$$Q=[k_1d_1+k_2,\ \ -1-k_1+k_1d_3+k_2d_2]$$

$$R=[l_1d_1+l_2,\ \ -1-l_1+l_1d_3+l_2d_2]$$

where $[k_1\ \ k_2]$, $[l_1\ \ l_2]$ are arbitrary polynomial matrices from $R_{12}\{d_1, d_2, d_3]$.

Conclusions

Sufficient conditions for the existence of a solution to the deadbeat servoproblem for multivariable *n-D* linear systems are given. An algorithm based on elementary column and row operations for finding the matrices P, Q, and R of the linear *n-D* controller is presented and illustrated by a simple 3-D example.

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Robust Controller Synthesis Using the Maximum Entropy Design Equations

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Abstract—**This note presents an application of the optimality conditions obtained in [1] for dynamic compensation in the presence of state-, control-, and measurement-dependent noise. By solving these equations, which represent a fundamental generalization of standard steady-state LQG theory, a series of increasingly robust control designs is obtained for the example considered in [2].**

I. Introduction

Perhaps the most significant aspect of LQG theory is the *explicit synthesis* of dynamic feedback compensators. In practice, however, LQG suffers from serious defects concerning closed-loop robustness with respect to plant deviations. In particular, LQG controllers may possess arbitrarily small stability margin with respect to parameter variations [2].

One approach to correcting this defect is to rederive the optimality conditions for dynamic compensation in the presence of state-, control-, and measurement-dependent noise [1]. Intuitively speaking, the quadratically optimal feedback controller designed in the presence of such multiplicative disturbances is *automatically desensitized* to *actual* parameter variations. The optimality conditions now comprise a system of *four* matrix equations, specifically, two modified Riccati equations and two modified Lyapunov equations, coupled by stochastic effects. This coupling is a graphic reminder of the breakdown of the separation principle in the uncertain plant case. When the uncertainty terms are absent, the equations immediately reduce to the standard pair of separated Riccati equations.

For the special case of full-order compensation in the presence of state-dependent noise only, versions of these equations were discovered independently by Hyland [3]-[5] and Mil'stein [6]. A crucial feature of [1], [3]-[5] is the interpretation of the closed-loop stochastic differential equation according to the Fisk-Stratonovich definition of stochastic integration. For modeling flexible mechanical structures, justification of this interpretation as an appropriate model for *a priori* parameter uncertainty was based upon the maximum entropy principle of Jaynes [1].

A time-varying version of these design equations involving uncorrelated state- and control-dependent noise has been given in [7]. The stochastic interpretation is in the sense of Ito as in [6].

The purpose of the present note is to summarize the *maximum entropy equations* for full-order dynamic feedback compensation. These equations are then applied to Doyle's example [2] to produce a series of quadratically optimal robust controllers. The full optimal projection/maximum entropy design equations, which also account for a constraint on controller order [1], [8], are applied to a more realistic design problem in [9].

II. Problem Statement and Maximum Entropy Design Equations

To state the optimal dynamic-compensation problem, we require the following notation. Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^l$, $u \in \mathbb{R}^m$, $A, A_1, \cdots, A_p \in \mathbb{R}^{n\times n}$, $B, B_1, \cdots, B_p \in \mathbb{R}^{n\times m}$, $C, C_1, \cdots, C_p \in \mathbb{R}^{l\times n}$, $R_1 \in \mathbb{R}^{n\times n}$, $R_1 \geq 0$, $R_2 \in \mathbb{R}^{m\times m}$, $R_2 > 0$. Furthermore, let $v_1, \cdots, v_p$ be unit-intensity, zero-mean, and mutually uncorrelated white noise processes and let $w_1 \in \mathbb{R}^n$ and $w_2 \in \mathbb{R}^l$ be zero-mean white noise processes with

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intensities $V_1 \geq 0$ and $V_2 > 0$, respectively, and cross intensity $V_{12} \in \mathbb{R}^{n\times l}$. It is further assumed that v_i, w_i and $x(0)$ are mutually uncorrelated. We require the technical assumption that, for each i, $B_i \neq 0$ implies $C_i = 0$, i.e., the control- and measurement-dependent noises are uncorrelated.

Optimal Dynamic-Compensation Problem

Given the controlled system

$$\dot{x} = \left(A + \sum_{i=1}^{p} v_i A_i\right)x + \left(B + \sum_{i=1}^{p} v_i B_i\right)u + w_1, \tag{2.1}$$

$$y = \left(C + \sum_{i=1}^{p} v_i C_i\right)x + w_2, \tag{2.2}$$

design an nth-order dynamic compensator

$$\dot{x}_c = A_c x_c + B_c y, \tag{2.3}$$

$$u = C_c x_c \tag{2.4}$$

which minimizes the performance criterion

$$J(A_c, B_c, C_c) = J_x(A_c, B_c, C_c) + J_{xu}(A_c, B_c, C_c) + J_u(A_c, B_c, C_c), \tag{2.5}$$

where

$$J_x(A_c, B_c, C_c) \triangleq \lim_{t\to\infty} \mathbb{E}[x^T R_1 x],$$

$$J_{xu}(A_c, B_c, C_c) \triangleq \lim_{t\to\infty} \mathbb{E}[2x^T R_{12} u],$$

$$J_u(A_c, B_c, C_c) \triangleq \lim_{t\to\infty} \mathbb{E}[u^T R_2 u].$$

To guarantee that J is finite and independent of initial conditions, we restrict (A_c, B_c, C_c) to the (open) set of second-moment-stabilizing triples

$$\mathbb{S} \triangleq \{(A_c, B_c, C_c) : \tilde{A}_s \oplus \tilde{A}_s + \sum_{i=1}^{p} \tilde{A}_i \otimes \tilde{A}_i \text{ is stable}\}$$

where $\oplus$ and $\otimes$ denote Kronecker sum and product and

$$\tilde{A}_s \triangleq \begin{bmatrix} A_s & B_s C_c \\ B_c C_s & A_c \end{bmatrix}, \quad \tilde{A}_i \triangleq \begin{bmatrix} A_i & B_i C_c \\ B_c C_i & 0 \end{bmatrix},$$

$$A_s \triangleq A + \frac{1}{2}\sum_{i=1}^{p} A_i^2, \quad B_s \triangleq B + \frac{1}{2}\sum_{i=1}^{p} A_i B_i, \quad C_s \triangleq C + \frac{1}{2}\sum_{i=1}^{p} C_i A_i.$$

For convenience in stating the optimality conditions, define the following notation for $Q, P, \hat{Q}, \hat{P} \in \mathbb{R}^{n\times n}$:

$$R_{2s} \triangleq R_2 + \sum_{i=1}^{p} B_i^T (P + \hat{P}) B_i, \quad V_{2s} \triangleq V_2 + \sum_{i=1}^{p} C_i (Q + \hat{Q}) C_i^T,$$

$$\mathcal{Q}_s \triangleq Q C_s^T + V_{12} + \sum_{i=1}^{p} A_i (Q + \hat{Q}) C_i^T, \quad \mathcal{P}_s \triangleq B_s^T P + R_{12}^T + \sum_{i=1}^{p} B_i^T (P + \hat{P}) A_i.$$

$$A_{Qs} \triangleq A_s - \mathcal{Q}_s V_{2s}^{-1} C_s, \quad A_{Ps} \triangleq A_s - B_s R_{2s}^{-1} \mathcal{P}_s.$$

Theorem 2.1: Suppose $(A_c, B_c, C_c) \in \mathbb{S}$ solves the optimal dynamic-compensation problem. Then there exist $n\times n$ nonnegative-definite matrices Q, P, $\hat{Q}$, and $\hat{P}$ such that A_c, B_c, C_c are given by

$$A_c = A_s - B_s R_{2s}^{-1} \mathcal{P}_s - \mathcal{Q}_s V_{2s}^{-1} C_s, \tag{2.6}$$

$$B_c = \mathcal{Q}_s V_{2s}^{-1}, \tag{2.7}$$

$$C_c = -R_{2s}^{-1} \mathcal{P}_s, \tag{2.8}$$

and such that the following conditions are satisfied:

$$0 = A_s Q + Q A_s^T + V_1 + \sum_{i=1}^{p} [A_i Q A_i^T + (A_i - B_i R_{2s}^{-1} \mathcal{P}_s) \hat{Q} (A_i - B_i R_{2s}^{-1} \mathcal{P}_s)^T] - \mathcal{Q}_s V_{2s}^{-1} \mathcal{Q}_s^T, \tag{2.9}$$

$$0 = A_s^T P + P A_s + R_1 + \sum_{i=1}^{p} [A_i^T P A_i + (A_i - \mathcal{Q}_s V_{2s}^{-1} C_i)^T \hat{P} (A_i - \mathcal{Q}_s V_{2s}^{-1} C_i)] - \mathcal{P}_s^T R_{2s}^{-1} \mathcal{P}_s, \tag{2.10}$$

$$0 = A_{Ps} \hat{Q} + \hat{Q} A_{Ps}^T + \mathcal{Q}_s V_{2s}^{-1} \mathcal{Q}_s^T, \tag{2.11}$$

$$0 = A_{Qs}^T \hat{P} + \hat{P} A_{Qs} + \mathcal{P}_s^T R_{2s}^{-1} \mathcal{P}_s. \tag{2.12}$$

Remark 2.1: Letting $A_i = 0$, $B_i = 0$ and $C_i = 0$, $i = 1, \cdots, p$, it can readily be seen that (2.11) and (2.12) are superfluous and that (2.9) and (2.10) yield the standard separated LQG Riccati equations.

Remark 2.2: Since $R_{2s} \geq R_2$, so that $R_{2s}^{-1} \leq R_2^{-1}$, it is clear that the control-dependent noise effectively suppresses the regulator gain C_c. Similarly, since $V_{2s} \geq V_2$, the measurement-dependent noise suppresses the observer gain B_c. The effect of the terms $A_i Q A_i^T$ is discussed in [1] for modal systems.

III. The Maximum Entropy Design Equations Applied to Doyle's Example

As shown in [2], LQG regulators for the example

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix}, \quad C = [1 \quad 0],$$

$$V_1 = \sigma \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, V_{12} = 0, \quad V_2 = 1,$$

$$R_1 = \rho \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad R_{12} = 0, \quad R_2 = 1,$$

have arbitrarily small stability margin with regard to variations $b + \Delta b$ when σ and ρ are sufficiently large and $b = 1$.

Setting $\sigma = \rho = 60$, it follows that the LQG regulator is only stable for $0.93 \leq b + \Delta b \leq 1.01$. Uncertainty in b can be modeled by setting $p = 1$, $A_1 = 0$, $B_1 = [0 \; b_1]^T$, and $C_1 = 0$. Solving the optimality conditions (2.9)–(2.12) with $b_1 = 0.05$, 0.10, 0.15, and 0.20 yields a series of increasingly robust controller designs with respect to both positive and negative variations Δb (see Table I and Figs. 1 and 2).

Conclusion

As demonstrated on the example of [2], the maximum entropy design equations provide a novel method for synthesizing robust feedback controllers. Since the design equations represent a fundamental generalization of standard LQG theory, the approach represents an alternative to LQG-modification techniques. Indeed, these equations are not intended as a device for recovering the gain and phase margins of LQ state-feedback regulators, but rather as a method for designing output-feedback dynamic compensators which are robust with respect to parametric deviations in

TABLE I
DYNAMIC COMPENSATOR GAINS FOR LQG AND MAXIMUM ENTROPY DESIGNS ($b = 1$, $\sigma = \rho = 60$)

b_1	A_c	B_c	C_c	Stability Range of $b + \Delta b$
0 (LQG)	$\begin{bmatrix} -9 & 1 \\ -20 & -9 \end{bmatrix}$	$\begin{bmatrix} 10 \\ 10 \end{bmatrix}$	[-10 -10]	(.93, 1.01)
.05	$\begin{bmatrix} -9.253 & 1.0 \\ -20.69 & -7.382 \end{bmatrix}$	$\begin{bmatrix} 10.25 \\ 12.31 \end{bmatrix}$	[-8.382 -8.382]	(.88, 1.03)
.10	$\begin{bmatrix} -9.639 & 1.0 \\ -23.27 & -6.318 \end{bmatrix}$	$\begin{bmatrix} 10.64 \\ 15.95 \end{bmatrix}$	[-7.318 -7.318]	(.82, 1.08)
.15	$\begin{bmatrix} -10.10 & 1.0 \\ -27.24 & -5.710 \end{bmatrix}$	$\begin{bmatrix} 11.10 \\ 20.53 \end{bmatrix}$	[-6.710 -6.710]	(.77, 1.13)
.20	$\begin{bmatrix} -10.69 & 1.0 \\ -32.97 & -5.295 \end{bmatrix}$	$\begin{bmatrix} 11.69 \\ 26.67 \end{bmatrix}$	[-6.295 -6.295]	(.72, 1.21)

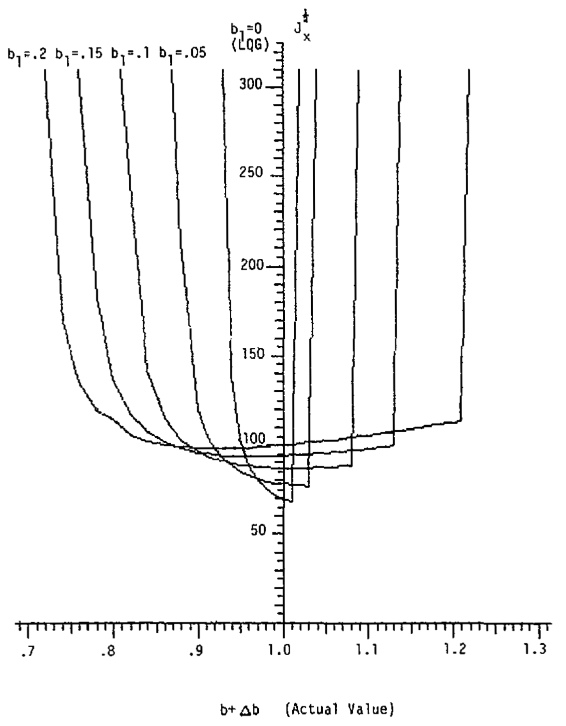

Fig. 1. Robustness of LQG versus maximum entropy designs (b_1 = 0.05, 0.1, 0.15, 0.2).

the plant model. As discussed in [10], these are significantly different objectives.

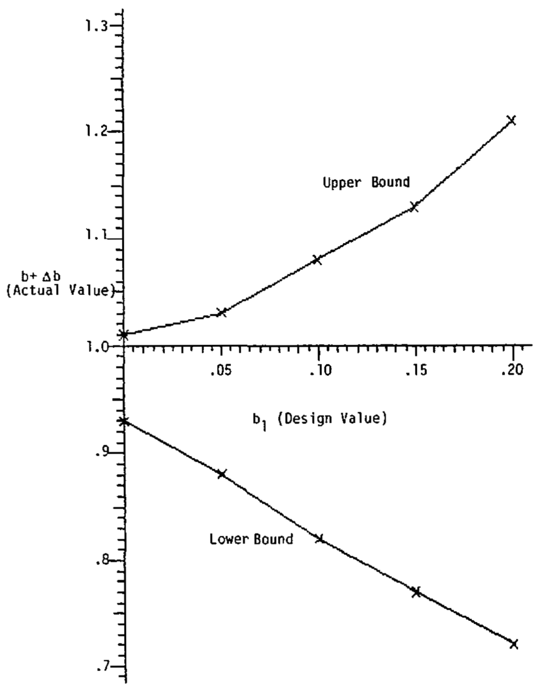

Fig. 2. Stability bounds for LQG and maximum entropy designs.

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Stability of Multiloop LQ Regulators with Nonlinearities—Part I: Regions of Attraction

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***Abstract*—The closed-loop stability of linear, time-invariant systems controlled by linear quadratic (LQ) regulators is investigated when there are nonlinearities in the control channels which lie outside the (0.5, ∞) stability sector in regions away from the origin (i.e., saturation-type nonlinearities). An estimate of the region of attraction is obtained which provides methods for selecting the performance function weights for more robust LQ designs.**

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