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ROBUST CONTROL WITH STRUCTURED PERTURBATIONS

by

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1. INTRODUCTION

This report treats two important problems in the area of control systems design and analysis. The first problem we treat here is the robust stability using characteristic polynomial. This problem is treated first in characteristic polynomial coefficient space with respect to perturbations in the coefficients of the characteristic polynomial, and then for a control system containing perturbed parameters in the transfer function description of the plant. In coefficient space, a simple expression is first given for the l^2 - stability margin for both monic and non-monic cases. Following this, a method is extended to reveal much larger stability region.

This result has been extended to the parameter space so that one can determine the stability margin, in terms of ranges of parameter variations, of the closed loop system when the nominal stabilizing controller is given. This stability margin can be enlarged by a choice of better stabilizing controller.

The second problem this report describes is the lower order stabilization problem. The motivation of the problem is as follows. Even though the wide range of stabilizing controller design methodologies are available in both the state space and the transfer function domains, all of these methods produce unnecessarily high order controllers.

In practice, the stabilization is only one of many requirements to be satisfied. Therefore, if the order of a stabilizing controller is excessively high, one can normally expect to have a even higher order controller upon the completion of design such as inclusion of dynamic response requirements, etc. Therefore, it is reasonable to have a lowest possible order stabilizing controller first and then adjust the controller to

meet additional requirements.

In this report, the algorithm of designing a lower order stabilizing controller is given. The algorithm is not necessarily produce the minimum order controller, however the algorithm is theoretically logical and some simulation results show that the algorithm works in general.

The above two problems have been solved and published. These are found in Appendix A and B. Finally, some remarks and on going research are briefly discussed.

2. PUBLICATIONS SUPPORTED BY THE GRANT

PUBLICATIONS

- H. Chapellat, S.P. Bhattacharyya, and L.H. Keel, "Stability Margin for Hurwitz Polynomials," *Proceedings of the 27th IEEE Conference on Decision and Control*, Austin, Texas, December 7 - 9, 1988.

- L.H. Keel and S.P. Bhattacharyya, "Matrix Equation Approach for Designing Lower Order Regulators," *SIAM Journal on Matrix Analysis and Applications*, To appear.

3. DISCUSSIONS AND DIRECTIONS OF RESEARCH

The stability margin in the parameter space is in general more practical than one in the coefficient space. When parameters enter into coefficients of characteristic polynomial linearly, the stability margin is exact. However, when parameters enter in nonlinear fashion which is the general facts in practical systems, the margin seems to be conservative.

The results show that the transfer function approach (or polynomial framework) has some difficulties because in most cases, the state space representation gives better description of the dynamic systems in terms of parameters. It is true in most cases because the state space description is obtained directly from the dynamic equations.

The next phase of research will be concentrated in developing new idea of robust control in the domain of state space. Even though some results of this problem are available, they are still primitive and not yet satisfactory.

APPENDIX A

STABILITY MARGINS FOR HURWITZ POLYNOMIALS[†]

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ABSTRACT

This paper treats the robust stability issue using the characteristic polynomial, for two different cases. First in coefficient space with respect to perturbations in the coefficients of the characteristic polynomial, and then for a control system containing perturbed parameters in the transfer function description of the plant. In coefficient space, a simple expression is first given for the l^2 -stability margin for both the monic and non-monic cases. Following this, a method is extended to reveal much larger stability regions. In parameter space we consider all single input (multi output) or single output (multi input) systems with a fixed controller and a plant described by a set of transfer functions which are ratios of polynomials with variable coefficients. The paper gives a procedure to calculate the radius of the largest stability ball in the space of these variable parameters. This calculation is important as it serves as a stability margin for the control system. The method is based on the application of the orthogonal projection theorem in the appropriate Euclidean vector space. The formulas that result are quasi closed form expressions for the stability margin, are computationally efficient and provide some insight. The paper is illustrated by numerical examples.

I. INTRODUCTION

Kharitonov's theorem was revealed to the control community in 1984 [1]. Since then there has been a growing interest in the theory of robust control with structured perturbations, [2]-[6]. Soh, Berger and Dabke introduced the largest stability hypersphere around a stable polynomial [2], and in [3] Biernacki, Hwang and Bhattacharyya extended their idea to the control problem and introduced the largest stability hypersphere in parameter space. This last result is applicable whenever the characteristic equation of the closed loop system is a linear or affine function of the transfer function subject to perturbation.

In particular, the latter situation always occurs in single input (multioutput) or single output(multiinput) plants, when the parameter vector is chosen to be the transfer function coefficients of the plant. These two results, however, suffer from the fact that the method given for the calculation of the radius of the stability hypersphere is not very satisfactory. In this paper we present new methods for this calculation, both in coefficient space and in parameter space, by using the orthogonal projection theorem in the appropriate Euclidean vector space. The formulas that result are quasi closed form expressions for the radius of the largest stability hypersphere. We also present some important simplifications of the scalar minimization problem that results. These results constitute a conceptual and computational improvement over the stability margin calculations given in [2] and [3].

The paper also considers the problem of computing the l^∞ margin around a Hurwitz polynomial and an interesting method is given which is based on Kharitonov's theorem and which greatly simplifies the one given in [1] and then [5]. Moreover it is shown by an example that this procedure is very simply extended and yields stability regions which are much larger than those provided by the simple computation of the l^∞ -margin. The paper is organized as follows. In section II, we present the results concerning the computation of stability margins in coefficient space, and in section III we introduce the general setting for the control problem considered in this paper, and present our result for the calculation of the l^2 -stability margin.

II. STABILITY MARGINS IN COEFFICIENT SPACE

In this section we first consider a stable polynomial $\delta(\cdot)$ of degree n , and we give an expression for the radius, in the space of coefficients, of the largest stability hypersphere around the nominal point, both in the non-monic and monic case.

II.1. l^2 stability margin

a) The non-monic case:

Let $\delta(\cdot)$ be an arbitrary stable polynomial of order n ,

$$\delta(s) = \delta_0 + \delta_1 s + \cdots + \delta_n s^n.$$

The l^2 norm of $\delta(s)$ is defined by,

$$\|\delta(s)\|_2^2 = \sum_{i=0}^n \delta_i^2.$$

Let us now separate $\delta(s)$ into its odd and even parts:

$$\delta(s) = \underbrace{\delta^{\text{even}}(s)}_{\text{even degree terms}} + \underbrace{\delta^{\text{odd}}(s)}_{\text{odd degree terms}}$$

and let us also define $\delta^e(\omega)$ and $\delta^o(\omega)$ as follows:

$$\begin{aligned} \delta^e(\omega) &= \delta^{\text{even}}(j\omega) = \delta_0 - \delta_2 \omega^2 + \delta_4 \omega^4 - \dots \\ \delta^o(\omega) &= \frac{\delta^{\text{odd}}(j\omega)}{j\omega} = \delta_1 - \delta_3 \omega^2 + \delta_5 \omega^4 - \dots \end{aligned} \quad (2.1)$$

Theorem 2.1

The radius of the largest stability hypersphere around $\delta(s)$ is given by:

$$\rho(\delta) = \min(|\delta_0|, |\delta_n|, \inf_{\omega \geq 0} d_\omega^n),$$

where d_ω^n is given by:

i) $n = 2p$

$$d_\omega^{n2} = \frac{[\delta^e(\omega)]^2}{1 + \omega^4 + \cdots + \omega^{4p}} + \frac{[\delta^o(\omega)]^2}{1 + \omega^4 + \cdots + \omega^{4(p-1)}} \quad (2.2)$$

ii) $n = 2p + 1$

$$d_\omega^{n2} = \frac{[\delta^e(\omega)]^2 + [\delta^o(\omega)]^2}{1 + \omega^4 + \cdots + \omega^{4p}} \quad (2.3)$$

b) The monic case:

In this section, we are given a stable monic polynomial

$$\beta(s) = \beta_0 + \beta_1 s + \cdots + \beta_n s^{n-1} + s^n,$$

and we want to find the radius of the largest stability hypersphere in the affine space of minoc polynomials of degree n . We have the following result.

Theorem 2.2

The radius of the largest stability hypersphere around $\beta(s)$ is given by

$$\rho(\beta) = \min(|\beta_0|, \inf_{\omega \geq 0} d_\omega^{n'})$$

where

$$d_\omega^{n'} = d_\omega^n$$

computed for

$$\beta(s) - s^n - \omega^2 s^{n-2},$$

that is

i) $n = 2p$

$$d_\omega^{n'2} = \frac{[\delta^e(\omega)]^2 + [\delta^o(\omega)]^2}{1 + \omega^4 + \cdots + \omega^{4(p-1)}} \quad (2.4)$$

ii) $n = 2p + 1$

$$d_\omega^{n'2} = \frac{[\delta^e(\omega)]^2}{1 + \omega^4 + \cdots + \omega^{4p}} + \frac{[\delta^o(\omega)]^2}{1 + \omega^4 + \cdots + \omega^{4(p-1)}} \quad (2.5)$$

we now show how the minimization problem that results from the application of formulas (2.2) – (2.5) can be simplified. Consider for example the calculation of

$$d_{\min} = \inf_{\omega \geq 0} d_\omega.$$

A simple manipulation we show that there is no need to carry out a minimization over the finite range $[0, \infty)$. We will consider the case when $n = 2p$, but a similar derivation holds if n is odd.

First it is clear that

$$d_{\min}^2 = \min\left(\inf_{\omega \in [0,1]} d_\omega^2, \inf_{\omega \in [0,1]} d_{\frac{1}{\omega}}^2\right),$$

and then we have

$$|\delta^e(\omega)|^2 = (\delta_0 - \delta_2\omega^2 + \cdots + (-1)^p \delta_{2p}\omega^{2p})^2$$

$$|\delta^o(\omega)|^2 = (\delta_1 - \delta_3\omega^2 + \cdots + (-1)^{p-1} \delta_{2p-1}\omega^{2p-2})^2,$$

which yields:

$$|\delta^e(\frac{1}{\omega})|^2 = \frac{1}{\omega^{4p}} (\delta_{2p} - \delta_{2p-2}\omega^2 + \cdots + (-1)^p \delta_0\omega^{2p})^2,$$

and

$$|\delta^o(\frac{1}{\omega})|^2 = \frac{1}{\omega^{4(p-1)}} (\delta_{2p-1} - \delta_{2p-3}\omega^2 + \cdots + (-1)^{p-1} \delta_1\omega^{2p-2})^2.$$

So that finally,

$$d_{\frac{1}{\omega}}^2 = \frac{\frac{1}{\omega^{4p}} (\delta_{2p} - \delta_{2p-2}\omega^2 + \cdots + (-1)^p \delta_0\omega^{2p})^2}{1 + \frac{1}{\omega^4} + \frac{1}{\omega^8} + \cdots + \frac{1}{\omega^{4p}}} + \frac{\frac{1}{\omega^{4(p-1)}} (\delta_{2p-1} - \delta_{2p-3}\omega^2 + \cdots + (-1)^{p-1} \delta_1\omega^{2p-2})^2}{1 + \frac{1}{\omega^4} + \frac{1}{\omega^8} + \cdots + \frac{1}{\omega^{4(p-1)}}}$$

This last expression however is nothing but

$$d_{\frac{1}{\omega}}^2 = \frac{(\delta_{2p} - \delta_{2p-2}\omega^2 + \delta_{2p-4}\omega^4 + \cdots + (-1)^p \delta_0\omega^{2p})^2}{1 + \omega^4 + \omega^8 + \omega^{4p}} + \frac{(\delta_{2p-1} - \delta_{2p-3}\omega^2 + \delta_{2p-5}\omega^4 + \cdots + (-1)^{p-1} \delta_1\omega^{2p-2})^2}{1 + \omega^4 + \omega^8 + \cdots + \omega^{4(p-1)}}$$

and we can see that $\delta_{\frac{1}{\omega}}^2$ has exactly the same structure as d_{ω}^2 . Can $\delta_{\frac{1}{\omega}}^2$ be considered as the " d_{ω}^2 " of some other polynomial? The answer is of course yes. Consider $\delta'(s) = s^n \delta(\frac{1}{s})$, which in our case is

$$\delta'(s) = s^{2p} \delta(\frac{1}{s}) = \delta_{2p} + \delta_{2p-1}s + \cdots + \delta_0 s^{2p},$$

then clearly

$$\delta'^e(\omega) = \delta_{2p} - \delta_{2p-2}\omega^2 + \cdots + (-1)^p \delta_0\omega^{2p}$$

and

$$\delta'^o(\omega) = \delta_{2p-1} - \delta_{2p-3}\omega^2 + \cdots + (-1)^{p-1} \delta_1\omega^{2p-2}.$$

Thus we see that in fact δ_{ω}^2 corresponds to d_{ω}^2 computed for $\delta'(\cdot)$. Suppose now that you have a subroutine $DMIN(\underline{\delta})$ that takes the vector of coefficients $\underline{\delta}$ as input and returns the minimum of δ_{ω}^2 over $[0, 1]$. Then the following algorithm will compute d_{\min} by simply calling $DMIN$ two times.

1. Set $\underline{\delta} = (\delta_0, \delta_1, \dots, \delta_n)$.
2. First call: $d_1 = DMIN(\underline{\delta})$.
3. Switch: set $\underline{\delta} = (\delta_n, \delta_{n-1}, \dots, \delta_0)$.
4. Second call: $d_2 = DMIN(\underline{\delta})$.
5. $d_{\min} = \min(d_1, d_2)$.

Incidentally, we already knew that the two polynomials $\delta(s)$ and $\delta'(s) = s^n \delta(\frac{1}{s})$ are stable together (i.e. one is stable if and only if the other one is stable). The development above tells us that moreover $\rho(\delta) = \rho(\delta')$.

It is to be noted that the results of this section were first proved in [4]. Recently similar results have been achieved in [8]. We now turn to the problem of finding the l^{∞} -stability margin.

II.2 l^{∞} stability margin

We consider here as given a stable nominal polynomial $\delta(s)$, as well as a set of non-negative numbers $\alpha_0, \alpha_1, \dots, \alpha_n$, and we want to find the largest box of the form

$$\mathcal{B}_{\rho} = (\delta_0 - \alpha_0 \rho, \delta_0 + \alpha_0 \rho) \times (\delta_1 - \alpha_1 \rho, \delta_1 + \alpha_1 \rho) \times \dots \times (\delta_n - \alpha_n \rho, \delta_n + \alpha_n \rho). \quad (2.6)$$

Kharitonov's theorem tells us that the closed box,

$$\mathcal{B}'_{\rho} = [\delta_0 - \alpha_0 \rho, \delta_0 + \alpha_0 \rho] \times [\delta_1 - \alpha_1 \rho, \delta_1 + \alpha_1 \rho] \times \dots \times [\delta_n - \alpha_n \rho, \delta_n + \alpha_n \rho]. \quad (2.7)$$

is stable if and only if its four Kharitonov polynomials are stable. Denoting

$$\begin{aligned} K^{\text{even}}(s) &= \alpha_0 - \alpha_2 s^2 + \alpha_4 s^4 - \dots, \\ K^{\text{odd}}(s) &= \alpha_1 s - \alpha_3 s^3 + \alpha_5 s^5 - \dots, \end{aligned} \quad (2.8)$$

and as in (2.1),

$$\begin{aligned} K^e(\omega) &= \alpha_0 + \alpha_2 \omega^2 + \alpha_4 \omega^4 + \dots, \\ K^o(\omega) &= \alpha_1 + \alpha_3 \omega^2 + \alpha_5 \omega^4 + \dots, \end{aligned} \quad (2.9)$$

the four Kharitonov polynomials associated with \mathcal{B}'_ρ can be written as,

$$\begin{aligned} K_\rho^1(s) &= \delta(s) - \rho K^{\text{even}}(s) - \rho K^{\text{odd}}(s), \\ K_\rho^2(s) &= \delta(s) - \rho K^{\text{even}}(s) + \rho K^{\text{odd}}(s), \\ K_\rho^3(s) &= \delta(s) + \rho K^{\text{even}}(s) - \rho K^{\text{odd}}(s), \\ K_\rho^4(s) &= \delta(s) + \rho K^{\text{even}}(s) + \rho K^{\text{odd}}(s). \end{aligned} \quad (2.10)$$

Thus we know that when increasing ρ , \mathcal{B}'_ρ will first contain an unstable polynomial or a polynomial of degree $< n$ when one of the four polynomials above becomes of degree $< n$ or acquires a $j\omega$ root or a root at the origin.

The case of a root at the origin or the case of a loss in degree is trivially achieved for

$$\rho = \frac{\delta_0}{\alpha_0} \quad \text{and} \quad \frac{\delta_n}{\alpha_n}, \quad \text{respectively.} \quad (2.11)$$

Let us now look at the case of the appearance of a $j\omega$ root for $\omega > 0$. If we consider for example $K_\rho^1(s)$, we know that this polynomial has a $j\omega$ root if and only if

$$\begin{cases} K_\rho^{1e}(\omega) = \delta^e(\omega) - \rho K^e(\omega) = 0 \\ \text{and} \\ K_\rho^{1o}(\omega) = \delta^o(\omega) - \rho K^o(\omega) = 0. \end{cases} \quad (2.12)$$

This of course is possible if and only if we have

$$\delta^e(\omega)K^o(\omega) - \delta^o(\omega)K^e(\omega) = 0. \quad (2.13)$$

This same polynomial is associated with $K_\rho^4(s)$ which has a $j\omega$ root if and only if

$$\begin{cases} K_\rho^{4e}(\omega) = \delta^e(\omega) + \rho K^e(\omega) = 0 \\ \text{and} \\ K_\rho^{4o}(\omega) = \delta^o(\omega) + \rho K^o(\omega) = 0. \end{cases} \quad (2.14)$$

Note that this polynomial is a polynomial of degree $n - 1$ in ω^2 and therefore we just have to find the positive roots of a polynomial of degree $n - 1$. Once these roots are found, we then look at the values (at these points) of the ratios

$$\frac{\delta^e(\omega)}{K^o(\omega)} = \frac{\delta^o(\omega)}{K^e(\omega)}.$$

The minimum positive value of these ratios is denoted ρ_1 and corresponds to $K^1(s)$, and the negative of the maximum negative value is denoted ρ_4 and corresponds to $K^4(s)$.

Similarly, the polynomial in ω^2 associated with $K^2(s)$ and $K^3(s)$ is

$$\delta^e(\omega)K^o(\omega) + \delta^o(\omega)K^e(\omega) = 0. \quad (2.15)$$

and ρ_2, ρ_3 are defined in a similar fashion. The maximum centered box is then obtained by taking ρ to be equal to the smallest the six positive numbers

$$\rho_1, \rho_2, \rho_3, \rho_4, \frac{\delta_0}{\alpha_0} \quad \text{and} \quad \frac{\delta_n}{\alpha_n}.$$

Once this first box is obtained it is also possible to further extend it, if the center is allowed to move, by freezing the coefficients corresponding to the 'closest' $K^i(s)$ in the first stage and using the same approach. An example will best explain the details of this extension procedure.

Example :

Consider the case treated in [7] of the polynomial,

$$\delta(s) = s^6 + 14.0s^5 + 80.25s^4 + 251.25s^3 + 502.75s^2 + 667.25s + 433.5,$$

and the set of parameters,

$$\alpha_0 = 92.32, \quad \alpha_1 = 33.36, \quad \alpha_2 = 38.28$$

$$\alpha_3 = 15.075, \quad \alpha_4 = 6.2, \quad \alpha_5 = 1.4, \quad \alpha_6 = 0.14$$

Here we have

$$K^e(\omega) = 92.32 + 38.28\omega^2 + 6.2\omega^4 + 0.14\omega^6$$

$$K^o(\omega) = 33.36 + 15.075\omega^2 + 1.4\omega^4.$$

Letting $t = \omega^2$, we have to find the positive roots of the two following polynomials,

$$\begin{aligned} P_1(t) &= \delta^e(t)K^o(t) - \delta^o(t)K^e(t) \\ &= -47138.96 - 12583.6575t - 106.49625t^2 + 1400.97375t^3 \\ &\quad + 45.65t^4 - 3.36t^5. \end{aligned}$$

and

$$\begin{aligned} P_2(t) &= \delta^e(t)K^o(t) + \delta^o(t)K^e(t) \\ &= 76062.08 - 7889.7975t - 8483.33625t^2 - 455.856254t^3 \\ &\quad + 148.9t^4 + 0.56t^5. \end{aligned}$$

It turns out that $P_1(t)$ has two positive roots,

$$t_1^1 = 4.020612 \quad \text{and} \quad t_2^1 = 28.1620029,$$

and $P_2(t)$ also has two positive roots which are

$$t_1^2 = 2.5415548 \quad \text{and} \quad t_2^2 = 9.0241863.$$

Based on these roots one computes the following values for the ρ_i 's,

$$\begin{aligned} \rho_1 &= 2.9937539, & \rho_2 &= 1.6229978, \\ \rho_3 &= 1.4757364, & \rho_4 &= 1.0001038. \end{aligned}$$

From this we conclude that the l^∞ - margin is obtained by putting $\rho = \rho_4 = 1.0001038$ in (2.7) giving,

$$\begin{aligned} \mathcal{B}'_1 &= [341.17042, 525.82957] \times [633.88654, 700.61346] \\ &\quad \times [464.46603, 541.03397] \times [236.17344, 266.32656] \\ &\quad \times [74.04936, 86.45064] \times [12.59986, 15.40014] \\ &\quad \times [0.85999, 1.14001]. \end{aligned}$$

We know that the Kharitonov polynomial $K^4(s)$ associated with \mathcal{B}'_1 is 'unstable'. However we can still increase \mathcal{B}'_1 in the other direction and thus consider,

$$\begin{aligned} \mathcal{B}'_{1\rho'} &= [\delta_0 - \alpha_0\rho', 525.82598] \times [\delta_1 - \alpha_1\rho', 700.61346] \\ &\quad \times [464.46603, \delta_2 + \alpha_2\rho'] \times [236.17344, \delta_3 + \alpha_3\rho'] \\ &\quad \times [\delta_4 - \alpha_4\rho', 86.45064] \times [\delta_5 - \alpha_5\rho', 15.40014] \\ &\quad \times [0.85999, \delta_6 + \alpha_6\rho']. \end{aligned}$$

The 3 remaining Kharitonov polynomials associated with $\mathcal{B}'_{1\rho'}$ can be rewritten as

$$\begin{aligned} K_{\rho'}^1(s) &= \delta(s) - \rho'K^e(s) - \rho'K^o(s), \\ K_{\rho'}^2(s) &= \delta(s) - \rho'K^e(s) + \rho K^o(s), \\ K_{\rho'}^3(s) &= \delta(s) + \rho K^e(s) - \rho'K^o(s). \end{aligned}$$

The smallest ρ' for which $K_{\rho'}^1(s)$ gets a $j\omega$ root is of course the same as the one calculated above, that is $\rho'_1 = 2.9937539$.

For $K_{\rho'}^2(s)$ on the other hand, ρ'_2 is simply given by the smallest positive value of the ratio $\delta^e(\omega)/K^e(\omega)$, at the real roots of,

$$\begin{aligned}\delta^o(\omega) + \rho K^o(\omega) &= K^{4o}(\omega) \\ &= 700.61346 - 236.17344\omega^2 + 15.40014\omega^4.\end{aligned}$$

this gives the value,

$$\rho'_2 = 2.3459473.$$

Similarly, ρ'_3 is given by the smallest positive value of the ration $\delta^o(\omega)/K^o(\omega)$, at the real roots of,

$$\begin{aligned}\delta^e(\omega) + \rho K^e(\omega) &= K^{4e}(\omega) \\ &= 525.82958 - 464.46603\omega^2 + 86.45064\omega^4 - 0.85999\omega^6.\end{aligned}$$

and this yields the value

$$\rho'_3 = 4.918607463.$$

Hence, the value of ρ' is $\rho' = \rho'_2 = 2.3459473$. This gives rise to the augmented box,

$$\begin{aligned}\mathcal{B}'_2 &= [216.92215, 525.82958] \times [588.98920, 700.61346] \\ &\times [464.46603, 592.55286] \times [236.1734, 286.61515] \\ &\times [65.70513, 86.45064] \times [10.71568, 15.40014] \\ &\times [0.85999, 1.32843].\end{aligned}$$

Here again we know that the Kharitonov polynomials $K^4(s)$ and $K^2(s)$ associated with \mathcal{B}'_2 are 'unstable'. However we can still increase \mathcal{B}'_2 by considering,

$$\begin{aligned}\mathcal{B}'_{2\rho''} &= [216.92215, 525.82958] \times [\delta_1 - \alpha_1\rho'', 700.61346] \\ &\times [464.46603, 592.55286] \times [236.17344, \delta_3 + \alpha_3\rho''] \\ &\times [65.70513, 86.45064] \times [\delta_5 - \alpha_5\rho'', 15.40014] \\ &\times [0.85999, 1.32843].\end{aligned}$$

The two remaining Kharitonov polynomials are

$$\begin{aligned}K_{\rho''}^1(s) &= \delta(s) - \rho'K^e(s) - \rho''K^o(s), \\ K_{\rho''}^3(s) &= \delta(s) + \rho K^e(s) - \rho''K^o(s).\end{aligned}$$

The value of ρ''_3 can be seen to be the same as ρ'_3 that is

$$\rho''_3 = 4.918607463.$$

As for ρ''_1 it is given by the smallest positive value of the ratio $\delta(\omega)/K^o(\omega)$, at the real roots of,

$$\begin{aligned} \delta^e(\omega) - \rho' K^e(\omega) = \\ 216.92215 - 592.55286\omega^2 + 65.70513\omega^4 - 1.32843\omega^6. \end{aligned}$$

This yields $\rho''_1 = 4.287652665$, which therefore gives the final answer,

$$\rho'' = \rho''_1 = 4.287652665,$$

so that we get another increase and a final

$$\begin{aligned} \mathcal{B}'_{2,\rho''} = & [216.92215, 525.82958] \times [524.21391, 700.6134] \\ & \times [464.46603, 592.55286] \times [236.17344, 315.88636] \\ & \times [65.70513, 86.45064] \times [7.99729, 15.40014] \\ & \times [0.85999, 1.32843], \end{aligned}$$

for which three of the four associated Kharitonov polynomials, namely $K^1(s)$, $K^2(s)$ and $K^4(s)$ are 'unstable'.

We now look at the control problem and consider the stability margin in parameter space.

III. THE l^2 - STABILITY MARGIN IN PARAMETER SPACE

In the standard feedback system of Figure 1, suppose that the plant is either single input (multioutput) or single output (multiinput). Since the formulation is similar for the two cases we restrict our considerations to the single input case.

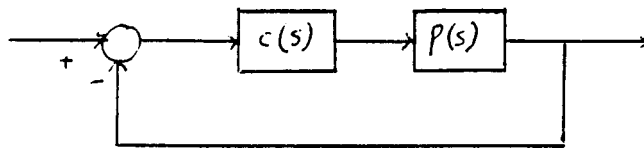


Figure 1: Unity Feedback System

Let therefore,

$$G(s) = \frac{1}{d^p(s)} \begin{bmatrix} n_1^p(s) \\ \vdots \\ n_m^p(s) \end{bmatrix},$$

be the transfer function of the system. The order of the plant is q and

$$d^p(s) = d_q^p s^q + \cdots + d_0^p,$$

while

$$n_i^p(s) = n_{i,q}^p s^q + \cdots + n_{i,0}^p.$$

The controller transfer function on the other hand is of order r and is described by

$$C(s) = \frac{1}{d^c(s)} [n_1^c(s) \quad \cdots \quad n_m^c(s)],$$

where

$$d^c(s) = d_r^c s^r + \cdots + d_0^c,$$

$$n_i^c(s) = n_{i,r}^c s^r + \cdots + n_{i,0}^c.$$

The characteristic polynomial of the closed loop system is then given by the polynomial $\delta(\cdot)$ of degree $n = q + r$

$$\delta(s) = d^c(s)d^p(s) + n_m^c(s)n_m^p(s) + \cdots + n_1^c(s)n_1^p(s). \quad (3.1)$$

In this paper the plant parameter vector is taken to be

$$\underline{P} := [n_1^p(s) \quad n_2^p(s) \quad \cdots \quad n_m^p(s) \quad d^p(s)].$$

The purpose of this section is to derive a measure of stability (stability margin) for plants with perturbed coefficients. This can be done by finding the largest stability hypersphere in parameter space. We give here a new method for computing the radius of such a sphere. For any l and n we denote by \mathcal{P}_n^l the vector space

$$\underbrace{\mathcal{P}_n \times \mathcal{P}_n \times \cdots \times \mathcal{P}_n}_{l \text{ times}}$$

where \times designates the Cartesian product and \mathcal{P}_n is the vector space of real polynomials of degree less than or equal to n . \mathcal{P}_n^l is supplied with the natural induced inner product defined as follows:

If

$$\underline{P} = [P_1(s) \quad P_2(s) \quad \cdot \quad \cdot \quad \cdot \quad P_l(s)],$$

and

$$\underline{R} = [R_1(s) \quad R_2(s) \quad \cdot \quad \cdot \quad \cdot \quad R_l(s)],$$

then

$$\langle\langle \underline{P}, \underline{R} \rangle\rangle = \sum_{i=1}^l \langle P_i(s), R_i(s) \rangle = \sum_{i=1}^l \sum_{j=0}^n p_{i,j} r_{i,j}. \quad (3.2)$$

The norm associated with this inner product corresponds to the Euclidean norm. As an example, for a plant of order q as defined above, we have

$$\underline{P} = [n_1^p(s) \quad n_2^p(s) \quad \cdot \quad \cdot \quad \cdot \quad n_m^p(s) \quad d^p(s)] \in \mathcal{P}_q^{m+1},$$

and

$$[\|\underline{P}\|]^2 = \sum_{i=1}^m \sum_{j=0}^q n_{i,j}^p{}^2 + \sum_{j=0}^q d_j^p{}^2.$$

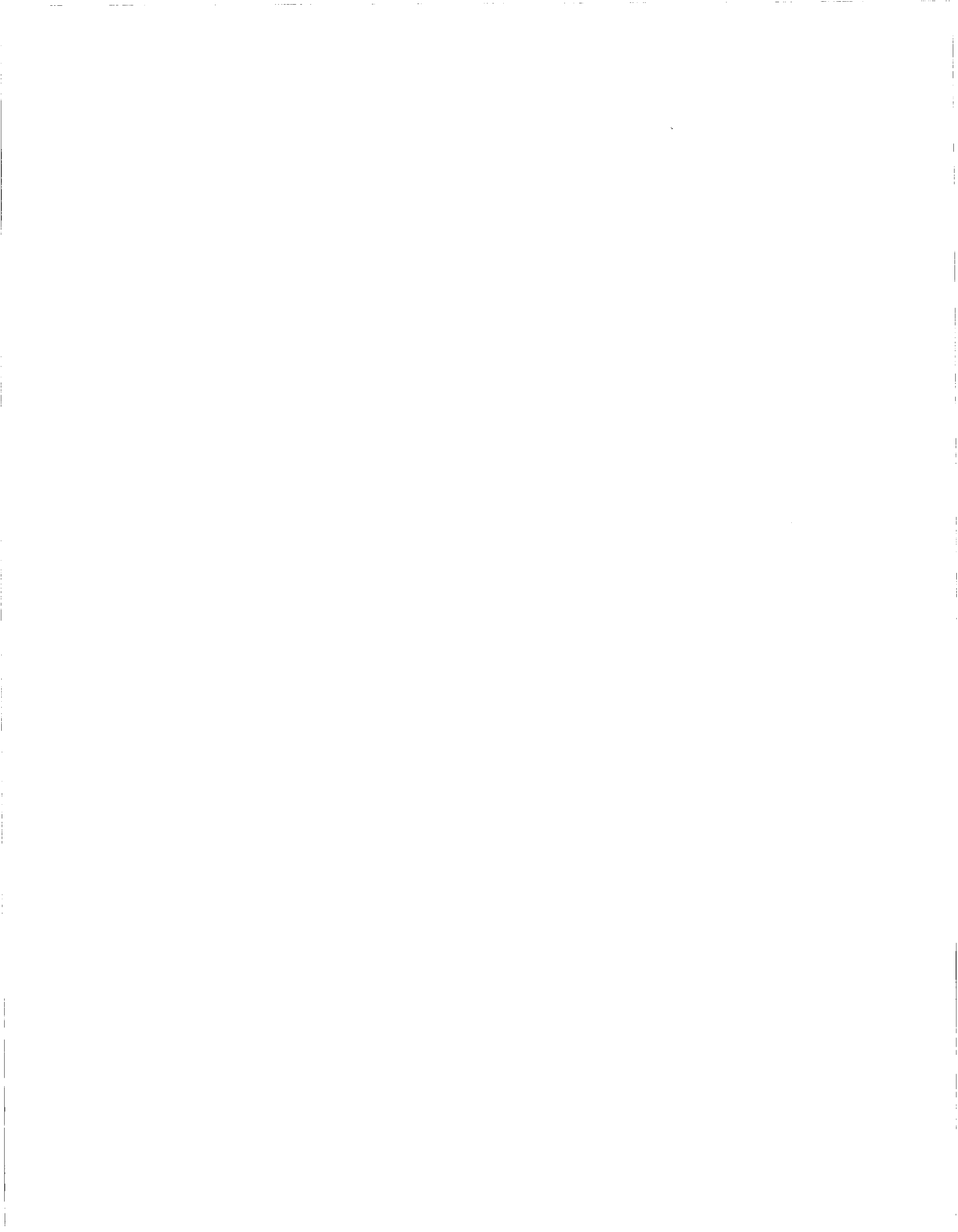
Here again, the question arises of being able to define a stability margin, but this time in parameter space. This stability margin will then tell us how much we can perturb the original plant \underline{P} and yet remain stable. The largest stability hypersphere as it was defined in [3] or [9] is characterized by the following theorem.

Theorem 3.1

Let $\underline{P} = [n_1^p(s) \quad \cdot \quad \cdot \quad n_m^p(s) \quad d^p(s)]$ be a given plant of order q , and $C(s)$ a stabilizing controller of order r described by $\underline{X} = [n_1^c(s) \quad \cdot \quad \cdot \quad n_m^c(s) \quad d^c(s)]$, and let $n = q + r$.

- a) There exists a largest stability hypersphere $\mathcal{S}(\underline{P}, \rho(\underline{P}))$ around \underline{P} , which is characterized by
 - For every plant \underline{P}' within the sphere, the closed loop characteristic polynomial $\delta_x(\underline{P}')$ is stable and of degree n .
 - At least one plant \underline{P}'' on the sphere itself is such that $\delta_x(\underline{P}'')$ is unstable or of degree less than n .
- b) Moreover if \underline{P}'' is any plant on the sphere such that $\delta_x(\underline{P}'')$ is unstable, then the unstable roots of $\delta_x(\underline{P}'')$ can only be pure imaginary or zero.

The radius $\rho(\underline{P})$ of the largest stability hypersphere around \underline{P} is now given by the following result.



Theorem 3.2

Let $\underline{P} = [n_1^p(s) \ \cdots \ n_m^p(s) \ d^p(s)]$ be a plant of order q and let $C(s)$ be a stabilizing controller of order r determined by

$$\underline{X} = [n_1^c(s) \ \cdots \ n_m^c(s) \ d^c(s)].$$

The radius of the largest stability hypersphere around \underline{P} is given by

$$\rho(\underline{P}) = \min(\inf_{\omega \geq 0} d_\omega^p, d_0^p, d_n^p),$$

where

a)

$$d_\omega^p = \frac{\lambda_1^2 [[\underline{Z}_1]]^2 + \lambda_2^2 [[\underline{Z}_2]]^2 - 2\lambda_1\lambda_2 \langle\langle \underline{Z}_1, \underline{Z}_2 \rangle\rangle}{[[\underline{Z}_1]]^2 [[\underline{Z}_2]]^2 - \langle\langle \underline{Z}_1, \underline{Z}_2 \rangle\rangle^2}$$

with

$$\begin{aligned} \lambda_1 &= \sum_{i=1}^m (n_i^{ce}(\omega)n_i^{pe}(\omega) - \omega^2 n_i^{co}(\omega)n_i^{po}(\omega)) \\ &\quad + d^{ce}(\omega)d^{pe}(\omega) - \omega^2 d^{co}(\omega)d^{po}(\omega), \\ \lambda_2 &= \sum_{i=1}^m (n_i^{ce}(\omega)n_i^{po}(\omega) + n_i^{co}(\omega)n_i^{pe}(\omega)) \\ &\quad + d^{ce}(\omega)d^{po}(\omega) + d^{co}(\omega)d^{pe}(\omega), \end{aligned}$$

and

$$\underline{Z}_1 = (n_1^{ce}(\omega)P_1(s) + n_1^{co}(\omega)P_2(s), \dots, d^{ce}(\omega)P_1(s) + d^{co}(\omega)P_2(s)),$$

$$\underline{Z}_2 = (n_1^{ce}(\omega)P_2(s) - \omega^2 n_1^{co}(\omega)P_1(s), \dots, d^{ce}(\omega)P_2(s) - \omega^2 d^{co}(\omega)P_1(s))$$

where $P_1(\cdot)$, $P_2(\cdot)$ are defined as follows (depending on the plant order q):

i) $\underline{q} = 2l$

$$P_1(s) = \begin{cases} s - \omega^2 s^2 + \cdots + (-1)^{l-1} \omega^{2l-2} s^{2l-1} & \text{if } q \neq 0 \\ 0 & \text{if } q = 0 \end{cases}$$

$$P_2(s) = 1 - \omega^2 s^2 + \cdots + (-1)^l \omega^{2l} s^{2l}.$$

ii) $\underline{q} = 2l + 1$



$$P_1(s) = s - \omega^2 s^3 + \dots + (-1)^l \omega^{2l} s^{2l+1},$$

$$P_2(s) = 1 - \omega^2 s^2 + \dots + (-1)^l \omega^{2l} s^{2l}.$$

b)

$$d_0^{p,2} = \frac{(\sum_{i=1}^m n_{i,0}^p n_{i,0}^c + d_0^p d_0^c)^2}{\sum_{i=1}^m n_{i,0}^c{}^2 + d_0^c{}^2}.$$

c)

$$d_n^{p,2} = \frac{(\sum_{i=1}^m n_{i,q}^p n_{i,r}^c + d_q^p d_r^c)^2}{\sum_{i=1}^m n_{i,r}^c{}^2 + d_r^c{}^2}.$$

The proof of this result relies on the application of the projection theorem in the Euclidean vector space \mathcal{P}_p^{m+1} and can be found in [9]. We now provide an example to show the applicability of this result.

Example :

Consider the following single input, single output plant of order $q = 3$,

$$G(s) = \frac{n^p(s)}{d^p(s)} = \frac{s}{1 - s + 4s^2 + s^3}$$

A stabilizing controller for $G(s)$ is

$$C(s) = \frac{n^c(s)}{d^c(s)} = \frac{3}{1 + s}$$

which is of order $r = 1$ and the resulting characteristic polynomial is

$$\begin{aligned} \delta(s) &= n^c(s)n^p(s) + d^c(s)d^p(s) \\ &= 1 + 3s + 3s^2 + 5s^3 + s^4. \end{aligned}$$

In this case we immediately have

$$d_0^{p,2} = \frac{1}{10} \quad \text{and} \quad d_n^{p,2} = 1.$$

And for $d_\omega^{p,2}$, we compute

$$P_1(s) = s - \omega^2 s^3 \quad \text{and} \quad P_2(s) = 1 - \omega^2 s^2,$$

$$n^{ce}(\omega) = 3, \quad n^{co}(\omega) = 0, \quad d^{ce}(\omega) = 1, \quad d^{co}(\omega) = 1,$$

$$n^{pe}(\omega) = 0, \quad n^{po}(\omega) = 1, \quad d^{pe}(\omega) = 1 - 4\omega^2, \quad d^{po}(\omega) = -1 - \omega^2$$

and then

$$\lambda_1 = \delta^e(\omega) = 1 - 3\omega^2 + \omega^4, \quad \lambda_2 = \delta^o(\omega) = 3 - 5\omega^2,$$

$$\underline{Z}_1 = (3P_1(s), P_1(s) + P_s(s))$$

$$= (3s - 3\omega^2 s^3, 1 + s - \omega^2 s^2 - \omega^2 s^3),$$

and

$$\underline{Z}_2 = (3P_2(s), P_2(s) - \omega^2 P_1(s))$$

$$= (3 - 3\omega^2 s^2, 1 - \omega^2 s - \omega^2 s^2 + \omega^4 s^3),$$

so that

$$[[\underline{Z}_1]]^2 = 11 + 11\omega^4$$

$$[[\underline{Z}_2]]^2 = 10 + 11\omega^4 + \omega^8,$$

and also

$$\langle\langle \underline{Z}_1, \underline{Z}_2 \rangle\rangle = 1 - \omega^2 + \omega^4 - \omega^6 = (1 - \omega^2)(1 + \omega^4).$$

Finally we get

$$d_\omega^p =$$

$$\frac{11(1 - 3\omega^2 + \omega^4)^2 + (10 + \omega^4)(3 - 5\omega^2)^2 - 2(1 - \omega^2)(1 - 3\omega^2)(1 - 3\omega^2 + \omega^4)(3 - 5\omega^2)}{(1 + \omega^4)(109 + 2\omega^2 + 10\omega^4)}$$

And the minimization of this function over $[0, +\infty)$ yields $d_\omega^p \simeq .012678$.

Some More Remarks: As before in the coefficient space, it is possible to carry out two similar minimizations on the finite range $[0, 1]$ in order to compute $d_{\min}^p = \inf_{\omega \leq 0} d_\omega^p$. After one minimization, one operates the following permutations.

$$n_i^p(s), d^p(s) \text{ are replaced by } s^q n_i^p\left(\frac{1}{s}\right) \text{ and } s^q d^p\left(\frac{1}{s}\right)$$

whereas

$$n_i^c(s), d^c(s) \text{ are replaced by } s^r n_i^c\left(\frac{1}{s}\right), s^r d^c\left(\frac{1}{s}\right).$$

For example in the case that we treated above, one just replaces $n^c(s) = 3$ by $n'^c(s) = 3s$, whereas

$d'^c(s) = d^c(s)$ is unchanged

and

$n^p(s) = s$ by $n'^p(s) = s^2$ and
 $d^p(s) = 1 - s + 4s^2 + s^3$ by $d'^p(s) = 1 + 4s - s^2 + s^3$.

Example 2 :

The Figure shows the experimental model which was developed by NASA Langley Research Center in order to demonstrate slewing flexible structures in a single axis while simultaneously suppressing vibration motion by the end of the maneuver. The detailed information of the model may be found in [10]. If we consider that the stiffness coefficient EI and motor viscous drag c are uncertain parameters, we can redefine parameters as follows.

$$d := (k_t k_b / R_a + c) N_g^2$$

$$e := (1.875)^2 \sqrt{EI / (\rho l)^4}$$

Now we introduce the zero-th order controller

$$C(s) = [n_1 \quad n_2 \quad n_3 \quad n_4]$$

Then the characteristic equation becomes

$$\begin{aligned} \delta(s) &= s^4 \\ &+ (0.032394788d + 1.688950471n_3 + 21.95051139n_4)s^3 \\ &+ (1.031307514e + 1.688950471n_1 + 21.95051139n_2)s^2 \\ &+ (0.032585741de + 1.69806062en_3)s \\ &+ 1.69806062en_1 \\ &= 0 \end{aligned}$$

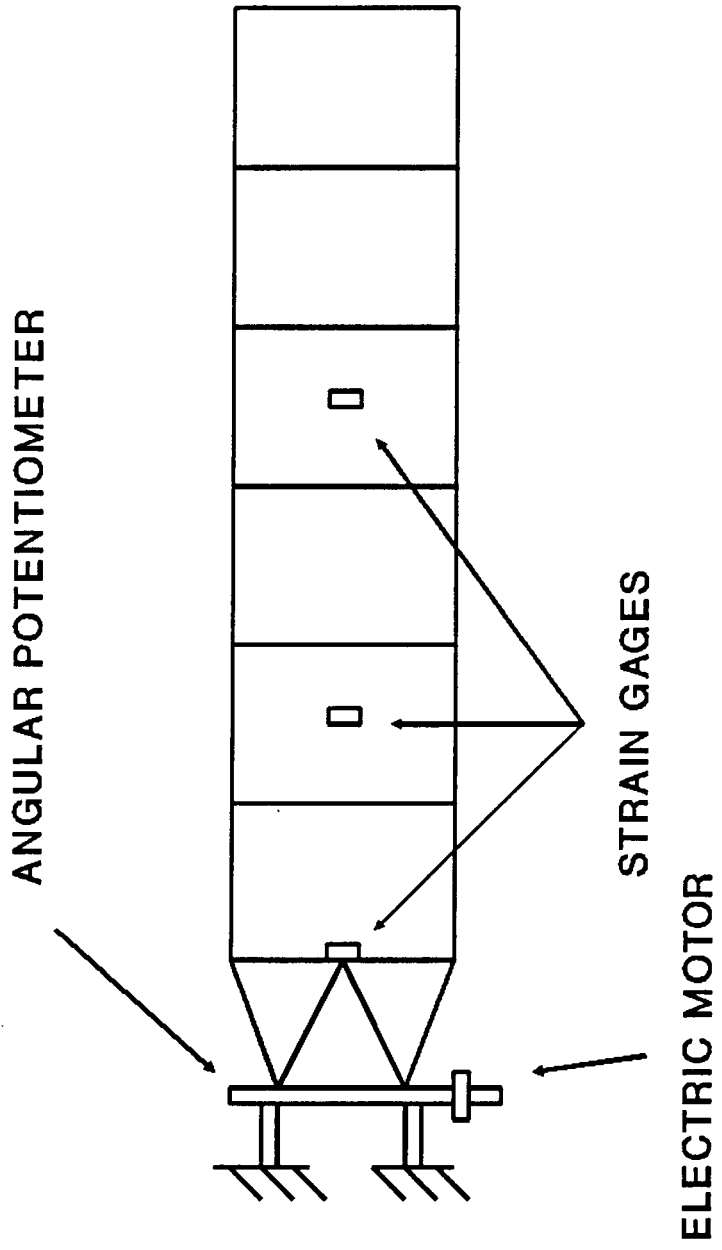


Figure. Experimental Model of Typical Setup

We can simply rewrite the equation to

$$\begin{aligned}
\delta(s) &= \underbrace{d}_{R_1(s)} \underbrace{(0.032394788s^3)}_{Q_1(s)} \\
&+ \underbrace{e}_{R_2(s)} \underbrace{(1.031307514s^2 + 1.69806062n_3s + 1.69806062n_1)}_{Q_2(s)} \\
&+ \underbrace{de}_{R_3(s)} \underbrace{(0.032585741s)}_{Q_3(s)} \\
&+ B(s) \\
&= 0
\end{aligned}$$

where

$$B(s) =$$

$$s^4 + (1.688950471n_3 + 21.95051139n_4)s^3 + (1.688950471n_1 + 21.95051139n_2)s^2$$

Thus

$$\delta(s) = R_1(s)Q_1(s) + R_2(s)Q_2(s) + R_3(s)Q_3(s) + B(s)$$

The value of d_n^p is in fact $+\infty$ since no matter what controller you choose, the characteristic polynomial remains of order 4. The same argument can be made for d_o^p . As long as the controller keeps the 0th and the highest order coefficients of the characteristic equation positive, these two coefficients will not effect the stability of the characteristic polynomial. Therefore, d_ω^p determines the stability of the polynomial. In order to determine d_ω^p , we have

$$P_1(s) = 0,$$

and,

$$P_2(s) = 1,$$

Then,

$$\begin{aligned}
\lambda_1 &= \delta^e(\omega) \\
&= 1.6898906162en_1 - (1.031307514e + 1.688950471n_1 + 21.95051139n_2)\omega^2 \\
&+ \omega^4
\end{aligned}$$

$$\begin{aligned}
\lambda_2 &= \delta^o(\omega) \\
&= (0.032585731de \\
&+ 1.69806062en_3) - (0.032394788d + 1.688950471n_3 + 21.95051139n_4)\omega^2
\end{aligned}$$

$$Z_1 = (Q_1^o(\omega), Q_2^o(\omega), Q_3^o(\omega)),$$

$$Z_2 = (Q_1^e(\omega), Q_2^e(\omega), Q_3^e(\omega)).$$

Thus,

$$Z_1 = (-0.032394788\omega^2, 1.69806062n_3, 0.032394788),$$

$$Z_2 = (0, 1.69806062n_1 - 1.031307514\omega^2, 0).$$

$$[[Z_1]]^2 = (-0.032394788\omega^2)^2 + (1.69806062n_3)^2 + (0.032394788)^2,$$

$$[[Z_2]]^2 = (1.69806062n_1 - 1.031307514\omega^2)^2.$$

$$\langle\langle Z_1, Z_2 \rangle\rangle = 1.69806062n_3(1.69806062n_1 - 1.031307514\omega^2).$$

Then, we can use d_w^p formula. The initial stabilizing controller was chosen to be

$$C(s) = [0.0369687 \quad -16.647767 \quad -29.677343 \quad 0.449643]$$

and it provides

$$d_w^p = 7.62351163$$

$$= \text{allowable perturbation of } \sqrt{(\Delta d)^2 + (\Delta e)^2 + (\Delta(de))^2}$$

The obtained robust controller is

$$C^*(s) = [369.73937 \quad 113.46374 \quad 48.22846 \quad 8.208517]$$

and it provides

$$d_w^p = 10248.294586$$

$$= \text{allowable perturbation of } \sqrt{(\Delta d)^2 + (\Delta e)^2 + (\Delta(de))^2}$$

This result is expected to be somewhat conservative because the nonlinear term de was treated as the separate independent parameter.

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APPENDIX B

A MATRIX EQUATION APPROACH TO THE DESIGN OF
LOW ORDER REGULATORS*

by

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ABSTRACT

This paper presents an algorithm for stabilizing a linear multivariable system with a controller of fixed dynamic order. This is an output feedback stabilization problem. An algorithm attempts to solve this via a sequence of approximate pole assignment problems. The approximation is driven by the optimization of a performance index consisting of a weighted sum of the condition number of the closed loop eigenvectors and the norm of the difference between the computed and actual controls.

The algorithm can be used for generating low order solutions to the regulator problem. The problem treated here is useful in design problems which involve parameter optimization and is also important in practical situations where stabilization is to be accomplished with a fixed number of available parameters.

1. INTRODUCTION

The regulator or feedback stabilization problem is the basic problem that control theory attempts to solve. Many design procedures can only be initiated after a nominal stabilizing controller has been found. However, except for very special cases, there are no direct procedures available to solve this problem when the controller order is fixed. Existing solutions to the regulator problem can only generate controllers that are of high enough order that arbitrary pole placement becomes possible. This includes the LQG theory [1], observed state feedback [2] and arbitrary pole placement approaches [3][4]. Controllers that are robust with respect to unstructured perturbations evidently suffer from the same difficulty of high order (see examples given in [5]). We also mention that adaptive control theory is notorious for producing high order solutions.

It is certainly essential in practice, to have low order solutions to the stabilization problem. This requirement arises because the controller must eventually carry out several functions such as tracking, disturbance rejection, desensitization against parameter variations, provide good transient response, small steady state error, prevent various signals from saturating etc., in addition to the basic task of stabilization. Many of these requirements are in conflict with each other in ways that cannot be handled analytically and the only recourse left to the designer is to iteratively redesign the controller using adhoc methods and graphical displays until a satisfactory solution is obtained. This redesign must be carried out in the parameter space of the stabilizing controller. If the basic stabilizing controller order is unnecessarily high this parameter space is also of high dimension and the subsequent design process can become unwieldy. From this prospective, the high order of

controllers produced by “modern” control theory is one of the severest limitations of this theory.

We attempt to alleviate this problem by presenting, in this paper a direct algorithm in the state space domain, for designing low order stabilizing controllers. This algorithm first attempts to stabilize the closed loop system with a fixed order controller. This corresponds to an extended output feedback stabilization problem for which no analytical solution is available. We attempt to solve this iteratively. At each iteration a state feedback matrix assigning a prescribed set of eigenvalues is found and this matrix is approximated by output feedback. This is done successively by readjusting the desired closed loop pole locations in the left half of the complex plane to minimize a performance index that measures the deviation of the actual eigenvalues from the desired ones. A low order solution is found by sequentially increasing the controller order until stabilization is achieved.

The algorithm that is given depends on the parameterization of the state feedback pole assignment problem derived in [6]. This is briefly described in the next section. In Section 3, the fixed order output feedback stabilization problem is formulated as an optimization problem and Section 4 describes how the performance index can be decreased by increasing the controller order. Examples are given in Section 5 and some of the gradient evaluations of Section 4 are derived in the Appendix.

2. THE SYLVESTER EQUATION FORMULATION

An algorithm was introduced in [6] for solving the pole assignment problem

using state feedback. This algorithm consists of solving for X and then for F

$$AX - X\tilde{A} = -BG \quad (2.1)$$

$$FX = G \quad (2.2)$$

for given (A, B, \tilde{A}) with an arbitrary choice of G . In (2.1) and (2.2) A , X and \tilde{A} are $n \times n$ matrices. From a result in [7] the solution X of (2.1) generically has full rank if (A, B) is controllable and (G, \tilde{A}) is observable. Let $\lambda_i(T)$ denote the i^{th} eigenvalue of T and $\lambda(T)$ the spectrum or eigenvalue set of T . It follows that if X has full rank the solution F has the property:

$$\lambda(A + BF) = \lambda(\tilde{A}) \quad (2.3)$$

The advantage of this algorithm are:

- a) The algebraic variety $F(\Lambda)$ of matrices F which assign a prescribed set of eigenvalues Λ can be obtained by setting $\Lambda = \lambda(\tilde{A})$ for a fixed \tilde{A} , and letting the free parameter G run through the set of all possible real values.
- b) Efficient numerical procedures [8] are available for the solution of Sylvester's equation (2.1).

Based on this parameterization of $F(\Lambda)$ algorithms were given [9] and [10] for optimizing the conditioning of the closed loop eigenvectors and [11] for minimizing the norm of the state feedback matrix F . Here, we extend these results by considering measurement rather than state feedback and by treating the problem of stabilization rather than arbitrary pole placement.

3. OUTPUT FEEDBACK CONTROLLERS

Consider the linear time invariant plant S cascaded with the p^{th} order feedback compensator C .

$$S : \dot{x} = Ax + Bu \quad (3.1)$$

$$y_m = Cx.$$

$$C : \dot{x}_c = A_c x_c + B_c y_m \quad (3.2)$$

$$u = C_c x_c + D_c y_m$$

The closed loop system is

$$\begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} = \begin{pmatrix} A + BD_c C & BC_c \\ B_c C & A_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} \quad (3.3)$$

or

$$\underbrace{\begin{pmatrix} \dot{x} \\ x_c \end{pmatrix}}_{\dot{x}_p} = \left\{ \underbrace{\begin{pmatrix} A & 0 \\ 0 & 0_p \end{pmatrix}}_{A_p} + \underbrace{\begin{pmatrix} B & 0 \\ 0 & I_p \end{pmatrix}}_{B_p} \underbrace{\begin{pmatrix} D_c & C_c \\ B_c & A_c \end{pmatrix}}_{K_p} \underbrace{\begin{pmatrix} C & 0 \\ 0 & I_p \end{pmatrix}}_{C_p} \right\} \underbrace{\begin{pmatrix} x \\ x_c \end{pmatrix}}_{x_p} \quad (3.4)$$

and the transfer function of the p^{th} order compensator is

$$C(s) := C_c(sI - A_c)^{-1} B_c + D_c \quad (3.5)$$

The formula (3.4) shows that any fixed order compensator design problem is equivalent to a static output feedback problem. In particular the problem of stabilization with a fixed order controller p is equivalent to that of stabilizing $A_p + B_p K_p C_p$ by choice of K_p . The general solution of this problem is unknown. The best available special results are those of Brasch and Pearson [3] and Kimura [4] which deal respectively with arbitrary eigenvalue assignment and "almost" arbitrary eigenvalue assignment.

Let Λ denote a symmetric set of $n + p$ complex numbers (i.e. complex numbers occur in complex conjugate pairs) and let

$$\underline{K}_p(\Lambda) := \{K_p | K_p \in R^{(m+p) \times (r+p)}, \sigma(A_p + B_p K_p C_p) \in \Lambda\} \quad (3.6)$$

where $A_p \in R^{(n+p) \times (n+p)}$, $B_p \in R^{(n+p) \times (m+p)}$, and $C_p \in R^{(r+p) \times (n+p)}$ are as in (3.4).

The result of Brasch and Pearson [3] states that if (A, B, C) is controllable and observable with controllability index ν_c and observability index ν_o , and $p \geq \min\{\nu_c, \nu_o\}$ then $K_p(\Lambda) \neq \emptyset$ for every choice of Λ . The result of Kimura [4] states that if $p \geq n - m - r + 1$ then $\sigma(A_p + B_p K_p C_p)$ can be made arbitrarily close to any set Λ of $n + p$ symmetric complex numbers.

The lower bound on the order of a stabilizing controller established by the above results is in general too conservative. This stems from the fact that both results essentially require arbitrary pole placement. In fact for specific choices of Λ , $K_p(\Lambda)$ will "almost always" be empty unless p the compensator order is high. To lower the compensator order we therefore relax the specification of Λ in (3.6) to a simply connected region $\Omega \subset C^-$ and consider the family

$$\underline{K}_p(\Omega) = \{K_p | K_p \in R^{(m+p) \times (r+p)}, \lambda(A_p + B_p K_p C_p) \subset \Omega \subset C^-\}. \quad (3.7)$$

It is reasonable to expect that $\underline{K}_p(\Omega)$ will in general be nonempty for values of p much less than the lower bounds given by the results of Brasch and Pearson or Kimura and numerical examples support this.

The effective characterization of the family $\underline{K}_p(\Omega)$ is an unsolved open problem. Our approach to this problem will be to consider the state feedback family

$$\underline{F}_p(\Omega) = \{F_p | F_p \in R^{(m+p) \times (n+p)}, \lambda(A_p + B_p F_p) \subset \Omega \subset C^-\} \quad (3.8)$$

and determine an $F_p \in \underline{F}_p(\Omega)$ and then find K_p such that $\|F_p - K_p C_p\|$ is small in the hope that such a $K_p \in \underline{K}_p(\Omega)$. The advantage of this approach is that the

family $\underline{E}_p(\Omega)$ can be characterized conveniently as shown later. For the remainder of this section we drop the subscript p for convenience.

In general, even if $\|F - KC\|$ is small it is not in general true that $\lambda(A + BF)$ and $\lambda(A + BKC)$ are close. The latter can be achieved by making the eigenstructure of $A + BF$ as orthonormal as possible. Let $\sigma_{max}(T)$ and $\sigma_{min}(T)$ denote the largest and smallest singular values of T . It is well known [8][12] that the perturbation of the eigenvalues of the diagonalisable matrix $(A + BF)$ for changes in the entries is small if the condition number $k(X) := \|X\|_2 \|X^{-1}\|_2$ of the eigenvector matrix X is small. Let $F - KC := T$ so that $A + BKC = A + BF - BT$. Then using the formula in [12] we have

$$\begin{aligned} |\lambda_i(A + BKC) - \lambda_j(A + BF)| &\leq \|BT\|_2 k(X) \\ &\leq \|B\|_2 \|T\|_2 k(X) \\ &\leq \|B\|_2 \|F - KC\|_F k(X) \end{aligned} \quad (3.9)$$

which shows that control over the eigenvalue locations of $A + BKC$ can be obtained only if both $\|F - KC\|$ and $k(X)$ are kept small. One way of doing this is to minimize

$$\begin{aligned} J &= \alpha_1 k(X) + \alpha_2 \|F - KC\|_F^2 \\ &= \alpha_1 \frac{\sigma_{max}(X)}{\sigma_{min}(X)} + \alpha_2 \text{Trace}\{(F - KC)^T (F - KC)\} \end{aligned} \quad (3.10)$$

by letting $\lambda(A + BF)$ range over the region $\Omega \subset C^-$. Similarly, by letting $A + F_D C = A + BK_D C$ a dual problem can be formulated as

$$J_D = \beta_1 \frac{\sigma_{max}(X_D)}{\sigma_{min}(X_D)} + \beta_2 \text{Trace}\{(F_D - BK_D)^T (F_D - BK_D)\} \quad (3.11)$$

The idea of improving the conditioning of the eigenstructure and of minimizing the norm of $F - KC$ was first introduced in Keel and Bhattacharyya [13][14]. Here

an improved version of this algorithm is presented. In particular we convert the constrained optimization problem to an unconstrained problem and extend the class of regions $\Omega \subset C^-$ to more general and useful regions. These details are given next.

4. STABILIZATION ALGORITHM

In the Sylvester equation approach described in Section 2,

$$AX - X\tilde{A} = -BG \tag{4.1}$$

$$FX = G \tag{4.2}$$

and let $\lambda(\tilde{A}) \subset \Omega \subset C^-$. Under the assumption $\lambda(A) \cap \lambda(\tilde{A}) = \emptyset$ and (A, B) controllable, (G, \tilde{A}) observable, the unique solution X will 'almost surely' be non-singular by deSouza and Bhattacharyya [7] and then $\lambda(A + BF) = \lambda(\tilde{A})$ with $F = GX^{-1}$. By letting $\lambda(\tilde{A})$ range over Ω this algorithm generates the family of $\underline{F}(\Omega)$, by letting G be a free parameter run through all possible values this formula generates the family $\underline{F}(\Omega)$ defined in (3.8).

If \tilde{A} is a complex diagonal matrix in (4.1), it is clear that X in (4.1) is the corresponding complex eigenvector matrix. However we want to treat these matrices as real for computational convenience. The following Lemma 4.5 shows that \tilde{A} can be taken as a real matrix without loss of generality. Before we state Lemma 4.3 it is necessary to introduce some facts.

Definition 4.1

A real square matrix A is called a pseudo diagonal matrix if it is of the form

$$A = \begin{pmatrix} \alpha_1 & \beta_1 & & & & \\ -\beta_1 & \alpha_1 & & & & \\ & & \alpha_2 & \beta_2 & & \\ & & -\beta_2 & \alpha_2 & & \\ & & & & \alpha_3 & \\ & & & & & \ddots \end{pmatrix} \quad (4.3)$$

with α_i, β_i real.

Definition 4.2

A complex square matrix is called normal if $A^*A = AA^*$.

Lemma 4.3 [15]

A complex square matrix is unitary similar to a diagonal complex matrix if and only if it is normal.

Lemma 4.4

Any real pseudo diagonal matrix is normal.

Proof

Taking the i^{th} block from (4.3) such as

$$A_i = \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix} \quad (4.4)$$

we have

$$A_i A_i^* = \begin{pmatrix} \alpha_i^2 + \beta_i^2 & 0 \\ 0 & \alpha_i^2 + \beta_i^2 \end{pmatrix} = A_i^* A_i. \quad (4.5)$$

Thus, each block is normal. Now let

$$A = \text{Diag}(A_1 \quad A_2 \quad \cdots \quad \cdots \quad A_n) \quad (4.6)$$

$$AA^* = \text{Diag}(A_1A_1^* \quad A_2A_2^* \quad \cdots \quad A_nA_n^*) \quad (4.7)$$

$$A^*A = \text{Diag}(A_1^*A_1 \quad A_2^*A_2 \quad \cdots \quad A_n^*A_n) \quad (4.8)$$

Since $AA^* = A^*A$, the statement is true.

Lemma 4.5

Let $(A + BF)X = X\tilde{A}$ and $(A + BF)Y = Y\hat{A}$ where

1. A, B, \tilde{A}, X and F are real matrices with appropriate dimensions.
2. \tilde{A} is real pseudo diagonal, \hat{A} is complex diagonal, and
3. X and Y are nonsingular. Then,

$$k(X) = k(Y) \quad (4.9)$$

Proof

From Lemma 4.1 and 4.2, \hat{A} is known to be normal and unitary similar to the complex diagonal matrix \tilde{A} . Thus

$$\tilde{A} = U\hat{A}U^*. \quad (4.10)$$

Write

$$(A + BF)X = X\tilde{A} = XU\hat{A}U^* \quad (4.11)$$

so that

$$(A + BF)XU = XU\hat{A} \quad (4.12)$$

and

$$XU = Y. \quad (4.13)$$

Now,

$$YY^* = XU U^* X^* = XX^* = XX^T. \quad \diamond \quad (4.14)$$

From this Lemma, minimizing $\sigma_{max}(X)/\sigma_{min}(X)$ in (3.10) is equivalent to minimizing $\sigma_{max}(Y)/\sigma_{min}(Y)$. Therefore we can henceforth take \tilde{A} as a real pseudo diagonal matrix without loss of generality. In fact the condition numbers of X and Y are equal, i.e. $k(X) = k(XU) = k(Y)$. In order to use a gradient based algorithm the closed form expression of the gradient of the performance index (3.10) with respect to the variables G, K and the variable elements of \tilde{A} denoted \tilde{a}_i is evaluated. The details of this derivation are given in Appendix A.

Theorem 4.6

Given the performance index J in (3.10), and constraints (4.1) and (4.2), the gradients of J with respect to the independent variables G, K , and \tilde{A} are as follows:

(a)

$$\frac{\partial J}{\partial G} = 2\{\alpha_2(F - KC)X^{-T} + B^T U^T\} \quad (4.15)$$

where U satisfies

$$\begin{aligned} \tilde{A}U - UA &= \frac{\alpha_1}{\sigma_{min}^2(X)} \{\sigma_{min}(X)v_a u_a^T - \sigma_{max}(X)v_i u_i^T\} \\ &\quad - 2\alpha_2 X^{-1}(F^T - (KC)^T)F \end{aligned} \quad (4.16)$$

where v_a and u_a are right and left singular vectors corresponding to $\sigma_{max}(X)$ and v_i and u_i are for $\sigma_{min}(X)$, respectively.

(b) Let \tilde{a}_i denote a variable element of \tilde{A} :

$$\frac{\partial J}{\partial \tilde{a}_i} = -\text{Trace}\{UX \frac{\partial \tilde{A}}{\partial \tilde{a}_i}\} \quad (4.17)$$

where U satisfies (4.16)

(c)

$$\frac{\partial J}{\partial K} = -2\alpha_2(F - KC)C^T \quad (4.18)$$

Equations (4.15) – (4.18) are used to devise a gradient algorithm that iterates on the free parameters G, K and the entries of \tilde{A} to reduce J . At each iteration of the algorithm we get \tilde{A}_i, F_i and K_i . Since $\sigma(\tilde{A}_i) \subset \Omega$ we have $\sigma(A + BF_i) \subset \Omega$ for each i . However $\sigma(A + BK_iC)$ may or may not be in Ω for each i , and the algorithm is designed to make $\sigma(A + BK_iC)$ close to $\sigma(\tilde{A}_i) = \sigma(A + BF_i)$ after some iterations.

The following structure of the closed loop eigenvalue matrix \tilde{A} ensures stability without constraints during the iterations.

$$\tilde{A} = \begin{pmatrix} -\tilde{a}_1^2 & \tilde{a}_2 & & & & \\ -\tilde{a}_2 & -\tilde{a}_1^2 & & & & \\ & & -\tilde{a}_3^2 & \tilde{a}_4 & & \\ & & -\tilde{a}_4 & -\tilde{a}_3^2 & & \\ & & & & \ddots & \\ & & & & & \ddots \end{pmatrix}$$

Note that \tilde{a}_i in the matrix \tilde{A} are the only nonzero parameters and furthermore the stability requirement, $\sigma(\tilde{A}) \subset C^-$, can be automatically satisfied without constraints, for all real values of \tilde{a}_i .

We can also parameterize \tilde{A} in such a way that the desired closed loop eigenvalue locations are automatically confined to some useful region Ω as in Figures 4.1 and 4.2.

In choosing \tilde{A} , a maximal number of 2×2 blocks are included in the initial choice. As the algorithm evolves some of the off diagonal terms may become very small. At that point we start to vary the corresponding diagonal terms independently. In the damping ration region described in Figure 4.2. θ is also a free parameter.

Marginal Stability Region

For this case we can simply modify the matrix \tilde{A} to

$$\tilde{A} = \begin{pmatrix} -(\tilde{a}_1^2 + \gamma) & \tilde{a}_2 & & & & \\ -\tilde{a}_2 & -(\tilde{a}_1^2 + \gamma) & & & & \\ & & -(\tilde{a}_3^2 + \gamma) & \tilde{a}_4 & & \\ & & -\tilde{a}_4 & -(\tilde{a}_3^2 + \gamma) & & \\ & & & & \ddots & \\ & & & & & \ddots & \\ & & & & & & \ddots & \end{pmatrix}$$

with \tilde{a}_i as the real variable parameters and γ is fixed. The eigenvalues of \tilde{A} are all to the left of the line $Re(s) = -\gamma$.

Damping Ratio Region

$$\tilde{A} = \begin{pmatrix} -(\tilde{a}_1^2 + \gamma) & (\tilde{a}_1^2 + \gamma)\tan\phi\sin\theta_1 & & & & \\ -(\tilde{a}_1^2 + \gamma)\tan\phi\sin\theta_1 & -(\tilde{a}_1^2 + \gamma) & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & \ddots & \end{pmatrix}$$

Now we discuss what happens when the proposed algorithm fails to find a stabilizing controller of order i . In this case, we increase the controller order to $i + 1$. It is then necessary to have a way to select the initial values of G_0 , \tilde{A}_0 and K_0 for the controller of order $i + 1$ to ensure that the performance index J keeps decreasing. The following theorem shows the way to select initial variables so that J always decreases with increasing controller order.

Theorem 4.7

Let J^* be the optimal performance index with optimal variables G^* , \tilde{A}^* and

K^* where

$$J^* = \frac{\sigma_{\max}(X^*)}{\sigma_{\min}(X^*)} + \|F^* - K^*C\|_F^2 \quad (4.19)$$

and X^* and F^* satisfy

$$AX^* - X^*\tilde{A}^* = -BG^*$$

$$F^* = G^*(X^*)^{-1}$$

Then for the extended system

$$A_e = \begin{pmatrix} A & 0 \\ 0 & I_i \end{pmatrix} \quad B_e = \begin{pmatrix} B & 0 \\ 0 & I_i \end{pmatrix} \quad C_e = \begin{pmatrix} C & 0 \\ 0 & I_i \end{pmatrix} \quad (4.20)$$

the value of its performance index J_e is equal to J^* if the set of initial variables are

$$G_e = \begin{pmatrix} G & 0 \\ 0 & X_3\tilde{A}_i \end{pmatrix} \quad K_e = \begin{pmatrix} K & 0 \\ 0 & X_3\tilde{A}_iX_3^{-1} \end{pmatrix} \quad (4.21)$$

where \tilde{A}_i is an arbitrary pseudo diagonal matrix of an extended matrix

$$\tilde{A}_e = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{A}_i \end{pmatrix}$$

and

$$X_3 = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \ddots & \\ & & & & \sigma_i \end{pmatrix}$$

for $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_i > 0$ with $\sigma_i \geq \sigma_{\min}(X^*)$ and $\sigma_1 \leq \sigma_{\max}(X^*)$.

Proof

Let the optimal values of J^* be obtained by G^* and K^* , then the extended system becomes

$$\underbrace{\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}}_{A_e} \underbrace{\begin{pmatrix} X^* & X_1 \\ X_2 & X_3 \end{pmatrix}}_{X_e} - \underbrace{\begin{pmatrix} X^* & X_1 \\ X_2 & X_3 \end{pmatrix}}_{X_e} \underbrace{\begin{pmatrix} \tilde{A}^* & 0 \\ 0 & \tilde{A}_i \end{pmatrix}}_{\tilde{A}_e}$$

$$\begin{aligned}
&= - \underbrace{\begin{pmatrix} B & 0 \\ 0 & I_i \end{pmatrix}}_{B_e} \underbrace{\begin{pmatrix} G^* & G_1 \\ G_2 & G_3 \end{pmatrix}}_{G_e} \\
&= \begin{pmatrix} AX^* - X^* \tilde{A}^* & AX_1 - X_1 \tilde{A}_i \\ -X_2 \tilde{A} & -X_3 \tilde{A}_i \end{pmatrix} = - \begin{pmatrix} BG^* & BG_1 \\ G_2 & G_3 \end{pmatrix} \quad (4.22)
\end{aligned}$$

if we pick $G_1 = 0$ and $G_2 = 0$, then $X_1 = 0$ and $X_2 = 0$ and $X_3 \tilde{A}_i = G_3$. Here we choose

$$X_3 = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_i \end{pmatrix} \quad (4.23)$$

for $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_i > 0$ with $\sigma_i \geq \sigma_{\min}(X^*)$ and $\sigma_1 \leq \sigma_{\max}(X^*)$. Such a X_3 is guaranteed by the choice of $G_3 = X_3 \tilde{A}_i$ and

$$\sigma_{\min}(X^*) = \sigma_{\min}(X_e) \quad (4.24)$$

$$\sigma_{\max}(X^*) = \sigma_{\max}(X_e) \quad (4.25)$$

Therefore,

$$\frac{\sigma_{\max}(X^*)}{\sigma_{\min}(X^*)} = \frac{\sigma_{\max}(X_e)}{\sigma_{\min}(X_e)} \quad (4.26)$$

Now consider the term $\|F - KC\|_F^2$. Since

$$X_e = \begin{pmatrix} X & 0 \\ 0 & X_3 \end{pmatrix},$$

we have

$$X_e^{-1} = \begin{pmatrix} X^{-1} & 0 \\ 0 & X_3^{-1} \end{pmatrix} \quad (4.27)$$

where

$$X_3^{-1} = \begin{pmatrix} \frac{1}{\sigma_1} & & & \\ & \ddots & & \\ & & & \frac{1}{\sigma_i} \end{pmatrix}.$$

Now

$$\begin{aligned} F_e = G_e X_e^{-1} &= \begin{pmatrix} G & 0 \\ 0 & X_3 \tilde{A}_i \end{pmatrix} \begin{pmatrix} X^{-1} & 0 \\ 0 & X_3^{-1} \end{pmatrix} \\ &= \begin{pmatrix} GX^{-1} & 0 \\ 0 & X_3 \tilde{A}_i X_3^{-1} \end{pmatrix} \end{aligned} \quad (4.28)$$

and let

$$K_e = \begin{pmatrix} K & K_1 \\ K_2 & K_3 \end{pmatrix} \quad (4.29)$$

then

$$F_e - K_e C_e = \begin{pmatrix} GX^{-1} - KC & -K_1 \\ -K_2 C & X_3 \tilde{A}_i X_3^{-1} - K_3 \end{pmatrix} \quad (4.30)$$

Here we choose $K_1 = 0$ and $K_2 = 0$. Also we can choose

$$K_3 = X_3 \tilde{A}_i X_3^{-1} \quad (4.31)$$

because X_3 and \tilde{A}_i are well defined. With such a K we have

$$F_e - K_e C_e = \begin{pmatrix} GX^{-1} - KC & 0 \\ 0 & 0 \end{pmatrix} \quad (4.32)$$

Thus,

$$\|F - KC\|_F^2 = \|F_e - K_e C_e\|_F^2 \quad (4.33)$$

Therefore, we conclude

$$\frac{\sigma_{\max}(X^*)}{\sigma_{\min}(X^*)} + \|F^* - K^* C\|_F^2 = \frac{\sigma_{\max}(X_e)}{\sigma_{\min}(X_e)} + \|F_e - K_e C_e\|_F^2 \quad (4.34)$$

with choices of

$$G_e = \begin{pmatrix} G^* & 0 \\ 0 & X_3 \tilde{A}_i \end{pmatrix} \quad \text{and} \quad K_e = \begin{pmatrix} K & 0 \\ 0 & X_3 \tilde{A}_i X_3^{-1} \end{pmatrix} \quad (4.35)$$

with X_3 as in (4.23). This concludes the proof. \diamond

This theorem is useful for finding a low order stabilizing controller because it shows how by sequentially increasing the order of the controller, J can be guaranteed

to decrease. Since a small enough value of each term of J confines the spectrum of $A + BKC$ to Ω (in accordance with (3.9)) the algorithm eventually stabilizes the system by sequentially increasing the order of controllers.

5. EXAMPLES

The algorithm developed in the last section is applied to several examples here. The gradient calculations of Theorem 4.7 are used along with the Harwell subroutine package.

Example 1

The first example is a simplified model of the NASA F-8 Digital Fly-By-Wire(DFBW) airplane[16] and its dynamic equation of lateral directional is as follows.

$$\frac{d}{dt} \begin{pmatrix} p \\ r \\ \beta \\ \phi \end{pmatrix} = \begin{pmatrix} -2.6 & 0.25 & -38. & 0 \\ -0.075 & -0.27 & 4.4 & 0 \\ 0.078 & -0.99 & -0.23 & 0.052 \\ 1.0 & 0.078 & 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ r \\ \beta \\ \phi \end{pmatrix} + \begin{pmatrix} 17. & 7. \\ 0.82 & -3.2 \\ 0 & 0.046 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_a \\ \delta_r \end{pmatrix}$$

$$\begin{pmatrix} r \\ \phi \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ r \\ \beta \\ \phi \end{pmatrix}$$

The given design specifications [16] are that the closed loop poles must be the left of the line $s = -0.2$ i.e. $\gamma = 0.2$, and the damping factor is ≥ 0.7 , i.e. $\phi = \frac{\pi}{4}$ in Figure 4.2. total equilibrium velocity $V_0 = 620 ft/s (Mach = 0.6)$ and equilibrium angle For the optimization problem initial values are chosen to be

$$\tilde{A}_0 = \begin{pmatrix} -3 & 2 & & \\ -2 & -3 & & \\ & & -5 & 3 \\ & & -3 & -5 \end{pmatrix}$$

$$G_0 = \begin{pmatrix} 1 & 1.5 & 0.5 & -2 \\ 5 & 1 & -0.25 & 0.5 \end{pmatrix}$$

$$K_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

After 41 gradient iterations minimizing J in (3.10) the following 0th order stabilizing compensator is obtained.

$$K^* = \begin{pmatrix} 4.60357 & -1.75629 \\ 5.21515 & -1.85922 \end{pmatrix}$$

Note that the order of pole placement compensators (both Brasch - Pearson and Kimura) is 1. The corresponding data is shown in Tables 1.1, 1.2 and Figure 5.1. For comparison, the same problem was run without including the condition number term in J (i.e. $\alpha_1 = 0$ in (3.10)). It is seen from the corresponding data, shown in Table 1.3, 1.4 and Figure 5.2 that the condition number increases significantly, and although stabilization is achieved the closed loop eigenvalues fail to be in Ω .

TABLE 1.1
Eigenvalues
($\alpha_1 = 1, \alpha_2 = 1, \phi = \frac{\pi}{4}, \zeta \geq 0.7, \gamma = -0.1$)

| A_0 | $A_0 + B_0 K_0^0 C$ | $A_0 + B_0 F_0^*$ | $A_0 + B_0 K_0^* C_0$ |
|-------------------|---------------------|-------------------|-----------------------|
| $-2.39 \pm j0.00$ | $-2.39 \pm j0.00$ | $-9.45 \pm j3.70$ | $-7.58 \pm j4.96$ |
| $+0.00 \pm j0.00$ | $+0.00 \pm j0.00$ | $-0.34 \pm j0.29$ | $-0.42 \pm j0.33$ |
| $-0.34 \pm j2.62$ | $-0.34 \pm j2.62$ | | |

TABLE 1.2
Objective Function.

| | J | $\ F_0 - K_0 C_0\ _F^2$ | $k(X_0)$ |
|---------|----------|-------------------------|----------|
| Initial | 155.9021 | 61.3301 | 94.572 |
| Optimal | 47.03439 | 0.06839 | 46.966 |

TABLE 1.3
Eigenvalues
($\alpha_1 = 0, \alpha_2 = 1, \phi = \frac{\pi}{4}, \zeta \geq 0.7, \gamma = 0.1$)

| A_0 | $A_0 + B_0 K_0^0 C_0$ | $A_0 + B_0 F_0^*$ | $A_0 + B_0 K_0^* C_0$ |
|-------------------|-----------------------|-------------------|-----------------------|
| $-2.39 \pm j0.00$ | $-2.39 \pm j0.00$ | $-2.39 \pm j0.01$ | $-1.44 \pm j2.54$ |
| $+0.00 \pm j0.00$ | $+0.00 \pm j0.00$ | $-2.42 \pm j0.32$ | $-3.44 \pm j0.00$ |
| $-0.34 \pm j2.62$ | $-0.34 \pm j2.62$ | | $-1.15 \pm j0.00$ |

TABLE 1.4
Objective Function

| | J | $\ F_0 - K_0 C_0\ _F^2$ | $k(X_0)$ |
|---------|-----------|-------------------------|----------|
| Initial | 155.9021 | 61.3301 | 94.572 |
| Optimal | 233089.02 | 0.01530 | 233089 |

Example 2

Consider the symmetric vibration model of the standard Draper/RPL satellite shown in Figure 5.3. The dynamic equations, taken from [17] are:

$$\frac{d}{dt} \begin{pmatrix} \theta \\ q_1 \\ q_2 \\ \dot{\theta} \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = A \begin{pmatrix} \theta \\ q_1 \\ q_2 \\ \dot{\theta} \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C \begin{pmatrix} \theta \\ q_1 \\ q_2 \\ \dot{\theta} \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 14.8732 & 32.8086 & 0 & 0 & 0 \\ 0 & -146.702 & -7476.64 & 0 & 0 & 0 \\ 0 & -41.8468 & -2699.36 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -0.04168 & 0.23623 \\ 10.38611 & -25.647 \\ 3.725120 & -9.1629 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

From the design specifications in [17], it follows that the closed loop system must have poles to the left of $s = -0.5$. For the minimization of J the initial values are chosen to be

$$\tilde{A}_0 = \begin{pmatrix} -0.2 & 2 & & & & \\ -2 & -0.2 & & & & \\ & & -1 & 10 & & \\ & & -10 & -1 & & \\ & & & & -0.5 & 1 \\ & & & & -1 & -0.5 \end{pmatrix}$$

$$G_0 = \begin{pmatrix} 1.125 & 1.5 & -0.5 & 3.5 & 1.5 & 2 \\ -1 & 2.5 & 1.6 & 4 & 0.5 & -1 \end{pmatrix}$$

$$K_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

After 67 iterations, the following 0th order stabilizing controller is obtained:

$$K^* = \begin{pmatrix} -90.97491 & 20.62868 \\ -197.646 & 5.326668 \end{pmatrix}$$

Note that the order of pole placement compensators (both Brasch - Pearson[3] and Kimua[4]) is 3. Tables 2.1, 2.2 and Figure 5.4 display the performance indices and

the corresponding eigenvalue locations. For the purpose of comparison, the problem was also run with the condition number term left out of the performance index (i.e. $\alpha_1 = 0$). In this case the algorithm fails to stabilize the system as shown in Table 2.3, 2.4 and Figure 5.5. This example illustrates that both terms of the performance index need to be considered in the stabilization procedure.

TABLE 2.1
Eigenvalues for Example 2
($\alpha_1 = 1, \alpha_2 = 1, \gamma = 0.5$)

| A_0 | $A_0 + B_0 K_0^0 C$ | $A_0 + B_0 F_0^*$ | $A_0 + B_0 K_0^* C_0$ |
|-------------------|---------------------|-------------------|-----------------------|
| $+0.00 \pm j53.1$ | $+0.00 \pm j53.1$ | $-2.89 \pm j36.7$ | $-3.58 \pm j30.7$ |
| $+0.00 \pm j5.43$ | $+0.00 \pm j5.43$ | $-2.18 \pm j0.30$ | $-2.41 \pm j0.67$ |
| $+0.00 \pm j0.00$ | $+0.00 \pm j0.00$ | $-0.88 \pm j5.82$ | $-1.45 \pm j6.08$ |
| $+0.00 \pm j0.00$ | $+0.00 \pm j0.00$ | | |

TABLE 2.2
Performance Indices

| | J | $\ F_0 - K_0 C_0\ _F^2$ | $k(X_0)$ |
|---------|----------|-------------------------|----------|
| Initial | 11965915 | 11965506 | 409.2925 |
| Optimal | 587.2232 | 45.63520 | 541.5880 |

TABLE 2.3
Eigenvalues for Example 2
($\alpha_0 = 0, \alpha_2 = 1, \gamma = 0.5$)

| A_0 | $A_0 + B_0 K_0^0 C_0$ | $A_0 + B_0 F_0^*$ | $A_0 + B_0 K_0^* C_0$ |
|-------------------|-----------------------|-------------------|-----------------------|
| $+0.00 \pm j53.1$ | $+0.00 \pm j53.1$ | $-0.63 \pm j0.05$ | $+178. \pm j0.00$ |
| $+0.00 \pm j5.43$ | $+0.00 \pm j5.43$ | $-0.66 \pm j3.41$ | $-2.57 \pm j6.19$ |
| $+0.00 \pm j0.00$ | $+0.00 \pm j0.00$ | $-0.59 \pm j7.86$ | $+1.61 \pm j0.00$ |
| $+0.00 \pm j0.00$ | $+0.00 \pm j0.00$ | | $+0.83 \pm j1.07$ |

TABLE 2.4
Performance Indices

| | J | $\ F_0 - K_0 C_0\ _F^2$ | $k(X_0)$ |
|---------|-----------|-------------------------|----------|
| Initial | 11965915 | 11965506 | 409.2925 |
| Optimal | 383859.66 | 490.5633 | 383369.1 |

6. CONCLUDING REMARKS

The results given here are algorithmic in nature and can be improved upon by developing constructive necessary and sufficient conditions for stabilizability with a fixed order controller. This in turn will require effective ways of characterizing the Hurwitz region. These problems are difficult and have received very little attention in the literature. Finally we mention that the algorithm guarantees neither a "global" minimum nor does it always find a stabilizing controller of a prescribed order whenever one exists. The existence of stabilizing controllers of a fixed order is still our unsolved problem.

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APPENDIX

Proof of Theorem 3.4

(a)

$$J = \alpha_1 \frac{\sigma_{max}(X)}{\sigma_{min}(X)} + \alpha_2 \text{Trace}\{(F - KC)^T(F - KC)\} \quad (\text{A.1})$$

Let

$$\begin{aligned} J_1 &:= \frac{\sigma_{max}(X)}{\sigma_{min}(X)} \\ &= \text{Trace}\left\{\frac{\sigma_{max}(X)}{\sigma_{min}(X)}\right\} \end{aligned} \quad (\text{A.2})$$

and

$$\Delta J_1 = \text{Trace}\left\{\frac{1}{\sigma_{min}^2(X)}(\sigma_{min}(X)\Delta\sigma_{max}(X)) - \sigma_{max}\Delta\sigma_{min}(X)\right\} \quad (\text{A.3})$$

Note that

$$\Delta\sigma_{max}(X) = u_a^T \Delta X v_a \quad (\text{A.4})$$

$$\Delta\sigma_{min}(X) = u_i^T \Delta X v_i \quad (\text{A.5})$$

where v_i and u_i are left and right singular vectors corresponding to σ_{min} and v_a and u_a are for σ_{max} . Thus,

$$\Delta J_1 = \frac{1}{\sigma_{min}^2(X)} \text{Trace}\{\sigma_{min}(X)v_i u_i^T - \sigma_{max}(X)v_a u_a^T\} \Delta X \quad (\text{A.6})$$

Now

$$\begin{aligned} J_2 &:= \text{Trace}\{(F - KC)^T(F - KC)\} \\ &= \text{Trace}\{F^T F - (KC)^T F - F^T(KC) + (KC)^T(KC)\} \\ &= \text{Trace}(F^T F) - 2\text{Trace}\{(KC)^T F\} + \text{Trace}\{(KC)^T(KC)\} \end{aligned} \quad (\text{A.7})$$

and

$$\begin{aligned}\Delta J_2 &= 2\text{Trace}(F^T \Delta F) - 2\text{Trace}\{(KC)^T \Delta F\} \\ &= 2\text{Trace}\{[F^T - (KC)^T] \Delta F\}\end{aligned}\tag{A.8}$$

Now we have

$$\begin{aligned}\Delta J &= \frac{\alpha_1}{\sigma_{\min}^2(X)} \\ &\quad \text{Trace}\{\sigma_{\min}(X)v_i u_i^T - \sigma_{\max}(X)v_a u_a^T\} \Delta X \\ &\quad + 2\alpha_2 \text{Trace}\{(F^T - (KC)^T) \Delta F\}.\end{aligned}\tag{A.9}$$

From $F = GX^{-1}$, the gradient of F with respect to G is given directly as

$$\begin{aligned}\Delta F &= \Delta GX^{-1} + G\Delta(X^{-1}) \\ &= \Delta GX^{-1} - GX^{-1}\Delta XX^{-1} \\ &= \Delta GX^{-1} - F\Delta XX^{-1} \\ &= (\Delta G - F\Delta X)X^{-1}.\end{aligned}\tag{A.10}$$

Substituting (A.10) into (A.9) we have

$$\begin{aligned}\Delta J &= 2\alpha_2 \text{Trace}\{(F^T - (KC)^T) \Delta GX^{-1}\} \\ &\quad + \text{Trace}\left\{\frac{\alpha_1}{\sigma_{\min}^2(X)}(\sigma_{\min}(X)v_i u_i^T - \sigma_{\max}(X)v_a u_a^T)\right. \\ &\quad \left. - 2\alpha_2 X^{-1}(F^T - (KC)^T)F\right\} \Delta X\end{aligned}\tag{A.11}$$

Using [14]

$$A\Delta X - \Delta X\tilde{A} = -B\Delta G\tag{A.12}$$

and

$$\Delta X = \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} A^{i-1} B \Delta G \tilde{A}^{j-1}.\tag{A.13}$$

Substituting (A.13) into the second term of (A.11) we have

$$\begin{aligned}
\Delta J &:= 2\alpha_2 \text{Trace}\{X^{-1}(F^T - (KC)^T)\Delta G\} \\
&\quad + \text{Trace}\left\{\sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \tilde{A}^{j-1} \right. \\
&\quad \left. \underbrace{\left(\frac{\alpha_1}{\sigma_{\min}^2(X)}(\sigma_{\min}(X)v_i u_i^T - \sigma_{\max}(X)v_a u_a^T) - 2\alpha_2 X^{-1}(F^T - (KC)^T)F\right)}_{X_f} \right. \\
&\quad \left. A^{i-1} B \Delta G\right\} \\
&= 2\alpha_2 \text{Trace}\{X^{-1}(F^T - (KC)^T)\Delta G\} \\
&\quad + \text{Trace}\left\{\underbrace{\sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \tilde{A}^{j-1} X_f A^{i-1} B}_{U} \Delta G\right\} \\
&= \text{Trace}\{[2\alpha_2 X^{-1}(F^T - (KC)^T) + BU]\Delta G\}
\end{aligned} \tag{A.14}$$

From (A.12) and (A.13) it follows that U is the unique solution of

$$\tilde{A}U - UA = X_f. \tag{A.15}$$

Therefore

$$\frac{\partial J}{\partial G} = 2\{\alpha_2(F - KC)X^{-T} + B^T U^T\} \tag{A.16}$$

where U satisfies

$$\tilde{A}U - UA = \frac{\alpha_1}{\sigma_{\min}^2(X)}\{\sigma_{\min}(X)v_i u_i^T - \sigma_{\max}(X)v_a u_a^T\} - 2\alpha_2 X^{-1}(F^T - (KC)^T)F \tag{A.17}$$

(b)

Now we evaluate the gradients of (3.10) with respect to the variable elements of \tilde{A} . Recall the equation (A.9)

$$\begin{aligned} \Delta J &= \frac{\alpha_1}{\sigma_{\min}^2(X)} \text{Trace}\{\sigma_{\min}(X)v_a u_a^T - \sigma_{\max}(X)v_i u_i^T\} \Delta X \\ &\quad + 2\alpha_2 \text{Trace}\{(F^T - (KC)^T)\Delta F\}. \end{aligned} \quad (\text{A.18})$$

From $F = GX^{-1}$, we compute (G is fixed)

$$\begin{aligned} \Delta F &= -GX^{-1}\Delta XX^{-1} \\ &= -F\Delta XX^{-1}. \end{aligned} \quad (\text{A.19})$$

Substituting ΔF into (A.18)

$$\begin{aligned} \Delta J &= \text{Trace} \\ &\quad \underbrace{\left\{ \frac{\alpha_1}{\sigma_{\min}^2(X)} (\sigma_{\min}(X)v_i u_i^T - \sigma_{\max}(X)v_a u_a^T) - 2\alpha_2 X^{-1}(F - (KC)^T)F \right\}}_{X_f} \Delta X \\ &= \text{Trace}\{X_f \Delta X\} \end{aligned} \quad (\text{A.20})$$

Since

$$A\Delta X - \Delta X\tilde{A} = X\Delta\tilde{A} \quad (\text{A.21})$$

$$\Delta X = \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} A^{i-1} (-X\Delta\tilde{A}) \tilde{A}^{j-1} \quad (\text{A.22})$$

Substituting (A.22) into (A.20)

$$\begin{aligned} \Delta J &= \text{Trace}\left\{ \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} X_f A^{i-1} (-X\Delta\tilde{A}) \tilde{A}^{j-1} \right\} \\ &= -\text{Trace}\left\{ \underbrace{\sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \tilde{A}^{j-1} X_f A^{i-1} X\Delta\tilde{A}}_U \right\} \end{aligned} \quad (\text{A.23})$$

It is clear that U is the unique solution of

$$\tilde{A}U - UA = X_f$$

as in (A.14).

$$\Delta J = -\text{Trace}\{UX\Delta\tilde{A}\} \quad (\text{A.24})$$

Therefore,

$$\frac{\partial J}{\partial \tilde{a}_i} = -\text{Trace}\left\{UX \frac{\partial \tilde{A}}{\partial \tilde{a}_i}\right\} \quad (\text{A.25})$$

As an example the following calculation is considered. Let

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \quad X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad \tilde{A} = \begin{pmatrix} \tilde{a}_1^2 & 0 \\ 0 & \tilde{a}_2^2 \end{pmatrix} \quad (\text{A.26})$$

Then

$$\begin{aligned} \frac{\partial J}{\partial \tilde{a}_1} &= 2\text{Trace}\left\{\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} 2\tilde{a}_1 & 0 \\ 0 & 0 \end{pmatrix}\right\} \\ &= 4\tilde{a}_1(u_{11}x_{11} + u_{12}x_{21}) \end{aligned} \quad (\text{A.27})$$

$$\begin{aligned} \frac{\partial J}{\partial \tilde{a}_2} &= 2\text{Trace}\left\{\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2\tilde{a}_2 \end{pmatrix}\right\} \\ &= 4\tilde{a}_2(u_{21}x_{12} + u_{22}x_{22}) \end{aligned} \quad (\text{A.28})$$

or

$$\begin{pmatrix} \frac{\partial J}{\partial \tilde{a}_1} \\ \frac{\partial J}{\partial \tilde{a}_2} \end{pmatrix} = 4 \begin{pmatrix} \tilde{a}_1(u_{11}x_{11} + u_{12}x_{21}) \\ \tilde{a}_2(u_{21}x_{12} + u_{22}x_{22}) \end{pmatrix} \quad (\text{A.29})$$

(c)

Finally the gradient of J with respect to K is easily derived.

$$\begin{aligned} \Delta J &= -2\alpha_2 \text{Trace}\{CF^T \Delta K - C(KC)^T \Delta K\} \\ &= -2\alpha_2 \text{Trace}\{(CF^T - C(KC)^T) \Delta K\} \end{aligned} \quad (\text{A.30})$$

Thus,

$$\frac{\partial J}{\partial K} = -2\alpha_2 [F - KC]C^T \quad (\text{A.31})$$

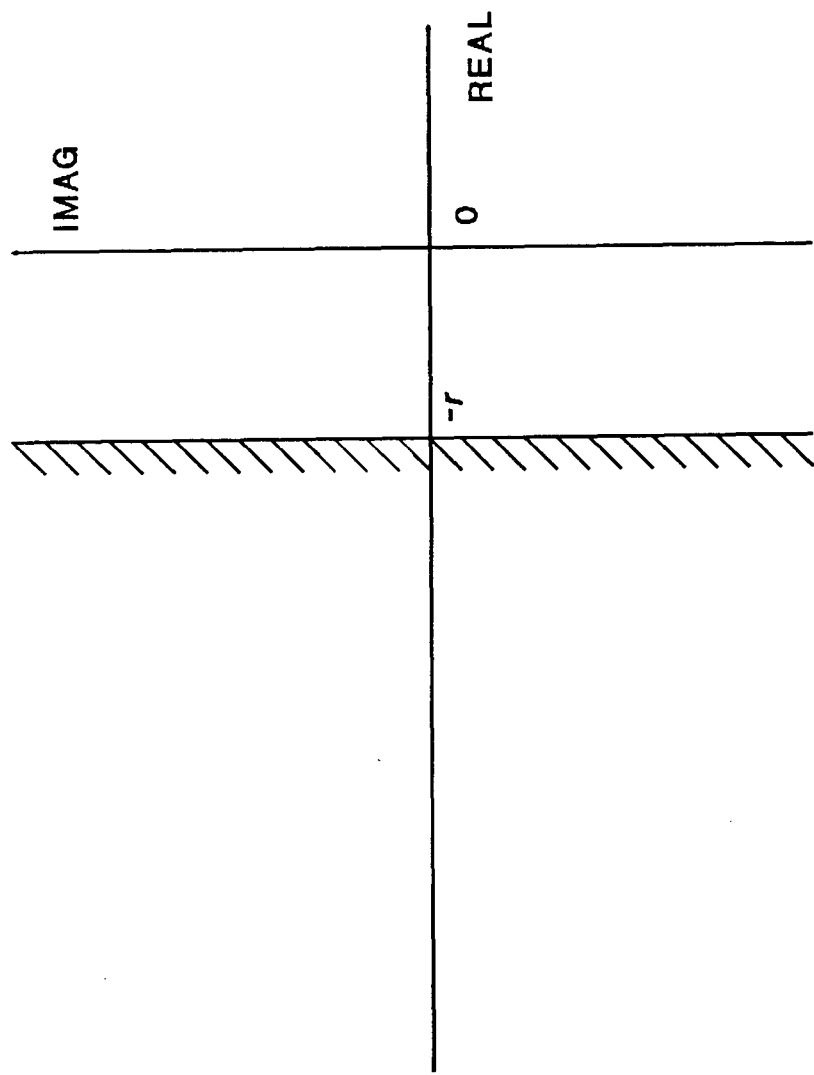


FIGURE 4.1 MARGINAL STABILITY REGION

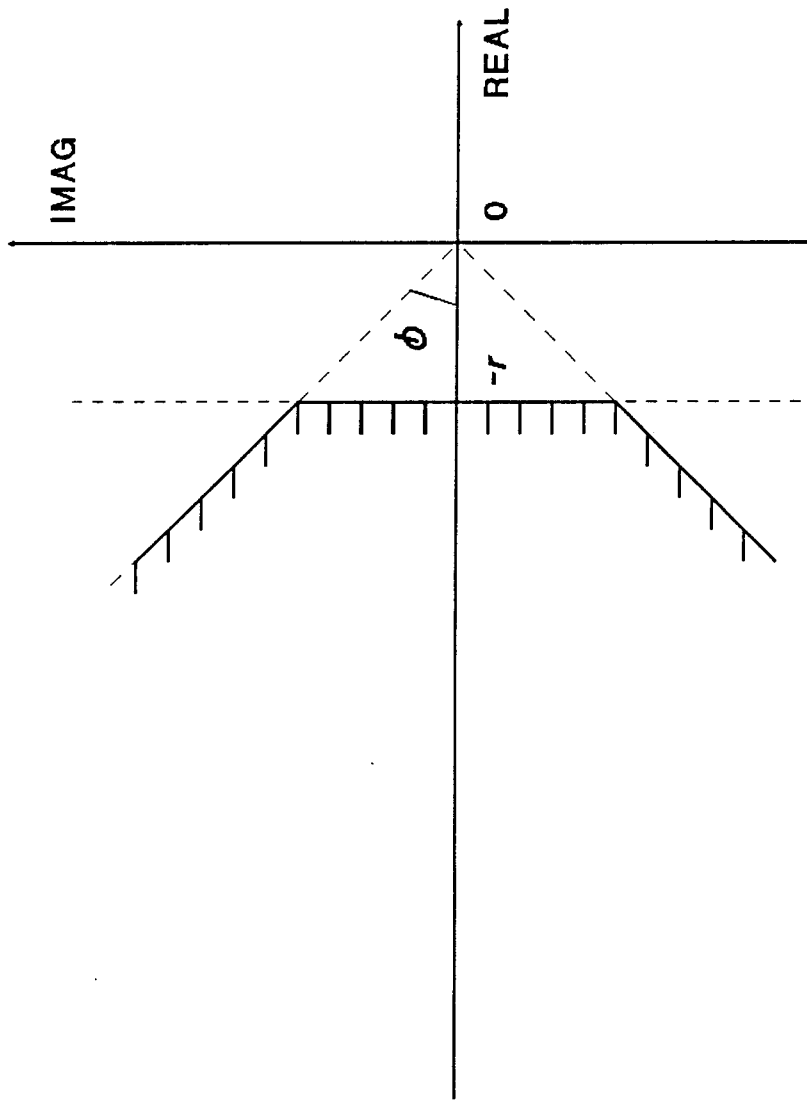


FIGURE 4.2 DAMPING RATIO REGION

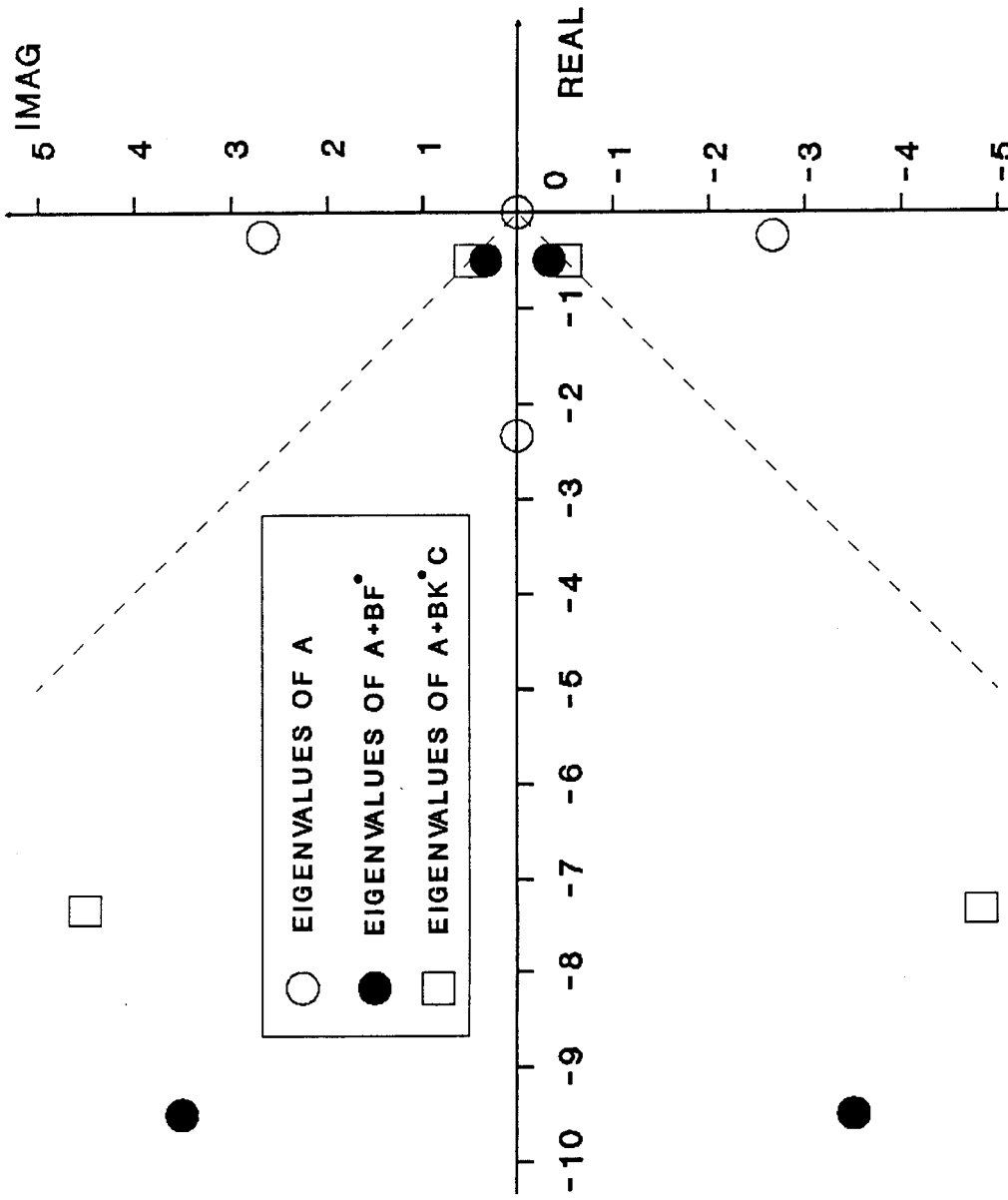


FIGURE 5.1 EIGENVALUE LOCATIONS CORRESPONDING TO TABLE 1.1

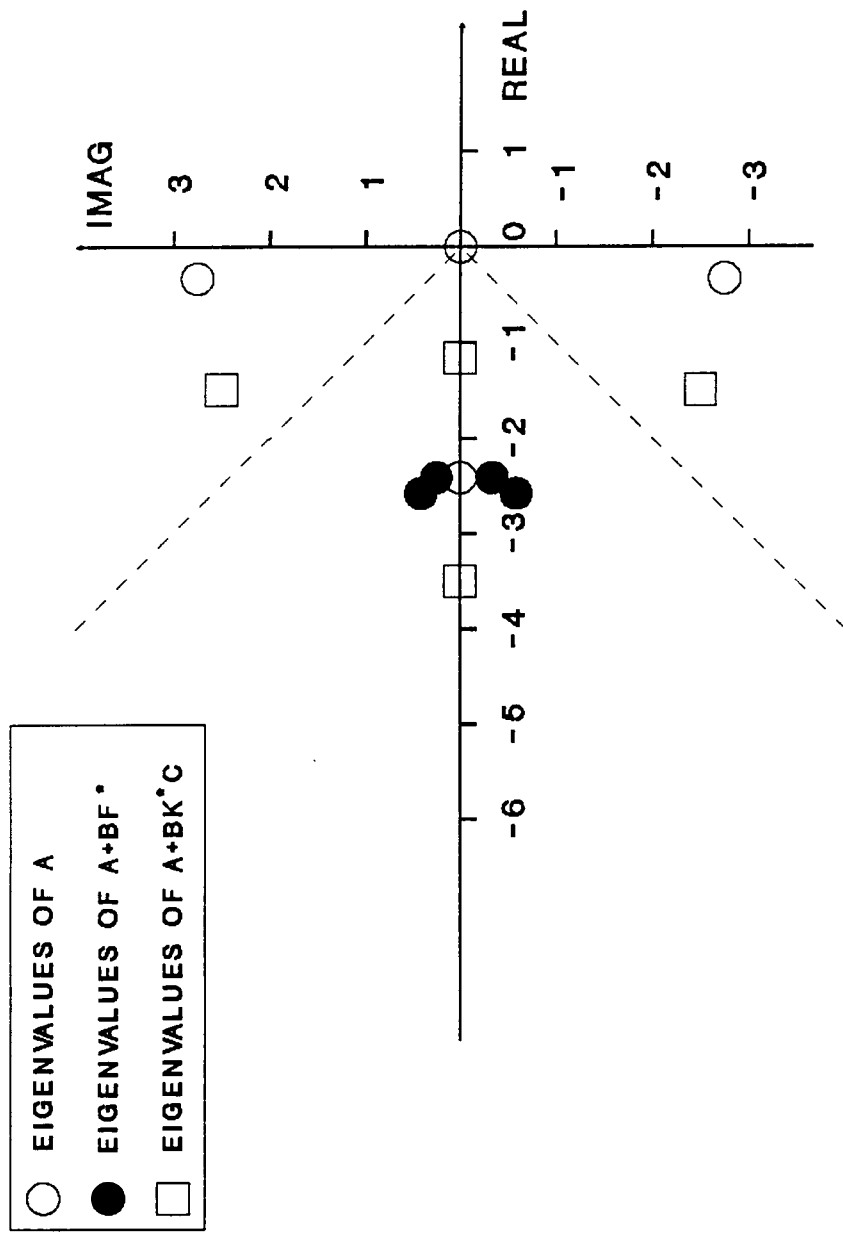


FIGURE 5.2 EIGENVALUE LOCATIONS CORRESPONDING TO TABLE 1.3

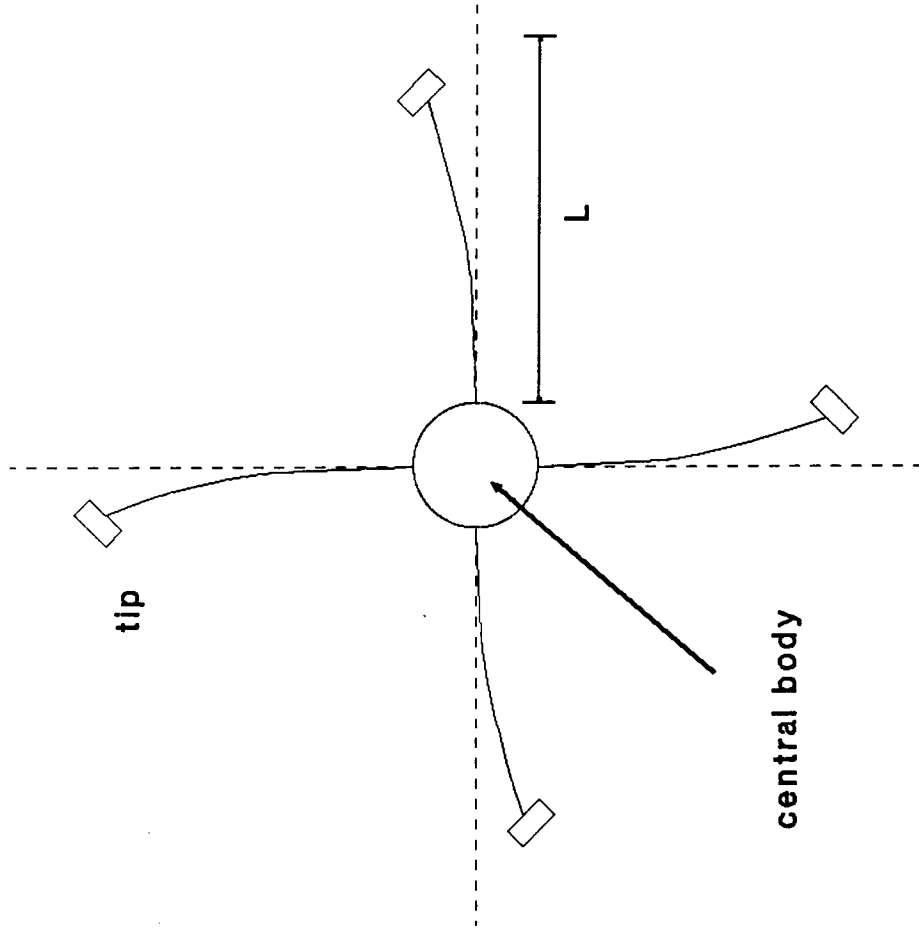


FIGURE 5.3 DRAPER/RPL SYMMETRIC VIBRATIONAL MODEL

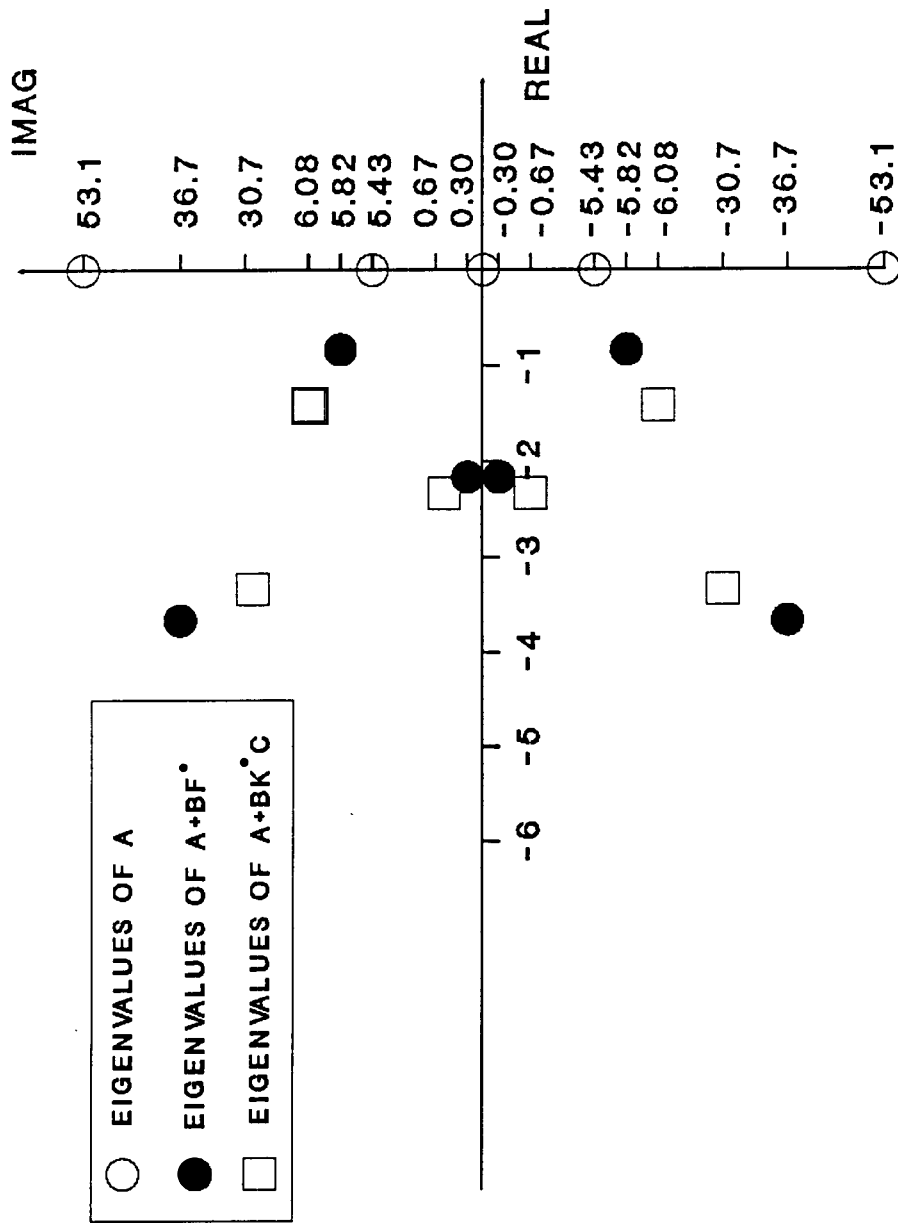


FIGURE 5.4 EIGENVALUE LOCATIONS CORRESPONDING TO TABLE 2.1

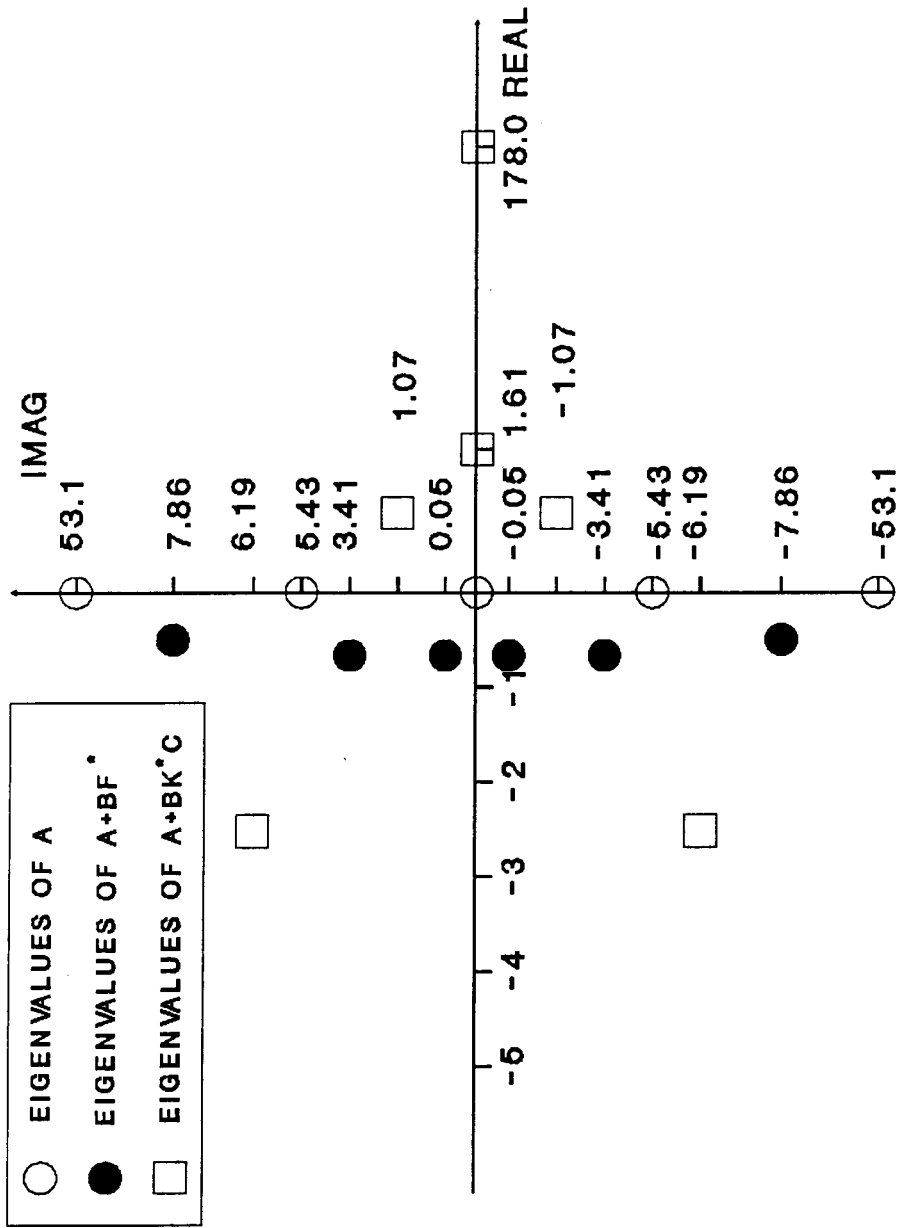


FIGURE 5.5 EIGENVALUE LOCATIONS CORRESPONDING TO TABLE 2.3