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Robust control and recursive utility

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Abstract. This paper shows that a finite-horizon version of the robust control criterion appearing in recent papers by Hansen, Sargent, and their coauthors can be described as recursive utility, which in continuous time takes the form of the Stochastic Differential Utility (SDU) of Duffie and Epstein (1992). While it has previously been noted that Bellman equations arising in robust control settings are of the same form as Bellman equations arising from SDU maximization, here this connection is shown directly without reference to any underlying dynamics, or Markov structure.

Key words: Recursive utility, robust control

JEL Classification: D81

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1 Introduction

This paper characterizes a robust control criterion recently introduced in economics by Anderson et al. (2000) as a form of the Stochastic Differential Utility (SDU) of Duffie and Epstein (1992). Maenhout (1999) and Hansen et al. (2001) note that the Bellman equation obtained in certain robust control specifications takes the same form as one would obtain for corresponding SDU specifications. Here this connection is shown directly without reference to any underlying dynamics, or indeed any Markov structure. Existence and the basic properties of the SDU obtained have

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been shown by Schroder and Skiadas (1999). Increasing robustness corresponds to higher risk aversion and preference for earlier resolution of uncertainty. Extensions of the main result that include Maenhout's (1999) formulation are also presented.

In order to introduce the robust control criterion analyzed in the main part of the paper, consider an agent with time-additive expected utility over consumption-rate paths: $E\left[\int_0^T e^{-\beta t} u\left(c_t\right) dt\right]$. (The case of recursive utility is outlined in the last section.) The underlying filtration is generated by some Brownian motion B under some underlying probability P. Suppose now that the agent contemplates the possibility that the underlying probability is not P, but some other probability P^x that is equivalent to P, meaning that it defines the same zero-probability events. It is well-known that one can find a process x such that $B_t^x = B_t - \int_0^t x_s\, ds$ is Brownian motion under P^x. The relative entropy of P^x with respect to P is a measure of how "close" P^x is to P and is defined quite generally in terms of the density of P^x with respect to P in the following section. In terms of x, the relative entropy can be expressed as the quadratic $\mathcal{R}^x = E^x\left[\int_0^T e^{-\beta t} \left|x_t\right|^2 dt\right]$, where E^x denotes an expectation under P^x. The robust control criterion we consider is of the form

$$\hat{V}_0 = \inf_x \left\{ E^x \left[\int_0^T e^{-\beta t} u\left(c_t\right) dt \right] + \theta \mathcal{R}^x \right\},$$

for some positive constant θ. For infinite θ, the infimum is achieved by the original measure P. As θ decreases, the agent assumes that the underlying probability could be one of increasingly larger relative entropy. We refer to Hansen et al. (2001), and Hansen and Sargent (2001) for further related discussion. The use of relative entropy as a measure of "distance" is motivated by tractability rather than some compelling decision theoretic foundation. Maenhout (1999) allows θ to be a function of the continuation utility, a case that will be covered in the last section of this paper.

Our main conclusion will be that $\hat{V}_0 = V_0$, where the process V is the unique (in a properly defined space) solution to the recursion

$$V_t = E_t \left[\int_t^T e^{-\beta(s-t)} \left(u\left(c_s\right) ds - \frac{1}{2\theta} d\left[V\right]_s \right) \right],$$

where $[V]$ denotes the quadratic variation process of V. This is an example of the stochastic differential utility (SDU) of Duffie and Epstein (1992), the specific functional form being analyzed by Schroder and Skiadas (1999). (For bounded u, existence also follows from Kobylanski 2000.) It offers the alternative view that robustness corresponds to a utility penalty that is proportional to the quadratic variation of the continuation utility. The higher θ, the higher this penalty. As Duffie and Epstein (1992) show, θ is also a measure of comparative risk-aversion: the higher θ, the more risk-averse the utility. Moreover, clearly θ has no effect on how the agent ranks deterministic consumption plans. Yet another interpretation of θ is provided in Skiadas (1998), where it is shown that the fact that θ is positive corresponds to the monotonicity of the utility in the underlying filtration. That is, more and earlier information increases the agent's utility.

The above recursive formulation allows the direct reinterpretation of available solutions for recursive utility in terms of robustness, to the extent that robustness is captured by the above criterion. For purposes of illustration, suppose $u(c_t) = \log(c_t)$, and consider the Merton (1971) problem of selecting an optimal lifetime consumption/portfolio strategy in a complete securities market. For the specific SDU form, a complete solution to this problem, under essentially arbitrary price dynamics, is provided by Schroder and Skiadas (1999). Assume first a constant investment opportunity set (that is, i.i.d. instantaneous returns). If $\theta = 0$, it is well-known that one obtains a myopic solution in which it is optimal to invest a fixed proportion of one's wealth in a mean-variance efficient mutual fund throughout one's life. However, if $\theta > 0$, Theorem 2 of Schroder and Skiadas (1999) shows that the time-t investment in the mutual fund must be scaled by the factor $\beta\theta/\left(1 + \beta\theta - e^{-\beta(T-t)}\right)$. As a result the investor increases the proportion of wealth invested in risky assets with age, as a direct implication of robustness considerations. Such a conclusion is of course a consequence of the specific parameterization of the utility function.[1]

A non-zero value of θ also has a strong effect on the composition of an optimal portfolio given a stochastic investment opportunity set, reflecting the well-documented predictability in asset returns and stochastic volatility. In the benchmark case of $\theta = 0$, the myopic nature of the solution implies that the agent will still invest in an instantaneously mean-variance efficient portfolio, despite the fact that the investment opportunity set may change stochastically over time. For $\theta > 0$, however, the optimal portfolio composition deviates from instantaneous mean-variance efficiency by an additional hedging term given in Theorem 3 of Schroder and Skiadas (1999). Moreover, the calibration of Campbell et al. (2001) shows that the additional hedging term is significant from a practical investment viewpoint. Further related results on portfolio selection under recursive utility can be found in Campbell and Viceira (2002), and Schroder and Skiadas (2002a).

The specific logarithmic SDU form is also adopted by Duffie et al. (1997) in a setting of imperfect information in which equilibrium state prices and the term structure of interest rates are computed in closed form, showing their dependence on the rate of information. Reinterpreting the model in terms of the robust control criterion provides an example in which equilibrium prices do not depend on the timing of resolution of uncertainty if there are no robustness concerns ($\theta = \infty$), but they do otherwise ($\theta < \infty$).

The robust control criterion analyzed in this paper differs only slightly from that discussed by Hansen et al. (2001) in the following ways. First, we adopt a finite horizon, the advantage of which is illustrated in the discussion of the lifetime portfolio selection problem. Taking a limit as the horizon stretches to infinity presents no problems. Second, we will allow a general stochastic discount process (in place of the constant β above), a generality that comes at small cost to the complexity of

[1] For time-additive expected utility and complete markets, Gollier and Zeckhauser (2000) relate the age-dependence of one's optimal allocation between a riskless and a risky asset to the curvature of the coefficient of absolute risk tolerance. Allowing θ to depend on the continuation utility, as in Maenhout (1999), can also result in different qualitative conclusions. Finally, all of these arguments do not cover the consequences of a nontradeable income stream.

the exposition. Third, Hansen et al. (2001) consider absolutely continuous alternative probabilities, and not just equivalent ones, but as will be noted in the following section this has no effect on the utility value. Finally, the technical construction of distributions is accomplished somewhat differently in Hansen et al. (2001). The two approaches are equivalent, and their exact relationship is spelled out in Chapt. 7 of Lipster and Shiryaev (2001).

Maenhout (1999) considered a variant of the above robust control formulation in which the parameter θ is stochastic and proportional to the continuation utility, and noted that the Bellman equation matches the one of the problem considered in Schroder and Skiadas (1999) (for different parameter values than those considered in the above discussion). In particular, with $u(c)$ being power utility, one obtains a reinterpretation of the remaining parametric solutions to the Merton problem in Schroder and Skiadas (1999). An extension of the main argument that includes Maenhout's setting, as well as a more general recursive utility specification, is outlined at the end of this paper.

This paper is also closely related to Chen and Epstein (2001), who relate a multiple priors utility to recursive utility. Finally, Geoffard (1996) and Dumas et al. (2000) consider a dual formulation of recursive utility that results in a minmax criterion. The minimization in their case is over stochastic discount factors, and not over alternative probabilities. There is a formal sense, however, in which all of the above papers can be viewed as instances of a general equivalence between minmax criteria and recursive utility.

The remainder of the paper is organized in five sections. Section 2 formally defines the robust control criterion. Section 3 computes the relative entropy quadratic expression summarized above. Section 4 presents the paper's main result, and Section 5 gives its proof. Finally, Sect. 6 presents extensions of the main result, including Maenhout's (1999) formulation.

2 The robust control criterion

We consider a finite time horizon $[0,T]$, and a probability space $(\Omega,\mathcal{F},P)$ supporting a n-dimensional Brownian motion $B=\{B_t : t\in[0,T]\}$. All stochastic processes are assumed progressively measurable relative to the augmented filtration generated by B, denoted $\{\mathcal{F}_t : t\in[0,T]\}$. We assume $\mathcal{F}=\mathcal{F}_T$. The expectation operator under P is denoted E, while the conditional expectation under P given $\mathcal{F}_t$ is denoted E_t.

We define the set of real-valued progressively measurable processes:

$$D_1^{\exp}=\left\{x : E\left[\exp\left(\alpha\int_0^T |x_t|\,dt\right)\right]<\infty \quad \text{for all } \alpha\in\mathbf{R}_+\right\},$$

and we take as primitive the following quantities:

1. A process, $U\in D_1^{\exp}$, representing a felicity process associated with a given consumption plan (suppressed in the notation). For example, in a time-additive specification, one could take $U_t=e^{-\delta t}u(c_t)$, where c_t is the time-t consumption rate, and u is a real-valued felicity function on $\mathbf{R}_+$. More generally, U

can be any function of the entire consumption plan, allowing, for example, the modeling of habits or durability.

2. A bounded process β such that $\beta_t \geq 0$ a.s. for all t.
3. A nonnegative constant $\theta \in \mathbf{R}_+$.

Let $\mathcal{P}$ be the set of all probabilities on $(\Omega, \mathcal{F})$ that are equivalent to P (that is, they define the same null events as P). Given $P^x \in \mathcal{P}$, the expectation operator under P^x is denoted E^x, with E_t^x being the corresponding time-t conditional expectation given $\mathcal{F}_t$. For each $P^x \in \mathcal{P}$, we define the conditional density process, a martingale, by

$$\xi_t^x = E_t\left[\frac{dP^x}{dP}\right], \quad t \in [0, T]. \tag{1}$$

The following well-known change of measure formula will be useful, where the notation Y^- refers to the negative part of a process Y, that is, $Y_t^- = \max\{-Y_t, 0\}$.

Lemma 1 *Suppose $P^x \in \mathcal{P}$ and Y is a progressively measurable process. Then $E^x\left[\int_0^T Y_s^- ds + Y_T^-\right] < \infty$ if and only if $E\left[\int_0^T Y_s^- \xi_s^x ds + Y_T^- \xi_T^x\right] < \infty$, in which case*

$$E_t^x\left[\int_t^T Y_s ds + Y_T\right] = \frac{1}{\xi_t^x} E_t\left[\int_t^T Y_s \xi_s^x ds + Y_T \xi_T^x\right], \quad t \in [0, T].$$

Letting $\phi : (0, \infty) \to \mathbf{R}$ be the function $\phi(\alpha) = \alpha \log(\alpha)$, we define the *relative entropy* process corresponding to $P^x \in \mathcal{P}$ by

$$\begin{aligned}\mathcal{R}_t^x &= E_t\left[\int_t^T \beta_s e^{-\int_t^s \beta_\tau d\tau} \phi\left(\frac{\xi_s^x}{\xi_t^x}\right) ds + e^{-\int_t^T \beta_\tau d\tau} \phi\left(\frac{\xi_T^x}{\xi_t^x}\right)\right] \\ &= E_t^x\left[\int_t^T \beta_s e^{-\int_t^s \beta_\tau d\tau} \log\left(\frac{\xi_s^x}{\xi_t^x}\right) ds + e^{-\int_t^T \beta_\tau d\tau} \log\left(\frac{\xi_T^x}{\xi_t^x}\right)\right], t \in [0, T].\end{aligned}$$

The second equality is a consequence of Lemma 1, assuming the first integral is well-defined.

Proposition 2 *$\mathcal{R}_t^x$ is well-defined and almost surely valued in* $[0, \infty]$.

Proof The function ϕ is convex with $\phi(1) = 0$ and $\phi'(1) = 1$. The supporting line at $\alpha = 1$ gives $\phi(\alpha) \geq \alpha - 1$. This shows that the integrand in the definition of relative entropy is bounded below by an integrable process, and therefore relative entropy is well-defined, although possibly infinite. The lower bound on ϕ can also be used to show non-negativity of $\mathcal{R}_t^x$, as follows. Letting $\mathcal{B}_t = \exp\left(-\int_0^t \beta_u \, du\right)$, and using Lemma 1,

$$\begin{aligned}\mathcal{B}_t \mathcal{R}_t^x &\geq \frac{1}{\xi_t^x} E_t\left[\int_t^T \beta_s \mathcal{B}_s \xi_s^x \, ds + \mathcal{B}_T \xi_T^x\right] - E_t\left[\int_t^T \beta_s \mathcal{B}_s \, ds + \mathcal{B}_T\right] \\ &= E_t^x\left[\int_t^T -d\mathcal{B}_s + \mathcal{B}_T\right] - E_t\left[\int_t^T -d\mathcal{B}_s + \mathcal{B}_T\right] = 0.\end{aligned}$$

□

The robust control criterion at time t is defined as

$$\hat{V}_t = \text{ess inf}\left\{V_t^x : P^x \in \mathcal{P}_U\right\},$$

where

$$V_t^x = E_t^x\left[\int_t^T e^{-\int_t^s \beta_u du} U_s \, ds\right] + \theta \mathcal{R}_t^x$$

and $\mathcal{P}_U$ is the set of measures $P^x \in \mathcal{P}$ such that $E^x\left[\int_0^T |U_t| \, dt\right] < \infty$. For example, if U is bounded, then $\mathcal{P}_U = \mathcal{P}$.

Remark 3 The value $\hat{V}$ does not change if, as in Hansen et. al. (2001), one takes an infimum over probabilities that are absolutely continuous with respect to P. Given any probability P^x that is absolutely continuous relative to P, one can define the sequence of measures $P^{x(n)} = n^{-1}P + \left(1 - n^{-1}\right) P^x$, $n = 1, 2, \ldots$, each one of which is equivalent to P. It is then not hard to show that, for every t, $V_t^{x(n)}$ converges to V_t^x almost surely.

3 An expression for relative entropy

Fixing some $P^x \in \mathcal{P}$, in this section we derive a convenient expression for $\mathcal{R}^x$. Let $\mathcal{L}_2$ be the set of n-dimensional progressively measurable processes x such that $\int_0^T x_s' x_s ds < \infty$ a.s. By the martingale representation theorem (see, for example, Protter (1990): Chapt. IV, Corollary 4 of Theorem 42), there exists $x \in \mathcal{L}_2$ such that $d\xi_t^x = \xi_t^x x_t' dB_t$, $\xi_0^x = 1$, or equivalently

$$\xi_t^x = \exp\left(\int_0^t x_s' \, dB_s - \frac{1}{2}\int_0^t x_s' x_s \, ds\right), \quad t \in [0, T]. \tag{2}$$

Conversely, let X be the set of all $x \in \mathcal{L}_2$ such that the process ξ^x defined by (2) is a martingale. Then x defines the probability $P^x \in \mathcal{P}$ by $P^x(F) = E\left[1_F \xi_T^x\right]$ for every $F \in \mathcal{F}$, and ξ^x satisfies (1). By Girsanov's theorem, for every $x \in X$,

$$B_t^x = B_t - \int_0^t x_s \, ds, \quad t \in [0, T], \tag{3}$$

defines a Brownian motion under P^x. Note that

$$\xi_t^x = \exp\left(\int_0^t x_s' \, dB_s^x + \frac{1}{2}\int_0^t x_s' x_s \, ds\right), \quad t \in [0, T]. \tag{4}$$

The following proposition extends a result by Hansen et al. (2001). Note that the claimed equation holds whether the quantities involved are finite or infinite.

Proposition 4 *For every $x \in X$,*

$$\mathcal{R}_t^x = \frac{1}{2} E_t^x\left[\int_t^T e^{-\int_t^u \beta_\tau d\tau} x_u' x_u \, du\right], \quad t \in [0, T].$$

Proof Fixing $x \in X$ and $t < T$, we first consider the event

$$F = \left\{ E_t^x \left[\int_t^T x_u' x_u \, du \right] < \infty \right\}.$$

We use the notation $\xi_{t,s}^x = \xi_s^x / \xi_t^x$ and $\mathcal{B}_{t,s} = \exp\left(-\int_t^s \beta_u \, du\right)$. Noting that $\int_s^T \beta_u \mathcal{B}_{t,u} \, du + \mathcal{B}_{t,T} = \mathcal{B}_{t,s}$, the definition of relative entropy implies

$$\mathcal{B}_{t,s} \left(\mathcal{R}_s^x + \log\left(\xi_{t,s}^x\right) \right) = - \int_t^s \beta_u \mathcal{B}_{t,u} \log\left(\xi_{t,u}^x\right) \, du + M_s \quad \text{on } F, \quad s \geq t,$$

where

$$M_s = E_s^x \left[1_F \left(\int_t^T \beta_u \mathcal{B}_{t,u} \log\left(\xi_{t,u}^x\right) \, du + \mathcal{B}_{t,T} \log\left(\xi_{t,T}^x\right) \right) \right].$$

From Eq. (4) and the definition of F, it follows that $\{M_s : s \geq t\}$ is a martingale under P^x. Applying integration by parts, we find that on the event F, and for $s \geq t$,

$$\begin{aligned} & -\beta_s \mathcal{B}_{t,s} \left(\mathcal{R}_s^x + \log\left(\xi_{t,s}^x\right) \right) ds + \mathcal{B}_{t,s} d\left(\mathcal{R}_s^x + \log\left(\xi_{t,s}^x\right) \right) \\ & = -\beta_s \mathcal{B}_{t,s} \log\left(\xi_{t,s}^x\right) ds + dM_s. \end{aligned}$$

Using Eq. (4), the above expression simplifies to

$$d\mathcal{R}_s^x = -\left(\frac{1}{2} x_s' x_s - \beta_s \mathcal{R}_s^x \right) ds + d\,(\text{martingale under } P^x) \quad \text{on } F, \quad s \geq t.$$

Applying integration by parts again and integrating, we obtain

$$\mathcal{R}_t^x = \frac{1}{2} E_t^x \left[\int_t^T e^{-\int_t^u \beta_\tau d\tau} x_u' x_u \, du \right] \quad \text{on } F.$$

There remains to show that $\mathcal{R}_t^x = \infty$ a.s. on F^c (the complement of F). For each $n = 1, 2, \ldots$, we define the stopping time

$$\tau_n = \min \left\{ s \geq t : \int_t^s x_u' x_u \, du = n \quad \text{or} \quad s = T \right\}.$$

The definition of relative entropy and the law of iterated expectations implies

$$\mathcal{R}_t^x = E_t^x \left[\int_t^{\tau_n} \beta_u \mathcal{B}_{t,u} \log\left(\xi_{t,u}^x\right) \, du + \mathcal{B}_{t,\tau_n} \left(\log\left(\xi_{t,\tau_n}^x\right) + \mathcal{R}_{\tau_n}^x \right) \right].$$

Since $\mathcal{R}^x$ is nonnegative,

$$\mathcal{R}_t^x \geq \mathcal{R}_t^n \triangleq E_t^x \left[\int_t^{\tau_n} \beta_u \mathcal{B}_{t,u} \log\left(\xi_{t,u}^x\right) \, du + \mathcal{B}_{t,\tau_n} \log\left(\xi_{t,\tau_n}^x\right) \right].$$

The same argument used above, but with τ_n in place of T, gives

$$\mathcal{R}_t^x \geq \mathcal{R}_t^n = \frac{1}{2} E_t^x \left[\int_t^{\tau_n} e^{-\int_t^u \beta_\tau d\tau} x_u' x_u \, du \right].$$

Monotone convergence implies

$$\mathcal{R}_t^x \geq \frac{1}{2} E_t^x \left[\int_t^T e^{-\int_t^u \beta_\tau d\tau} x_u' x_u \, du \right].$$

Therefore, $\mathcal{R}_t^x = \infty$ a.s. on the event F^c. □

4 Main result

Letting

$$X_U = \left\{ x \in X : E^x \left[\int_0^T |U_t| \, dt \right] < \infty \right\},$$

the robust control criterion can be reformulated as

$$\hat{V}_t = \text{ess inf} \left\{ V_t^x : x \in X_U \right\},$$

where

$$V_t^x = E_t^x \left[\int_t^T e^{-\int_t^s \beta_u du} \left(U_s + \frac{\theta}{2} x_s' x_s \right) ds \right]. \tag{5}$$

The paper's main result is stated below, while extensions are outlined in the last section. We use the space of progressively measurable processes

$$D_0^{\exp} = \left\{ x : E \left[\exp \left(\alpha \, \text{ess sup}_t \; |x_t| \right) \right] < \infty \quad \text{for all } \alpha \in \mathbf{R}_+ \right\},$$

and the space D^n of all $\mathbf{R}^n$-valued progressively measurable processes x such that $E \left[\int_0^T x_t' x_t \, dt \right] < \infty$.

Theorem 5 *There exists a unique pair* $(V, \sigma) \in D_0^{\exp} \times D^n$ *such that*

$$dV_t = -\left(U_t - \beta V_t - \frac{1}{2\theta} \sigma_t' \sigma_t \right) dt + \sigma_t' \, dB_t, \quad V_T = 0. \tag{6}$$

For every $x \in X_U$*,*

$$V_t^x = V_t + \frac{\theta}{2} E_t^x \left[\int_t^T e^{-\int_t^s \beta_u \, du} \left(x_s + \frac{1}{\theta} \sigma_s \right)' \left(x_s + \frac{1}{\theta} \sigma_s \right) ds \right]. \tag{7}$$

Finally, if $\hat{x} = -\theta^{-1} \sigma$*, then* $\hat{x} \in X_U$*, and*

$$\hat{V} = V^{\hat{x}} = V.$$

From now on, (V, σ) are the processes characterized in the above theorem. The detailed proof of the theorem is provided in the following section. The claim on the existence and uniqueness of a solution to Eq. (6) follows from Theorem A1 of Schroder and Skiadas (1999), after a simple rescaling. The same theorem implies that the solution V as a function of the parameter U is monotonically increasing and concave, and, by Lemma A3 of Schroder and Skiadas (1999), V is bounded if U is bounded. (For the case of bounded U, existence and uniqueness also follows from Kobylanski's 2000 results. Note that the functional form considered violates the technical restrictions imposed by Pardoux and Peng 1990 and Duffie and Epstein 1992.)

Another useful characterization of V, proved as part of Lemma A1 of Schroder and Skiadas (1999), is that V is the unique process in $D_0^{\exp}$ that solves

$$V_t = -\theta \log \left(E_t \left[\exp \left(-\frac{1}{\theta} \int_t^T (U_s + \beta_s V_s)\, ds \right) \right] \right), \quad t \in [0, T].$$

This equation delivers a closed-form expression for the case of no discounting:

$$V_t = -\theta \log \left(E_t \left[\exp \left(-\frac{1}{\theta} \int_t^T U_s\, ds \right) \right] \right), \quad \text{if} \quad \beta = 0.$$

Yet another characterization of V (with or without discounting) is obtained after the monotone transformation

$$Y_t = \theta \left(1 - \exp \left(-\frac{1}{\theta} V_t \right) \right).$$

Applying Ito's lemma gives the recursion:

$$Y_t = E_t \left[\int_t^T \left(1 - \frac{1}{\theta} Y_s \right) \left(U_s + \theta \beta_t \log \left(1 - \frac{1}{\theta} Y_s \right) \right) ds \right].$$

The details of this argument are as in the proof of Theorem 1 of Schroder and Skiadas (1999).

As noted in the introduction, as θ increases the SDU becomes comparatively more risk averse (Duffie and Epstein 1992). Another interesting property concerns preferences toward the timing of resolution of uncertainty. Kreps and Porteus (1978) first introduced the idea that an agent with recursive utility (in discrete time) may have preferences for earlier or later resolution of uncertainty, depending on the curvature of an intertemporal aggregator. This idea is extended in Skiadas (1998), where a utility is defined to exhibit preferences for early (late) resolution, or more generally preferences for more (less) information, if the utility is monotonically increasing (decreasing) in the underlying information filtration (partially ordered by inclusion of the corresponding sigma algebras). Skiadas (1998) also characterizes this property of an SDU in terms of the convexity or concavity of the intertemporal aggregator. The particular recursive utility form introduced above exhibits preferences for more information, and hence early resolution of uncertainty, whether or not there are any planning advantages from more information. This is a consequence of the convexity of the integrand in the last expression relative to Y. (The proof of this claim follows easily along the lines of the proof in Skiadas, 1998, using the comparison lemmas of Schroder and Skiadas, 1999.) Duffie et al. (1997) illustrate the role of preferences for the timing of resolution of uncertainty in an equilibrium setting using an SDU of the form introduced above.

5 Proof of main result

This section contains a proof of the main theorem. It can be skipped without loss of continuity.

Lemma 6 *For any $x \in X_U$ with finite entropy, Eq. (7) holds.*

Proof We fix any $x \in X_U$ with finite relative entropy, or equivalently

$$E^x\left[\int_0^T x_t' x_t\, dt\right] < \infty.$$

Given any process X, we let $\tilde{X}$ denote the discounted version:

$$\tilde{X}_t = \exp\left(-\int_0^t \beta_s\, ds\right) X_t,$$

and we also let $\tilde{\theta}_t = \exp\left(-\int_0^t \beta_s\, ds\right)\theta$. By Ito's lemma, Eq. (6) implies

$$\begin{aligned} d\tilde{V}_t &= -\left(\tilde{U}_t - \frac{1}{2\tilde{\theta}_t}\tilde{\sigma}_t'\tilde{\sigma}_t\right) dt + \tilde{\sigma}_t'\, dB_t \\ &= -\left(\tilde{U}_t - \frac{1}{2\tilde{\theta}_t}\tilde{\sigma}_t'\tilde{\sigma}_t - \tilde{\sigma}_t' x_t\right) dt + \tilde{\sigma}_t'\, dB_t^x. \end{aligned}$$

Note that, given any $x \in X_U$ with finite entropy, Eq.(5) and the martingale representation theorem imply that there exists a $\sigma^x \in \mathcal{L}_2$ such that

$$dV_t^x = -\left(U_t + \frac{\theta}{2}x_t'x_t - \beta_t V_t^x\right) dt + \sigma_t^{x\prime}\, dB_t^x.$$

Therefore,

$$d\tilde{V}_t^x = -\left(\tilde{U}_t + \frac{\tilde{\theta}_t}{2}x_t'x_t\right) dt + \tilde{\sigma}_t^{x\prime}\, dB_t^x.$$

Combining the dynamics for $\tilde{V}$ and $\tilde{V}^x$, we obtain

$$d\left(\tilde{V}_t^x - \tilde{V}_t\right) = -\frac{\tilde{\theta}_t}{2}\left(x_t + \frac{1}{\theta}\sigma_t\right)'\left(x_t + \frac{1}{\theta}\sigma_t\right) dt + dM_t,$$

where $M_t = \int_0^t (\tilde{\sigma}_s^x - \tilde{\sigma}_s)'\, dB_s^x$. We fix some time $t \in [0, T]$. Since M is a local martingale under P^x, we can select an increasing sequence of stopping times

$\{\tau_n : n = 1, 2, \dots\}$ such that $\tau_n \geq t$ a.s., $\lim_{n\to\infty} \tau_n = T$ a.s., and the stopped process $\{M_{s\wedge\tau_n} : s \geq t\}$ is a martingale under P^x. Integrating the dynamics of $\tilde{V}^x - \tilde{V}$ from t to τ_n, and applying the operator E^x_t, we obtain

$$\tilde{V}^x_t - \tilde{V}_t = E^x_t \left[\int_t^{\tau_n} \frac{\tilde{\theta}_s}{2} \left(x_s + \frac{1}{\theta}\sigma_s \right)' \left(x_s + \frac{1}{\theta}\sigma_s \right) ds \right] + E^x_t \left[\tilde{V}^x_{\tau_n} - \tilde{V}_{\tau_n} \right].$$

Letting n approach infinity gives

$$\tilde{V}^x_t - \tilde{V}_t = E^x_t \left[\int_t^{T} \frac{\tilde{\theta}_s}{2} \left(x_s + \frac{1}{\theta}\sigma_s \right)' \left(x_s + \frac{1}{\theta}\sigma_s \right) ds \right],$$

which implies the claimed equation. Taking the limit under the integral is justified as follows. The first expectation converges by monotone convergence, since the integrand is nonnegative. The term $E^x_t \left[\tilde{V}_{\tau_n} \right]$ converges to zero as $n \to \infty$ almost surely by dominated convergence, using the fact that $V \in D^{\exp}_0$. Finally, $E^x_t \left[V^x_{\tau_n} \right]$ converges to zero as $n \to \infty$ almost surely. This follows from the expression

$$E^x_t \left[\tilde{V}^x_{\tau_n} \right] = E^x_t \left[\int_{\tau_n}^{T} \left(\tilde{U}_s + \frac{\tilde{\theta}_s}{2} x'_s x_s \right) ds \right],$$

and dominated convergence, since $x \in X_U$ implies $E^x_t \left[\int_t^T \left| \tilde{U}_s \right| ds \right] < \infty$, and x having finite entropy implies $E^x_t \left[\int_t^T x'_s x_s \, ds \right] < \infty$. □

The above lemma suggests that V^x is minimized for $x = -\theta^{-1}\sigma$. This raises the question of whether x is an element of X_U. The following lemma will help us show that it is.

Lemma 7 *Let* $x = -\theta^{-1}\sigma$. *Then*

$$\xi^x_t = \exp\left(\frac{1}{\theta} \left(V_0 - V_t - \int_0^t (U_s - \beta V_s) \, ds \right) \right).$$

Proof Integrating Eq. (6) from 0 to t, and using the fact that $\sigma = -\theta x$, we obtain

$$\begin{aligned} V_t - V_0 &= -\int_0^t \left(U_s - \beta V_s - \frac{1}{2\theta} \sigma'_s \sigma_s \right) ds + \int_0^t \sigma'_s \, dB_s \\ &= -\int_0^t (U_s - \beta V_s) \, ds - \theta \left(\int_0^t x'_s \, dB_s - \frac{1}{2} \int_0^t x'_s x_s \, ds \right). \end{aligned}$$

Combining this expression with Eq. (2) gives the result. □

Suppose now that $x = -\theta^{-1}\sigma$. There remains to show that $x \in X_U$. We know that ξ^x is a local martingale. Let $\{\tau_n\}$ be a corresponding increasing sequence so that ξ^x stopped at τ_n is a martingale. Then $E\xi^x_{\tau_n} = 1$ for every n. The last lemma implies that there exist constants, A, B, C, such that

$$\xi^x_{\tau_n} \leq \exp\left(A + B \sup_t V_t + C \int_0^T |U_s|\; ds\right), \quad n = 1, 2, \ldots$$

Since $U \in D_1^{\exp}$ and $V \in D_0^{\exp}$, dominated convergence implies that $E\xi^x_T = 1$, which is known to imply (see, for example, Karatzas and Shreve 1988) that ξ^x is a martingale. Finally, by the Cauchy-Schwarz inequality,

$$\begin{aligned} E^x\left[\int_0^T |U_s|\; ds\right] &= E\left[\xi^x_T \int_0^T |U_s|\; ds\right] \\ &\leq E\left[(\xi^x_T)^2\right]^{\frac{1}{2}} E\left[\left(\int_0^T |U_s|\; ds\right)^2\right]^{\frac{1}{2}} < \infty, \end{aligned}$$

where the last inequality is again a consequence of the last lemma and the fact that $U \in D_1^{\exp}$ and $V \in D_0^{\exp}$.

Remark 8 Lazrak and Quenez (2002) comment on an earlier version of this paper by providing a variation of the proof that $\hat{V} = V$ for the case of bounded U. The difference of the two arguments is small. While in the proof of Lemma 6 we compute the drift of $\tilde{V}^x - \tilde{V}$, which is obviously minimized and is zero if $x = -\theta^{-1}\sigma$, Lazrak and Quenez (2002) compute the drift of V^x and use a comparison lemma by Kobylanski (2000). The point of the above argument is that the use of a comparison lemma becomes trivialized, while the integrated expression for $V^x - V$ is interesting in itself. Finally, it should be noted that if we make the assumption of bounded U, the results of Kobylanski (2000) and Schroder and Skiadas (1999) imply that V is also bounded, and as a result several of the technical details above simplify.

6 Generalizations

This concluding section presents a natural generalization of the utility function in which U and θ are replaced by, possibly time and state-dependent, functions of V^x. The paper's main result applies with such a more general specification, with only minor extensions to the central argument.

We begin with the special case of Maenhout (1999), who allowed the coefficient θ (which is $1/\theta$ in his paper) to be a function of the value function in the context of a Bellman equation. The current setting allows us to specify Maenhout's utility function independently of any dynamics, and to state a general equivalence with recursive utility.

We assume that $\hat{V}_t = \text{ess}\inf_{x \in X_U} V^x_t$, where

$$V^x_t = E^x\left[\int_t^T e^{-\beta(s-t)} U_s\, ds\right] + \psi\left(V^x_t\right) \mathcal{R}^x_t,$$

for some function $\psi : \mathbf{R} \to \mathbf{R}_{++}$. Suppose also that the progressively measurable pair (V, σ) satisfies

$$dV_t = -\left(U_t - \beta V_t - \frac{1}{2\psi\left(V_t\right)}\sigma_t'\sigma_t\right)\,dt + \sigma_t'\,dB_t, \quad V_T = 0.$$

Then the argument of Lemma 6 gives

$$V_t^x = V_t + \frac{\psi\left(V_t^x\right)}{2} E_t^x\left[\int_t^T e^{-\int_t^s \beta_u\,du}\left(x_s + \frac{1}{\psi\left(V_t^x\right)}\sigma_s\right)'\left(x_s + \frac{1}{\psi\left(V_t^x\right)}\sigma_s\right)\,ds\right].$$

Therefore $V^x \geq V$ for all x, and $V^{\hat{x}} = V$ if $\hat{x}_t = -\psi\left(V_t\right)^{-1}\sigma_t$, provided that $\hat{x} \in X_U$. (For example, if ψ takes values in a positive interval, a simple comparison lemma confirms this conclusion.)

Maenhout (1999) considers the case $\psi\left(V_t^x\right) = \rho V_t^x$ for some positive constant ρ, assuming $U_t > 0$. In this case, the BSDE defining V is also studied in Appendix A of Schroder and Skiadas (1999) (see Theorem A2). Letting $U_t = c_t^{\gamma} / \left|\gamma\right|$ leads to the homothetic class of SDE used by Schroder and Skiadas (1999) (see Theorem 1 and its proof), who provide optimal consumption-portfolio rules for this class of preferences. Maenhout is therefore able to reinterpret these solutions in terms of the robustness interpretation. An equivalent formulation is obtained in an extension of the Chen and Epstein (2001) formulation of "κ-ignorance" in which the bound κ is allowed to be proportional to the diffusion coefficient of the value function.

Finally, we outline the main argument for the more general utility function $\hat{V}_t = \operatorname{ess\,inf}_{x \in X_U} V_t^x$, where

$$V_t^x = E^x\left[\int_t^T e^{-\beta(s-t)} f\left(s, V_s^x\right)\,ds\right] + g\left(t, V_t^x\right)\mathcal{R}_t^x,$$

where $f, g : \Omega \times [0,T] \times \mathbf{R} \to \mathbf{R}$ are (possibly) state and time-dependent progressively measurable functions of the continuation utility, such that $g\left(\omega, t, v\right) \geq 0$ for all (ω, t, v), and either $f\left(\omega, t, \cdot\right)$ is concave for all (ω, t), or $f\left(\omega, t, \cdot\right)$ is convex for all (ω, t). As always, the notation of the reference consumption plan is suppressed here. For example, one could specify $f\left(t, V_t\right) = \phi\left(t, c_t, V_t\right)$ for some function $\phi : [0,T] \times \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$, where c_t represents the time-t consumption rate. The set X_U is defined as the set of all $x \in X$ for which the process V^x is well-defined by the above recursion. Without the relative entropy term, V^x is the usual SDU form of Duffie and Epstein (1992). The above formulation can therefore be viewed as a robust version of SDU.

We will see that, under some regularity assumptions, $\hat{V}$ equals the SDU V that satisfies the BSDE:

$$dV_t = -\left(f\left(t, V_t\right) - \beta V_t - \frac{1}{2g\left(t, V_t\right)}\sigma_t'\sigma_t\right)\,dt + \sigma_t'\,dB_t, \quad V_T = 0.$$

We henceforth assume that the progressively measurable pair (V, σ) solves the above BSDE. Essentially the same argument used in Lemma 6 leads to the identity:

$$V_t^x = V_t + E_t^x \left[\int_t^T e^{-\int_t^s \beta_u \, du} \left[f\left(s, V_s^x\right) - f\left(s, V_s\right) + Q^x\left(s, V_s^x\right) \right] ds \right],$$

where

$$Q^x\left(s, V_s^x\right) = \frac{g\left(s, V_s^x\right)}{2} \left(x_s + \frac{1}{g\left(s, V_s^x\right)} \sigma_s \right)' \left(x_s + \frac{1}{g\left(s, V_s^x\right)} \sigma_s \right) \geq 0.$$

We also assume that $\hat{x}_t = -g^{-1}\left(t, V_t\right) \sigma_t$ defines an element of X_U. Clearly, $V^{\hat{x}} = V$. Finally, we claim that $V_t^x \geq V_t$ a.s. for all t, and therefore $\hat{V} = V$.

If f is convex in its utility argument, the gradient inequality and the fact that Q^x is nonnegative, gives

$$V_t^x - V_t \geq E_t^x \left[\int_t^T e^{-\int_t^s \beta_u \, du} f_v\left(t, V_t\right) \left(V_t^x - V_s\right) ds \right],$$

where for simplicity we have assumed the derivative f_v of f with respect to V exists. Similarly, if f is concave in its utility argument, the gradient inequality implies

$$V_t^x - V_t \geq E_t^x \left[\int_t^T e^{-\int_t^s \beta_u \, du} \left(-f_v\left(s, V_s^x\right)\right) \left(V_s^x - V_s\right) ds \right].$$

In either case, the "stochastic Gronwall-Bellman inequality" (see Duffie and Epstein 1992, and Schroder and Skiadas 1999 for more general versions) implies that $V_t^x \geq V_t$ a.s. for all t, provided the corresponding f_v term is sufficiently integrable (for example the result holds if f_v is bounded).

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