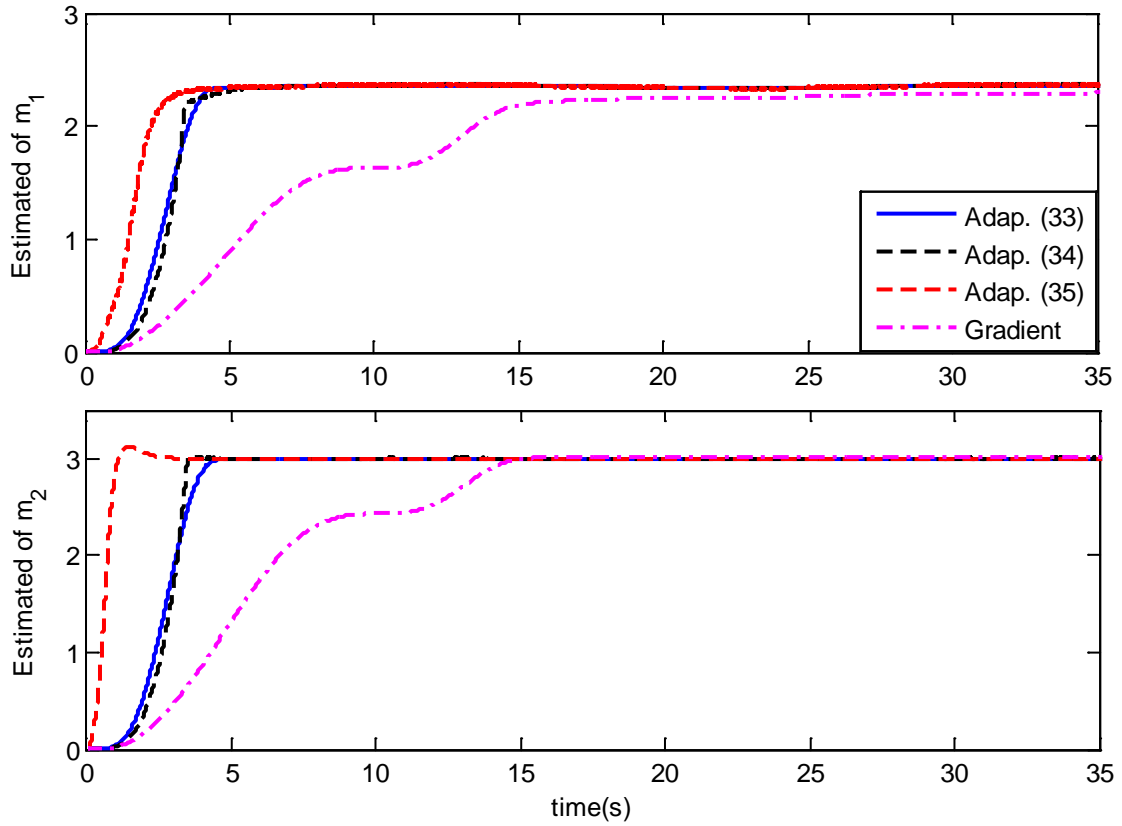


Fig.2 Parameter estimation with adaptation (33), (34) and (35)

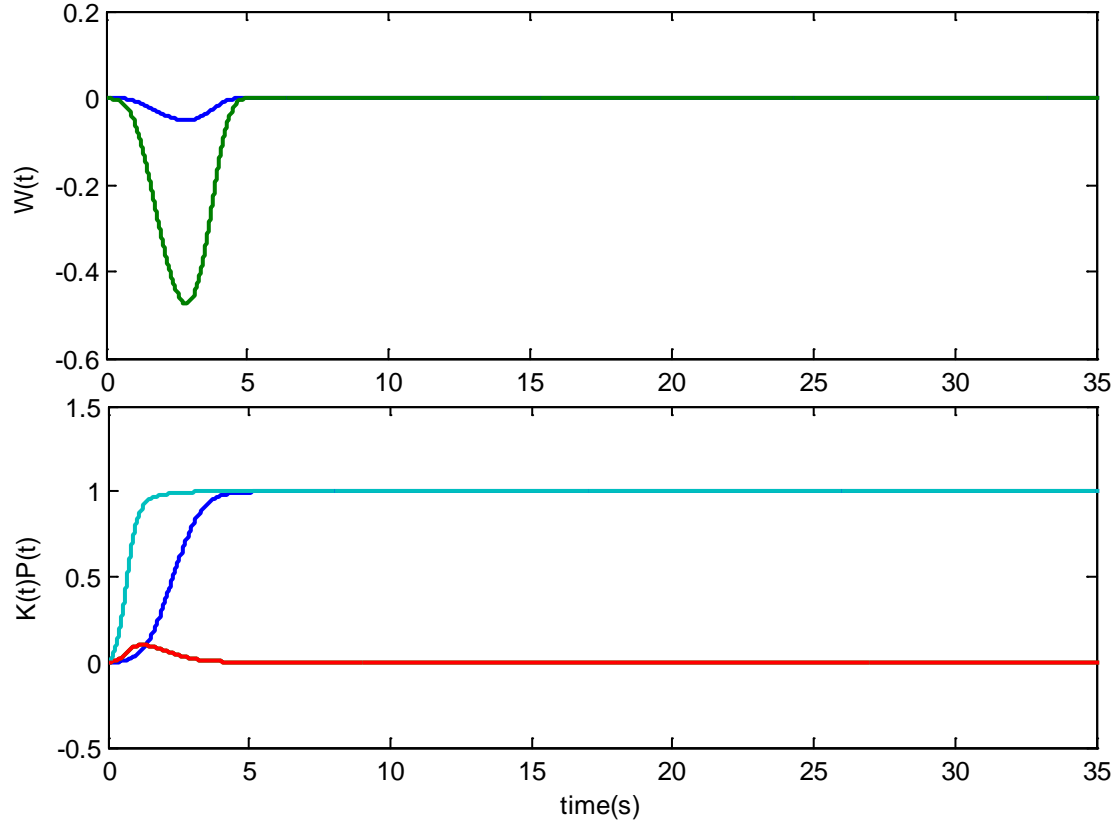
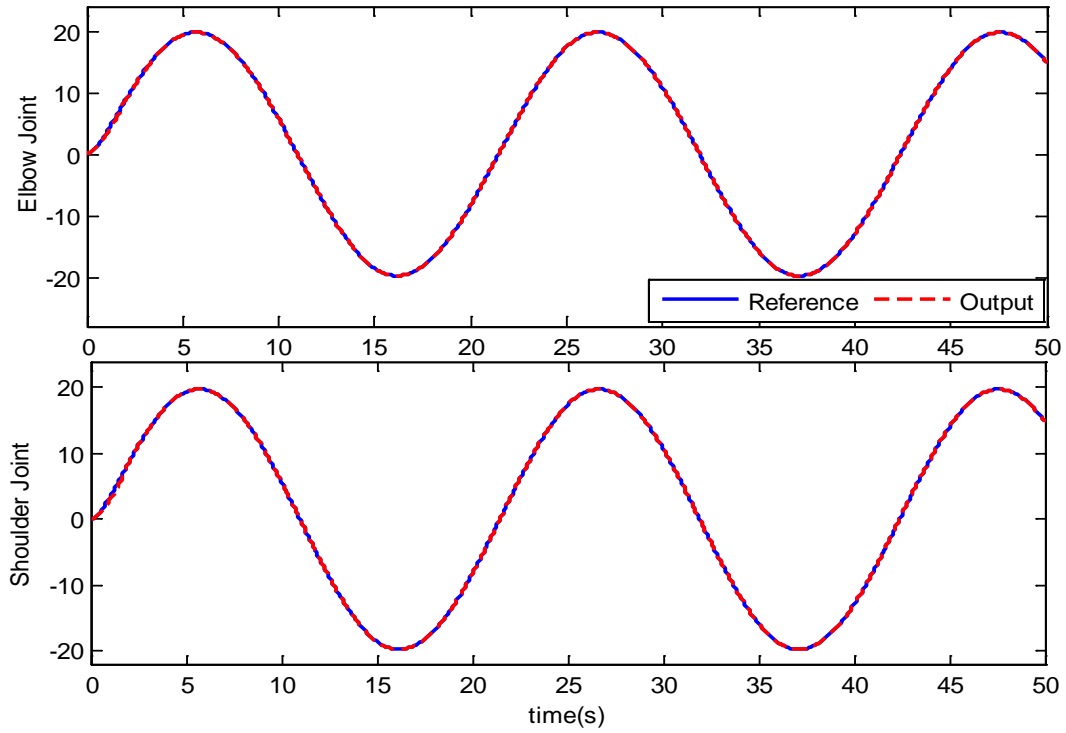
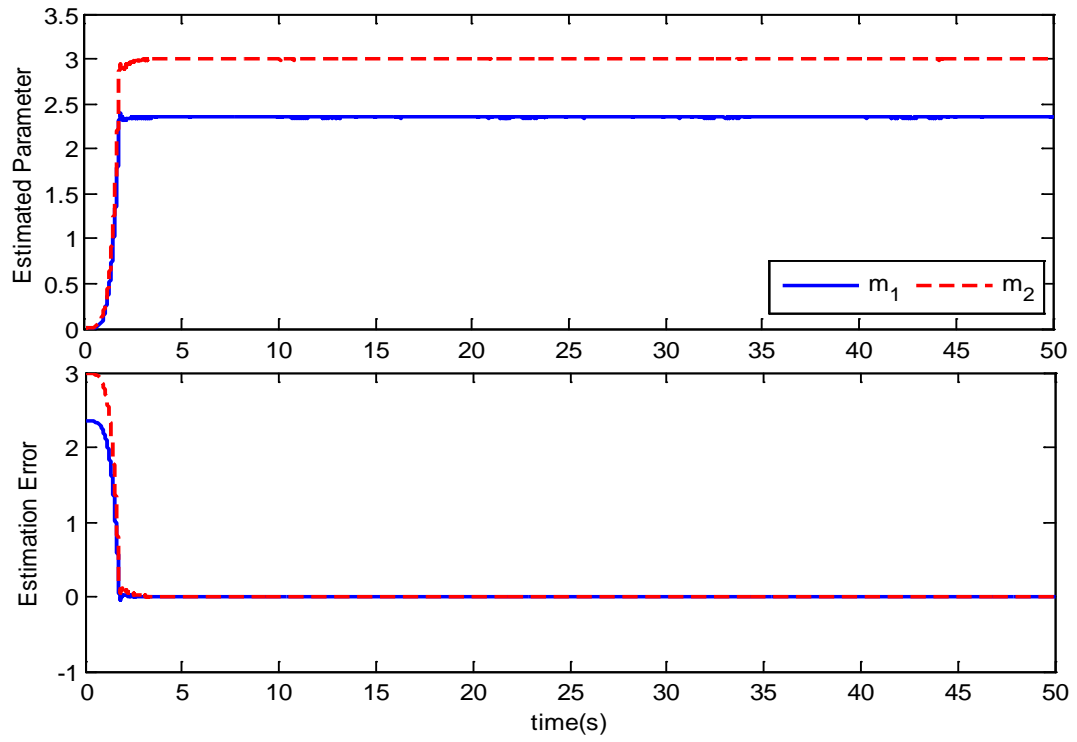


Fig.3 Parameter estimation error W_1 and matrix elements of K_1P_1 .

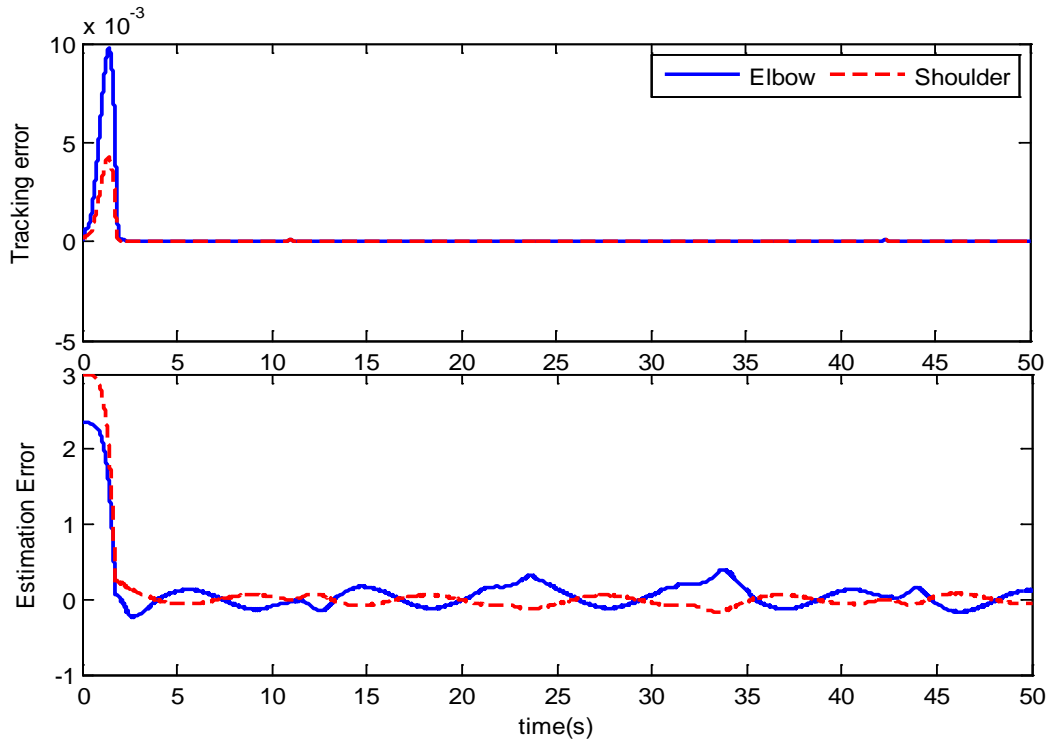


(a) Output tracking performance with control (52).

Fig.4 Adaptive finite-time control and parameter estimation.



(b) Parameter estimation with adaptation (60).
Fig.4 Adaptive finite-time control and parameter estimation.



(c) Tracking error and estimation error with disturbance.
Fig.4 Adaptive finite-time control and parameter estimation.