Robust Approximation to Multi-Period Inventory Management*

Chuen-Teck See† Melvyn Sim‡

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Abstract

We propose a robust optimization approach to address a multi-period, inventory control problem under ambiguous demands, that is, only limited information of the demand distributions such as mean, support and some measures of deviations. Our framework extends to correlated demands and is developed around a factor-based model, which has the ability to incorporate business factors as well as time series forecast effects of trend, seasonality and cyclic variations. We can obtain the parameters of the replenishment policies by solving a tractable deterministic optimization problem in the form of second order cone optimization problem (SOCP), with solution time, unlike dynamic programming approaches, is polynomial and independent on parameters such as replenishment lead time, demand variability, correlations, among others. The proposed truncated linear replenishment policy (TLRP), which is piecewise linear with respect to demand history, improves upon static and linear policies. Our computational studies also suggest that it performs better than simple heuristics derived from dynamic programming.

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†Department of Industrial and Systems Engineering, National University of Singapore. Email: see_chuenteck@yahoo.com.sg

‡NUS Business School, National University of Singapore and Singapore MIT Alliance (SMA). Email: dscsimm@nus.edu.sg. The research of the author is supported by SMA, NUS Risk Management Institute and NUS academic research grants R-314-000-066-122 and R-314-000-068-122.

1 Introduction

Inventory ties up working capital and incurs holding costs, reducing profit every day excess stock is held. Good inventory management has hence become crucial to businesses as they seek to continually improve their customer service and profit margins, in the heat of global competition and demand variability. Baldenius and Reichelstein [4] offered perhaps the most convincing study of the contribution of good inventory management to profitability. They studied inventories of publicly traded American manufacturing companies between 1981 and 2000, and they concluded that "Firms with abnormally high inventories have abnormally poor long-term stock returns. Firms with slightly lower than average inventories have good stock returns, but firms with the lowest inventories have only ordinary returns".

The ability to incorporate more realistic assumptions about product demand into inventory models is one key factor to profitability. Practical models of inventory would need to address the issue of demand forecasting while staying sufficiently immunized against uncertainty and maintaining tractability. In most industrial contexts, demand is uncertain. Many demand histories have factors that behave like random walks that evolve over time with frequent changes in their directions and rates of growth or decline. In practice, for such demand processes, inventory managers often rely on forecasts based on a time series of prior demands, which are often correlated over time. For example, a product demand may depend on factors such as market outlook, oil prices and so forth, and contains effects of trend, seasonality, cyclic variation and randomness.

In this paper, we address the problem of optimizing multi-period inventory using factor-based stochastic demand models, where the coefficients of the random factors can be forecasted statistically, perhaps using historical time-series data. We assume that the demands may be correlated and are ambiguous, that is, limited information of the demand distributions (only the mean, support and some measures of deviations) are available. Using robust optimization techniques, we develop a tractable methodology that uses past demand history to adaptively control multi-period inventory. Our model also includes a range of features such as delivery delay and capacity limit on order quantity.

Our work is closely related to the multi-period inventory control problem, a well studied problem in operations research. For the single product inventory control problem, it is well-known that the base-stock policy based on a critical fractile is optimum. See Scarf [40, 41], Azoury [3], Miller [36] and Zipkin [44]. For correlated demands, Veinott [43], characterized conditions under which a myopic policy is optimum. Extending the results of Veinott, Johnson and Thompson [31] considered an autoregressive, moving-average (ARMA) demand process, zero replenishment lead-time and no backlogs, and showed the optimality of a myopic policy when there demand in each period is bounded. Lovejoy [33] showed that a myopic critical-fractile policy is optimum or near optimum in some inventory models with adaptive demand processes, citing exponential smoothing on the demand process and Bayesian updating on uniformly distributed demand as examples. Song and Zipkin [42] addressed the case of Poisson demand, where the transition rate between states is governed by a Markov process.

Although optimum policies can be characterized in many interesting variants of inventory control problems, it is not easy to compute them efficiently, that is, in polynomial time with respect to the input size of the problem. In this paper, we use the term *tractable replenishment policy* if the parameters of the policy are polynomial in size and can be obtained in polynomial time. For instance, the celebrated optimum base-stock policy may not necessarily be a tractable one. Sampling-based approximation have been applied to the inventory control problem. Levi, Roundy, and Shmoys [35] gave theoretical results on the sample size required to achieve ϵ-optimum solution. They showed that when the sample size is greater than $\frac{9}{2\epsilon^2}\left(\frac{\min(b,h)}{b+h}\right)^{-2}\ln(\frac{2}{\delta})$, the solution of the sample average approximation is at most $1+\epsilon$ times the optimum solution with probability of at least $1-\delta$, with b being the backlog cost and h being the holding cost. For instance, for the case of $b = 100$ and $h = 2$, it would require an exuberant 6×10^8 independent demand samples to ensure a 99%-optimum solution with 99% confidence. Using marginal cost accounting and cost-balancing techniques, Levi et. al. [34] proposed an elegant 2-approximation algorithm for the inventory control problem. However, there is a lack of computational studies demonstrating the effectiveness of the approximation algorithm. Other sampling-based approaches include infinitesimal perturbation analysis (see Glasserman et. al. [28]) which uses stochastic gradient estimation technique, and the concave adaptive value estimation procedure, which successively approximates the objective cost function with a sequence of piecewise linear functions (see Godfrey et. al., [29] and Powell et. al. [38]).

One of the fundamental assumptions of stochastic models, which has recently been challenged, is the availability of probability distributions in characterizing uncertain parameters. Bertsimas and Thiele [16] illustrated that an optimum inventory control policy that is heavily tuned to a particular demand distribution may perform poorly against another demand distribution bearing the same mean and variance. One approach to account for distributional ambiguity is to consider a family of demand distributions, which can be characterized by their descriptive statistics such as partial moments information, support and so forth. Research on inventory control under ambiguous demand distributions dates back to Scarf [39], where he considered a newsvendor problem and determined orders that minimize the maximum expected cost over all possible demand distributions with the same first and second moments and with non-negative support. Various extensions of Scarf's single period results have been studied by Gallego and Moon [26]. Although the solutions to these single period models are in the form of second order cone optimization problem (SOCP), which are polynomial time solvable, the minimax approach does not scale well computationally with the number of periods. Nevertheless, the optimum policies for multi-period inventory control problems under various forms of demand ambiguity have been characterized by Kasugai and Kasegai [32] and Gallego, Ryan and Simchi-Levi [27].

In recent years, robust optimization has witnessed an explosive growth and has become a dominant approach to address optimization problem under uncertainty. Traditionally, the goal of robust optimization is to immunize uncertain mathematical optimization problems against infeasibility while preserving the tractability of the models. See, for instance, Ben-Tal and Nemirovski [7, 8, 9], Bertsimas and Sim [13, 14], Bertsimas, et al. [12], El-Ghaoui and Lebret [23], and El-Ghaoui, et al. [24]. Many robust optimization approaches have the following two important characteristics:

(a) The model of data uncertainty in robust optimization permits distributional ambiguity. Data uncertainty can also be completely distributional-free and specified by an uncertainty set parameterized by the "Budget of Uncertainty", which controls the size of uncertainty set. Another model of uncertainty is to consider uncertain parameters with unknown distributions but having the same descriptive statistics such as known means and variances.

(b) The solution (or approximate solution) to a robust optimization model can be obtained by solving a tractable deterministic mathematical optimization problem such as SOCP, whose associated solvers are commercially available, robust and efficiently optimized. Robust optimization methodology often decouples model formulation from the optimization engine, which enables the modeler to focus on modeling the actual problem and not to be hindered by algorithm design.

Based on the framework of robust optimization, Bertsimas and Thiele [16] developed a new approach to address demand ambiguity in a multi-period inventory control problem, which has the advantage of being computationally tractable. They considered a family of demand distributions similar to Scarf and enforced independence across time periods. Bertsimas and Thiele mapped the demand uncertainty model into a "Budget of Uncertainty" model of Bertsimas and Sim [14] and proposed an open-loop inventory control approach in which the solutions can be obtained by solving a tractable linear optimization problem. They showed that the optimum solution of robust model has a base-stock structure and the tractability of the problem readily extends to problems with capacities and over networks, and their paper characterized the optimum policies for these cases. The analysis of the robust models and computational experiments for independent demands suggest that robust approaches compare well against an optimum model under exact distribution and is yet immunized against distributional ambiguity. Using similar approach, Adida and Perakis [1] proposed a deterministic robust optimization formulation for dealing with demand uncertainty in a dynamic pricing and inventory control problem for a make-to-stock manufacturing system. They developed a demand-based fluid model and showed that the robust formulation is not much harder to solve than the nominal problem. Other related work in the robust inventory control literature includes Beinstock and Ozbay [18], where they proposed a robust model focusing on base-stock policy structure.

To address the inadequacy of open-loop robust optimization models involving multistage decision process, Ben-Tal et al. [5] introduced the concept of adjustable robust counterpart, which permits decisions to be delayed until the availability of information. Unfortunately, with the additional flexibility in modeling, adjustable robust counterpart models are generally NP-hard and the authors have proposed and advocated the use of linear decision rule as a tractable approximation. Ben-Tal et.al. [6] applied their model to a multi-period inventory control problem and showed by means of computational studies, the advantages of the linear replenishment policy over the open-loop model in which the replenishment policy is static. We emphasize that in contrast to stochastic models, the uncertainty considered in adjustable robust counterpart is completely distribution-free, that is, the data uncertainty is characterized only by its support. Due to the different models of uncertainty, it is meaningless to compare adjustable robust counterpart models vis-a-vis stochastic ones.

To bridge the gap between robust optimization and stochastic models, Chen et al. [20] introduced the notions of directional deviations known as *forward and backward deviations* and proposed computationally tractable robust optimization models for immunizing linear optimization problems against infeasibility, which enhanced the modeling power of robust optimization in the characterization of ambiguous distributions. In a parallel work, Chen et al. [21] proposed several piecewise linear decision rules for approximating stochastic linear optimization problems that improve upon linear rules. These approaches have been unified by Chen and Sim [22][1], where they proposed a general family of distributions characterized by the mean, covariance, directional deviations and support and showed how it can be extended to approximate solution for a two period stochastic model under a satisficing objective.

In this paper, we extend these new ideas of robust optimization to the multi-period inventory control

[1] Note that the first alphabetically ordered author bearing the same last name in Chen et al. [20] and Chen et al. [21] is a different person from that of Chen and Sim [22].

problem. Instead of the "Budget of Uncertainty" demand model, we focus on uncertain demands being robustly characterized by their descriptive statistics. The former requires specification on the size of uncertainty set, which, as exemplified in Berstsimas and Sim [15], can be dependent on the types of stochastic optimization problem we are addressing. We feel that the "Budget of Uncertainty" approach to uncertainty, though has its strengths, is less appealing when we compare vis-a-vis stochastic demand models, which is the case in this paper. Our contributions over the related works of Bertsimas and Thiele [16] and Ben-Tal et.al. [6] can be summarized as follows:

(a) Our proposed robust optimization approximation is based upon a comprehensive factor-based demand model that can capture correlations such as the autoregressive nature of demand, the effect of external factors, as well as trends and seasonality, among others. In addition, we provide for distributional ambiguity in the underlying factors by considering a family of distributions characterized by the mean, covariance, support and directional deviations. In contrast, the robust optimization model of Bertsimas and Thiele [16] is restricted to independent demands with an identical mean and variance, while the model of Ben-Tal et.al. [6] is confined to completely distribution-free demand uncertainty.

(b) We propose a new policy called the truncated linear replenishment policy and show that it gives improved approximation to the multi-period inventory control problem over static and linear decision rules used in the robust optimization proposals of Bertsimas and Thiele [16] and Ben-Tal et.al. [6] respectively. We also *do not* restrict the policy structure to base-stock. We develop a new bound on a nested sum of expected positive values of random variables and show that the parameters of the truncated linear replenishment policy can be obtain by solving a tractable deterministic mathematical optimization problem in the form of SOCP, whose solution time is independent on replenishment lead time, demand variability, correlations, among others. Although we are unable to quantify the level of approximation, to the best of knowledge, this is the best tractable deterministic approximation of the multi-period inventory control problem to date.

(c) We study the computational performance of the static, linear and truncated linear replenishment policies against the optimum history dependent policy and two dynamic programming based heuristics, namely, the myopic policy and a history independent base-stock policy. We analyze the impact of the solutions over realistic ranges of planing horizon, cost parameters and demand correlations. In contrast, the computational experiment of Bertsimas and Thiele [16] is confined to independent demands, while the experiment considered in Ben-Tal et.al. [6] does not benchmark against stochastic demand models. Our computational results show that even under an unrealistic assumption that the demand distributions are available, the truncated linear replenishment policy, together with information on the directional deviations, yield reasonably good solutions against the optimum and give the best overall performance among tractable policies and simple dynamic programming based heuristics.

This paper is organized as follows. In Section 2 we describe a stochastic inventory model. We formulate our robust inventory models in Section 3 and discuss extensions in Section 4. We describe computational results in Section 5 and conclude the paper in Section 6.

Notations: Throughout this paper, we denote a random variable with the tilde sign such as $\tilde{y}$ and vectors with bold face lower case letters such as $\boldsymbol{y}$. We use $\mathbf{y}'$ to denote the transpose of vector $\mathbf{y}$. Also, denote $y^+ = \max(y, 0)$, $y^- = \max(-y, 0)$, and $\|\boldsymbol{y}\|_2 = \sqrt{\sum y_i^2}$.

2 Stochastic Inventory Model

The stochastic inventory model involves the derivation of replenishment decisions over a discrete planning horizon consisting of a finite number of periods under stochastic demand. The demand for each period is usually a sequence of random variables which are *not* necessarily identically distributed and *not* necessarily independent. We consider an inventory system with T planning horizons from $t = 1$ to $t = T$. External demands arrive at the inventory system and the system replenishes its inventory from some central warehouse (or supplier) with ample supply. The time line of events is as follows:

1. At the beginning of the tth time period, before observing the demand, the inventory manager places an order of x_t at unit cost c_t for the product to be arrived after a (fixed) order lead-time of L periods. Orders placed at the *beginning* of the tth time period will arrive at the *beginning* of $t + L$th period. We assume that replenishment ceases at the end of the planning horizon, so that the last order is placed in period $T - L$. Without loss of generality, we assume that purchase costs for inventory are charged at the time of order. The case where purchase costs are charged at the time of delivery can be represented by a straight-forward shift of cost indices.

2. At the beginning of each time period t, the inventory manager faces an initial net-inventory y_t and receives an order of x_{t-L}. The demand of inventory for the period is realized at the end of the time period. After receiving a demand of d_t, the net-inventory at the end of the period is $y_t + x_{t-L} - d_t$.

3. Excess inventory is carried to the next period incurring a per-unit overage (holding) cost. On the other hand, each unit of unsatisfied demand is backlogged (carried over) to the next period with a per-unit underage (backlogging) penalty cost. At the last period, $t = T$, the penalty of lost sales can be accounted through the underage cost.

We assume a risk neutral inventory manager whose objective is to determine the dynamic ordering quantities x_t from period $t = 1$ to period $t = T - L$ so as to minimize the total expected ordering, inventory overage (holding) and underage (backlog) costs in response to the uncertain demands. Observe that for $L \geq 1$, the quantities x_{t-L}, $t = 1, \ldots, L$ are known values. They denote orders made before period $t = 1$ and are inventories in the delivery pipeline when the planning horizon starts.

We introduce the following notations:

- $\tilde{d}_t$: stochastic exogenous demand at period t
- $\tilde{\boldsymbol{d}}_t$: a vector of random demands from period 1 to t, that is, $\tilde{\boldsymbol{d}}_t = (\tilde{d}_1, \ldots, \tilde{d}_t)$
- $x_t(\tilde{\boldsymbol{d}}_{t-1})$: order placed at the beginning of the tth time period after observing $\tilde{\boldsymbol{d}}_{t-1}$. The first period inventory order is denoted by $x_1(\tilde{\boldsymbol{d}}_0) = x_1^0$
- $y_t(\tilde{\boldsymbol{d}}_{t-1})$: net-inventory at the beginning of the tth time period
- h_t: unit inventory overage (holding) cost charged on excess inventory at the end of the tth time period
- b_t: unit underage (backlog) cost charged on backlogged inventory at the end of the tth time period
- c_t: unit purchase cost of inventory for orders placed at the tth time period
- S_t: the maximum amount that can be ordered at the tth time period.

We use $x_t(\tilde{\boldsymbol{d}}_{t-1})$ to represent the non-anticipative replenishment policy at the beginning of period t. That is, the replenishment decision is based solely on the observed information available at the beginning of period t, which is given by the demand vector $\tilde{\boldsymbol{d}}_{t-1} = (\tilde{d}_1, \ldots, \tilde{d}_{t-1})$. Given the order quantity $x_{t-L}(\tilde{\boldsymbol{d}}_{t-L-1})$ and stochastic exogenous demand $\tilde{d}_t$, the net-inventory at the *end* of the t time period (which is also the net-inventory at start of $t + 1$ time period) is given by

$$y_{t+1}(\tilde{\boldsymbol{d}}_t) = y_t(\tilde{\boldsymbol{d}}_{t-1}) + x_{t-L}(\tilde{\boldsymbol{d}}_{t-L-1}) - \tilde{d}_t, \qquad t = 1, \ldots, T. \tag{1}$$

In resolving the initial boundary conditions, we adopt the following notations:

- The initial net-inventory of the system is $y_1(\tilde{\boldsymbol{d}}_0) = y_1^0$.
- When $L \geq 1$, the orders that are placed before the planning horizon starts are denoted by

$$x_t(\tilde{\boldsymbol{d}}_{t-1}) = x_t^0, \qquad t = 1-L, \ldots, 0.$$

Note that Equation (1) can be written using the cumulative demand up to period t and cumulative order received as follows:

$$y_{t+1}(\tilde{\boldsymbol{d}}_t) = \underbrace{y_1^0}_{\text{initial inventory}} + \underbrace{\sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0}_{\text{committed orders}} + \underbrace{\sum_{\tau=L+1}^{t} x_{\tau-L}(\tilde{\boldsymbol{d}}_{\tau-L-1})}_{\text{order decisions}} - \underbrace{\sum_{\tau=1}^{t} \tilde{d}_\tau}_{\text{cumulative demands}}. \tag{2}$$

Observe that positive (respectively negative) value of $y_{t+1}(\tilde{\boldsymbol{d}}_t)$ represents the total amount of inventory overage (respectively underage) at the end of the period t after meeting demand. Thus, the total expected cost, including ordering, inventory overage and underage charges is equal to

$$\sum_{t=1}^{T}\left(\mathrm{E}\left(c_t x_t(\tilde{\boldsymbol{d}}_{t-1})\right) + \mathrm{E}\left(h_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^+\right) + \mathrm{E}\left(b_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^-\right)\right).$$

Therefore, the multi-period inventory problem can be formulated as a T stage stochastic optimization model as follows:

$$\begin{aligned} Z_{STOC} = \min \quad & \sum_{t=1}^{T}\left(\mathrm{E}\left(c_t x_t(\tilde{\boldsymbol{d}}_{t-1})\right) + \mathrm{E}\left(h_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^+\right) + \mathrm{E}\left(b_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^-\right)\right). \\ \text{s.t.} \quad & y_{t+1}(\tilde{\boldsymbol{d}}_t) = y_t(\tilde{\boldsymbol{d}}_{t-1}) + x_{t-L}(\tilde{\boldsymbol{d}}_{t-L-1}) - \tilde{d}_t \qquad t = 1, \ldots, T \\ & 0 \leq x_t(\tilde{\boldsymbol{d}}_{t-1}) \leq S_t \qquad t = 1, \ldots, T-L \end{aligned} \tag{3}$$

The aim of the stochastic optimization model is to derive a feasible replenishment policy that minimizes the expected ordering and inventory costs. That is, we seek a sequence of action rules that advises the inventory manager the action to take in time t as a function of demand history. Unfortunately, the decision variables in Problem (3), $x_t(\tilde{\boldsymbol{d}}_{t-1}), t = 1 \ldots T-L$ and $y_t(\tilde{\boldsymbol{d}}_{t-1}), t = 2 \ldots T+1$ are functionals, which means that Problem (3) is an optimization problem with infinite number of variables and constraints, and hence generally intractable.

The stochastic optimization problem (3) can also be formulated as a dynamic programming problem. For simplicity, assuming zero lead-time, the dynamic programming requires the following updates on the value function:

$$\begin{aligned} J_t(y_t, d_1, \ldots, d_{t-1}) \quad = \quad & \min_{x \in [0, S_t]} \mathrm{E}\Big(c_t x + r_t(y_t + x - \tilde{d}_t) + \\ & J_{t+1}(y_t + x - \tilde{d}_t, d_1, \ldots, d_{t-1}, \tilde{d}_t) \mid \tilde{d}_1 = d_1, \ldots, \tilde{d}_{t-1} = d_{t-1}\Big), \end{aligned}$$

where $r_t(u) = h_t \max(u, 0) + b_t \max(-u, 0)$. Maintaining the value function $J_t(\cdot)$ is computationally prohibitive, and hence most inventory control literatures identify conditions such that the value functions are not dependent on past demand history, so that the state space is computationally amenable. For instance, it is well known that when the lead-time is zero and the demands are independently distributed across time periods, there exists base-stock levels, q_t such that the following replenishment policy,

$$x_t(\tilde{\boldsymbol{d}}_{t-1}) = \min\left\{\max\left\{q_t - y_t(\tilde{\boldsymbol{d}}_{t-1}), 0\right\}, S_t\right\} \tag{4}$$

is optimum. Hence, instead of being a function of the entire demand history, the optimum demand policy can be characterized by the net-inventory level as follow:

$$x_t(y_t) = \min\left\{\max\left\{q_t - y_t, 0\right\}, S_t\right\}.$$

Note that in order to obtain an optimum history independent base-stock policy for positive lead-time, $L > 0$, we require some restrictions on the cost parameters. See for instance, Zipkin [44].

3 Robust Inventory Model

Stochastic inventory control problem requires full information of the demand distributions, which is practically prohibitive. Furthermore, even if the probability distributions are known, due to computational complexity, we may not be able to obtain the optimum solution based on the risk neutral preference. The robust optimization approach we are proposing aims to address these issues collectively. As such, the modeler remains risk neutral but he/she is uncertain of the underlying probability distributions and that computing the optimum policy is computationally prohibitive. We first relax the assumption of full distributional knowledge and modify the representation of uncertain demands with the aim of producing a computationally tractable model.

3.1 Factor-based Demand Model

We introduce a factor-based demand model in which the uncertain demand are affinely dependent on zero mean random factors $\tilde{\boldsymbol{z}} \in \Re^N$ as follows:

$$d_t(\tilde{\boldsymbol{z}}) \triangleq \tilde{d}_t = d_t^0 + \sum_{k=1}^{N} d_t^k \tilde{z}_k, \qquad t = 1, \ldots, T,$$

where

$$d_t^k = 0 \qquad \forall k \geq N_t + 1,$$

and $1 \leq N_1 \leq N_2 \leq \cdots \leq N_T = N$. Under a factor-based demand model, the random factors, $\tilde{z}_k$, $k = 1, \ldots, N$ are realized sequentially. At period t, the factors, $\tilde{z}_k$, $k = 1, \ldots, N_t$ has already been unfolded. In progressing to period $t+1$, the new factors $\tilde{z}_k$, $k = N_t + 1, \ldots, N_{t+1}$ are made available.

Demand that is affected by random noise or shocks can be represented by the factor-based demand model. For independently distributed demand, which is assumed in most inventory models, we have

$$d_t(\tilde{\boldsymbol{z}}) = d_t^0 + \tilde{z}_t, \qquad t = 1, \ldots, T,$$

in which $\tilde{z}_t$ are independently distributed. However, in many industrial contexts, demands across periods may be correlated. In fact, many demand histories behave more like random walks over time with frequent changes in directions and rate of growth or decline. See Johnson and Thompson [31] and Graves [30]. In those settings, we may consider standard forecasting techniques such as an ARMA(p, q) demand process (see Box et al. [19]) as follows:

$$d_t(\tilde{\boldsymbol{z}}) = \begin{cases} d_t^0 & \text{if } t \leq 0 \\ \displaystyle\sum_{j=1}^{p} \phi_i d_{t-j}(\tilde{\boldsymbol{z}}) + \tilde{z}_t + \sum_{j=1}^{\min\{q,t-1\}} \theta_i \tilde{z}_{t-j} & \text{otherwise,} \end{cases}$$

where $\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q$ are known constants. Indeed, its is easy to show by induction that $d_t(\tilde{\boldsymbol{z}})$ can be expressed in the form of a factor-based demand model. Song and Zipkin [42] presented a world driven demand model where the demand is Poisson with rate controlled by finite Markov states representing different business environments. However, it may be difficult to determine exhaustively the business states and their state transition probabilities. On the other hand, factor-based models have been used extensively in finance for modeling returns as affine functions of external factors, in which the coefficients of the factors can be determined statistically. In the same way, we can apply the factor-based demand model to characterize the influence of demands with external factors such as market outlook, oil prices and so forth. Effects of trend, seasonality, cyclic variation, and randomness can also be incorporated.

Instead of assuming full distributions on the factors, which is practically prohibitive, we adopt a modest distributional assumption on the random factors, such as known means, supports and some aspects of deviations. The factors may be partially characterized using the directional deviations, which are recently introduced by Chen et al. [20].

Definition 1 *(Directional deviations) Given a random variable $\tilde{z}$ with zero mean, the forward deviation is defined as*

$$\sigma_f(\tilde{z}) \triangleq \sup_{\theta > 0} \left\{ \sqrt{2 \ln(\mathrm{E}(\exp(\theta \tilde{z})))/\theta^2} \right\} \tag{5}$$

and backward deviation is defined as

$$\sigma_b(\tilde{z}) \stackrel{\Delta}{=} \sup_{\theta>0} \left\{ \sqrt{2\ln(\mathrm{E}(\exp(-\theta\tilde{z})))/\theta^2} \right\}. \tag{6}$$

Given a sequence of independent samples, we can essentially estimate the magnitude of the directional deviations from (5) and (6). Some of the properties of the directional deviations include:

Proposition 1 *(Chen et al. [20])*

Let σ, p and q be respectively the standard, forward and backward deviations of a random variable, $\tilde{z}$ with zero mean.

(a)

$$p \geq \sigma \qquad q \geq \sigma.$$

If $\tilde{z}$ is normally distributed, then $p = q = \sigma$.

(b) *For all $\theta \geq 0$,*

$$\mathrm{P}(\tilde{z} \geq \theta p) \leq \exp(-\theta^2/2);$$

$$\mathrm{P}(\tilde{z} \leq -\theta q) \leq \exp(-\theta^2/2).$$

Proposition 1(a) shows that the directional deviations are no less than the standard deviation of the underlying distribution, and under the normal distribution, these two values coincide with the standard deviation. As exemplified in Proposition 1(b), the directional deviations provide an easy bound on the distributional tails. The advantage of using the directional deviations is the ability to capture distributional asymmetry and stochastic independence, while keeping the resultant optimization model computationally amicable. We refer the reader to the paper by Natarajan et al. [37] for the computational experience of using directional derivations derived from real-life data.

In this paper, we adopt the random factor model introduced by Chen and Sim [22], which encompasses most of the uncertainty models found in the literatures of robust optimization.

Assumption U: We assume that the uncertainties $\{\tilde{z}_j\}_{j=1:N}$ are zero mean random variables, with positive definite covariance matrix, $\mathbf{\Sigma}$. Let $\mathcal{W}$ be the smallest convex set containing the support of $\tilde{\boldsymbol{z}}$. We denote a subset, $\mathcal{I} \subseteq \{1, \ldots, N\}$, which can be an empty set, such that $\tilde{z}_j$, $j \in \mathcal{I}$ are stochastically independent. Moreover, the corresponding forward and backward deviations are given by $p_j = \sigma_f(\tilde{z}_j)$ and $q_j = \sigma_b(\tilde{z}_j)$ respectively for $j \in \mathcal{I}$ and that $p_j = q_j = \infty$ for $j \notin \mathcal{I}$.

The choice of the support set, $\mathcal{W}$ can influence the computational tractability of the problem. Henceforth, we assume that the support set is a second order conic representable set (a.k.a conic quadratic representable set) proposed in Ben-Tal and Nemirovski [7], which includes polyhedral and ellipsoidal sets. A common support set is the interval set, which is given by $\mathcal{W} = [-\underline{\boldsymbol{z}}, \bar{\boldsymbol{z}}]$, in which $\underline{\boldsymbol{z}}, \bar{\boldsymbol{z}} > \mathbf{0}$.

For notational convenience, we define the following sets:

$$\begin{array}{ll} \mathcal{I}_1 \triangleq \{i \ : \ p_j < \infty\} & \bar{\mathcal{I}}_1 \triangleq \{i \ : \ p_j = \infty\} \\ \mathcal{I}_2 \triangleq \{i \ : \ q_j < \infty\} & \bar{\mathcal{I}}_2 \triangleq \{i \ : \ q_j = \infty\}. \end{array}$$

Furthermore, if $p_j = \infty$ (respectively $q_j = \infty$), its product with zero remains zero, that is, $p_j \times 0 = 0$ (respectively $q_j \times 0 = 0$).

3.2 Bound on $\mathrm{E}((\cdot)^+)$

In the absence of full distributional information, it would be meaningless to evaluate the optimum objective as depicted in Problem (3). Instead, we aim to minimize a good upper bound on the objective function. Such approach of soliciting inventory decisions based on partial demand information is not new. In the 50s, Scarf [39] considered a min-max newsvendor problem with uncertain demand $\tilde{d}$ given by only its mean and standard deviations. Scarf was able to obtain solutions to the tight upper bound of the newsvendor problem. The central idea in addressing such problem is to solicit a good upper bound on $\mathrm{E}((\cdot)^+)$, which appears at the objective of the newsvendor problem and also in Problem (3). The following result is well known:

Proposition 2 *(Scarf's upper bound [39]) Let $\tilde{z}$ be a random variable in $[-\mu, \infty)$ with mean μ and standard deviation σ, then for all $a \geq -\mu$,*

$$\mathrm{E}((\tilde{z} - a)^+) \leq \begin{cases} \dfrac{1}{2}\left(-a + \sqrt{\sigma^2 + a^2}\right) & \text{if } a \geq \frac{\sigma^2 - \mu^2}{2\mu} \\ -a\dfrac{\mu^2}{\mu^2 + \sigma^2} + \mu\dfrac{\sigma^2}{\mu^2 + \sigma^2} & \text{if } a < \frac{\sigma^2 - \mu^2}{2\mu} \end{cases}.$$

Moreover, the bound is achievable.

Interestingly, Bertsimas and Thiele [16] used the bound of Proposition 2 to calibrate the "Budget of Uncertainty" parameter in their robust inventory models. Unfortunately, it is generally computationally intractable to evaluate tight probability bounds involving multivariate random variables with known moments and support information (see Bertsimas and Popescus [17]). We adopt the bounds of Chen and Sim [22] to evaluate the expected positive part of an affine sum of random variables under the Assumption U.

Definition 2 *We say a function, $f(\boldsymbol{z})$ is* non-zero crossing *with respect to $\boldsymbol{z} \in \mathcal{W}$ if at least one of the following conditions hold:*

1. $f(\boldsymbol{z}) \geq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W}$
2. $f(\boldsymbol{z}) \leq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W}.$

Theorem 1 *(Chen and Sim [22]) Let $\tilde{\boldsymbol{z}} \in \Re^N$ be a multivariate random variable under the Assumption U. Then*

$$\mathrm{E}((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+) \leq \pi(y_0, \boldsymbol{y}),$$

where $\pi(y_0, \boldsymbol{y})$ is given by

$$\begin{array}{rll}
\pi(y_0, \boldsymbol{y}) = \min & r_1 + r_2 + r_3 + r_4 + r_5 & \\
\text{s.t.} & y_{10} + \max_{\boldsymbol{z} \in \mathcal{W}} \boldsymbol{z}'\boldsymbol{y}_1 \leq r_1 & \\
& 0 \leq r_1 & \\
& \max_{\boldsymbol{z} \in \mathcal{W}} \boldsymbol{z}'(-\boldsymbol{y}_2) \leq r_2 & \\
& y_{20} \leq r_2 & \\
& \frac{1}{2} y_{30} + \frac{1}{2} \|(y_{30}, \boldsymbol{\Sigma}^{1/2}\boldsymbol{y}_3)\|_2 \leq r_3 & \\
& \inf_{\mu > 0} \frac{\mu}{e} \exp\left(\frac{y_{40}}{\mu} + \frac{\|\boldsymbol{u}\|_2^2}{2\mu^2}\right) \leq r_4 & \\
& u_j \geq p_j y_{4j} \ \forall j \in \mathcal{I}_1, \qquad y_{4j} \leq 0 \ \forall j \in \bar{\mathcal{I}}_1 & (7) \\
& u_j \geq -q_j y_{4j} \ \forall j \in \mathcal{I}_2, \qquad y_{4j} \geq 0 \ \forall j \in \bar{\mathcal{I}}_2 & \\
& y_{50} + \inf_{\mu > 0} \frac{\mu}{e} \exp\left(-\frac{y_{50}}{\mu} + \frac{\|\boldsymbol{v}\|_2^2}{2\mu^2}\right) \leq r_5 & \\
& v_j \geq q_j y_{5j} \ \forall j \in \mathcal{I}_2, \qquad y_{5j} \leq 0 \ \forall j \in \bar{\mathcal{I}}_2 & \\
& v_j \geq -p_j y_{5j} \ \forall j \in \mathcal{I}_1, \qquad y_{5j} \geq 0 \ \forall j \in \bar{\mathcal{I}}_1 & \\
& y_{10} + y_{20} + y_{30} + y_{40} + y_{50} = y_0 & \\
& \boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 + \boldsymbol{y}_4 + \boldsymbol{y}_5 = \boldsymbol{y}. & \\
& r_i, y_{i0} \in \Re, \boldsymbol{y}_i \in \Re^N, i = 1, \ldots, 5, \boldsymbol{u}, \boldsymbol{v} \in \Re^N &
\end{array}$$

Moreover, the bound is tight if $y_0 + \boldsymbol{y}'\boldsymbol{z}$ is a non-zero crossing function with respect to $\boldsymbol{z} \in \mathcal{W}$. That is, if

$$y_0 + \boldsymbol{y}'\boldsymbol{z} \geq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W}$$

we have $\mathrm{E}\left((\, y_0 + \boldsymbol{y}'\boldsymbol{z}\,)^+\right) = \pi(y_0, \boldsymbol{y}) = y_0$. Likewise, if

$$y_0 + \boldsymbol{y}'\boldsymbol{z} \leq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W},$$

we have $\mathrm{E}\left((\, y_0 + \boldsymbol{y}'\boldsymbol{z}\,)^+\right) = \pi(y_0, \boldsymbol{y}) = 0.$

Remark 1: Due to the presence of the constraints, $\inf_{\mu>0} \mu \exp\left(\frac{a}{\mu} + \frac{b^2}{\mu^2}\right) \leq c$, the set of constraints in Problem (7) is not exactly second order cone representable (see Ben-Tal and Nemirovski [10]). Fortunately, using a few second order cones, we can accurately approximate such constraints to a good level of numerical precision. The interested readers can refer to Chen and Sim [22].

Remark 2: Note that the first and third constraints involving the support set, $\mathcal{W}$ take the form of

$$v_0 + \max_{\boldsymbol{z} \in \mathcal{W}} \boldsymbol{v}'\boldsymbol{z} \leq 0$$

or equivalently as

$$v_0 + \boldsymbol{v}'\boldsymbol{z} \leq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W}.$$

Such a constraint is known as the robust counterpart whose explicit formulation under difference choices of tractable support set, $\mathcal{W}$ is well discussed in Ben-Tal and Nemirovski [7, 10]. Since $\mathcal{W}$ is a second order conic representable set, the robust counterpart is also second order cone representable. For instance, if $\mathcal{W} = [-\underline{\boldsymbol{z}}, \bar{\boldsymbol{z}}]$, the corresponding robust counterpart is representable by the following linear inequalities,

$$v^0 + \underline{\boldsymbol{z}}'\boldsymbol{t} + \bar{\boldsymbol{z}}'\boldsymbol{s} \leq 0$$

for some $\boldsymbol{s}, \boldsymbol{t} \geq \boldsymbol{0}$ satisfying $\boldsymbol{s} - \boldsymbol{t} = \boldsymbol{v}$.

Remark 3: Note that under the Assumption U, it is not necessary to provide all the information such as the directional deviations. So, whenever such information is unavailable, we can assign an infinite value to the corresponding parameter. For instance, suppose the factor $\tilde{z}_j$ has standard deviation, σ and unknown directional deviations, we would set $p_j = q_j = \infty$. With more information on the random factors, the bound of Problem (7) is never worse off.

Remark 4: In the absence of uncertainty, the non-zero crossing condition ensures that the bound is tight. That is, $y^+ = \mathrm{E}(y^+) = \pi(y, \boldsymbol{0})$.

The robust model of Bertsimas and Thielle [16] uses Proposition 2. We next show that for a univariate random variable with one-sided support, the bound of Theorem 1 is as tight.

Proposition 3 *Let $\tilde{z}$ be a random variable in $[-\mu, \infty)$ with mean μ and standard deviation σ, then for all $a \geq -\mu$,*

$$\mathrm{E}((\tilde{z} - a)^+) \leq \pi(-a, 1) = \begin{cases} \frac{1}{2}\left(-a + \sqrt{\sigma^2 + a^2}\right) & \text{if } a \geq \frac{\sigma^2 - \mu^2}{2\mu} \\ -a\frac{\mu^2}{\mu^2 + \sigma^2} + \mu\frac{\sigma^2}{\mu^2 + \sigma^2} & \text{if } a < \frac{\sigma^2 - \mu^2}{2\mu} \end{cases}$$

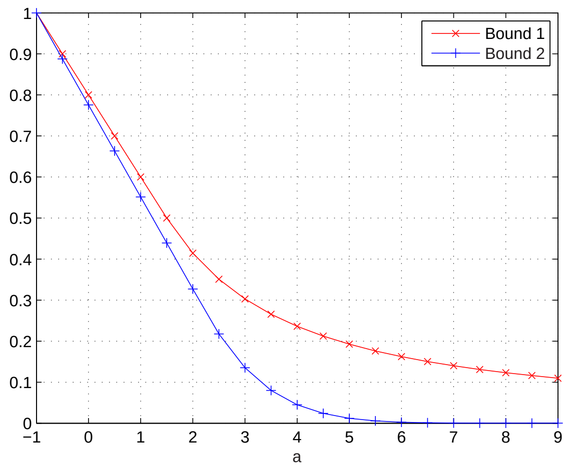

Figure 1: Comparing bounds of $\mathrm{E}((\tilde{z}-a)^+)$

Proof : See Appendix A.

We can further improve the bound if the distribution of the random variable $\tilde{z}$ is sufficiently light tailed such that the directional deviations are close to its standard deviation, such as those of normal and uniform distributions. Figure 1 compares the bounds of $\mathrm{E}((\tilde{z}-a)^+)$ in which $\mu = 1$ and $\sigma = \sigma_f(\tilde{z}) = \sigma_b(\tilde{z}) = 2$. Bound 1 corresponds to the bound of Proposition 2, while Bound 2 corresponds to the bound of Theorem 1. Clearly, despite the lack of tightness results, incorporating the directional deviations can potentially improve the bound on $\mathrm{E}((\tilde{z}-a)^+)$. We will further demonstrate the benefits in our computational experiments.

3.3 Tractable Replenishment Policies

Having introduced the demand uncertainty model, a suitable approximation of the replenishment policy $x_t(\tilde{\boldsymbol{d}}_{t-1})$ is needed to obtain a tractable formulation. That is, we seek a formulation in which the policy can be obtained by solving a optimization problem that runs in polynomial time and scalable across time period. We review two tractable replenishment policies, static as well as linear with respect to the random factors of demand, which are decision rules prevalent in the context of robust optimization. We introduce a new replenishment policy, known as the truncated linear replenishment policy, which improves over these policies.

Static replenishment policy

The static replenishment policy, a.k.a the open-loop policy, has order decisions not being influenced by the random factors of demand as follows:

$$x_t(\tilde{\boldsymbol{d}}_{t-1}) = x_t^0. \tag{8}$$

A tractable model under such replenishment policy is as follows:

$$\begin{aligned} Z_{SRP} = \min \quad & \sum_{t=1}^{T}\left(c_t x_t^0 + h_t \pi\left(y_{t+1}^0, \boldsymbol{y}_{t+1}\right) + b_t \pi\left(-y_{t+1}^0, -\boldsymbol{y}_{t+1}\right)\right) \\ \text{s.t.} \quad & y_{t+1}^0 = y_t^0 + x_{t-L}^0 - d_t^0 \qquad t = 1, \dots, T \\ & y_{t+1}^k = y_t^k - d_t^k \qquad k = 1, \dots, N, t = 1, \dots, T \\ & 0 \le x_t^0 \le S_t \qquad t = 1, \dots, T-L, \end{aligned} \tag{9}$$

with y_1^0 being the initial net-inventory and $y_1^k = 0$ for all $k = 1, \dots, N$. For $L \geq 1$, x_t^0 are the known committed orders made at time periods $t = 1-L, \dots, 0$.

Under Equation (8), it is evident from Equation (1) that the net-inventory level also takes an affine structure,

$$y_{t+1}(\tilde{\boldsymbol{d}}_t) = y_{t+1}^0 + \sum_{k=1}^{N} y_{t+1}^k \tilde{z}_k. \tag{10}$$

Using Theorem 1, we can bound the excess net-inventory at time period t, that is, $\mathrm{E}\left((y_{t+1}(\tilde{\boldsymbol{d}}_t))^+\right) \leq \pi(y_{t+1}^0, \boldsymbol{y}_{t+1})$. Proceeding similarly for the backlog inventory gives the objective function of Problem (9). Equating the coefficients of the constant and $\tilde{z}_k$ term of Equation (1) gives the first two sets of constraints in Problem (9) respectively. The last set of constraints enforce the range on order quantity, that is, non-negativity and upper limit.

Theorem 2 *The expected cost of the stochastic inventory problem under the static replenishment policy,*

$$x_t^{SRP}(\tilde{\boldsymbol{d}}_{t-1}) = x_t^{0*} \qquad t = 1, \dots, T-L$$

in which x_t^{0*}, $t = 1, \dots, T-L$ are the optimum solution of Problem (9), is at most Z_{SRP}.

Proof : See Appendix B.

Linear replenishment policy

A more refined replenishment policy introduced in Ben Tal et. al [6], and Chen et al. [20] is the linear replenishment policy where the order decisions are affinely dependent on the random factors of demand, that is,

$$x_t^{LRP}(\tilde{\boldsymbol{d}}_{t-1}) = x_t^0 + \boldsymbol{x}_t'\tilde{\boldsymbol{z}}, \tag{11}$$

in which the vector $\boldsymbol{x}_t = (x_t^1, \dots, x_t^N)$ satisfies the following non-anticipative constraints:

$$x_t^k = 0 \qquad \forall k \geq N_{t-1} + 1. \tag{12}$$

Since the order decision is made at the beginning of the tth period, the non-anticipative constraints ensure that the linear replenishment policy is not influenced by demand factors that are unavailable up to the beginning of the tth period. The model for the linear replenishment policy is as follows:

$$\begin{array}{rll} Z_{LRP} = \min & \displaystyle\sum_{t=1}^{T} \Big(c_t x_t^0 + h_t \pi\left(y_{t+1}^0, \boldsymbol{y}_{t+1}\right) + b_t \pi\left(-y_{t+1}^0, -\boldsymbol{y}_{t+1}\right) \Big) & \\ \text{s.t.} & y_{t+1}^k = y_t^k + x_{t-L}^k - d_t^k \qquad k = 0, \dots, N, t = 1, \dots, T & \\ & x_t^k = 0 \qquad\qquad\qquad \forall k \geq N_{t-1} + 1, t = 1, \dots, T - L & \\ & 0 \leq x_t^0 + \boldsymbol{x}_t'\boldsymbol{z} \leq S_t \qquad\qquad \forall \boldsymbol{z} \in \mathcal{W} \qquad t = 1, \dots, T - L & \end{array} \tag{13}$$

with y_1^0 being the initial net-inventory and $y_1^k = 0$ for all $k = 1, \dots, N$. For $L \geq 1$, x_t^0 are the known committed orders made at time periods $t = 1 - L, \dots, 0$.

Under Equation (11), the net-inventory level has a structure similar to Equation (10). The objective function and the first set of constraints are hence obtained in similar manner as Problem (9). The last set of constraints ensures that the linear replenishment policy is confined within the ordering capacity for all possible states of random factors. Observe that under the assumption that $\mathcal{W}$ is tractable conic representable uncertainty set, the robust counterpart

$$0 \leq x_t^0 + \boldsymbol{x}_t'\boldsymbol{z} \leq S_t \qquad \forall \boldsymbol{z} \in \mathcal{W}$$

can be represented concisely as tractable conic constraints. Therefore, Problem (13) is essentially a tractable conic optimization problem.

Theorem 3 *The expected cost of the stochastic inventory problem under the linear replenishment policy,*

$$x_t^{LRP}(\tilde{\boldsymbol{d}}_{t-1}) = x_t^{0*} + \boldsymbol{x}_t^{*\prime}\tilde{\boldsymbol{z}} \qquad t = 1, \dots, T - L$$

in which x_t^{k*}, $k = 0, \dots, N, t = 1, \dots, T - L$ are the optimum solution of Problem (13), is at most Z_{LRP}. Moreover, $Z_{LRP} \leq Z_{SRP}$.

Proof : See Appendix C.

Truncated linear replenishment policy

Chen et al. [21] studied the weakness of linear decision rules (or policy) and showed that carefully chosen piecewise linear decision rules can strengthen the approximation of stochastic optimization problems. Indeed, a base-stock policy such as Equation (4), can be shown by induction to be piecewise linear with respect to the historical demands. In the same spirit, we introduce a new piecewise linear replenishment policy, which we call the truncated linear replenishment policy. It takes the following form:

$$x_t^{TLRP}(\tilde{\boldsymbol{d}}_{t-1}) = \min\left\{\max\left\{x_t^0 + \boldsymbol{x}_t'\tilde{\boldsymbol{z}}, 0\right\}, S_t\right\}, \tag{14}$$

where the vector $\boldsymbol{x}_t = (x_t^1, \ldots, x_t^N)$ satisfies the following non-anticipative constraints:

$$x_t^k = 0 \qquad \forall k \geq N_{t-1} + 1. \tag{15}$$

Note that the truncated linear replenishment policy is piecewise linear and directly satisfies the ordering range constraint as follows:

$$0 \leq x_t^{TLRP}(\tilde{\boldsymbol{d}}_{t-1}) \leq S_t.$$

Before introducing the model, we present the following bound on a nested sum of expected positive values of random variables:

Theorem 4 *Let $\tilde{\boldsymbol{z}} \in \Re^N$ be a multivariate random variable under Assumption U. Then*

$$\mathrm{E}\left(\left(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + \sum_{i=1}^{p}\left(x_i^0 + \boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}\right)^+\right)^+\right) \leq \eta((y^0, \boldsymbol{y}), (x_1^0, \boldsymbol{x}_1), \ldots, (x_p^0, \boldsymbol{x}_p)) \tag{16}$$

where

$$\begin{aligned} & \eta((y^0, \boldsymbol{y}), (x_1^0, \boldsymbol{x}_1), \ldots, (x_p^0, \boldsymbol{x}_p)) \\ = & \min_{w_i^0, \boldsymbol{w}_i, i=1,\ldots,p}\left\{\pi\left(y^0 + \sum_{i=1}^{p} w_i^0, \boldsymbol{y} + \sum_{i=1}^{p}\boldsymbol{w}_i\right) + \sum_{i=1}^{p}\left(\pi(-w_i^0, -\boldsymbol{w}_i) + \pi(x_i^0 - w_i^0, \boldsymbol{x}_i - \boldsymbol{w}_i)\right)\right\}. \end{aligned}$$

Moreover, the bound is tight if $y^0 + \boldsymbol{y}'\boldsymbol{z} + \sum_{i=1}^{p}\left(x_i^0 + \boldsymbol{x}_i{}'\boldsymbol{z}\right)^+$ and $x_i^0 + \boldsymbol{x}_i{}'\boldsymbol{z}$, $i = 1, \ldots, p$ are non-zero crossing functions with respect to $\boldsymbol{z} \in \mathcal{W}$.

Proof : See Appendix D.

Remark : It is easy to establish that

$$\begin{aligned} \mathrm{E}\left(\left(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + \sum_{i=1}^{p}\left(x_i^0 + \boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}\right)^+\right)^+\right) & \leq \mathrm{E}\left(\left(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}}\right)^+\right) + \sum_{i=1}^{p}\mathrm{E}\left(\left(x_i^0 + \boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}\right)^+\right) \\ & \leq \pi\left(y^0, \boldsymbol{y}\right) + \sum_{i=1}^{p}\pi(x_i^0, \boldsymbol{x}_i). \end{aligned}$$

However, this is a weaker bound, considering the fact that

$$
\begin{aligned}
& \eta((y^0, \boldsymbol{y}), (x_1^0, \boldsymbol{x}_1), \ldots, (x_p^0, \boldsymbol{x}_p)) \\
= & \min_{w_i^0, \boldsymbol{w}_i, i=1,\ldots,p} \left\{ \pi\left(y^0 + \sum_{i=1}^{p} w_i^0, \boldsymbol{y} + \sum_{i=1}^{p} \boldsymbol{w}_i \right) + \sum_{i=1}^{p} \left(\pi(-w_i^0, -\boldsymbol{w}_i) + \pi(x_i^0 - w_i^0, \boldsymbol{x}_i - \boldsymbol{w}_i) \right) \right\} \\
\leq & \ \pi\left(y^0, \boldsymbol{y}\right) + \sum_{i=1}^{p} \pi(x_i^0, \boldsymbol{x}_i).
\end{aligned}
$$

The model for the truncated linear replenishment policy can be formulated as follows:

$$
\begin{aligned}
Z_{TLRP} = \min \quad & \sum_{t=1}^{T} c_t \pi(x_t^0, \boldsymbol{x}_t) + \sum_{t=1}^{L} \Big(h_t \pi(y_{t+1}^0, \boldsymbol{y}_{t+1}) + b_t \pi(-y_{t+1}^0, -\boldsymbol{y}_{t+1}) \Big) + \\
& \sum_{t=L+1}^{T} \Big(h_t \eta\left((y_{t+1}^0, \boldsymbol{y}_{t+1}), (-x_1^0, -\boldsymbol{x}_1), \ldots, (-x_{t-L}^0, -\boldsymbol{x}_{t-L})\right) + \\
& \qquad b_t \eta\left((-y_{t+1}^0, -\boldsymbol{y}_{t+1}), (x_1^0 - S_t, \boldsymbol{x}_1), \ldots, (x_{t-L}^0 - S_t, \boldsymbol{x}_{t-L})\right) \Big) \\
\text{s.t.} \quad & y_{t+1}^k = y_t^k + x_{t-L}^k - d_t^k \qquad k = 0, \ldots, N, t = 1, \ldots, T \\
& x_t^k = 0 \qquad\qquad \forall k \geq N_{t-1} + 1, t = 1, \ldots, T - L
\end{aligned}
\tag{17}
$$

with y_1^0 being the initial net-inventory and $y_1^k = 0$ for all $k = 1, \ldots, N$. For $L \geq 1$, x_t^0 are the known committed orders made at time periods $t = 1 - L, \ldots, 0$.

Under Equation (14), the net-inventory levels, $y_{t+1}(\tilde{\boldsymbol{d}}_t)$ are no longer affinely dependent on $\tilde{\boldsymbol{z}}$. The terms at the objective function account for the costs associated with excess net-inventory and backlog, taking into considerations of the piecewise linear policy. It can be shown that the truncated linear replenishment policy dominates over the linear replenishment policy as follows:

Theorem 5 *The expected cost of the stochastic inventory problem under the truncated linear replenishment policy,*

$$
x_t^{TLRP}(\tilde{\boldsymbol{d}}_{t-1}) = \min\left\{\max\left\{x_t^{0*} + \boldsymbol{x}_t^{*\prime}\tilde{\boldsymbol{z}}, 0\right\}, S_t\right\} \qquad t = 1, \ldots, T - L
$$

in which x_t^{k*}, $k = 0, \ldots, N$, $t = 1, \ldots, T - L$ are the optimum solution of Problem (17), is at most Z_{TLRP}. Moreover, $Z_{TLRP} \leq Z_{LRP}$.

Proof : See Appendix E.

Remark : For the case of unbounded ordering quantity, that is, $S_t = \infty$, the truncated linear replenishment policy becomes,

$$
x_t^{TLRP}(\tilde{\boldsymbol{d}}_{t-1}) = \max\left\{x_t^0 + \boldsymbol{x}_t'\tilde{\boldsymbol{z}}, 0\right\},
$$

and we can simplify Problem (17) as follows:

$$
\begin{aligned}
Z_{TLRP} = \min \quad & \sum_{t=1}^{T} c_t \pi(x_t^0, \boldsymbol{x}_t) + \sum_{t=1}^{L} \Big(h_t \pi(y_{t+1}^0, \boldsymbol{y}_{t+1}) + b_t \pi(-y_{t+1}^0, -\boldsymbol{y}_{t+1}) \Big) + \\
& \qquad \sum_{t=L+1}^{T} \Big(h_t \eta \left((y_{t+1}^0, \boldsymbol{y}_{t+1}), (-x_1^0, -\boldsymbol{x}_1), \ldots, (-x_{t-L}^0, -\boldsymbol{x}_{t-L}) \right) + \\
& \qquad\qquad b_t \pi \left(-y_{t+1}^0, -\boldsymbol{y}_{t+1} \right) \Big) \\
\text{s.t.} \quad & y_{t+1}^k = y_t^k + x_{t-L}^k - d_t^k \qquad k = 0, \ldots, N, t = 1, \ldots, T \\
& x_t^k = 0 \qquad\qquad\qquad\qquad \forall k \geq N_{t-1} + 1, t = 1, \ldots, T - L.
\end{aligned}
\tag{18}
$$

We have shown that $Z_{STOC} \leq Z_{TLRP} \leq Z_{LRP} \leq Z_{SRP}$. The linear replenishment policy improves over the static replenishment policy because it is able to adapt to demand history. Since setting the coefficient of the random factors $\boldsymbol{x}_t$ to be zero in Problem (13) gives Problem (9), it is evident from Equation (11) that the linear replenishment policy subsumes the static replenishment policy. Observe that in Problem (13) from which the solution of the linear replenishment policy is derived, the set of constraints restricting the ordering quantity,

$$
0 \leq x_t^0 + \boldsymbol{x}_t' \boldsymbol{z} \leq S_t \qquad \forall \boldsymbol{z} \in \mathcal{W} \qquad t = 1, \ldots, T - L
$$

can be over constraining on the decision policy. For the case when the uncertainty set $\mathcal{W}$ is unbounded, such as $\mathcal{W} = \{\boldsymbol{z} \ : \ \boldsymbol{z} \geq -\underline{\boldsymbol{z}}\}$, the decision variables $\boldsymbol{x}_t$ will be driven to zeros. This means that the order decision of Problem (13) degenerates to a static replenishment policy, losing the ability to adapt to the history of random factors. The truncated linear replenishment policy, on the other hand, avoids this issue. Moreover, we also note that in Problem (13), information of mean, variance, directional deviations are not utilized at the set of constraints restricting the ordering quantity. In contrast, the truncation linear replenishment policy is defined to satisfy the ordering constraint. Hence, the robust model of Problem (18) does not have the explicit constraints on ordering levels and is able to utilize the additional information via the π and η functions for improving the bound.

It should be noted that establishing the bounds does not necessarily imply the superiority of truncated linear replenishment policy over static and linear ones. Nevertheless, this behavior is observed throughout our computational studies.

4 Other Extensions

In this section, we discuss some extension to the basic model.

4.1 Fixed Ordering Cost

Unfortunately, with fixed ordering cost, the inventory replenishment problem becomes nonconvex and is much harder to address. Using the idea of Bertsimas and Thiele [16], we can formulate a restricted problem where the time period in which the orders that can be placed is determined at the start of the planning horizon as follows:

$$\begin{array}{rll} Z_{STOCF} = \min & \displaystyle\sum_{t=1}^{T}\left(\mathrm{E}\left(c_t x_t(\tilde{\boldsymbol{d}}_{t-1}) + K_t r_t + h_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^+\right) + \mathrm{E}\left(b_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^-\right)\right). & \\ \text{s.t.} & y_{t+1}(\tilde{\boldsymbol{d}}_t) = y_t(\tilde{\boldsymbol{d}}_{t-1}) + x_{t-L}(\tilde{\boldsymbol{d}}_{t-L-1}) - \tilde{d}_t \qquad t = 1, \ldots, T & (19) \\ & 0 \le x_t(\tilde{\boldsymbol{d}}_{t-1}) \le S_t r_t \qquad t = 1, \ldots, T-L & \\ & r_t \in \{0, 1\} \qquad t = 1, \ldots, T-L. & \end{array}$$

In Problem (19), inventory can only be replenished at period where the corresponding binary variable r_t takes the value of one. We can then incorporate the tractable replenishment policies developed in the previous section. The resulting optimization model is a conic integer program, which is already addressed in commercial solvers such as CPLEX 10.1. Admittedly, algorithms for solving conic integer program are still at their infancy. On the theoretical front, Atamtürk and Narayanan [2] recently developed general-purpose conic mixed-integer rounding cuts based on polyhedral conic substructures of second-order conic sets, which can be readily incorporated in branch-and-bound algorithms that solve continuous conic optimization problems at the nodes of the search tree. Their preliminary computational experiments suggest that the new cuts are quite effective in reducing the integrality gap of continuous relaxations of conic mixed-integer programs.

4.2 Supply Chain Networks

The models we have presented in the preceding section can also be extended to more complex supply chain networks such as the series system, or more generally the tree network. These are multi-stage system where goods transit from one stage to the next stage, each time moving closer to their final destination. In many supply chains, the main storage hubs, or the sources of the network, receive their supplies from outside manufacturing plants in a tree-like hierarchical structure and send items throughout the network until they finally reach the stores, or the sinks of the network. The extension to tree structure uses the concept of echelon inventory and closely follows Bertsimas and Thiele [16]. We refer interested readers to their paper.

5 Computational Studies

We studied the computational performance of the static, linear and truncated linear replenishment policies against the optimum history dependent policy and two dynamic programming based heuristics, namely, the myopic policy and a history independent base-stock policy. We also analyzed the impact of the solutions over realistic ranges of planing horizon, cost parameters and demand correlations. To benchmark the performance, we have to assume knowledge of the underlying the distribution. We did not conduct experiments to test robustness of policies against distributional ambiguity such as those studied in Bertsimas and Thiele [16] and Chen and Sim [22]. Instead, we have focused on how good or bad the tractable replenishment policies perform against the optimum policy obtained by dynamic programming, as well as against common heuristics used in inventory control.

We are aware of the folding horizon implementation, where the replenishment policy can be enhanced by solving repeatedly with updated demand information. For instance, the static replenishment policy proposed by Bertsimas and Thiele [16] has a base-stock structure under the folding horizon implementation. Since more accurate information is used each time the model is solved, the results will only improve. Unfortunately, due to the computational intensiveness of the evaluation, we have excluded folding horizon implementations from our computational studies. For instance, under the folding horizon implementation, it would typically take about four minutes to evaluate the sample path of a ten period model based on the truncated linear replenishment policy. Through sizing experiments, we envisaged that it would require about 100,000 sample paths to reduce the standard error of the estimated objective value to less than 1%, which amounts to about 280 days of computational time.

5.1 Experimental Setup

The demand process we considered is motivated by Graves [30] as follows:

$$d_t(\tilde{\boldsymbol{z}}) = \tilde{z}_t + \alpha\tilde{z}_{t-1} + \alpha\tilde{z}_{t-2} + \cdots + \alpha\tilde{z}_1 + \mu, \tag{20}$$

where the mean and shocks factors $\tilde{z}_t$ are independently uniform distributed random variables in $[-\bar{z}, \bar{z}]$, and have standard deviations and directional deviations numerically close to $0.58\bar{z}$.

Observe that the demand process of Equation (20) for $t \geq 2$ can be expressed recursively as

$$d_t(\tilde{\boldsymbol{z}}) = d_{t-1}(\tilde{\boldsymbol{z}}) - (1-\alpha)\tilde{z}_{t-1} + \tilde{z}_t. \tag{21}$$

Hence, this demand process is an integrated moving average (IMA) process of order $(0,1,1)$. See also Box et al. [19]. Note that given $\bar{\mu} = d_{t-1}(\tilde{\boldsymbol{z}}) - (1-\alpha)\tilde{z}_{t-1}$ at time period t, the distribution of $d_t(\tilde{\boldsymbol{z}})$ is uniform in $[-\bar{z}+\bar{\mu}, \bar{z}+\bar{\mu}]$.

A range of demand processes can be modeled by varying α. With $\alpha = 0$, the demand process follows an i.i.d process of uniformly distributed random variables. As α grows, the demand process becomes non-stationary and less stable with increasing variance. When $\alpha = 1$, the demand process is a random walk on a continuous state space.

We considered problems with $T = 5, 10, 20$ and 30, and selected parameters so that the demand, $d_t(\tilde{\boldsymbol{z}})$ is nonnegative for all $\alpha \in [0,1]$. The lead-time L is zero, $S_t = 260$, unit ordering cost $c_t = 0.1$, and unit holding cost $h_t = 0.02$ for all periods $t = 1, \ldots, T$. In view of the long computational time for dynamic programming, especially for $T = 20$ and 30, we have used more manageable parameters for the demand process as follows:

- For $T = 5$, we used $\mu = 200$, and $\bar{z} = 20$.
- For $T = 10$, we used $\mu = 200$, and $\bar{z} = 10$.
- For $T = 20$, we used $\mu = 240$, and $\bar{z} = 6$.
- For $T = 30$, we used $\mu = 240$, and $\bar{z} = 4$.

Since unfulfilled demands are lost at the end of T, we set a relatively high backlog cost, $b_T = 10b_1$, to heavily penalize unmet demand at the last period throughout our experiments. For notational convenience, we use b and h to denote the backlog and holding cost from $t = 1 \ldots T-1$. In our study, we varied α from 0 to 1 in steps of 0.25 and set b/h to range from 10 to 50.

We benchmarked our solutions against solution based on dynamic programming, where the optimum replenishment policy can be characterized by the following backward recursion:

$$\begin{aligned} J_t(y_t, d_{t-1}, z_{t-1}) &= \min_{0 \le x \le S_t} \mathrm{E}\Big(c_t x + r_t(y_t + x - \delta_t(d_{t-1}, z_{t-1}, \tilde{z}_t)) + \\ &\qquad J_{t+1}(y_t + x - \delta_t(d_{t-1}, z_{t-1}, \tilde{z}_t), \delta_t(d_{t-1}, z_{t-1}, \tilde{z}_t), \tilde{z}_t)\Big) \end{aligned}$$

where $\delta_t(d_{t-1}, z_{t-1}, \tilde{z}_t) = d_{t-1} - (1-\alpha)z_{t-1} + \tilde{z}_t$ and $r_t(u) = h_t \max(u, 0) + b_t \max(-u, 0)$. By letting $v_t = d_t - (1-\alpha)z_t$, we have equivalently

$$J_t(y_t, v_{t-1}) = \min_{0 \le x \le S_t} \mathrm{E}\Big(c_t x + r_t(y_t + x - v_{t-1} - \tilde{z}_t) + J_{t+1}(y_t + x - v_{t-1} - \tilde{z}_t, v_{t-1} + \alpha\tilde{z}_t)\Big),$$

which reduces the state space by one dimension. The optimum replenishment policy at time t is a function of the current net-inventory y_t and v_{t-1} as follows:

$$x_t^{OPT}(y_t, v_{t-1}) = \arg \min_{0 \le x \le S_t} \mathrm{E}\Big(c_t x + r_t(y_t + x - v_{t-1} - \tilde{z}_t) + J_{t+1}(y_t + x - v_{t-1} - \tilde{z}_t, v_{t-1} + \alpha \tilde{z}_t)\Big).$$

In our implementation, we discretized the value functions uniformly and used linear interpolations for evaluating the intermediate points. The underlying expectations were computed using the well-known Simpson's rule of numerical integration. To obtain a near optimum policy within reasonable time, we adjusted the level of discretization such that when the discretization is increased by two, the improvement in objective value is less than 1%.

We also considered two heuristics. The first being a history independent base-stock policy (BSP), where we computed the replenishment policy recursively by ignoring the dependency of previous demands as follows:

$$J_t^{BSP}(y_t) = \min_{0 \le x \le S_t} \mathrm{E}_{\tilde{d}_t}\Big(c_t x + r_t(y_t + x - \tilde{d}_t) + J_{t+1}^{BSP}(y_t + x - \tilde{d}_t)\Big),$$

where $\tilde{d}_t = \tilde{z}_t + \alpha \tilde{z}_{t-1} + \alpha \tilde{z}_{t-2} + \cdots + \alpha \tilde{z}_1 + \mu$. The replenishment policy is given by

$$x_t^{BSP}(y_t) = \arg \min_{0 \le x \le S_t} \mathrm{E}_{\tilde{d}_t}\Big(c_t x + r_t(y_t + x - \tilde{d}_t) + J_{t+1}^{BSP}(y_t + x - \tilde{d}_t)\Big).$$

Under capacity limit on order quantities, the modified history independent base-stock policy is optimum if the demands are independently distributed, which occurs only when $\alpha = 0$. See Federgruen and Zipkin [25]. Note that when $\alpha > 0$, evaluating the expectation exactly involves multi-dimensional integration, which can be computationally prohibitive. Therefore, at every dynamic programming recursion, we computed the value functions approximately using sampling approximations from 500 instances of demand realizations instead.

The other heuristic we considered is an adaptive myopic policy (MP), where the replenishment level is derived by minimizing the following one-period expected cost as described below:

$$x_t^{MP}(y_t, v_{t-1}) = \arg \min_{0 \le x \le S_t} \mathrm{E}\Big(c_t x + r_t(y_t + x - v_{t-1} - \tilde{z}_t)\Big).$$

Under the uniform distribution, the myopic policy can be obtained using the critical fractile as follows:

$$x_t^{MP}(y_t, v_{t-1}) = \min \left\{ \left(v_{t-1} - \bar{z} + 2\bar{z}\left(1 - \frac{c_t + h_t}{b_t + h_t}\right) - y_t \right)^+, S_t \right\}.$$

In contrast with the optimum dynamic programming recursion, the adaptive myopic policy optimizes only the current period expected cost and ignores all subsequent costs.

After obtaining the policies, we compared them using $100,000$ simulated inventory runs and reported the sample means over all the runs. The results for the $T = 5, 10, 20$ and 30 problems solved are given in Table 1, Table 2, Table 3 and Table 4 respectively. The robust policies were obtained using the bounds of Theorem 1 and Theorem 4 where the support, covariance, directional deviations associated with random factors are specified. In the tables, we have used TLRP, LRP, SRP, BSP, MP to denote the sample mean of the expected cost under the simulated runs when the replenishment policies are the truncated linear replenishment policy, linear replenishment policy, static replenishment policy, history independent base-stock policy and adaptive myopic policy respectively. Correspondingly, we used OPT to denote the values derived from the optimum policy. For convenience, we used these abbreviations to denote the respective policies throughout this section. We also provided in the parenthesis, the performance of the corresponding policy with respect to the optimum value. For example, the performance of TLRP given in parenthesis shows the value of TLRP/OPT. A value of 1.05 hence shows that the deviation from OPT is 5%. We also reported the model objective values for the robust models as Z_{TLRP}, Z_{LRP} and Z_{SRP} to four significant places. Throughout the tables, the sample errors of the mean are less than 1%, and the sample means are shown to three significant places.

5.2 Comparison of Policies

In all the cases tested, TLRP deviates from the optimum answer by not more than 7%, whereas LRP is observed to deviate by as much as 29%, SRP by as much as 48%, MP by as much as 26%, and BSP by as much as 20% from OPT.

For $\alpha = 0$, TLRP and LRP perform well, coming within 1% from OPT. We observed that when α is small, the model objective values of TLRP and LRP, Z_{TLRP}, Z_{LRP}, come near to the simulated inventory runs, indicating the closeness of the bound. MP and BSP perform reasonably well for $\alpha \leq 0.5$ with deviation of not more than 10%. However, for large α, the deviation can exceed 20%. We observed that TLRP is never worst off against LRP, SRP, and outperforms BSP and MP in most of the cases. Moreover, TLRP has the sharpest lead against LRP, SRP and MP when the α is high. It is also interesting to know that when $\alpha = 1$, the bounds of LRP and SRP are rather close, while TLRP has much better performance.

Overall, the out-performance of TLRP over the rest of the non-optimum policies can be as high as 14%. In relatively few cases, BSP and MP may outperform TLRP. However, the margins do not exceed 1%. The results suggest that TLRP has the best overall performance.

	Simulated Inventory Runs						Objective Value		
b/h	TLRP	LRP	SRP	MP	BSP	OPT	Z_{TLRP}	Z_{LRP}	Z_{SRP}
	$\alpha = 0$								
10	$108_{(1)}$	$108_{(1)}$	$121_{(1.1)}$	$115_{(1.07)}$	$107_{(1)}$	108	108.0	108.0	120.8
30	$108_{(1)}$	$108_{(1)}$	$124_{(1.13)}$	$110_{(1.02)}$	$108_{(1)}$	108	108.0	108.0	124.4
50	$108_{(1)}$	$108_{(1)}$	$126_{(1.14)}$	$109_{(1.01)}$	$108_{(1)}$	108	108.0	108.0	125.8
	$\alpha = 0.25$								
10	$108_{(1.01)}$	$109_{(1.01)}$	$130_{(1.18)}$	$116_{(1.08)}$	$109_{(1.01)}$	107	108.3	109.1	130.3
30	$108_{(1)}$	$109_{(1.01)}$	$136_{(1.22)}$	$111_{(1.03)}$	$110_{(1.02)}$	108	108.6	109.2	135.5
50	$108_{(1)}$	$109_{(1.01)}$	$138_{(1.24)}$	$110_{(1.02)}$	$110_{(1.02)}$	108	108.8	109.2	137.6
	$\alpha = 0.50$								
10	$110_{(1.02)}$	$118_{(1.06)}$	$141_{(1.25)}$	$119_{(1.1)}$	$112_{(1.04)}$	108	111.2	117.7	140.5
30	$111_{(1.02)}$	$125_{(1.1)}$	$148_{(1.31)}$	$114_{(1.05)}$	$115_{(1.06)}$	109	114.3	125.0	147.5
50	$112_{(1.03)}$	$130_{(1.12)}$	$150_{(1.33)}$	$113_{(1.04)}$	$117_{(1.07)}$	109	116.7	129.6	150.5
	$\alpha = 0.75$								
10	$113_{(1.03)}$	$133_{(1.14)}$	$151_{(1.31)}$	$126_{(1.15)}$	$117_{(1.07)}$	110	119.0	133.3	151.1
30	$118_{(1.05)}$	$153_{(1.22)}$	$163_{(1.35)}$	$125_{(1.12)}$	$124_{(1.1)}$	112	131.9	152.5	162.9
50	$122_{(1.06)}$	$166_{(1.25)}$	$173_{(1.34)}$	$130_{(1.14)}$	$130_{(1.14)}$	114	142.7	166.2	172.7
	$\alpha = 1$								
10	$118_{(1.04)}$	$152_{(1.21)}$	$163_{(1.35)}$	$137_{(1.21)}$	$126_{(1.12)}$	113	132.3	152.3	163.3
30	$131_{(1.06)}$	$191_{(1.28)}$	$193_{(1.31)}$	$151_{(1.22)}$	$145_{(1.18)}$	123	164.8	191.0	193.3
50	$140_{(1.06)}$	$223_{(1.28)}$	$223_{(1.29)}$	$168_{(1.28)}$	$158_{(1.2)}$	132	195.2	222.9	223.3

Table 1: Performance of truncated linear replenishment policy $T = 5$

	Simulated Inventory Runs						Objective Value		
b/h	TLRP	LRP	SRP	MP	BSP	OPT	Z_{TLRP}	Z_{LRP}	Z_{SRP}
	$\alpha = 0$								
10	$206_{(1)}$	$206_{(1)}$	$220_{(1.06)}$	$214_{(1.04)}$	$206_{(1)}$	206	206.0	206.0	220.2
30	$206_{(1)}$	$206_{(1)}$	$224_{(1.08)}$	$209_{(1.01)}$	$206_{(1)}$	206	206.0	206.0	223.8
50	$206_{(1)}$	$206_{(1)}$	$225_{(1.08)}$	$208_{(1.01)}$	$206_{(1)}$	206	206.0	206.0	225.3
	$\alpha = 0.25$								
10	$206_{(1)}$	$206_{(1)}$	$240_{(1.14)}$	$214_{(1.04)}$	$207_{(1.01)}$	206	206.0	206.1	239.5
30	$206_{(1)}$	$206_{(1)}$	$247_{(1.18)}$	$209_{(1.01)}$	$208_{(1.01)}$	206	206.0	206.1	246.7
50	$206_{(1)}$	$206_{(1)}$	$250_{(1.19)}$	$208_{(1.01)}$	$209_{(1.02)}$	206	206.0	206.1	249.7
	$\alpha = 0.5$								
10	$206_{(1)}$	$213_{(1.03)}$	$260_{(1.23)}$	$214_{(1.04)}$	$210_{(1.02)}$	206	206.3	213.0	260.0
30	$206_{(1)}$	$215_{(1.04)}$	$271_{(1.28)}$	$209_{(1.01)}$	$212_{(1.03)}$	206	207.0	215.1	270.9
50	$206_{(1)}$	$216_{(1.04)}$	$275_{(1.3)}$	$208_{(1.01)}$	$214_{(1.04)}$	206	207.5	216.0	275.5
	$\alpha = 0.75$								
10	$207_{(1.01)}$	$232_{(1.1)}$	$281_{(1.31)}$	$215_{(1.04)}$	$214_{(1.04)}$	206	210.5	231.6	280.8
30	$211_{(1.02)}$	$242_{(1.14)}$	$296_{(1.38)}$	$211_{(1.02)}$	$218_{(1.05)}$	207	215.4	241.9	295.6
50	$213_{(1.03)}$	$247_{(1.16)}$	$302_{(1.41)}$	$211_{(1.02)}$	$221_{(1.07)}$	207	218.2	247.4	301.8
	$\alpha = 1$								
10	$213_{(1.02)}$	$257_{(1.18)}$	$302_{(1.39)}$	$220_{(1.06)}$	$221_{(1.06)}$	208	220.6	257.4	301.8
30	$222_{(1.05)}$	$281_{(1.25)}$	$322_{(1.46)}$	$222_{(1.05)}$	$231_{(1.1)}$	210	235.5	281.1	321.7
50	$228_{(1.07)}$	$296_{(1.29)}$	$331_{(1.48)}$	$229_{(1.08)}$	$240_{(1.13)}$	212	245	296.0	331.5

Table 2: Performance of truncated linear replenishment policy $T = 10$

	Simulated Inventory Runs						Objective Value		
b/h	TLRP	LRP	SRP	MP	BSP	OPT	Z_{TLRP}	Z_{LRP}	Z_{SRP}
	$\alpha = 0$								
10	$486_{(1)}$	$486_{(1)}$	$506_{(1.04)}$	$496_{(1.02)}$	$486_{(1)}$	486	486.0	486.0	506.3
30	$486_{(1)}$	$486_{(1)}$	$511_{(1.05)}$	$489_{(1.01)}$	$486_{(1)}$	486	486.0	486.0	511.2
50	$486_{(1)}$	$486_{(1)}$	$513_{(1.05)}$	$488_{(1)}$	$486_{(1)}$	486	486.0	486.0	513.2
	$\alpha = 0.25$								
10	$488_{(1)}$	$520_{(1.06)}$	$556_{(1.13)}$	$497_{(1.02)}$	$489_{(1.01)}$	486	490.7	520.0	556.1
30	$490_{(1.01)}$	$532_{(1.08)}$	$570_{(1.15)}$	$491_{(1.01)}$	$491_{(1.01)}$	487	495.7	532.0	570.3
50	$492_{(1.01)}$	$538_{(1.09)}$	$576_{(1.17)}$	$490_{(1.01)}$	$493_{(1.01)}$	487	499.0	537.9	576.4
	$\alpha = 0.50$								
10	$507_{(1.02)}$	$588_{(1.14)}$	$609_{(1.19)}$	$515_{(1.04)}$	$507_{(1.02)}$	496	528.0	587.7	609.3
30	$534_{(1.05)}$	$636_{(1.17)}$	$643_{(1.21)}$	$536_{(1.05)}$	$536_{(1.05)}$	511	569.4	635.9	642.5
50	$550_{(1.05)}$	$667_{(1.19)}$	$668_{(1.2)}$	$564_{(1.08)}$	$562_{(1.08)}$	522	600.0	667.3	667.9
	$\alpha = 0.75$								
10	$549_{(1.04)}$	$674_{(1.18)}$	$677_{(1.2)}$	$562_{(1.07)}$	$552_{(1.05)}$	527	601.4	673.7	677.1
30	$620_{(1.05)}$	$818_{(1.17)}$	$818_{(1.17)}$	$670_{(1.14)}$	$654_{(1.11)}$	590	754.2	817.8	817.8
50	$686_{(1.05)}$	$959_{(1.15)}$	$959_{(1.15)}$	$788_{(1.21)}$	$756_{(1.16)}$	652	898.2	958.5	958.5
	$\alpha = 1$								
10	$604_{(1.04)}$	$780_{(1.19)}$	$780_{(1.19)}$	$631_{(1.09)}$	$614_{(1.06)}$	578	708.0	780.1	780.1
30	$773_{(1.05)}$	$1120_{(1.14)}$	$1120_{(1.14)}$	$876_{(1.19)}$	$828_{(1.12)}$	739	1057	1118	1119
50	$935_{(1.04)}$	$1460_{(1.11)}$	$1460_{(1.11)}$	$1130_{(1.25)}$	$1040_{(1.15)}$	899	1398	1457	1457

Table 3: Performance of truncated linear replenishment policy $T = 20$

	Simulated Inventory Runs						Objective Value		
b/h	TLRP	LRP	SRP	MP	BSP	OPT	Z_{TLRP}	Z_{LRP}	Z_{SRP}
	$\alpha = 0$								
10	$726_{(1)}$	$726_{(1)}$	$749_{(1.03)}$	$736_{(1.01)}$	$726_{(1)}$	725	725.6	725.6	748.9
30	$726_{(1)}$	$726_{(1)}$	$754_{(1.03)}$	$729_{(1)}$	$727_{(1)}$	726	725.6	725.6	754.4
50	$726_{(1)}$	$726_{(1)}$	$757_{(1.04)}$	$728_{(1)}$	$729_{(1)}$	726	725.6	725.6	756.7
	$\alpha = 0.25$								
10	$726_{(1)}$	$766_{(1.05)}$	$830_{(1.12)}$	$736_{(1.01)}$	$729_{(1)}$	725	726.8	765.6	829.6
30	$727_{(1)}$	$778_{(1.06)}$	$850_{(1.15)}$	$729_{(1)}$	$731_{(1.01)}$	726	728.5	777.7	850.4
50	$727_{(1)}$	$783_{(1.07)}$	$860_{(1.17)}$	$728_{(1)}$	$732_{(1.01)}$	726	729.7	783.3	859.2
	$\alpha = 0.50$								
10	$738_{(1.01)}$	$862_{(1.14)}$	$913_{(1.21)}$	$746_{(1.02)}$	$742_{(1.01)}$	732	755.7	861.6	913.4
30	$762_{(1.03)}$	$909_{(1.18)}$	$953_{(1.25)}$	$757_{(1.02)}$	$763_{(1.03)}$	743	792.5	908.6	952.7
50	$778_{(1.04)}$	$936_{(1.19)}$	$972_{(1.26)}$	$767_{(1.03)}$	$778_{(1.04)}$	750	815.6	935.6	972.0
	$\alpha = 0.75$								
10	$787_{(1.03)}$	$976_{(1.21)}$	$1000_{(1.26)}$	$789_{(1.03)}$	$786_{(1.03)}$	763	840.3	976.1	1004
30	$862_{(1.06)}$	$1100_{(1.24)}$	$1100_{(1.25)}$	$886_{(1.09)}$	$888_{(1.09)}$	816	963.8	1102	1103
50	$902_{(1.06)}$	$1190_{(1.23)}$	$1190_{(1.23)}$	$974_{(1.15)}$	$970_{(1.14)}$	849	1064	1194	1194
	$\alpha = 1$								
10	$857_{(1.05)}$	$1110_{(1.24)}$	$1120_{(1.26)}$	$868_{(1.06)}$	$863_{(1.06)}$	818	965.4	1115	1119
30	$1020_{(1.06)}$	$1412_{(1.21)}$	$1412_{(1.21)}$	$1119_{(1.17)}$	$1100_{(1.15)}$	957	1286	1412	1412
50	$1150_{(1.06)}$	$1700_{(1.18)}$	$1700_{(1.18)}$	$1370_{(1.26)}$	$1310_{(1.2)}$	1090	1587	1704	1704

Table 4: Performance of truncated linear replenishment policy $T = 30$

		Simulated Inventory Runs			Objective Value	
α	b/h	TLRP with directional dev.	TLRP without directional dev.	OPT	Z_{TLRP} with directional dev.	Z_{TLRP} without directional dev.
0	10	$108_{(1.01)}$	$108_{(1.01)}$	107	108.0	108.0
	50	$108_{(1)}$	$108_{(1)}$	108	108.0	180.0
0.5	10	$110_{(1.02)}$	$110_{(1.02)}$	108	111.2	122.0
	50	$112_{(1.02)}$	$113_{(1.03)}$	109	116.7	175.0
1	10	$118_{(1.04)}$	$122_{(1.07)}$	113	132.3	159.4
	50	$140_{(1.06)}$	$163_{(1.24)}$	132	195.2	347.1

Table 5: Performance of truncated linear replenishment policy $T = 5$ with and without directional deviations

5.3 Influence of Directional Deviations

Table 5 shows a comparison of the TLRP with and without information on the directional deviations. In the latter case, the robust policies were obtained using the bound of Theorem 1 with information only on the support and covariance associated with the random factors. When $\alpha = 0$, information on directional deviations has little impact on the model objective. It is observed that when α is high, the directional deviations can significantly improve the performance of TLRP.

5.4 Effects of Demand Variability

We also investigated the influence of demand variability on the performance of the best robust policy, namely, TLRP. Shown in Table 6 are results of TLRP for the $T = 5$ model, with b/h of 50, for $\alpha = 0$, $\alpha = 0.25$, $\alpha = 0.5$ and various degree of variability, as reported by $\bar{z}$, the half range of the random factor. It is observed that the bound of Z_{TLRP} degrades significantly as demand variability increases. However, the impact on the performance against the optimum policy is marginal.

5.5 Analysis of Policies

Although the robust models appear to be complex, implementing the policy derived from the model is extremely easy. The truncated linear replenishment policy is computed simply by taking an affine sum of random factors using weights given by the TLRP model solution and then restricting the range of

α	$\bar{z}$	Simulated Inventory Runs TLRP	OPT	Objective Value Z_{TLRP}
0	10	$104_{(1)}$	104	104.0
	20	$108_{(1)}$	108	108.0
	30	$112_{(1)}$	112	112.0
	40	$125_{(1.03)}$	118	131.8
	50	$136_{(1.04)}$	131	161.3
0.25	10	$102_{(1)}$	102	102.0
	20	$108_{(1)}$	108	108.8
	30	$119_{(1.04)}$	114	128.9
	40	$142_{(1.05)}$	135	190.7
	50	$195_{(1.04)}$	187	306.4
0.5	10	$104_{(1)}$	104	104.0
	20	$112_{(1.03)}$	132	116.7
	30	$139_{(1.06)}$	131	187.9

Table 6: Performance of truncated linear replenishment policy $T = 5$ with various demand range

t	z_t	d_t	x_t^{TRLP}	y_{t+1}
1	13.3	213.3	260.0	46.7
2	6.2	212.9	214.5	48.4
3	14.5	224.2	224.8	48.9
4	-24.1	192.9	249.6	105.6
5	-32.3	172.7	139.5	72.5

Table 7: A sample path of the truncated linear replenishment policy

the order quantity. For example, a sample problem where $\alpha = 0.5$ has the following model solution:

$$\begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \\ x_4^0 \\ x_5^0 \end{bmatrix} = \begin{bmatrix} 260.00 \\ 191.93 \\ 218.29 \\ 243.57 \\ 126.31 \end{bmatrix} \qquad \begin{bmatrix} \boldsymbol{x}_1' \\ \boldsymbol{x}_2' \\ \boldsymbol{x}_3' \\ \boldsymbol{x}_4' \\ \boldsymbol{x}_5' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0 & 0 & 0 & 0 \\ 1.7 & 0 & 0 & 0 & 0 \\ 0 & 1.04 & 0 & 0 & 0 \\ 1.29 & 1.44 & 1.6 & 1.5 & 0 \end{bmatrix}$$

Table 7 shows the sample path, constructed using weights from the model solution, and then applying the relevant capacity constraints,

$$x_i^{TLRP}(\boldsymbol{z}) = \min\{(x_i^0 + \boldsymbol{x}_i'\boldsymbol{z})^+, 260\}.$$

In the above example, the inventory manager would order a quantity of $260, 215, 225, 250$ and 140 for periods 1 to 5 respectively.

Ben Tal et. al. [6] showed that the linear replenishment policy is equivalent to a history independent base-stock if and only if it exhibits Markovian behavior and takes the form $x_t(\tilde{\boldsymbol{d}}_{t-1}) = x_t^0 + \tilde{z}_{t-1}$. The truncated linear replenishment policy has a different structure and in general, we are unable to show the connection with a base-stock structure. When the demands are independent, that is, $\alpha = 0$, it is observed that TLRP exhibits Markovian behavior for most input parameters. There are also instances that LRP is Markovian while TLRP is not. For example, for $T = 20, \alpha = 0, b/h = 40, \bar{z} = 20$, TLRP and LRP are the same and having a Markovian structure. See Table 8. However, when $\bar{z} = 40$, the TLRP and LRP policies presented in Table 9 and Table 10 respectively, show a difference in the structure. For the case of correlated demands, we did not observe any Markovian structure in our experiments.

$$\begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \\ x_4^0 \\ x_5^0 \\ x_6^0 \\ x_7^0 \\ x_8^0 \\ x_9^0 \\ x_{10}^0 \end{bmatrix} = \begin{bmatrix} 240 \\ 220 \\ 220 \\ 220 \\ 220 \\ 220 \\ 220 \\ 220 \\ 220 \\ 220 \end{bmatrix} \qquad \begin{bmatrix} \boldsymbol{x}_1' \\ \boldsymbol{x}_2' \\ \boldsymbol{x}_3' \\ \boldsymbol{x}_4' \\ \boldsymbol{x}_5' \\ \boldsymbol{x}_6' \\ \boldsymbol{x}_7' \\ \boldsymbol{x}_8' \\ \boldsymbol{x}_9' \\ \boldsymbol{x}_{10}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Table 8: TLRP and LRP for $\alpha = 0, T = 10, b/h = 40, \bar{z} = 20, S_t = 240$

$$\begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \\ x_4^0 \\ x_5^0 \\ x_6^0 \\ x_7^0 \\ x_8^0 \\ x_9^0 \\ x_{10}^0 \end{bmatrix} = \begin{bmatrix} 240.0 \\ 232.9 \\ 229.9 \\ 227.2 \\ 225.8 \\ 224.9 \\ 224.4 \\ 227.5 \\ 228.2 \\ 229.7 \end{bmatrix} \qquad \begin{bmatrix} \boldsymbol{x}_1' \\ \boldsymbol{x}_2' \\ \boldsymbol{x}_3' \\ \boldsymbol{x}_4' \\ \boldsymbol{x}_5' \\ \boldsymbol{x}_6' \\ \boldsymbol{x}_7' \\ \boldsymbol{x}_8' \\ \boldsymbol{x}_9' \\ \boldsymbol{x}_{10}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.04 & 0.21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.03 & 0.03 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.03 & 0.03 & 0.03 & 0.28 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.02 & 0.02 & 0.02 & 0.02 & 0.30 & 0 & 0 & 0 & 0 & 0 \\ 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.31 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.31 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.30 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.26 & 0 \end{bmatrix}$$

Table 9: TLRP for $\alpha = 0, T = 10, b/h = 40, \bar{z} = 40, S_t = 240$

$$\begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \\ x_4^0 \\ x_5^0 \\ x_6^0 \\ x_7^0 \\ x_8^0 \\ x_9^0 \\ x_{10}^0 \end{bmatrix} = \begin{bmatrix} 240.0 \\ 240.0 \\ 240.0 \\ 240.0 \\ 229.9 \\ 228.6 \\ 227.9 \\ 227.3 \\ 229.3 \\ 240.0 \end{bmatrix} \qquad \begin{bmatrix} \boldsymbol{x}_1' \\ \boldsymbol{x}_2' \\ \boldsymbol{x}_3' \\ \boldsymbol{x}_4' \\ \boldsymbol{x}_5' \\ \boldsymbol{x}_6' \\ \boldsymbol{x}_7' \\ \boldsymbol{x}_8' \\ \boldsymbol{x}_9' \\ \boldsymbol{x}_{10}' \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

Table 10: LRP for $\alpha = 0, T = 10, b/h = 40, \bar{z} = 40, S_t = 240$

5.6 Computational Time

We formulated the robust models using an in-house developed software, *PROF* (Platform for Robust Optimization Formulation). The Matlab based software is essentially a algebraic modeling language for robust optimization that contains reusable functions for modeling multi-period robust optimization using decision rules. After formulating the model, it calls upon a commercial SOCP solver, MOSEK 5.0 for solution. We have implemented bounds for $\pi(\cdot)$ of Theorem 1 and $\eta(\cdot)$ of Theorem 4. The sample formulation of Problem (17) provided in Appendix F shows the ease of formulating the TLRP model using the software. The size of the problem we considered is presented in Table 11. Our computation was carried out on a 2.4GHz desktop with 2Mb memory. The computational time depends on the number of periods. It typically takes less than 0.3 seconds to solve the TLRP model for $T = 5$. For $T = 10, 20$ and 30, the times taken were 3 seconds, 30 seconds and 3 minutes respectively, suggesting that the computational time scales reasonably well with respect to the size of the problem. Moreover, the time needed for computation does not depend on the replenishment lead time, demand variability, correlations, among others. On the other hand, much of the computational effort lies in solving the optimum history dependent policy using dynamic programming. In the experiments, we have customized and optimized the dynamic programming algorithm so that we can reduce the computational time to less than three hours. For instance, we implemented the Golden-section search method and exploited the fact that $v_t = \mu + \alpha\tilde{z}_1 + \cdots + \alpha\tilde{z}_t \in [\mu - t\alpha\underline{z}, \mu + t\alpha\bar{z}]$ to reduce the size of the state space. Table 12 compares the computational times of the TLRP model against the optimum dynamic programming model.

6 Conclusions

In this paper, we propose a robust optimization approach to address a multi-period, inventory control problem under ambiguous demands. Interestingly, even though the best robust policy does not necessarily have a base-stock structure, our computational studies suggest that it can perform better than simple heuristics derived from dynamic programming. Perhaps against popular beliefs, the structural behavior of an optimum policy derived from a simplified model may not necessarily lead to better inventory control in practice.

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	$T = 5$	$T = 10$	$T = 20$	$T = 30$
Affine constraints	5911	24721	125041	348961
Free variables	3366	15806	89911	266316
Non-negative variables	1700	9000	56400	174200

	$T = 5$	$T = 10$	$T = 20$	$T = 30$		$T = 5$	$T = 10$	$T = 20$	$T = 30$
$\mathcal{L}^2$ cones	12	12	12	12	$\mathcal{L}^{18}$ cones	-	-	206	206
$\mathcal{L}^3$ cones	1226	4026	14426	31226	$\mathcal{L}^{19}$ cones	-	-	218	218
$\mathcal{L}^4$ cones	38	38	38	38	$\mathcal{L}^{20}$ cones	-	-	230	230
$\mathcal{L}^5$ cones	50	50	50	50	$\mathcal{L}^{21}$ cones	-	-	282	242
$\mathcal{L}^6$ cones	72	62	62	62	$\mathcal{L}^{22}$ cones	-	-	102	254
$\mathcal{L}^7$ cones	27	74	74	74	$\mathcal{L}^{23}$ cones	-	-	-	255
$\mathcal{L}^8$ cones	-	86	86	86	$\mathcal{L}^{24}$ cones	-	-	-	278
$\mathcal{L}^9$ cones	-	98	98	98	$\mathcal{L}^{25}$ cones	-	-	-	290
$\mathcal{L}^{10}$ cones	-	110	110	110	$\mathcal{L}^{26}$ cones	-	-	-	302
$\mathcal{L}^{11}$ cones	-	142	122	122	$\mathcal{L}^{27}$ cones	-	-	-	314
$\mathcal{L}^{12}$ cones	-	52	134	134	$\mathcal{L}^{28}$ cones	-	-	-	326
$\mathcal{L}^{13}$ cones	-	-	146	146	$\mathcal{L}^{29}$ cones	-	-	-	338
$\mathcal{L}^{14}$ cones	-	-	158	158	$\mathcal{L}^{30}$ cones	-	-	-	350
$\mathcal{L}^{15}$ cones	-	-	170	170	$\mathcal{L}^{31}$ cones	-	-	-	422
$\mathcal{L}^{16}$ cones	-	-	182	182	$\mathcal{L}^{32}$ cones	-	-	-	152
$\mathcal{L}^{17}$ cones	-	-	194	194					

Table 11: Size of the TLRP model, where $\mathcal{L}^n = \{(x_0, \boldsymbol{x}) \in \Re \times \Re^{n-1} \ : \ \|\boldsymbol{x}\|_2 \leq x_0\}$.

	$T = 5$	$T = 10$	$T = 20$	$T = 30$
TLRP	0.3 sec	3 sec	30 sec	3 min
OPT, $\alpha = 0$	5 sec	18 sec	25 sec	85 sec
OPT, $\alpha = 1$	12 min	30 min	1.5 hr	2.5 hr

Table 12: Computation time

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A Proof of Proposition 3

Proof : The bound $\mathrm{E}((\tilde{z}-a)^+) \leq \pi(-a,1)$ follows directly from Theorem 1. Since the bound of Proposition 2 is tight, it suffices to show

$$\pi(-a,1) \leq \begin{cases} \frac{1}{2}\left(-a+\sqrt{\sigma^2+a^2}\right) & \text{if } a \geq \frac{\sigma^2-\mu^2}{2\mu} \\ -a\frac{\mu^2}{\mu^2+\sigma^2}+\mu\frac{\sigma^2}{\mu^2+\sigma^2} & \text{if } a < \frac{\sigma^2-\mu^2}{2\mu} \end{cases}$$

With $\underline{z}=\mu$ and $p=q=\bar{z}=\infty$, we first simplify the bound as follows:

$$\begin{aligned}
\pi(y_0,\boldsymbol{y}) = \min \quad & r_1+r_2+r_3 \\
\text{s.t.} \quad & y_{10}+t_1\mu \leq r_1 \\
& 0 \leq r_1 \\
& -t_1 = y_{11} \\
& t_1 \geq 0 \\
& h_1\mu \leq r_2 \\
& y_{20} \leq r_2 \\
& h_1 = y_{21} \\
& h_1 \geq 0 \\
& \tfrac{1}{2}y_{30}+\tfrac{1}{2}\sqrt{y_{30}^2+\sigma^2 y_{31}^2} \leq r_3 \\
& y_{10}+y_{20}+y_{30} = -a \\
& y_{11}+y_{21}+y_{31} = 1 \\
= \min \quad & (y_{10}-y_{11}\mu)^+ + \max\{y_{21}\mu, y_{20}\} + \tfrac{1}{2}y_{30}+\tfrac{1}{2}\sqrt{y_{30}^2+\sigma^2 y_{31}^2} \\
\text{s.t.} \quad & y_{11} \leq 0 \\
& y_{21} \geq 0 \\
& y_{10}+y_{20}+y_{30} = -a \\
& y_{11}+y_{21}+y_{31} = 1.
\end{aligned} \tag{22}$$

Clearly, with $y_{10}=y_{20}=0$, $y_{30}=-a$, $y_{11}=y_{21}=0$ and $y_{31}=1$, we see that $\pi(y_0,\boldsymbol{y}) \leq -\frac{1}{2}a+\frac{1}{2}\sqrt{a^2+\sigma^2}$. Now for $a<\frac{\sigma^2-\mu^2}{2\mu}$, we let $y_{10}=y_{11}=0$,

$$\begin{aligned}
y_{20} &= \mu\tfrac{\sigma^2-\mu^2-2\mu a}{\mu^2+\sigma^2}, \\
y_{21} &= \tfrac{\sigma^2-\mu^2-2\mu a}{\mu^2+\sigma^2} \geq 0, \\
y_{30} &= (\mu+a)\tfrac{\mu^2-\sigma^2}{\mu^2+\sigma^2}, \\
y_{31} &= 2\mu\tfrac{\mu+a}{\mu^2+\sigma^2}.
\end{aligned}$$

which are feasible in Problem (22). Hence,

$$\begin{aligned}
\pi(-a,1) &\le (y_{10}-y_{11}\mu)^+ + \max\{y_{21}\mu, y_{20}\} + \tfrac{1}{2}y_{30} + \tfrac{1}{2}\sqrt{y_{30}^2+\sigma^2 y_{31}^2} \\
&= -a - \frac{1}{2}(\mu+a)\frac{\mu^2-\sigma^2}{\mu^2+\sigma^2} + \frac{1}{2}\underbrace{\sqrt{(a+\mu)^2}}_{=a+\mu} \\
&= -a\frac{\mu^2}{\mu^2+\sigma^2} + \mu\frac{\sigma^2}{\mu^2+\sigma^2}.
\end{aligned}$$

■

B Proof of Theorem 2

Proof : Under the static replenishment policy and using the factor-based demand model, the net-inventory at the end of period t is given by

$$\begin{aligned}
y_{t+1}^{SRP}(\tilde{\boldsymbol{d}}_t) &= y_1^0 + \sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} x_{\tau-L}^{SRP}(\tilde{\boldsymbol{d}}_{\tau-L-1}) - \sum_{\tau=1}^{t} d_\tau(\tilde{\boldsymbol{z}}) \\
&= y_1^0 + \sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} x_{\tau-L}^{0*} - \sum_{\tau=1}^{t} d_\tau^0 - \sum_{\tau=1}^{t}\sum_{k=1}^{N} d_\tau^k \tilde{z}_k \\
&= \underbrace{y_1^0 + \sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} x_{\tau-L}^{0*} - \sum_{\tau=1}^{t} d_\tau^0}_{=y_{t+1}^{0*}} + \sum_{k=1}^{N}\underbrace{\left(\sum_{\tau=1}^{t}(-d_\tau^k)\right)}_{=y_{t+1}^{k*}}\tilde{z}_k \\
&= y_{t+1}^{0*} + \sum_{\tau=1}^{N} y_{t+1}^{k*}\tilde{z}_k
\end{aligned}$$

where y_{t+1}^{k*} $k=0,\ldots,N$, $t=1,\ldots,T$ are the optimum solutions of Problem (9). Clearly, the static replenishment policy, $x_t^{SRP}(\tilde{\boldsymbol{d}}_{t-1})$ is feasible in Problem (3). Moreover, by Theorem 1, we have

$$\begin{aligned}
&\mathrm{E}\left(c_t x_t^{SRP}(\tilde{\boldsymbol{d}}_{t-1}) + h_t\left(y_{t+1}^{SRP}(\tilde{\boldsymbol{d}}_t)\right)^+ + b_t\left(y_{t+1}^{SRP}(\tilde{\boldsymbol{d}}_t)\right)^-\right) \\
=\ &\mathrm{E}\left(c_t x_t^{0*} + h_t\left(y_{t+1}^{0*} + \sum_{k=1}^{N} y_{t+1}^{k*}\tilde{z}_k\right)^+ + b_t\left(-y_{t+1}^{0*} - \sum_{k=1}^{N} y_{t+1}^{k*}\tilde{z}_k\right)^+\right) \qquad (23) \\
\le\ &c_t x_t^{0*} + h_t\pi\left(y_{t+1}^{0*}, \boldsymbol{y}_{t+1}^*\right) + b_t\pi\left(-y_{t+1}^{0*}, -\boldsymbol{y}_{t+1}^*\right).
\end{aligned}$$

Hence, $Z_{STOC} \le Z_{SRP}$. ■

C Proof of Theorem 3

Proof : Observe that Problem (13) with additional constraints $x_t^k = 0$, $k=1,\ldots,N$, $t=1\ldots,T-L$ gives the same feasible constraint set as Problem (9). Moreover, the objective functions of both problems are the same. Hence, $Z_{LRP} \leq Z_{SRP}$. Under the linear replenishment policy, the net-inventory at the end of period t is given by

$$\begin{aligned}
y_{t+1}^{LRP}(\tilde{\boldsymbol{d}}_t) &= y_1^0 + \sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} x_{\tau-L}^{LRP}(\tilde{\boldsymbol{d}}_{\tau-L-1}) - \sum_{\tau=1}^{t} d_\tau(\tilde{\boldsymbol{z}}) \\
&= y_1^0 + \sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} \left(x_{\tau-L}^{0*} + \sum_{k=1}^{N} x_{\tau-L}^{k*}\tilde{z}_k \right) - \sum_{\tau=1}^{t} d_\tau^0 - \sum_{\tau=1}^{t}\sum_{k=1}^{N} d_\tau^k \tilde{z}_k \\
&= \underbrace{y_1^0 + \sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} x_{\tau-L}^{0*} - \sum_{\tau=1}^{t} d_\tau^0}_{=y_{t+1}^{0*}} + \sum_{k=1}^{N} \underbrace{\left(\sum_{\tau=1}^{t} (x_{\tau-L}^{k*} - d_\tau^k) \right)}_{=y_{t+1}^{k*}} \tilde{z}_k \\
&= y_{t+1}^{0*} + \sum_{\tau=1}^{N} y_{t+1}^{k*}\tilde{z}_k
\end{aligned}$$

where y_{t+1}^{k*} $k = 0, \ldots, N$, $t = 1, \ldots, T$ are the optimum solutions of Problem (13). Clearly, the linear replenishment policy, $x_t^{LRP}(\tilde{\boldsymbol{d}}_{t-1})$ is feasible in Problem (3). Moreover, by Theorem 1 and that $\tilde{\boldsymbol{z}}$ being zero mean random variables, we have

$$\begin{aligned}
& \mathrm{E}\left(c_t x_t^{LRP}(\tilde{\boldsymbol{d}}_{t-1}) + h_t \left(y_{t+1}^{LRP}(\tilde{\boldsymbol{d}}_t) \right)^+ + b_t \left(y_{t+1}^{LRP}(\tilde{\boldsymbol{d}}_t) \right)^- \right) \\
= \; & \mathrm{E}\left(c_t \left(x_t^{0*} + \boldsymbol{x}_t^{*\prime}\tilde{\boldsymbol{z}} \right) + h_t \left(y_{t+1}^{0*} + \sum_{k=1}^{N} y_{t+1}^{k*}\tilde{z}_k \right)^+ + b_t \left(-y_{t+1}^{0*} - \sum_{k=1}^{N} y_{t+1}^{k*}\tilde{z}_k \right)^+ \right) \qquad (24) \\
\leq \; & c_t x_t^{0*} + h_t \pi\left(y_{t+1}^{0*}, \boldsymbol{y}_{t+1}^* \right) + b_t \pi\left(-y_{t+1}^{0*}, -\boldsymbol{y}_{t+1}^* \right).
\end{aligned}$$

Hence, $Z_{STOC} \leq Z_{LRP}$. ■

D Proof of Theorem 4

Proof : We first show the following bound:

$$\left(y + \sum_{i=1}^{p} x_i^+ \right)^+ \leq \left(y + \sum_{i=1}^{p} w_i \right)^+ + \sum_{i=1}^{p} \left((-w_i)^+ + (x_i - w_i)^+ \right) \qquad (25)$$

for all w_i, $i = 1, \ldots, p$. Note that for any scalars a, b

$$a^+ + b^+ \geq (a+b)^+ \qquad (26)$$

$$a^+ + b^+ = a^+ + (b^+)^+ \geq (a + b^+)^+. \qquad (27)$$

Therefore, we have

$$\begin{aligned}
& \left(y + \sum_{i=1}^{p} w_i\right)^+ + \sum_{i=1}^{p} \left((-w_i)^+ + (x_i - w_i)^+\right) \\
\geq\ & \left(y + \sum_{i=1}^{p} \left(w_i + (-w_i)^+ + (x_i - w_i)^+\right)\right)^+ && \text{from Inequality (27)} \\
=\ & \left(y + \sum_{i=1}^{p} \left(w_i^+ + (x_i - w_i)^+\right)\right)^+ \\
\geq\ & \left(y + \sum_{i=1}^{p} x_i^+\right)^+ && \text{from Inequality (26).}
\end{aligned}$$

For notational convenience, we denote $y(\tilde{\boldsymbol{z}}) = y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}}$, $x_i(\tilde{\boldsymbol{z}}) = x_i^0 + \boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}$ and $w_i(\tilde{\boldsymbol{z}}) = w_i^0 + \boldsymbol{w}_i{}'\tilde{\boldsymbol{z}}$. To prove Inequality (16), it suffices to show that for any $w_i^0, \boldsymbol{w}_i, i = 1, \ldots, p$, we have

$$\begin{aligned}
& \pi\left(y^0 + \sum_{i=1}^{p} w_i^0, \boldsymbol{y} + \sum_{i=1}^{p} \boldsymbol{w}_i\right) + \sum_{i=1}^{p} \left(\pi(-w_i^0, -\boldsymbol{w}_i) + \pi(x_i^0 - w_i^0, \boldsymbol{x}_i - \boldsymbol{w}_i)\right) \\
\geq\ & \mathrm{E}\left(\left(y(\tilde{\boldsymbol{z}}) + \sum_{i=1}^{p} w_i(\tilde{\boldsymbol{z}})\right)^+\right) + \sum_{i=1}^{p} \left(\mathrm{E}\left((-w_i(\tilde{\boldsymbol{z}}))^+\right) + \mathrm{E}\left((x_i(\tilde{\boldsymbol{z}}) - w_i(\tilde{\boldsymbol{z}}))^+\right)\right) \\
\geq\ & \mathrm{E}\left(\left(y(\tilde{\boldsymbol{z}}) + \sum_{i=1}^{p} x_i(\tilde{\boldsymbol{z}})^+\right)^+\right),
\end{aligned}$$

where the first inequality follows from Theorem 1 and the last inequality follows from Inequality (25).

To prove the tightness of the bound, we consider the case when $x_i^0 + \boldsymbol{x}_i{}'\boldsymbol{z}$, $i = 1, \ldots, p$ are non-zero crossing functions with respect to $\boldsymbol{z} \in \mathcal{W}$. Let

$$\mathcal{K} = \{k \ :\ x_k^0 + \boldsymbol{x}_k{}'\boldsymbol{z} \geq 0 \ \forall \boldsymbol{z} \in \mathcal{W}\}.$$

Hence,

$$y^0 + \boldsymbol{y}'\boldsymbol{z} + \sum_{i=1}^{p} \left(x_i^0 + \boldsymbol{x}_i{}'\boldsymbol{z}\right)^+ = y^0 + \boldsymbol{y}'\boldsymbol{z} + \sum_{i \in \mathcal{K}} \left(x_i^0 + \boldsymbol{x}_i{}'\boldsymbol{z}\right) \qquad \forall \boldsymbol{z} \in \mathcal{W}.$$

Therefore, if

$$y^0 + \boldsymbol{y}'\boldsymbol{z} + \sum_{i=1}^{p} \left(x_i^0 + \boldsymbol{x}_i{}'\boldsymbol{z}\right)^+ \geq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W}$$

or equivalently,

$$y^0 + \boldsymbol{y}'\boldsymbol{z} + \sum_{i \in \mathcal{K}} \left(x_i^0 + \boldsymbol{x}_i{}'\boldsymbol{z}\right) \geq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W},$$

we have

$$\begin{aligned}
& \mathrm{E}\left(\left(y^0+\boldsymbol{y}'\tilde{\boldsymbol{z}}+\sum_{i=1}^{p}\left(x_i^0+\boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}\right)^+\right)^+\right) \\
= & \mathrm{E}\left(\left(y^0+\boldsymbol{y}'\tilde{\boldsymbol{z}}+\sum_{i\in\mathcal{K}}\left(x_i^0+\boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}\right)\right)^+\right) \\
= & y^0+\sum_{i\in\mathcal{K}}x_i^0.
\end{aligned}$$

Likewise, if

$$y^0+\boldsymbol{y}'\boldsymbol{z}+\sum_{i=1}^{p}\left(x_i^0+\boldsymbol{x}_i{}'\boldsymbol{z}\right)^+\leq 0 \qquad \forall\boldsymbol{z}\in\mathcal{W}$$

or equivalently,

$$y^0+\boldsymbol{y}'\boldsymbol{z}+\sum_{i\in\mathcal{K}}\left(x_i^0+\boldsymbol{x}_i{}'\boldsymbol{z}\right)\leq 0 \qquad \forall\boldsymbol{z}\in\mathcal{W},$$

we have

$$\mathrm{E}\left(\left(y^0+\boldsymbol{y}'\tilde{\boldsymbol{z}}+\sum_{i=1}^{p}\left(x_i^0+\boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}\right)^+\right)^+\right)=\mathrm{E}\left(\left(y^0+\boldsymbol{y}'\tilde{\boldsymbol{z}}+\sum_{i\in\mathcal{K}}\left(x_i^0+\boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}\right)\right)^+\right)=0.$$

Indeed, for all $k\in\mathcal{K}$, let $(w_i^0,\boldsymbol{w}_i)=(x_i^0,\boldsymbol{x}_i)$ and for all $k\notin\mathcal{K}$, $(w_i^0,\boldsymbol{w}_i)=(0,\boldsymbol{0})$. Therefore, using the tightness result of Theorem 1, we have

$$\begin{aligned}
& \mathrm{E}\left(\left(y^0+\boldsymbol{y}'\tilde{\boldsymbol{z}}+\sum_{i=1}^{p}\left(x_i^0+\boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}\right)^+\right)^+\right) \\
\leq & \eta((y^0,\boldsymbol{y}),(x_1^0,\boldsymbol{x}_1),\ldots,(x_p^0,\boldsymbol{x}_p)) \\
= & \min_{w_i^0,\boldsymbol{w}_i,i=1,\ldots,p}\left\{\pi\left(y^0+\sum_{i=1}^{p}w_i^0,\boldsymbol{y}+\sum_{i=1}^{p}\boldsymbol{w}_i\right)+\sum_{i=1}^{p}\left(\pi(-w_i^0,-\boldsymbol{w}_i)+\pi(x_i^0-w_i^0,\boldsymbol{x}_i-\boldsymbol{w}_i)\right)\right\} \\
\leq & \pi\left(y^0+\sum_{i\in\mathcal{K}}x_i^0,\boldsymbol{y}+\sum_{i\in\mathcal{K}}\boldsymbol{x}_i\right)+\sum_{i\in\mathcal{K}}\left(\underbrace{\pi(-x_i^0,-\boldsymbol{x}_i)}_{=0}+\pi(0,\boldsymbol{0})\right)+ \\
& \sum_{i\notin\mathcal{K}}\left(\pi(-0,-\boldsymbol{0})+\underbrace{\pi(x_i^0,\boldsymbol{x}_i)}_{=0}\right) \\
= & \pi\left(y^0+\sum_{i\in\mathcal{K}}x_i^0,\boldsymbol{y}+\sum_{i\in\mathcal{K}}\boldsymbol{x}_i\right) \\
= & \begin{cases} y^0+\sum_{i\in\mathcal{K}}x_i^0 & \text{if } y^0+\boldsymbol{y}'\tilde{\boldsymbol{z}}+\sum_{i\in\mathcal{K}}\left(x_i^0+\boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}\right)\geq 0\ \forall\boldsymbol{z}\in\mathcal{W} \\ 0 & \text{if } y^0+\boldsymbol{y}'\tilde{\boldsymbol{z}}+\sum_{i\in\mathcal{K}}\left(x_i^0+\boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}\right)\leq 0\ \forall\boldsymbol{z}\in\mathcal{W} \end{cases} \\
= & \mathrm{E}\left(\left(y^0+\boldsymbol{y}'\tilde{\boldsymbol{z}}+\sum_{i=1}^{p}\left(x_i^0+\boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}\right)^+\right)^+\right).
\end{aligned}$$

$\blacksquare$

E Proof of Theorem 5

Proof : We first show that $Z_{TLRP} \leq Z_{LRP}$. Let $x_t^{k\dagger}$, $k = 0, \ldots, N$, $t = 1, \ldots, T-L$ and $y_{t+1}^{k\dagger}$, $k = 0, \ldots, N$, $t = 1, \ldots, T$ be the optimum solution to Problem (13), which is also feasible in Problem (17). Based on the following inequality,

$$\begin{aligned} & \eta((y^0, \boldsymbol{y}), (x_1^0, \boldsymbol{x}_1), \ldots, (x_p^0, \boldsymbol{x}_p)) \\ = & \min_{w_i^0, \boldsymbol{w}_i, i=1,\ldots,p} \left\{ \pi\left(y^0 + \sum_{i=1}^p w_i^0, \boldsymbol{y} + \sum_{i=1}^p \boldsymbol{w}_i\right) + \sum_{i=1}^p \left(\pi(-w_i^0, -\boldsymbol{w}_i) + \pi(x_i^0 - w_i^0, \boldsymbol{x}_i - \boldsymbol{w}_i)\right)\right\} \\ \leq & \pi\left(y^0, \boldsymbol{y}\right) + \sum_{i=1}^p \pi(x_i^0, \boldsymbol{x}_i), \end{aligned} \tag{28}$$

we have

$$\begin{aligned} Z_{TLRP} \leq & \sum_{t=1}^T c_t \pi(x_t^{0\dagger}, \boldsymbol{x}_t^\dagger) + \sum_{t=1}^L \left(h_t \pi(y_{t+1}^{0\dagger}, \boldsymbol{y}_{t+1}^\dagger) + b_t \pi(-y_{t+1}^{0\dagger}, -\boldsymbol{y}_{t+1}^\dagger)\right) + \\ & \sum_{t=L+1}^T \Big(h_t \eta\left((y_{t+1}^{0\dagger}, \boldsymbol{y}_{t+1}^\dagger), (-x_1^{0\dagger}, -\boldsymbol{x}_1), \ldots, (-x_{t-L}^{0\dagger}, -\boldsymbol{x}_{t-L}^\dagger)\right) + \\ & \quad b_t \eta\left((-y_{t+1}^{0\dagger}, -\boldsymbol{y}_{t+1}^\dagger), (x_1^{0\dagger} - S_t, \boldsymbol{x}_1^\dagger), \ldots, (x_{t-L}^{0\dagger} - S_t, \boldsymbol{x}_{t-L}^\dagger)\right) \Big) \\ \leq & \sum_{t=1}^T c_t \pi(x_t^{0\dagger}, \boldsymbol{x}_t^\dagger) + \sum_{t=1}^L \left(h_t \pi(y_{t+1}^{0\dagger}, \boldsymbol{y}_{t+1}^\dagger) + b_t \pi(-y_{t+1}^{0\dagger}, -\boldsymbol{y}_{t+1}^\dagger)\right) + \\ & \sum_{t=L+1}^T \Big(h_t \pi\left(y_{t+1}^{0\dagger}, \boldsymbol{y}^\dagger\right) + h_t \sum_{i=1}^{t-L} \pi(-x_i^{0\dagger}, -\boldsymbol{x}_i^\dagger) \quad + \\ & \quad b_t \pi\left(-y_{t+1}^{0\dagger}, -\boldsymbol{y}_{t+1}^\dagger\right) + b_t \sum_{i=1}^{t-L} \pi(-x_i^{0\dagger} - S_t, \boldsymbol{x}_i^\dagger) \Big). \end{aligned}$$

Observe that since $x_t^{0\dagger} + \boldsymbol{x}_t^{\dagger\prime} \boldsymbol{z} \geq 0$, $-x_t^{0\dagger} - \boldsymbol{x}_t^{\dagger\prime} \boldsymbol{z} \leq 0$ and $x_t^{0\dagger} - S_t + \boldsymbol{x}_t^{\dagger\prime} \boldsymbol{z} \leq 0$ for all $\boldsymbol{z} \in \mathcal{W}$, we have from Theorem 1, $\pi(x_i^{0\dagger}, \boldsymbol{x}_i^\dagger) = x_i^{0\dagger}$,$\pi(-x_i^{0\dagger}, -\boldsymbol{x}_i^\dagger) = 0$ and $\pi(x_i^{0\dagger} - S_t, \boldsymbol{x}_i^\dagger) = 0$ for all $i = 1, \ldots, T-L$. Hence,

$$Z_{TLRP} \leq \sum_{t=1}^T \left(c_t x_t^{0\dagger} + h_t \pi\left(y_{t+1}^{0\dagger}, \boldsymbol{y}^\dagger\right) + b_t \pi\left(-y_{t+1}^{0\dagger}, -\boldsymbol{y}_{t+1}^\dagger\right)\right) = Z_{LRP}.$$

We next show that $Z_{STOC} \leq Z_{TLRP}$. Under the truncated linear replenishment policy, the net-inventory at the end of period t is given by

$$y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t) = y_1^0 + \sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0 + \sum_{\tau=L+1}^t x_{\tau-L}^{TLRP}(\tilde{\boldsymbol{d}}_{\tau-L-1}) - \sum_{\tau=1}^t d_\tau(\tilde{\boldsymbol{z}}).$$

Let x_t^{k*}, $k = 0, \ldots, N$, $t = 1, \ldots, T-L$ and y_{t+1}^{k*}, $k = 0, \ldots, N$, $t = 1, \ldots, T$ be the optimum solution to Problem (17). It suffices to show that the following bounds:

(a)

$$\mathrm{E}\left(x_t^{TLRP}(\tilde{\boldsymbol{d}}_{t-1})\right) \leq \pi(x_t^{0*}, \boldsymbol{x}_t^*).$$

(b) For $t = 1, \ldots, L$,

$$\mathrm{E}\left(\left(y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t)\right)^+\right) \leq \pi\left(y_{t+1}^{0*}, \boldsymbol{y}_{t+1}^*\right)$$

and

$$\mathrm{E}\left(\left(y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t)\right)^-\right) \leq \pi\left(-y_{t+1}^{0*}, -\boldsymbol{y}_{t+1}^*\right)$$

(c) For $t = L+1, \ldots, T$,

$$\mathrm{E}\left(\left(y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t)\right)^+\right) \leq \eta\left((y_{t+1}^{0*}, \boldsymbol{y}_{t+1}^*), (-x_1^{0*}, -\boldsymbol{x}_1^*), \ldots, (-x_{t-L}^{0*}, -\boldsymbol{x}_{t-L}^*)\right)$$

and

$$\mathrm{E}\left(\left(y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t)\right)^-\right) \leq \eta\left((-y_{t+1}^{0*}, -\boldsymbol{y}_{t+1}^*), (x_1^{0*} - S_t, \boldsymbol{x}_1^*), \ldots, (x_{t-L}^{0*} - S_t, \boldsymbol{x}_{t-L}^*)\right).$$

For Bound (a), we note that

$$\begin{aligned}
\mathrm{E}\left(x_t^{TLRP}(\tilde{\boldsymbol{d}}_{t-1})\right) &= \mathrm{E}\left(\min\left\{\max\left\{x_t^{0*} + \boldsymbol{x}_t^{*\prime}\tilde{\boldsymbol{z}}, 0\right\}, S_t\right\}\right) \\
&\leq \mathrm{E}\left(\max\left\{x_t^{0*} + \boldsymbol{x}_t^{*\prime}\tilde{\boldsymbol{z}}, 0\right\}\right) \\
&= \mathrm{E}\left(\left(x_t^{0*} + \boldsymbol{x}_t^{*\prime}\tilde{\boldsymbol{z}}\right)^+\right) \\
&\leq \pi(x_t^{0*}, \boldsymbol{x}_t^*).
\end{aligned}$$

We focus on deriving Bound (c), as the exposition of Bound (b) is similar. Indeed, using the bound of

Theorem 4, we have for $t \geq L+1$,

$$
\begin{aligned}
& \mathrm{E}\left(\left(y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t)\right)^+\right) \\
= \; & \mathrm{E}\left(\left(y_1^0 + \sum_{\tau=1}^{L} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} x_{\tau-L}^{TLRP}(\tilde{\boldsymbol{d}}_{\tau-L-1}) - \sum_{\tau=1}^{t} d_\tau(\tilde{\boldsymbol{z}})\right)^+\right) \\
= \; & \mathrm{E}\left(\left(y_1^0 + \sum_{\tau=1}^{L} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} \min\left\{\max\left\{x_{\tau-L}^{0*} + \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}, 0\right\}, S_t\right\} - \sum_{\tau=1}^{t} d_\tau^0 - \sum_{\tau=1}^{t}\sum_{k=1}^{N} d_\tau^k \tilde{z}_k\right)^+\right) \\
\leq \; & \mathrm{E}\left(\left(y_1^0 + \sum_{\tau=1}^{L} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} \max\left\{x_{\tau-L}^{0*} + \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}, 0\right\} - \sum_{\tau=1}^{t} d_\tau^0 - \sum_{k=1}^{N}\sum_{\tau=1}^{t} d_\tau^k \tilde{z}_k\right)^+\right) \\
= \; & \mathrm{E}\Bigg(\Bigg(y_1^0 + \sum_{\tau=1}^{L} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} \left(x_{\tau-L}^{0*} + \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}\right) + \\
& \qquad \sum_{\tau=L+1}^{t} \max\left\{-x_{\tau-L}^{0*} - \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}, 0\right\} - \sum_{\tau=1}^{t} d_\tau^0 - \sum_{k=1}^{N}\sum_{\tau=1}^{t} d_\tau^k \tilde{z}_k\Bigg)^+\Bigg) \\
= \; & \mathrm{E}\Bigg(\Bigg(\underbrace{y_1^0 + \sum_{\tau=1}^{t} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} x_{\tau-L}^{0*} - \sum_{\tau=1}^{t} d_\tau^0}_{=y_{t+1}^{0*}} + \\
& \qquad \sum_{\tau=L+1}^{t} \left(-x_{\tau-L}^{0*} - \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}\right)^+ + \sum_{k=1}^{N} \underbrace{\left(\sum_{\tau=1}^{t}(x_{\tau-L}^k - d_\tau^k)\right)}_{=y_{t+1}^{k*}} \tilde{z}_k\Bigg)^+\Bigg) \\
= \; & \mathrm{E}\left(\left(y_{t+1}^{0*} + \boldsymbol{y}_{t+1}^{*}{}'\tilde{\boldsymbol{z}} + \sum_{\tau=L+1}^{t} \left(-x_{\tau-L}^{0*} - \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}\right)^+\right)^+\right) \\
\leq \; & \eta\left((y_{t+1}^{0*}, \boldsymbol{y}_{t+1}^{*}), (-x_1^{0*}, -\boldsymbol{x}_1^{*}), \ldots, (-x_{t-L}^{0*}, -\boldsymbol{x}_{t-L}^{*})\right).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
& \mathrm{E}\left(\left(y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t)\right)^-\right) \\
= \;& \mathrm{E}\left(\left(-y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t)\right)^+\right) \\
= \;& \mathrm{E}\left(\left(-y_1^0 - \sum_{\tau=1}^{L} x_{\tau-L}^0 - \sum_{\tau=L+1}^{t} x_{\tau-L}^{TLRP}(\tilde{\boldsymbol{d}}_{\tau-L-1}) + \sum_{\tau=1}^{t} d_\tau(\tilde{\boldsymbol{z}})\right)^+\right) \\
= \;& \mathrm{E}\left(\left(-y_1^0 - \sum_{\tau=1}^{L} x_{\tau-L}^0 - \sum_{\tau=L+1}^{t} \min\left\{\max\left\{x_{\tau-L}^{0*} + \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}, 0\right\}, S_t\right\} - \sum_{\tau=1}^{t} d_\tau^0 + \sum_{\tau=1}^{t}\sum_{k=1}^{N} d_\tau^k \tilde{z}_k\right)^+\right) \\
\leq \;& \mathrm{E}\left(\left(-y_1^0 - \sum_{\tau=1}^{L} x_{\tau-L}^0 - \sum_{\tau=L+1}^{t} \min\left\{x_{\tau-L}^{0*} + \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}, S_t\right\} + \sum_{\tau=1}^{t} d_\tau^0 - \sum_{k=1}^{N}\sum_{\tau=1}^{t} d_\tau^k \tilde{z}_k\right)^+\right) \\
= \;& \mathrm{E}\Bigg(\Bigg(-y_1^0 - \sum_{\tau=1}^{L} x_{\tau-L}^0 - \sum_{\tau=L+1}^{t} \left(x_{\tau-L}^{0*} - \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}\right) + \\
& \qquad \sum_{\tau=L+1}^{t} \left(-\min\left\{S_t - x_{\tau-L}^{0*} - \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}, 0\right\}\right) + \sum_{\tau=1}^{t} d_\tau^0 + \sum_{k=1}^{N}\sum_{\tau=1}^{t} d_\tau^k \tilde{z}_k\Bigg)^+\Bigg) \\
= \;& \mathrm{E}\Bigg(\Bigg(\underbrace{-y_1^0 - \sum_{\tau=1}^{L} x_{\tau-L}^0 - \sum_{\tau=L+1}^{t} x_{\tau-L}^{0*} + \sum_{\tau=1}^{t} d_\tau^0}_{=-y_{t+1}^{0*}} + \\
& \qquad \sum_{\tau=L+1}^{t} \left(x_{\tau-L}^{0*} - S_t + \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}\right)^+ + \sum_{k=1}^{N} \underbrace{\left(\sum_{\tau=1}^{t}(-x_{\tau-L}^{k*} + d_\tau^k)\right)}_{=-y_{t+1}^{k*}} \tilde{z}_k\Bigg)^+\Bigg) \\
= \;& \mathrm{E}\left(\left(-y_{t+1}^{0*} - \boldsymbol{y}_{t+1}^{*}{}'\tilde{\boldsymbol{z}} + \sum_{\tau=L+1}^{t} \left(x_{\tau-L}^{0*} - S_t + \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}\right)^+\right)^+\right) \\
\leq \;& \eta\left((-y_{t+1}^{0*}, -\boldsymbol{y}_{t+1}^{*}), (x_1^{0*} - S_t, \boldsymbol{x}_1^{*}), \ldots, (x_{t-L}^{0*} - S_t, \boldsymbol{x}_{t-L}^{*})\right).
\end{aligned}
$$

■

F Sample Formulation in *PROF*

The following is a sample formulation of Problem (17) in *PROF* is presented in Table 13. Note that the function `meanpositivebound()` implements $\pi(\cdot)$ of Equation (7), and `meannestedposbound()` implements $\eta(\cdot)$ of Theorem 4.

```
% Input Model of Uncertainty
Range = 20;
Z.zlow = Range*ones(N,1);
Z.zupp = Range*ones(N,1);
Z.p = .58*Range*ones(N,1);
Z.q = .58*Range*ones(N,1);
Z.sigma =.58*Range*ones(N,1);
Ny =[0 1:T];
Nx = [zeros(1,L) 0:T-L-1];
Nxms = [zeros(1,L) 0:T-L-1];
zcoef = eye(T,T);
MeanD = mu*ones(T,1);
for n = 2:T
     zcoef(1:n-1,n)= alpha;
end

% Start PROF
startmodel
x = linearrule(T,N,Nx)
xms = linearrule(T,N,Nxms)
y = linearrule(T+1,N,Ny);

for i=1:T
    addconst(xms(i,:)  == x(i,:)-S*ldrdata([0 1],N));
end

hbound=0;
sbound=0;
for t=1:T
    if L+1 ≤ t
        hbound = hbound+ h*meannestedposbound(Z,y(t+1,0:t),-x(L+1:t,0:t),t);
        sbound = sbound + b(t)*meannestedposbound(Z,-y(t+1,0:t),xms(L+1:t,0:t),t);
     else
         hbound = hbound+h*meanpositivebound(Z,y(t+1,:),1,N);
         sbound = sbound + b(t)*meanpositivebound(Z,-y(t+1,:),1,N);
     end
end

minimize (sbound+hbound + c*sum(meanpositivebound(Z,x(L+1:T,:),T-L,N)))
addconst(x(1:L,0)==initx); addconst(y(1,0)==inity);
for i=1:T
    t addconst(y(i+1,:)==y(i,:)+x(i,:)-ldrdata([0 MeanD(i);(1:N)' zcoef(:,i)],N));
end
m=endmodel;
s = m.solve('MOSEK')
```

Table 13: Formulation of Problem (17) using *PROF*