

Robust control of networked control systems, with uncertain time-varying delays (draft)

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Robust control of Networked Control Systems, with uncertain
time-varying delays
(draft)

DCT report 2006-121

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Chapter 1

Introduction

Networks and control is a challenging and promising direction for current and future research in the area of control engineering, as described by the expert panel on future directions in control, dynamics and systems [1]. Two different research areas are distinguished in [2]: control over networks and networked control systems (NCSs). Control over networks studies the control problems in networks, such as congestion control and is also part of the field of information technology. Networked control systems consist of one or more control loops that are closed over a communication network. One of the research issues is the stability analysis of the NCS. In this report, we focus on the effect of time-varying delays on NCSs.

An NCS exists of a system coupled over a communication network to a controller. The system behaves in continuous-time, while the controller is executed on a processor in discrete-time. A schematic overview of an NCS is depicted in Figure 1.1.

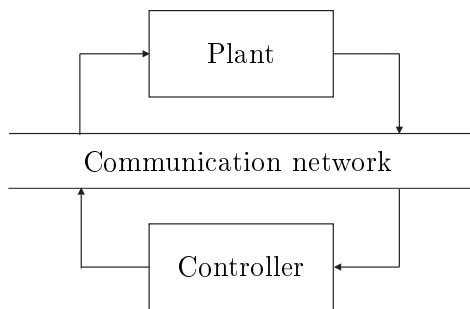


Figure 1.1: Schematic overview of an NCS.

Compared to the traditional point-to-point control systems, where the controller is directly linked to the actuators and sensors of the plant, advantages of an NCS are increased flexibility, the possibility to use decentralized control, decreased maintenance costs and reduction of the system wiring [3], [4]. A disadvantage is the complicated analysis, due to the use of the communication network. Four different aspects of the network need to be taken into account [4], [5]. First, time-varying delays between the controller and system (plant) and vice-versa occur. Second, package loss, i.e. data that does not arrive at the controller or plant, occurs. Third, multiple packets to send all data over the network may be needed, which may result in delivery of part of the data. Fourth, variations in the sample-time may occur. Here,

we assume that all data is sent in one packet, that all packets arrive and that the sample-time is constant.

The network induced time-varying delays, in combination with the computation time, consumed by the controller, affect the stability of the controlled system. Different approaches to model an NCS and investigate its stability are described in literature.

Probably, the first NCS model with time-delay, is proposed in [6]. They propose a finite-dimensional, time-varying discrete-time model, based on a time-driven sensor and controller and an event-driven actuator. The model is based on describing the available control inputs, during one constant sample-interval. The stability of constant delays and known periodic time-varying delays can be determined, based on checking the eigenvalues of the corresponding systems.

A comparable model is given in e.g. [7], [8], and [9]. They use a discrete-time representation, but assume a time-driven sensor and an event-driven controller and actuator. These assumptions allow to sum all three time-delays together, which simplifies the model, because only one total time-delay is used, instead of three separate terms. In [7] the model and stability analysis are limited to constant time-delays. In [8] an example is presented where bounded time-varying delays result in instability, while the controller was designed to stabilize the system for all constant delays within the same bound. This example shows the need of robust control analysis and synthesis for systems with time-varying delays. In [9] the variation of the time-delay is modeled, using either a probability distribution or a Markov chain. For both cases the stability is analyzed and optimal controllers are designed.

In this technical report, we adopt the model of [7] and derive stability conditions for uncertain time-varying delays that are upperbounded by the sample-time. The stability conditions are based on a Jordan form representation of the continuous-time system matrices. Here, the Jordan based representation is derived for the case of only real eigenvalues, only complex eigenvalues and a combination of real and complex eigenvalues.

In Section 2, the real Jordan form and the Jordan canonical form are discussed in general. In Section 3, the Jordan form representation of the NCS is derived and stability conditions are proposed.

Chapter 2

Preliminaries

2.1 Jordan Canonical Form

For every square matrix $A \in \mathbb{R}^{n \times n}$, there exists a Jordan Canonical Form $J \in \mathbb{R}^{n \times n}$, given by [10], [11]:

$$J = Q^{-1}AQ, \tag{2.1}$$

with $Q \in \mathbb{R}^{n \times n}$ a matrix that contains the generalized eigenvectors of A and

$$J = [J_1 \oplus \dots \oplus J_m] = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & J_p \end{pmatrix}, \tag{2.2}$$

where J_i is called a Jordan block, which has a block diagonal form, represented by one of the following matrices:

$$\lambda_i, \begin{pmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{pmatrix}, \begin{pmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{pmatrix}, \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & \dots & 0 & \lambda_i \end{pmatrix}, \tag{2.3}$$

with λ_i the i^{th} eigenvalue of the matrix A . Therefore each J_i in (2.2), with $i \in \{1, 2, \dots, p\}$ corresponds to one distinct eigenvalue. If the geometric multiplicity g_i of the λ_i^{th} eigenvalue is equal to one, then the dimension of the i^{th} Jordan block is equal to the algebraic multiplicity m_i of the λ_i^{th} eigenvalue. If the geometric multiplicity (g_i) is unequal to one, then g_i Jordan blocks describe the Jordan block associated with λ_i :

$$J_i = [J_{i,1} \oplus J_{i,2}, \dots, \oplus J_{i,g_i}]. \tag{2.4}$$

The largest Jordan block in J_i determines, the number of different parameters (see e.g. (2.7)) that can be obtained from the Jordan block J_i . Therefore, we define:

$$c_i = \max_{\tilde{j}=\{1,2,\dots,g_i\}} \dim(J_{i,\tilde{j}}), \tag{2.5}$$

where $\dim(J)$ denotes the dimension of the square matrix J .

The dimension of the combined g_i Jordan blocks in J_i in (2.4) is equal to the algebraic multiplicity m_i of the i^{th} eigenvalue¹. Note that the geometric multiplicity can never exceed the algebraic multiplicity of λ_i , so $1 \leq g_i \leq m_i$ [12], and [13].

For the exponential of A it holds that:

$$e^{tA} = Q[e^{tJ_1} \oplus \dots \oplus e^{tJ_m}]Q^{-1}. \quad (2.6)$$

The exponential functions of the Jordan blocks of (2.3) are, respectively, given by:

$$e^{\lambda_i t}, e^{\lambda_i t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, e^{\lambda_i t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, e^{\lambda_i t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{(k-1)}}{(k-1)!} \\ 0 & 1 & t & \dots & \frac{t^{(k-2)}}{(k-2)!} \\ \vdots & \ddots & & & \vdots \\ 0 & 0 & 1 & t & \\ 0 & 0 & 0 & 1 & \end{pmatrix}, \quad (2.7)$$

with k the dimension of the corresponding Jordan block.

2.1.1 Real Jordan Form

If the matrix exhibits complex eigenvalues $\lambda = a \pm bj$, the Real Jordan Form (RJF) [14], [15] is more useful, because it avoids the occurrence of complex matrices J and Q in (2.1). A complex Jordan block $J_i(a + bj)$ can be replaced by a Jordan block $K_i(a, b)$, of the form:

$$D, \begin{pmatrix} D & I \\ 0 & D \end{pmatrix}, \begin{pmatrix} D & I & 0 \\ 0 & D & I \\ 0 & 0 & D \end{pmatrix}, \begin{pmatrix} D & I & 0 & \dots & 0 \\ 0 & D & I & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & D & I \\ 0 & 0 & \dots & 0 & D \end{pmatrix}, \quad (2.8)$$

with the matrix $D(a, b)$, defined as

$$D = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = aI + bL_r, \quad (2.9)$$

with I the identity matrix and $L_r^2 = -I$.

Every square matrix A can be written in the Real Jordan Form as:

$$K = R^{-1}AR, \quad (2.10)$$

with $K \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{n \times n}$, and the Real Jordan Form $K(a, b)$ defined as:

$$K = [K_1 \oplus \dots \oplus K_m] \quad (2.11)$$

¹The algebraic multiplicity describes the number of times that an eigenvalue occurs. The geometric multiplicity is equal to the dimension of the nullspace of $(\lambda_i I - A)$, and can be computed as: $\text{nullity}(\lambda_i I - A) = n - \text{rank}(\lambda_i I - A)$, with n the dimension of A .

and K_i defined in (2.8) and (2.9). Note that for real eigenvalues the Real Jordan Form is equal to the Jordan Canonical Form.

The exponential function of the Real Jordan blocks, given in (2.8), is:

$$e^{Dt}, \begin{pmatrix} e^{Dt} & e^{Dt} \frac{t}{1!} \\ 0 & e^{Dt} \end{pmatrix}, \begin{pmatrix} e^{Dt} & e^{Dt} \frac{t}{1!} & e^{Dt} \frac{t^2}{2!} \\ 0 & e^{Dt} & e^{Dt} \frac{t}{1!} \\ 0 & 0 & e^{Dt} \end{pmatrix}, \begin{pmatrix} e^{Dt} & e^{Dt} \frac{t}{1!} & e^{Dt} \frac{t^2}{2!} & \cdots & e^{Dt} \frac{t^{k-1}}{(k-1)!} \\ 0 & e^{Dt} \frac{t}{1!} & e^{Dt} \frac{t^2}{2!} & \cdots & e^{Dt} \frac{t^{k-2}}{(k-2)!} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & e^{Dt} & e^{Dt} \frac{t}{1!} \\ 0 & 0 & \cdots & 0 & e^{Dt} \end{pmatrix}, \quad (2.12)$$

with

$$e^{Dt} = e^{at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}. \quad (2.13)$$

Obviously, if both real and complex eigenvalues occur, combinations of the exponential of the Real Jordan blocks (2.8) and Jordan blocks as in (2.3) are used.

Chapter 3

Robust stability of an NCS with time-varying delays

This chapter deals with the stability analysis of Networked Control Systems with small time-delays, i.e. the time-delay is upperbounded by the sample-time. First, a standard discrete-time NCS-model is described in Section 3.1. Next, this model is rewritten in a Jordan form, where both the cases with real and complex eigenvalues are investigated. In Section 3.2 the robust stability problem is solved for systems with time-varying delays upperbounded by the sample-time.

3.1 NCS model

In this chapter, the discrete-time description of an NCS of [8] and [7], will be used. The NCS is schematically depicted in Figure 3.1. It consists of a continuous-time plant and a discrete-time controller, which receives information from the plant at the sampling instants t_k , only. Additionally, in the model, the computation time and the networked induced delays, i.e. the sensor-to-controller delay and the controller-to-actuator delay, are taken into account. The sensor acts in a time-driven fashion, while the controller and actuator act in an event-driven fashion. Under these assumptions, all delays can be represented by a single delay τ_k that delays the control input u_k with respect to the measurement y_k [9]. The sampling moments k are determined by the time-driven sensor output y_k . The continuous-time model of the NCS is given by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu^*(t) \\ y_k &= Cx_k \\ u^*(t) &= u_k, \quad \text{for } t \in [kh + \tau_k, (k+1)h + \tau_{k+1}] \end{aligned} \tag{3.1}$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{n \times n}$ the continuous-time system matrices; $x(t) \in \mathbb{R}^n$ the state; $t \in \mathbb{R}$ the time; τ_k the delay for sampling moment k ; $y_k \in \mathbb{R}^n$ the discrete-time measurement; and $u_k \in \mathbb{R}$ the (delayed) discrete-time input for sample moment k . For simplicity, we assume that we measure the entire state at the sampling instants, i.e. $y_k = Cx_k$, with C the identity matrix. Note that the sensor-to-controller delay is not present in this output equation, because it was already accounted for in the total time-varying delay τ_k .

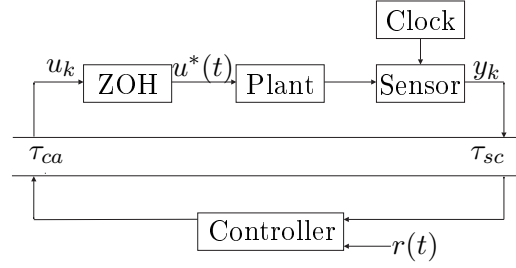


Figure 3.1: Schematic overview of the networked control system.

If we assume that the total delay τ_k is smaller than the constant sample-time h at every sampling moment k , the discretization of (3.1) gives the NCS model:

$$x_{k+1} = e^{Ah}x_k + \int_0^{h-\tau_k} e^{As}dsBu_k + \int_{h-\tau_k}^h e^{As}dsBu_{k-1}, \quad (3.2)$$

with x_k , u_k the discretized state and control input, respectively. Defining the extended state vector $\xi_k = [x_k^T \ u_{k-1}^T]^T$ results in the following state-space model, given a maximum delay $\tau_{max} \in [0, h]$:

$$\xi_{k+1} = \tilde{A}(\tau_k)\xi_k + \tilde{B}(\tau_k)u_k, \quad \tau_k \in [0, \tau_{max}], \quad (3.3)$$

with $\tilde{A}(\tau_k) = \begin{pmatrix} e^{Ah} & \int_{h-\tau_k}^h e^{As}dsB \\ 0 & 0 \end{pmatrix}$ and $\tilde{B}(\tau_k) = \begin{pmatrix} \int_0^{h-\tau_k} e^{As}dsB \\ I \end{pmatrix}$.

3.1.1 Jordan forms of the NCS model

To perform analysis on system (3.3), the system can be rewritten as a combination of constant matrices that are multiplied by time-varying delay τ_k dependent parameters. Here, we will use the Jordan forms, as presented in Section 2.1.

The Jordan form representation of the continuous-time matrix A is given by:

$$A = QJQ^{-1}, \quad (3.4)$$

with J the Jordan Canonical Form, Real Jordan Form or a combination¹. System (3.3) can be rewritten as:

$$\xi_{k+1} = \begin{pmatrix} Qe^{Jh}Q^{-1} & \int_{h-\tau_k}^h Qe^{Js}dsQ^{-1}B \\ 0 & 0 \end{pmatrix} \xi_k + \begin{pmatrix} \int_0^{h-\tau_k} Qe^{Js}dsQ^{-1}B \\ I \end{pmatrix} u_k, \quad \tau_k \in [0, \tau_{max}]. \quad (3.5)$$

Real eigenvalues

For simplicity, first, we will consider a situation with only real eigenvalues that can be multiplicative. If none of the eigenvalues are equal to zero, $J = J_{NZ}$ is invertible and the integrals

¹The Real Jordan Form is equal to the Jordan Canonical Form if the eigenvalues are real.

in (3.4) can be solved, which gives in general:

$$\begin{aligned} \xi_{k+1} = & \begin{pmatrix} Qe^{J_{NZ}h}Q^{-1} & QJ_{NZ}^{-1}e^{J_{NZ}h}Q^{-1}B \\ 0 & 0 \end{pmatrix} \xi_k + \begin{pmatrix} 0 & -QJ_{NZ}^{-1}e^{J_{NZ}(h-\tau_k)}Q^{-1}B \\ 0 & 0 \end{pmatrix} \xi_k + \\ & \begin{pmatrix} -QJ_{NZ}^{-1}Q^{-1}B \\ I \end{pmatrix} u_k + \begin{pmatrix} QJ_{NZ}^{-1}e^{J_{NZ}(h-\tau_k)}Q^{-1}B \\ 0 \end{pmatrix} u_k. \end{aligned} \quad (3.6)$$

The matrix $e^{J_{NZ}(h-\tau_k)}$ contains the time-varying parameters τ_k , and can be rewritten as:

$$e^{J_{NZ}(h-\tau_k)} = \sum_{i=1}^{p_{NZ}} \sum_{j=0}^{(c_{i_{NZ}}-1)} \frac{(h-\tau_k)^j}{j!} e^{\lambda_i(h-\tau_k)} \hat{S}_{i,j}, \quad (3.7)$$

where p_{NZ} denotes the number of different eigenvalues of A and $c_{i_{NZ}}$ is defined in (2.5) as the dimension of the largest Jordan block of the i^{th} non-zero eigenvalue. The matrix $\hat{S}_{i,j} \in \mathbb{R}^{n \times n}$ is an appropriate matrix, with a one at the matrix entries of $e^{J_{NZ}(h-\tau_k)}$ dependent

of $\frac{(h-\tau_k)^j}{j!} e^{\lambda_i(h-\tau_k)}$ and a zero at all other matrix entries. For example, if $J_{NZ} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$,

it holds that $p_{NZ} = 1$, $c_1 = 3$, which gives

$$e^{J_{NZ}(h-\tau_k)} = e^{(h-\tau_k)} \hat{S}_{1,0} + (h-\tau_k) e^{(h-\tau_k)} \hat{S}_{1,1} + \frac{(h-\tau_k)^2}{2} e^{(h-\tau_k)} \hat{S}_{1,2},$$

with $\hat{S}_{1,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\hat{S}_{1,1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, and $\hat{S}_{1,2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Another example is

given to show the differences in $\hat{S}_{i,j}$ that can occur. If $J_{NZ} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$, it holds that

$p_{NZ} = 2$, $c_1 = 1$ and $c_2 = 2$, which gives

$$e^{J_{NZ}(h-\tau_k)} = e^{(h-\tau_k)} \hat{S}_{1,0} + e^{2(h-\tau_k)} \hat{S}_{2,0} + (h-\tau_k) e^{2(h-\tau_k)} \hat{S}_{2,1},$$

with $\hat{S}_{1,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\hat{S}_{2,0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\hat{S}_{2,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. From these examples, it

is obvious that the shape of $\hat{S}_{i,j}$ depends on the used combination of Jordan blocks.

For eigenvalues unequal to zero, system (3.6) can be rewritten in a form where all matrices are independent of the time-varying delay:

$$\begin{aligned} \xi_{k+1} = & \hat{\Phi}_0 \xi_k + \left(\sum_{i=1}^{p_{NZ}} \left(\sum_{j=0}^{(c_{i_{NZ}}-1)} \frac{(h-\tau_k)^j}{j!} \right) e^{\lambda_i(h-\tau_k)} \right) \hat{\Phi}_{i,j} \xi_k + \\ & \hat{\Gamma}_0 u_k + \sum_{i=1}^{p_{NZ}} \left(\sum_{j=0}^{(c_{i_{NZ}}-1)} \frac{(h-\tau_k)^j}{j!} \right) e^{\lambda_i(h-\tau_k)} \hat{\Gamma}_{i,j} u_k, \end{aligned} \quad (3.8)$$

with $\hat{\Phi}_0 = \begin{pmatrix} Qe^{J_{NZ}h}Q^{-1} & QJ_{NZ}^{-1}e^{J_{NZ}h}Q^{-1}B \\ 0 & 0 \end{pmatrix}$, $\hat{\Phi}_{i,j} = \begin{pmatrix} 0 & -QJ_{NZ}^{-1}\hat{S}_{i,j}Q^{-1}B \\ 0 & 0 \end{pmatrix}$, $\hat{\Gamma}_0 = \begin{pmatrix} -QJ_{NZ}^{-1}Q^{-1}B \\ I \end{pmatrix}$,

and $\hat{\Gamma}_{i,j} = \begin{pmatrix} QJ_{NZ}^{-1}\hat{S}_{i,j}Q^{-1}B \\ 0 \end{pmatrix}$.

If eigenvalues equal to zero occur, the inverse of the Jordan matrix does not exist and description (3.8) can not be used. For simplicity, a system with all eigenvalues equal to zero will be described first ($J = J_Z$). System (3.5) can be rewritten as:

$$\xi_{k+1} = \begin{pmatrix} Qe^{J_Z h} Q^{-1} & \sum_{j=1}^{c_Z} \left(\frac{h^j}{j!} - \frac{(h-\tau_k)^j}{j!} \right) Q\check{S}_j Q^{-1} B \\ 0 & 0 \end{pmatrix} \xi_k + \begin{pmatrix} \sum_{j=1}^{c_Z} \frac{(h-\tau_k)^j}{j!} Q\check{S}_j Q^{-1} B \\ I \end{pmatrix} u_k, \quad \tau_k \in [0, \tau_{max}], \quad (3.9)$$

with c_Z the size of the maximum Jordan block for the zero eigenvalues defined in (2.5), \check{S}_j a matrix with all zeros, except ones at the matrix entries of $e^{J_Z(h-\tau_k)}$ corresponding to $\frac{(h-\tau_k)^j}{j!}$. For example, if $J_Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, this gives $c_Z = 2$ (because $g_Z = 1$), which results in $\check{S}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\check{S}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in (3.9).

Splitting up (3.9) in a constant and a delay-dependent part gives:

$$\xi_{k+1} = \check{\Phi}_0 \xi_k + \sum_{j=1}^{c_Z} \frac{(h-\tau_k)^j}{j!} \check{\Phi}_j \xi_k + \check{\Gamma}_0 u_k + \sum_{j=1}^{c_Z} \frac{(h-\tau_k)^j}{j!} \check{\Gamma}_j u_k, \quad \tau_k \in [0, \tau_{max}], \quad (3.10)$$

with $\check{\Phi}_0 = \begin{pmatrix} Qe^{J_Z h} Q^{-1} & \sum_{j=1}^{a_{jZ}} \frac{h^j}{j!} Q\check{S}_j Q^{-1} B \\ 0 & 0 \end{pmatrix}$, $\check{\Phi}_j = \begin{pmatrix} 0 & -Q\check{S}_j Q^{-1} B \\ 0 & 0 \end{pmatrix}$, $\check{\Gamma}_0 = \begin{pmatrix} 0 \\ I \end{pmatrix}$, and $\check{\Gamma}_j = \begin{pmatrix} Q\check{S}_j Q^{-1} B \\ 0 \end{pmatrix}$.

If the system has real eigenvalues that are both zero and non-zero, then a combination of (3.8) and (3.10) is needed. The Jordan matrix becomes:

$$J = J_{NZ} \oplus J_Z, \quad (3.11)$$

with J_{NZ} the combination of Jordan blocks for all non-zero eigenvalues, and J_Z the combination of Jordan blocks for all zero eigenvalues. System (3.3) is now represented by:

$$\begin{aligned} \xi_{k+1} &= \tilde{\Phi}_0 \xi_k + \sum_{i=1}^{p_{NZ}} \sum_{j=0}^{(c_{iNZ}-1)} \frac{(h-\tau_k)^j}{j!} e^{\lambda_i(h-\tau_k)} \tilde{\Phi}_{i,j} \xi_k \\ &+ \tilde{\Gamma}_0 u_k + \sum_{i=1}^{p_{NZ}} \sum_{j=0}^{(c_{iNZ}-1)} \frac{(h-\tau_k)^j}{j!} e^{\lambda_i(h-\tau_k)} \tilde{\Gamma}_{i,j} u_k \\ &+ \sum_{j=1}^{c_Z} \frac{(h-\tau_k)^j}{j!} \tilde{\Phi}_j \xi_k + \sum_{j=1}^{c_Z} \frac{(h-\tau_k)^j}{j!} \tilde{\Gamma}_j u_k, \quad \tau_k \in [0, \tau_{max}], \end{aligned} \quad (3.12)$$

with $\tilde{\Phi}_0 = \begin{pmatrix} Qe^{Jh} Q^{-1} & Q(J_{NZ}^{-1} e^{J_{NZ} h} \oplus \sum_{j=1}^{c_Z} \frac{h^j}{j!} \check{S}_j) Q^{-1} B \\ 0 & 0 \end{pmatrix}$, $\tilde{\Gamma}_0 = \begin{pmatrix} Q(J_{NZ}^{-1} \oplus O_Z) Q^{-1} B \\ I \end{pmatrix}$;

$\tilde{\Phi}_{i,j} = \hat{\Phi}_{i,j}$ and $\tilde{\Gamma}_{i,j} = \hat{\Gamma}_{i,j}$ as defined in (3.8), where $\hat{S}_{i,j}$ is replaced by $\tilde{S}_{i,j} = \hat{S}_{i,j} \oplus O_Z$; $\tilde{\Phi}_i = \check{\Phi}_i$ and $\tilde{\Gamma}_i = \check{\Gamma}_i$ as defined in (3.10), where \check{S}_j is replaced by $\tilde{S}_j = O_{NZ} \oplus \check{S}_j$. Herein, O_Z is a matrix with only zeros and has a dimension equal to J_Z , and O_{NZ} is a matrix with only zeros and has a dimension equal to J_{NZ} . Note that the matrices $\tilde{S}_{i,j}$ and \tilde{S}_i have the same dimension as Q .

Complex eigenvalues

If A has only complex eigenvalues, the Real Jordan Form can be applied to avoid the occurrence of complex elements in the system matrices. First, we will consider a situation with one pair of complex conjugated eigenvalues, thus $J_C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. The exponential of the Real Jordan

Matrix $e^{J_C s}$ is then given by:

$$e^{J_C s} = e^{as} \begin{pmatrix} \cos(bs) & -\sin(bs) \\ \sin(bs) & \cos(bs) \end{pmatrix}.$$

To determine the matrices in (3.5) the corresponding integrals need to be solved. Note that it holds that:

$$\int e^{as} \cos(bs) ds = \frac{e^{as}}{a^2 + b^2} (a \cos(bs) + b \sin(bs)), \quad (3.13)$$

and

$$\int e^{as} \sin(bs) ds = \frac{e^{as}}{a^2 + b^2} (a \sin(bs) - b \cos(bs)). \quad (3.14)$$

Then, system (3.3) can be rewritten with matrices $\bar{\Phi}_i$ and $\bar{\Gamma}_i$ that are independent of the time-varying delay τ_k :

$$\begin{aligned} \xi_{k+1} &= \left(\bar{\Phi}_{1,0} + \frac{e^{a(h-\tau_k)}}{a^2+b^2} \cos(b(h-\tau_k)) \bar{\Phi}_{1,0,1} + \frac{e^{a(h-\tau_k)}}{a^2+b^2} \sin(b(h-\tau_k)) \bar{\Phi}_{1,0,2} \right) \xi_k \\ &+ \left(\bar{\Gamma}_0 + \frac{e^{a(h-\tau_k)}}{a^2+b^2} \cos(b(h-\tau_k)) \bar{\Gamma}_{1,0,1} + \frac{e^{a(h-\tau_k)}}{a^2+b^2} \sin(b(h-\tau_k)) \bar{\Gamma}_{1,0,2} \right) u_k, \end{aligned} \quad (3.15)$$

with

$$\begin{aligned} \bar{\Phi}_{1,0} &= \begin{pmatrix} e^{ah} Q \begin{pmatrix} \cos(bh) & -\sin(bh) \\ \sin(bh) & \cos(bh) \end{pmatrix} Q^{-1} & \frac{e^{ah}}{a^2+b^2} Q \begin{pmatrix} a \cos(bh) + b \sin(bh) & -a \sin(bh) + b \cos(bh) \\ a \sin(bh) - b \cos(bh) & a \cos(bh) + b \sin(bh) \end{pmatrix} Q^{-1} B \\ 0 & 0 \end{pmatrix}, \\ \bar{\Phi}_{1,0,1} &= \begin{pmatrix} 0 & -QT_{1,0,1}Q^{-1}B \\ 0 & 0 \end{pmatrix}, \quad \bar{\Phi}_{1,0,2} = \begin{pmatrix} 0 & -QT_{1,0,2}Q^{-1}B \\ 0 & 0 \end{pmatrix}, \\ \bar{\Gamma}_{1,0} &= \begin{pmatrix} -QT_{1,0}Q^{-1}B \\ I \end{pmatrix}, \\ \bar{\Gamma}_{1,0,1} &= \begin{pmatrix} QT_{1,0,1}Q^{-1}B \\ 0 \end{pmatrix}, \quad \bar{\Gamma}_{1,0,2} = \begin{pmatrix} QT_{1,0,2}Q^{-1}B \\ 0 \end{pmatrix}, \end{aligned} \quad (3.16)$$

with

$$T_{1,0} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad T_{1,0,1} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad T_{1,0,2} = \begin{pmatrix} b & -a \\ a & b \end{pmatrix}. \quad (3.17)$$

If a pair of complex conjugated eigenvalues occurs twice, another Real Jordan Matrix needs to be used. In this case, besides the integral of (3.13) and (3.14), two additional integrals need to be solved:

$$\int s e^{as} \cos(bs) ds = \frac{s e^{as}}{a^2 + b^2} (a \cos(bs) + b \sin(bs)) + \frac{e^{as}}{(a^2 + b^2)^2} ((b^2 - a^2) \cos(bs) - 2ab \sin(bs)),$$

$$(3.18)$$

and

$$\int se^{as} \sin(bs) ds = \frac{se^{as}}{a^2 + b^2} (a \sin(bs) - b \cos(bs)) + \frac{e^{as}}{(a^2 + b^2)^2} ((b^2 - a^2) \sin(bs) + 2ab \cos(bs)). \quad (3.19)$$

System (3.3) can now be formulated as:

$$\begin{aligned} \xi_{k+1} &= \left(\bar{\Phi}_{1,0} + \frac{e^{a(h-\tau_k)}}{a^2+b^2} \cos(b(h-\tau_k)) \bar{\Phi}_{1,0,1} + \frac{e^{a(h-\tau_k)}}{a^2+b^2} \sin(b(h-\tau_k)) \bar{\Phi}_{1,0,2} \right. \\ &+ (h-\tau_k) \frac{e^{a(h-\tau_k)}}{a^2+b^2} \cos(b(h-\tau_k)) \bar{\Phi}_{1,1,1} + (h-\tau_k) \frac{e^{a(h-\tau_k)}}{a^2+b^2} \sin(b(h-\tau_k)) \bar{\Phi}_{1,1,2} \left. \right) \xi_k \\ &+ \left(\bar{\Gamma}_{1,0} + \frac{e^{a(h-\tau_k)}}{a^2+b^2} \cos(b(h-\tau_k)) \bar{\Gamma}_{1,0,1} + \frac{e^{a(h-\tau_k)}}{a^2+b^2} \sin(b(h-\tau_k)) \bar{\Gamma}_{1,0,2} \right. \\ &+ \left. \frac{e^{a(h-\tau_k)}}{a^2+b^2} (h-\tau_k) \cos(b(h-\tau_k)) \bar{\Gamma}_{1,1,1} + \frac{e^{a(h-\tau_k)}}{a^2+b^2} (h-\tau_k) \sin(b(h-\tau_k)) \bar{\Gamma}_{1,1,2} \right) u_k, \end{aligned} \quad (3.20)$$

The matrices $\bar{\Phi}_i$ and $\bar{\Gamma}_i$ are all functions that are dependent of the parameters a, b, h and can be determined similar to the situation with only one complex pair of eigenvalues, they are:

$$\begin{aligned} \bar{\Phi}_{1,0} &= \begin{pmatrix} \begin{pmatrix} \cos bh & -\sin bh & h \cos bh & -h \sin bh \\ \sin bh & \cos bh & h \sin bh & h \cos bh \\ 0 & 0 & \cos bh & -\sin bh \\ 0 & 0 & \sin bh & \cos bh \end{pmatrix} Q^{-1} & QH_1Q^{-1}B \\ & 0 \end{pmatrix}, \\ \bar{\Phi}_{1,0,1} &= \begin{pmatrix} 0 & -QT_{1,0,1}Q^{-1}B \\ 0 & 0 \end{pmatrix}, \quad \bar{\Phi}_{1,0,2} = \begin{pmatrix} 0 & -QT_{1,0,2}Q^{-1}B \\ 0 & 0 \end{pmatrix}, \\ \bar{\Phi}_{1,1,1} &= \begin{pmatrix} 0 & -QT_{1,1,1}Q^{-1}B \\ 0 & 0 \end{pmatrix}, \quad \bar{\Phi}_{1,1,2} = \begin{pmatrix} 0 & -QT_{1,1,2}Q^{-1}B \\ 0 & 0 \end{pmatrix}, \\ \bar{\Gamma}_{1,0} &= \begin{pmatrix} -QT_{1,0}Q^{-1}B \\ I \end{pmatrix}, \quad \bar{\Gamma}_{1,0,1} = \begin{pmatrix} QT_{1,0,1}Q^{-1}B \\ 0 \end{pmatrix}, \quad \bar{\Gamma}_{1,0,2} = \begin{pmatrix} QT_{1,0,2}Q^{-1}B \\ 0 \end{pmatrix}, \\ \bar{\Gamma}_{1,1,1} &= \begin{pmatrix} QT_{1,1,1}Q^{-1}B \\ 0 \end{pmatrix}, \quad \bar{\Gamma}_{1,1,2} = \begin{pmatrix} QT_{1,1,2}Q^{-1}B \\ 0 \end{pmatrix}, \end{aligned} \quad (3.21)$$

with

$$H_1 = \frac{e^{ah}}{a^2 + b^2} \begin{pmatrix} a \cos bh + b \sin bh & -a \sin bh + b \cos bh & l_1 & l_2 \\ a \sin bh - b \cos bh & a \cos bh + b \sin bh & l_3 & l_4 \\ 0 & 0 & a \cos bh + b \sin bh & -a \sin bh + b \cos bh \\ 0 & 0 & a \sin bh - b \cos bh & a \cos bh + b \sin bh \end{pmatrix}, \quad (3.22)$$

with

$$\begin{aligned} l_1 &= h(a \cos bh + b \sin bh) + \frac{1}{a^2+b^2} ((b^2 - a^2) \cos bh - 2ab \sin bh), \\ l_2 &= h(-a \sin bh + b \cos bh) - \frac{1}{a^2+b^2} (2ab \cos bh + (b^2 - a^2) \sin bh), \\ l_3 &= h(a \sin bh - b \cos bh) + \frac{1}{a^2+b^2} (2ab \cos bh + (b^2 - a^2) \sin bh), \\ l_4 &= h(a \cos bh + b \sin bh) + \frac{1}{a^2+b^2} ((b^2 - a^2) \cos bh - 2ab \sin bh), \end{aligned}$$

and

$$\begin{aligned}
T_{1,0} &= \frac{1}{a^2+b^2} \begin{pmatrix} a & b & \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} \\ -b & a & \frac{2ab}{a^2+b^2} & \frac{b^2-a^2}{a^2+b^2} \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{pmatrix}, \quad T_{1,0,1} = \begin{pmatrix} a & b & \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} \\ -b & a & \frac{2ab}{a^2+b^2} & \frac{b^2-a^2}{a^2+b^2} \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{pmatrix}, \\
T_{1,0,2} &= \begin{pmatrix} b & -a & \frac{-2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} \\ a & b & \frac{b^2-a^2}{a^2+b^2} & \frac{-2ab}{a^2+b^2} \\ 0 & 0 & b & -a \\ 0 & 0 & a & b \end{pmatrix}, \quad T_{1,1,1} = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & -b & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
T_{1,1,2} &= \begin{pmatrix} 0 & 0 & b & -a \\ 0 & 0 & a & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{3.23}$$

Previous obtained results for complex eigenvalues are only valid for conjugated pairs of eigenvalues for which c_{iC} , defined in (2.5), is equal to 2 or 4. Generalizing, dependent on the number of different complex conjugated pairs of eigenvalues p_C , and their resulting number of variable parameters c_i , the following general equation holds:

$$\begin{aligned}
\xi_{k+1} &= \bar{\Phi}_0 \xi_k + \sum_{i=1}^{p_C} \sum_{j=0}^{\tilde{c}_{iC}-1} \frac{(h-\tau_k)^j}{j!} e^{a_i(h-\tau_k)} \cos(b_i(h-\tau_k)) \bar{\Phi}_{i,j,1} \xi_k \\
&+ \sum_{i=1}^{p_C} \sum_{j=0}^{\tilde{c}_{iC}-1} \frac{(h-\tau_k)^j}{j!} e^{a_i(h-\tau_k)} \sin(b_i(h-\tau_k)) \bar{\Phi}_{i,j,2} \xi_k \\
&+ \bar{\Gamma}_0 u_k + \sum_{i=1}^{p_C} \sum_{j=0}^{\tilde{c}_{iC}-1} \frac{(h-\tau_k)^j}{j!} e^{a_i(h-\tau_k)} \cos(b_i(h-\tau_k)) \bar{\Gamma}_{i,j,1} u_k \\
&+ \sum_{i=1}^{p_C} \sum_{j=0}^{\tilde{c}_{iC}-1} \frac{(h-\tau_k)^j}{j!} e^{a_i(h-\tau_k)} \sin(b_i(h-\tau_k)) \bar{\Gamma}_{i,j,2} u_k,
\end{aligned} \tag{3.24}$$

with $\tilde{c}_{iC} = \frac{c_{iC}}{2}$ and

$$\begin{aligned}
\bar{\Phi}_0 &= \begin{pmatrix} Qe^{J_C h} Q^{-1} & Qf(a, b, h) Q^{-1} B \\ 0 & 0 \end{pmatrix}, \quad \bar{\Phi}_{i,j,l} = \begin{pmatrix} 0 & -QT_{i,j,l} Q^{-1} B \\ 0 & 0 \end{pmatrix}, \\
\bar{\Gamma}_0 &= \begin{pmatrix} -QT_0 Q^{-1} B \\ I \end{pmatrix}, \quad \bar{\Gamma}_{i,j,l} = \begin{pmatrix} QT_{i,j,l} Q^{-1} B \\ 0 \end{pmatrix}.
\end{aligned} \tag{3.25}$$

The matrices $T_{i,j,l}$ are defined similar to (3.17) or (3.23). Note that $l = \{1, 2\}$, dependent on the cos or sin part of the equation, respectively, i denotes the i^{th} eigenvalue and j the number of uncertain parameters that depend in this eigenvalue. The function $f(a, b, h)$ is equal to the primitive of $e^{J_C s}$, evaluated at $s = h$. For one complex pair of eigenvalues, with algebraic multiplicity 1, it is equal to the corresponding part in $\bar{\Phi}_0$ in (3.16), and if the algebraic multiplicity is equal to 2, it is equal to H_1 , as defined in (3.22). The matrix T_0 is equal to the primitive of $e^{J_C s}$, evaluated at $s = 0$, for a system with multiple complex eigenvalues it holds that $T_0 = T_{1,0} \oplus T_{2,0} \oplus \dots \oplus T_{p_C,0}$, with $T_{i,0}$ defined in (3.17) or (3.23) for eigenvalues with an algebraic multiplicity 1 or 2, respectively.

Both complex and real eigenvalues

If both complex and real eigenvalues occur, a combination of previous obtained results can be used. The Jordan matrix can be written as:

$$J = [J_{NZ} \oplus J_Z \oplus J_C], \tag{3.26}$$

where J_{NZ} represents the Jordan block with all real, non-zero eigenvalues, J_Z the Jordan block for all real, zero eigenvalues, J_C the real Jordan block with all complex eigenvalues.

A general notation of system (3.3) is thus a combination of the previous obtained results:

$$\begin{aligned}
\xi_{k+1} &= \Phi_0 \xi_k + \sum_{i=1}^{p_{NZ}} \sum_{j=0}^{(c_{i_{NZ}}-1)} \frac{(h-\tau_k)^j}{j!} e^{\lambda_i(h-\tau_k)} \Phi_{i,j} \xi_k \\
&+ \Gamma_0 u_k + \sum_{i=1}^{p_{NZ}} \sum_{j=0}^{(c_{i_{NZ}}-1)} \frac{(h-\tau_k)^j}{j!} e^{\lambda_i(h-\tau_k)} \Gamma_{i,j} u_k \\
&+ \sum_{i=1}^{c_Z} \frac{(h-\tau_k)^i}{i!} \Phi_i \xi_k + \sum_{i=1}^{c_Z} \frac{(h-\tau_k)^i}{i!} \Gamma_i u_k \\
&+ \sum_{i=1}^{p_C} \sum_{j=0}^{(c_{i_C}-1)} \frac{(h-\tau_k)^j}{j!} e^{a_i(h-\tau_k)} \cos(b_i(h-\tau_k)) \Phi_{i,j,1} \xi_k \\
&+ \sum_{i=1}^{p_C} \sum_{j=0}^{(c_{i_C}-1)} \frac{(h-\tau_k)^j}{j!} e^{a_i(h-\tau_k)} \sin(b_i(h-\tau_k)) \Phi_{i,j,2} \xi_k \\
&+ \sum_{i=1}^{p_C} \sum_{j=0}^{(c_{i_C}-1)} \frac{(h-\tau_k)^j}{j!} e^{a_i(h-\tau_k)} \cos(b_i(h-\tau_k)) \Gamma_{i,j,1} u_k \\
&+ \sum_{i=1}^{p_C} \sum_{j=0}^{(c_{i_C}-1)} \frac{(h-\tau_k)^j}{j!} e^{a_i(h-\tau_k)} \sin(b_i(h-\tau_k)) \Gamma_{i,j,2} u_k,
\end{aligned} \tag{3.27}$$

with

$$\Phi_0 = \begin{pmatrix} \Theta_1 & \Theta_2 \\ 0 & 0 \end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix} \Psi_1 \\ I \end{pmatrix} \tag{3.28}$$

and

$$\begin{aligned}
\Theta_1 &= Q (e^{J_{NZ}h} \oplus e^{J_Zh} \oplus e^{J_Ch}) Q^{-1} \\
\Theta_2 &= Q \left(J_{NZ}^{-1} e^{J_{NZ}h} \oplus \sum_{i=1}^{m_Z-g_Z+1} \frac{h^i}{i!} \check{S}_i \oplus f(a, b, h) \right) Q^{-1} B \\
\Psi_1 &= -Q (J_{NZ}^{-1} \oplus O_Z \oplus T_0) Q^{-1} B.
\end{aligned} \tag{3.29}$$

The function $f(a, b, h)$ and matrix T_0 correspond to their definitions in (3.25). It holds that $\Phi_{i,j} = \hat{\Phi}_{i,j}$ and $\Gamma_{i,j} = \hat{\Gamma}_{i,j}$, as defined in (3.8), with $\check{S}_{i,j}$ replaced by $\hat{S}_{i,j} \oplus O_Z \oplus O_C$. Herein, O_Z and O_C are zero-matrices with the same dimension as J_Z and J_C , respectively. Similar, it holds $\Phi_i = \check{\Phi}_i$ and $\Gamma_i = \check{\Gamma}_i$, as defined in (3.10), where \check{S}_i is replaced by $O_{NZ} \oplus \check{S}_i \oplus O_C$, with O_{NZ} a zero-matrix with the same dimension as J_{NZ} . Moreover, it holds that $\Phi_{i,j,l} = \bar{\Phi}_{i,j,l}$ and $\Gamma_{i,j,l} = \bar{\Gamma}_{i,j,l}$, as defined in (3.25), with the matrix $T_{i,j,l}$ replaced by $O_{NZ} \oplus O_Z \oplus T_{i,j,l}$. The parameters p_{NZ} , p_Z and p_C denote the number of real non-zero, zero and pairs of complex eigenvalues, respectively. $c_{i_{NZ}}$, c_Z and c_{i_C} denote the maximum Jordan block of the i^{th} real non-zero, zero or complex eigenvalue, respectively.

The complicated, general Jordan representation of the NCS (3.2), presented in (3.27) can be simplified to:

$$\xi_{k+1} = \left(F_0 + \sum_{i=1}^{\nu} \alpha_i(\tau_k, h, \dots) F_i \right) \xi_k + \left(G_0 + \sum_{i=1}^{\nu} \alpha_i(\tau_k, h, \dots) G_i \right) u_k, \tag{3.30}$$

with α_i functions dependent on the time-delay τ_k , the sample-time h , and the eigenvalue

parameters λ_i or a_i, b_i . It is defined as:

$$\alpha_i = \begin{cases} \sum_{j=0}^{c_{1NZ}-1} \frac{(h-\tau_k)^j}{j!} e^{\lambda_{1NZ}(h-\tau_k)}, & \text{for } i = j + 1, j = \{0, 1, \dots, c_{1NZ} - 1\} \\ \sum_{j=0}^{c_{2NZ}-1} \frac{(h-\tau_k)^j}{j!} e^{\lambda_{2NZ}(h-\tau_k)}, & \text{for } i = j + 1 + c_{1NZ}, j = \{0, 1, \dots, c_{2NZ} - 1\} \\ \vdots \\ \sum_{j=0}^{c_{p,NZ}} \frac{(h-\tau_k)^j}{j!} e^{\lambda_{pNZ}(h-\tau_k)}, & \text{for } i = j + \sum_{s=1}^{pNZ-1} c_s + 1, j = \{0, 1, \dots, c_{p,NZ} - 1\} \\ \sum_{j=1}^{c_{pZ}} \frac{(h-\tau_k)^j}{j!} e^{\lambda_1(h-\tau_k)}, & \text{for } i = j + q_{NZ}, j = \{1, 2, \dots, c_Z\} \\ \sum_{j=0}^{c_{pC}-1} \frac{(h-\tau_k)^j}{j!} e^{a_{1C}(h-\tau_k)} \cos(b_{1C}(h-\tau_k)), & \text{for } i = j + 1 + q_Z + q_{NZ}, j = \{0, 1, \dots, c_Z\} \\ \vdots \\ \sum_{j=0}^{c_{pC}-1} \frac{(h-\tau_k)^j}{j!} e^{a_{pC}(h-\tau_k)} \cos(b_{pC}(h-\tau_k)), & \text{for } i = j + q_Z + q_{NZ} + \sum_{s=1}^{pC-1} \frac{c_s}{2} + 1, j = \{0, 1, \dots, c_{pC}\} \\ \sum_{j=0}^{c_{1,C}-1} \frac{(h-\tau_k)^j}{j!} e^{a_{1C}(h-\tau_k)} \sin(b_{1C}(h-\tau_k)), & \text{for } i = j + q_Z + q_{NZ} + \frac{q_C}{2} + 1, j = \{0, 1, \dots, c_{p,C}\} \\ \vdots \\ \sum_{j=0}^{c_{p,C}-1} \frac{(h-\tau_k)^j}{j!} e^{a_{pC}(h-\tau_k)} \sin(b_{pC}(h-\tau_k)), & \text{for } i = j + q_Z + q_{NZ} + \frac{q_C}{2} + \sum_{s=1}^{pC-1} \frac{c_s}{2}, j = \{0, 1, \dots, c_{p,C}\} \end{cases} \quad (3.31)$$

with q_{NZ} the total number of time-varying parameters that depend on the real non-zero eigenvalues:

$$q_{NZ} = \sum_{i=1}^{pNZ} c_i,$$

with c_i defined in (2.5) for the i^{th} real non-zero eigenvalue, 1_{NZ} the first real non-zero eigenvalue and p_{NZ} the number of distinct real non-zero eigenvalues. q_Z denotes the number of time-varying parameters that depend on the zero eigenvalues:

$$q_Z = c_Z,$$

with c_Z defined according to (2.5). q_C denotes the total number of time-varying parameters that depend on the complex eigenvalues:

$$q_C = \sum_{i=1}^{pC} c_i,$$

where p_C denotes the number of distinct pairs of complex conjugated eigenvalues, 1_C the first pair of complex eigenvalues and c_i defined in (2.5), which denotes the number of distinct eigenvectors. Note that c_i is always an even number, because it represents two eigenvectors for one pair of complex eigenvalues.

For the matrices, $F_i, G_i, i = \{0, 1, 2, \dots, \nu\}$ it holds that:

$$F_0 = \Phi_0, F_i = \begin{pmatrix} 0 & \Theta_{3,i} \\ 0 & 0 \end{pmatrix}, G_0 = \Gamma_0, G_i = \begin{pmatrix} -\Theta_{3,i} \\ 0 \end{pmatrix}, \quad (3.32)$$

with Φ_0 and Γ_0 defined in (3.28). To define $\Theta_{3,i}$, we first define O_Z , which is a zero-matrix with the same dimension as J_Z , O_{NZ} a zero matrix with the same dimension as J_{NZ} and O_C

a zero-matrix with the same dimension as J_C . Then $\Theta_{3,i}$ is given by:

$$\Theta_{3,i} = \begin{cases} \begin{pmatrix} 0 & -QJ_{NZ}^{-1}(\hat{S}_{1,j} \oplus O_Z \oplus O_C)Q^{-1}B \\ 0 & 0 \end{pmatrix}, & \text{for } i = j + 1, j = \{0, 1, \dots, c_{1NZ} - 1\} \\ \begin{pmatrix} 0 & -QJ_{NZ}^{-1}(\hat{S}_{2,j} \oplus O_Z \oplus O_C)Q^{-1}B \\ 0 & 0 \end{pmatrix}, & \text{for } i = j + c_{1NZ} + 1, j = \{0, 1, \dots, c_{2NZ} - 1\} \\ \vdots & \\ \begin{pmatrix} 0 & -QJ_{NZ}^{-1}(\hat{S}_{pNZ,j} \oplus O_Z \oplus O_C)Q^{-1}B \\ 0 & 0 \end{pmatrix}, & \text{for } i = j + \sum_{s=1}^{pNZ-1} c_s + 1, j = \{0, 1, \dots, c_{pNZ} - 1\} \\ \begin{pmatrix} 0 & -Q(O_{NZ} \oplus \check{S}_j \oplus O_C)Q^{-1}B \\ 0 & 0 \end{pmatrix}, & \text{for } i = j + q_{NZ}, j = \{1, \dots, c_{pZ}\} \\ \begin{pmatrix} 0 & -QJ_{NZ}^{-1}(O_{NZ} \oplus O_Z \oplus T_{1,j,1})Q^{-1}B \\ 0 & 0 \end{pmatrix}, & \text{for } i = j + q_{NZ} + q_Z + 1, j = \{1, \dots, c_{1C}\} \\ \vdots & \\ \begin{pmatrix} 0 & -QJ_{NZ}^{-1}(O_{NZ} \oplus O_Z \oplus T_{1,j,1})Q^{-1}B \\ 0 & 0 \end{pmatrix}, & \text{for } i = j + q_{NZ} + q_Z + \sum_{s=1}^{(p_C-1)C} (c_s) + 1, j = \{1, \dots, c_{p_C}\} \\ \begin{pmatrix} 0 & -QJ_{NZ}^{-1}(O_{NZ} \oplus O_Z \oplus T_{1,j,2})Q^{-1}B \\ 0 & 0 \end{pmatrix}, & \text{for } i = j + q_{NZ} + q_Z + \frac{q_C}{2} + 1, j = \{1, \dots, c_{1C}\} \\ \vdots & \\ \begin{pmatrix} 0 & -QJ_{NZ}^{-1}(O_{NZ} \oplus O_Z \oplus T_{1,j,2})Q^{-1}B \\ 0 & 0 \end{pmatrix}, & \text{for } i = j + q_{NZ} + q_Z + \frac{q_{NZ}}{2} + \sum_{s=1}^{(p_C-1)C} (c_s) + 1, j = \{1, \dots, c_{p_C}\} \end{cases} \quad (3.33)$$

The parameter ν gives the number of different uncertain time-varying functions and is equal to:

$$\nu = q_{NZ} + q_Z + q_C, \quad (3.34)$$

with ν smaller than or equal to n (obtained from $A \in \mathbb{R}^{n \times n}$).

3.2 Robust stability

For system (3.3) a full-state feedback control law is adopted, $u_k = -Kx_k$. In this case, we can simplify (3.3) to:

$$\chi_{k+1} = \hat{A}(\tau_k)\chi_k, \quad (3.35)$$

with $\chi_k = (x_k \ x_{k-1})^T$, $\hat{A}(\tau_k) = \begin{pmatrix} e^{Ah} - \int_0^{h-\tau_k} e^{As} ds BK & - \int_{h-\tau_k}^h e^{As} ds BK \\ I & 0 \end{pmatrix}$ and $\tau_k \in [0, \tau_{max}]$, $\tau_{max} \leq h$.

For this system we can investigate the stability of the equilibrium point $x_{k+1} = 0$. In this stability analysis, the previously derived Jordan forms are used, where the matrix A can consist of real and complex eigenvalues.

Note that, due to the time-varying delay τ_k , a set of matrices $\hat{A}(\tau_k)$ can be constructed:

$$\hat{A} = \left\{ \hat{A}(\tau_k) : \tau_k \in [0, \tau_{max}], k \in \mathbb{Z}^+ \right\}. \quad (3.36)$$

To find a general solution of the integrals in the system matrix $\hat{A}(\tau_k)$, system (3.35) is rewritten in its Jordan form representation, similar to (3.30):

$$\chi_{k+1} = \left(\hat{F}_0 + \sum_{i=1}^{\nu} \alpha_i(\tau_k, h, \dots) \hat{F}_i \right) \chi_k, \quad (3.37)$$

with the matrices \hat{F}_i derived similar to (3.30), as:

$$\hat{F}_0 = \begin{pmatrix} \Theta_1 - \Psi_1 K & -\Theta_2 K \\ I & 0 \end{pmatrix}, \quad \hat{F}_i = \begin{pmatrix} \Theta_{3,i} K & -\Theta_{3,i} K \\ 0 & 0 \end{pmatrix}, \quad (3.38)$$

with $i = \{1, 2, \dots, \nu\}$ and all other parameters equal to (3.29) and (3.33).

System (3.37) contains the uncertain time-varying parameters α_i , which may be nonlinear in τ_k . These parameters form, together with the matrices \hat{F}_i , $i = \{1, 2, \dots, \nu\}$ a set of system matrices for (3.37) that can be tested to derive the stability of the system. This set $\hat{\mathcal{F}}$ is represented by:

$$\hat{\mathcal{F}} = \left\{ \hat{F}_0 + \sum_{i=1}^{\nu} \alpha_i(\tau_k, h, \dots) \hat{F}_i : \tau_k \in [0, \tau_{max}] \right\}. \quad (3.39)$$

For each uncertainty parameter an overestimation can be derived, based on

$$\alpha_i = \underline{\alpha}_i + \delta_i(\bar{\alpha}_i - \underline{\alpha}_i), \quad i = \{1, 2, \dots, \nu\}, \quad (3.40)$$

where δ_i can take any value in the interval $[0, 1]$ and $\bar{\alpha}_i$, $\underline{\alpha}_i$ are defined as the maximum and minimum of the function α_i , respectively:

$$\begin{aligned} \bar{\alpha}_i &= \max_{\tau_k \in [0, \tau_{max}]} \alpha_i(\tau_k, h, \dots) \\ \underline{\alpha}_i &= \min_{\tau_k \in [0, \tau_{max}]} \alpha_i(\tau_k, h, \dots), \end{aligned} \quad (3.41)$$

for given h and λ_i , a_i , b_i .

Then, the set $\hat{\mathcal{F}}$, as in (3.39), can be overestimated by:

$$\bar{\mathcal{F}} = \left\{ \bar{F}_0 + \sum_{i=1}^{\nu} \delta_i \bar{F}_i : \delta_i \in [0, 1], i = \{1, 2, \dots, \nu\} \right\}, \quad (3.42)$$

with $\bar{F}_0 = \hat{F}_0 + \sum_{i=1}^{\nu} \underline{\alpha}_i \hat{F}_i$ and $\bar{F}_i = (\bar{\alpha}_i - \underline{\alpha}_i) \hat{F}_i$. Note that each δ_i can be chosen individually from the set $[0, 1]$.

Now, it is obvious that it holds that:

$$\hat{A} = \hat{\mathcal{F}} \subset \bar{\mathcal{F}}. \quad (3.43)$$

For stability of the origin of (3.35) it is sufficient to prove stability for $\chi_{k+1} = \bar{F} \chi_k$, for any $\bar{F} \in \bar{\mathcal{F}}$. The set of matrices $\bar{\mathcal{F}}$ is infinite, but every matrix in this set can be written as a convex combination of the generators (corner points) of this set. First, we define the set of generators of $\bar{\mathcal{F}}$, which is given by:

$$\mathcal{H} = \left\{ \bar{F}_0 + \sum_{i=1}^{\nu} \delta_i \bar{F}_i : \delta_i = \{0, 1\}, i = \{1, 2, \dots, \nu\} \right\}. \quad (3.44)$$

Note that the set \mathcal{H} consists of 2^ν matrices, which we will denote individually by H_j , $j = \{1, 2, \dots, 2^\nu\}$. Second, we define the convex overapproximation of $\bar{\mathcal{F}}$, based on this set of generators, by

$$\bar{\mathcal{H}} = \left\{ \sum_{j=1}^{2^\nu} (\mu_j H_j) : \sum_{j=1}^{2^\nu} \mu_j = 1, \mu_j \in [0, 1], j = \{1, 2, \dots, 2^\nu\} \right\}. \quad (3.45)$$

Note that this set is again infinite, due to all allowable values of μ_j , $j = \{1, 2, \dots, 2^\nu\}$. It is obvious that it holds that:

$$\bar{\mathcal{F}} \subseteq \bar{\mathcal{H}}. \quad (3.46)$$

Now, we formulate a result, posing sufficient conditions, based on LMIs, for the asymptotic stability of the origin of (3.35) for time-varying delays $\tau_k \in [0, \tau_{max}]$ with $\tau_{max} \leq h$.

Theorem 3.2.1 *Consider the networked control system (3.35) with time-varying delays taken from a bounded set $\tau_k \in [0, \tau_{max}]$, with $\tau_{max} \leq h$. Consider the set of matrices \mathcal{H} defined by (3.44), (3.42), (3.39), (3.38) and parameterized by τ_{max} . If there exists a $P \in \mathbb{R}^{2n \times 2n}$ such that the following LMI conditions are satisfied:*

$$\begin{aligned} P = P^T &> 0 \\ H_j^T P H_j - P &< 0, \forall H_j \in \mathcal{H}, \end{aligned} \quad (3.47)$$

then (3.35) is robustly GAS for any time-varying delay τ_k satisfying $\tau_k \in [0, \tau_{max}] \forall k \in \mathbb{Z}^+$.

Proof Consider the quadratic Lyapunov function $V = \chi_k^T P \chi_k$. The existence of a common quadratic Lyapunov function for all $\hat{A}(\tau_k) \in \hat{\mathcal{A}}$ is sufficient for the asymptotic stability of the origin of system (3.35). Therefore, we require that:

$$\Delta V = \chi_k^T \left(\hat{A}(\tau_k)^T P \hat{A}(\tau_k) - P \right) \chi_k < 0, \forall \tau_k \in [0, \tau_{max}], \tau_{max} \in [0, h].$$

Due to the overapproximation of $\hat{\mathcal{A}}$ by $\bar{\mathcal{H}}$, as shown in (3.46) and (3.43), it is sufficient to require:

$$\Delta V = \chi_k^T \left(\sum_{j=1}^{2^\nu} \mu_j H_j^T P \sum_{j=1}^{2^\nu} \mu_j H_j - P \right) \chi_k < 0, \forall \mu_j \in [0, 1], j = \{1, 2, \dots, 2^\nu\}, \quad (3.48)$$

satisfying $\sum_{j=1}^{2^\nu} \mu_j = 1$. The use of Schur's complement [16] allows us to rewrite the inequality in (3.48) in the form of the following LMI:

$$\begin{pmatrix} P & \sum_{j=1}^{2^\nu} \mu_j H_j^T P \\ P \sum_{j=1}^{2^\nu} \mu_j H_j & P \end{pmatrix} > 0,$$

which can be rewritten as:

$$\sum_{j=1}^{2^\nu} \mu_j \begin{pmatrix} P & H_j^T P \\ P H_j & P \end{pmatrix} > 0.$$

Reapplying Schur's complement, in combination with $\mu_j > 0, \forall j \in \{1, 2, \dots, \nu\}$ gives (3.47).

□

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