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Robust decentralized control design using integral sliding mode control

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Abstract— **The problem of robust decentralization of uncertain inter-connected systems is concerned with the goal of de-coupling a Lipschitz non-linear systems into individual "decentralized" subsystems satisfying security and fault-tolerance objectives. This work proposes a new strategy for robust decentralized control in which each subsystem uses an observer-based state estimate structure invoking an approach to *separation principle recovery*, based on Integral Sliding Model Control (ISMC) with careful consideration of both matched and unmatched uncertainties arising from inter-connections and disturbances. The proposed design strategy for the linear observer and uncertainty de-coupling designs involves a single LMI. An example of 3 unstable inter-connected non-linear systems is used to illustrate the power of the approach.**

Keywords Decentralized control, Integral sliding mode control

I. INTRODUCTION

The study of control of non-linear inter-connected systems has received considerable interest in recent years [1], [2], [3]. Some research focuses on interconnected systems with uncertainties, e.g. unknown nonlinear interconnections and disturbances, presenting robustness design challenges involving control specifications for each subsystem. These systems are particularly difficult to design when faced with limitations arising from uncertainty matching conditions and lack of available state information [4].

In most cases the design of robust decentralized systems focuses on state feedback problems. However, in reality only output information is available and this adds a further challenge to the robust design problem. It is often the case that the controller designs must depend to a degree on estimated states, and hence it is common practice in the literature to investigate the observer based feedback control approach with state estimates based on local information [5], [6]. In fact, the derivation of robust output feedback for decentralized control systems with uncertain inter-connection remains a difficult challenge in the literature [7].

Observer-based strategies represent a commonly used way of dealing with output feedback design and there are two observer-based control paradigms for decentralized systems. Firstly, a separate "decentralized" observer is designed for each subsystem, taking account of local information. The second approach involves the use of "inter-connected observers" [8] in which each observer measurement and input information is shared with observers from other local subsystems.

In many branches of control systems there is a need to compensate robustly for effects of system uncertainties or effects of input disturbances or even faults, to maintain required closed-loop performance and stability. One such approach is the use of sliding mode control (SMC) in which the system dynamic behavior can be forced to be independent of inputs, and certain disturbances and modeling uncertainties, once the so-called sliding regime has been reached. Several studies of inter-connected decentralized systems have focused on the use of SMC as a basis for solving robustness [9]. However, the classical approach to SMC requires (i) a reachability condition to guarantee that the SMC sliding or switching motion in state space can be reached from arbitrary initial conditions, and (ii) that two separate controllers be designed to achieve reachability and satisfy the sliding mode design objectives [10]. In the case of ISMC the requirements for both (i) & (ii) above are obviated, since the sliding motion is reached from initial time, making the use of ISMC very attractive for robust control of decentralized systems [11].

This paper focuses on the use of ISMC for decentralized control, based on state estimate feedback. It is assumed that the local system states are not measurable and hence the *decentralized observer* approach outlined above is used as a part of a state-estimate feedback design problem. Decentralized observers are used as a part of the strategy to de-couple the effects of inter-connections between subsystems. Although, each observer has linear feedback structure the observer-based control is formulated using a single LMI procedure to satisfy both *Lyapunov stability* and performance of the augmented state space form of the observer-controller state space system. This relates to the classical Separation Principle only in the sense that objective for each subsystem is to provide effective recovery of the Separation Principle and hence also effective decentralization. This is achieved through the use of the single LMI approach involving the feedback designs for each observer and controller [12]. The system description involves both matched and unmatched uncertainty components (arising from inter-connections and disturbance) and the paper deals with both forms of uncertainty.

The paper is structured as follows. Section II describes the problem formulation. Then section III considered the proposed control approach that includes output integral sliding mode

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control (OISLMC) in the first part and LMI observer–based control design in the second part. Section IV describes a numerical example with three interconnected systems to illustrate the design approach and simulation performance. Section V gives some conclusions.

II. PROBLEM FORMULATION

Consider an interconnected system comprising subsystems described by:

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + Z_i(x_i,t) + W_i(x_i,t) + E_i d_i(t) + B_i f_i(t)$$
$$y_i(t) = c_i x_i(t) \qquad i = 1,2,\square,n \qquad (1)$$

where $x_i(t) \in \mathbb{R}^n$ is the state vector, $u_i(t) \in \mathbb{R}^m$ are the control inputs and $y_i(t) \in \mathbb{R}^p$ is the vector of system outputs. A_i, B_i, C_i and E_i are known matrices of appropriate dimension. $Z_i(x_i,t) \in \mathbb{R}^n$ represent the unknown time-varying interactions between the subsystems, containing matched and unmatched components. Hence, $Z_i = Z_{mi} + Z_{ui}$ where Z_{mi} is a matched component of Z_i and Z_{ui} is the unmatched components [13].

Dropping the subscripts in $Z_i(x_i,t)$ and using the identity $I_n = BB^+ + B^\perp B^{\perp +}$ where $B^+ = (B^TB)^{-1}B^T$, $Z_i = B_i{B_i}^+Z_i + {B_i}^\perp {B_i}^{\perp +}Z_i$ and $B^TB^\perp = 0$, then $Z_i = B_i{B_i}^+Z_i + \zeta_i$ where $\zeta_i = {B_i}^\perp {B_i}^{\perp +}Z_i$ contains the unmatched uncertainty components.

$W_i(x_i,t)$ represents the subsystem unknown modeling uncertainties that satisfies the matching condition $W_i(x_i,t) = B_iQ_i(x_i,t)$ are unknown, $d_i(t)$ is an unknown bounded process disturbance, $f_i(t) \in \mathbb{R}^k$ denotes the actuator faults, where $f_i(t) = -K(t)u_i$ and $K(t) = diag(K_i)$ with $0 \le K_i \le 1$, $K_i = 0$ means that the actuator is working perfectly and if $K_i = 1$ the actuator has failed completely otherwise the fault is present.

***Assumptions*:**

- **A1**-The pair (A_i, B_i) is controllable and (C_i, A_i) is an observable pair.
- **A2**-B_i has full rank m_i.
- **A3**-The initial state $x_i(t_o)$ is bounded.
- **A4**- The $Z_i(x_i,t)$ are Euclidean bounded norms as: $\|Z_i(x_i,t)\| \le \beta_i(x_i,t)$ where $\beta_i(x_i,t)$ is a known nonlinear function [14].
- **A5**- The $Q_i(x_i,t)$ are bounded as: $\|Q_i(x_i)\| \le \kappa_i\|x_i\|$ where $\kappa_i > 0$ are known Lipschitz constants [15].
- **A6**- The $d_i(t)$ are Euclidean bounded norms as: $\|d_i(t)\| \le \gamma_i\,\|x_i\|$ where $\gamma_i > 0$ are known constants.
- **A7**- $f_i(t)$ are Euclidean bounded norms as: $\|f_i(t)\| \le \eta_i\,\|x_i\|$ where $\eta_i > 0$ are known Lipschitz constants.

All these assumptions are applicable for real control system problems, since all designs are made off-line.

Following the assumptions above Eq. (1) becomes:

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + B_iB_i^+Z_i(t) + \zeta_i(t) + B_iQ_i(x_i,t) + E_i d_i(t) + B_i f_i(t)$$
$$y_i(t) = c_i x_i(t) \qquad i = 1,2,\ldots\ldots,n \qquad (2)$$

The control signal contains *two* components:

$$u_i(t) = u_i^{OBC}(t) + u_i^{ISM}(t) \qquad (3)$$

where u_i^{OBC} is responsible for stabilizing the system and affects the desired performance and decreases the affect of unmatched components where the state is not available. u_i^{ISM} is a discontinuous control responsible for rejecting the effects of matched components (uncertainties and actuator faults).

III. CONTROL DESIGN

As described in (3) the subsystem control signal includes two parts with each part designed using a different method where (i) $u_i^{ISM}(t)$ is designed by output integral sliding mode control OISMC where the state is not available and only the estimated state is obtainable, and (ii) $u_i^{OBC}(t)$ depends on the estimated state as shown in Fig. 1.

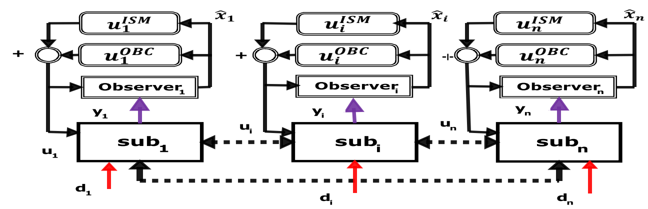

Figure 1 output control of interconnected systems via LMI+ISMC

A. *OUTPUT INTEGRAL SLIDING MODE CONTROL (OISLMC)*

As outlined in Section 1 the integral sliding control can be used to remove the reachability problem. The output feedback case of integral sliding mode control can be developed by defining the following integral sliding switching surface:

$$\sigma_i(y_i,\hat{x}_i,t) = G_i[y_i(t) - y_i(t_o) - \int_{t_o}^{t}(C_iA_i\hat{x}_i(t) + C_iB_iu_i^{LMI}(t))dt] \qquad (4)$$

where $G_i \in \mathbb{R}^{mxp}$ is a design freedom matrix that must satisfy the invertibility of $G_iC_iB_i$.

The two ISMC design steps are as:

1- Sliding surface design that satisfies the system performance and ensures that the system has required performance when the state trajectory is on the sliding surface.

2- Appropriate discontinuous control to maintain the system trajectory close to or on the sliding surface.

In the ISMC the design freedom of the integral action can be used to design a control law that satisfies the prescribed closed-loop performance.

The equivalent control $u_{eqi}(t)$ can be maintained on the sliding surface by forcing the time derivative of $\sigma_i(y_i,\hat{x}_i,t)$ in (4) to be zero-valued [16], i.e.:

$$\dot{\sigma}_i(y_i,\hat{x}_i,t) = G_i\dot{y}_i(t) - G_iC_iA_i\hat{x}_i(t) - G_iC_iB_iu_i^{OBC}(t) = 0 \qquad (5)$$

Then substituting (2) and (3) into (5) yields:

$$G_iC_iA_ix_i(t) + G_iC_iB_iu_i^{OBC} + G_iC_iB_iu_i^{ISM} + G_iC_iB_iB_i^+Z_i(t) + G_iC_i\zeta_i(t) + G_iC_iB_iQ_i(x_i,t) + G_iC_iE_id_i(t) + G_iC_iB_if_i(t) - G_iC_iA_i\hat{x}_i(t) - G_iC_iB_iu_i^{OBC} = 0 \qquad (6)$$

Hence, the so-called *equivalent control* for the output feedback case is:

$$u_{eqi}(t) = u_i^{ISM} = -(G_iC_iB_i)^{-1}G_iC_iA_i(x_i(t) - \hat{x}_i(t)) - B_i^{+}Z_i(t) - (G_iC_iB_i)^{-1}G_iC_i\zeta_i(t) - Q_i(x_i,t) - (G_iC_iB_i)^{-1}G_iC_iE_id_i(t) - f_i(t) \quad (7)$$

Substituting (7) into (2) gives the i^{th} subsystem state equation as:

$$\dot{x}_i(t) = A_ix_i(t) + B_iu_i^{OBC}(t) + [I_i - B_i(G_iC_iB_i)^{-1}G_iC_i]\zeta_i(t) + [I_i - B_i(G_iC_iB_i)^{-1}G_iC_i]E_id_i(t) - (G_iC_iB_i)^{-1}G_iC_iA_i(x_i(t) - \hat{x}_i(t) \quad (8)$$

From (8) the unknown matched uncertainties and actuator faults are completely nulled but the dynamics on the sliding surface contains the unknown unmatched uncertainties, disturbance and the state error. The unknown unmatched uncertainties and disturbances are multiplied by a matrix:

$$\Psi_i = [I_i - B_i(G_iC_iB_i)^{-1}G_iC_i]$$

To simplify the notation Eq. (10) can now be re-written as:

$$\dot{x}_i(t) = A_ix_i(t) + B_iu_i^{OBC}(t) + \Psi_i\zeta_i(t) + \Psi_iE_id_i(t) - M_ie_i(t) \quad (9)$$

where $M_i = (G_iC_iB_i)^{-1}G_iC_iA_i$ and $e_i(t) = x_i(t) - \hat{x}_i(t) \in \mathbb{R}^n$ is the estimation error. The proposed discontinuous control is:

$$u_i^{ISM}(t) = -\mu_i \frac{\sigma_i(y_i,\hat{x}_i,t)}{\|\sigma_i(y_i,\hat{x}_i,t)\| + \mathfrak{z}_i} \quad (10)$$

The parameters $\mathfrak{z}_i > 0$ are chosen to reduce the amount of "chattering" of the motion around the sliding surface [15]. To satisfy subsystem stability the positive scalar μ_i is chosen according to the following derivation:

$$\mu_i > (G_iC_iB_i)^{-1}G_iC_i\beta_i(x_i,t) + \kappa_i\|x_i\| + (G_iC_iB_i)^{-1}G_iC_iE_i\gamma_i\|x_i\| + \eta_i\|x_i\| + (G_iC_iB_i)^{-1}G_iC_iA_i\|e_i(t)\| \quad (11)$$

To ensure sliding motion let $\sigma_i(y_i,\hat{x}_i,t) = 0$. Furthermore, the stability of the inter-connected system (1) is considered in terms of a positive definite summation of individual Lyapunov subsystems components as:

$$\sum_{i=1}^{n} V_i(\sigma_i(y_i,\hat{x}_i,t)) = \sum_{i=1}^{n}\|\sigma_i(y_i,\hat{x}_i,t)\| > 0:$$

The derivative of the subsystem Lyapunov functions are:

$$\dot{V}_i(\sigma_i(y_i,\hat{x}_i,t)) = \frac{\sigma_i(y_i,\hat{x}_i,t)^T\dot{\sigma}_i(y_i,\hat{x}_i,t)}{\|\sigma_i(y_i,\hat{x}_i,t)\|} \quad (12)$$

Hence, from (4), (5) & (12) it can be shown that:

$$\sum_{i=1}^{n}\dot{V}_i(\sigma_i(y_i,\hat{x}_i,t)) = \sum_{i=1}^{n}[-G_iC_iB_i\mu_i + \frac{\sigma_i(y_i,\hat{x}_i,t)}{\|\sigma_i(y_i,\hat{x}_i,t)\|}G_iC_iZ_i(t) + \frac{\sigma_i(y_i,\hat{x}_i,t)}{\|\sigma_i(y_i,\hat{x}_i,t)\|}G_iC_iB_iQ_i(x_i,t) + \frac{\sigma_i(y_i,\hat{x}_i,t)}{\|\sigma_i(y_i,\hat{x}_i,t)\|}G_iC_iE_id_i(t) + \frac{\sigma_i(y_i,\hat{x}_i,t)}{\|\sigma_i(y_i,\hat{x}_i,t)\|}G_iC_iB_if_i(t) + \frac{\sigma_i(y_i,\hat{x}_i,t)}{\|\sigma_i(y_i,\hat{x}_i,t)\|}G_iC_iA_ie_i(t)] \quad (13)$$

which can be re-written as:

$$\sum_{i=1}^{n}\dot{V}_i(\sigma_i(y_i,\hat{x}_i,t)) \le \sum_{i=1}^{n}[-(G_iC_iB)[\mu_i - (G_iC_iB_i)^{-1}G_iC_i\|Z_i\| - \|Q_i\| - (G_iC_iB_i)^{-1}G_iC_iE_i\|d_i\| - \|f_i\| + (G_iC_iB_i)^{-1}G_iC_iA_i\|e_i(t)\|] \quad (14)$$

Then, according to A4, A5, A6 & A7:

$$\sum_{i=1}^{n}\dot{V}_i(\sigma_i(y_i,\hat{x}_i,t)) \le \sum_{i=1}^{n}[-(G_iC_iB)[\mu_i - (G_iC_iB_i)^{-1}G_iC_i\beta_i(x_i,t) - \kappa_i\|x_i\| - (G_iC_iB_i)^{-1}G_iC_iE_i\gamma_i\|x_i\| - \eta_i\|x_i\| + (G_iC_iB_i)^{-1}G_iC_iA_i\|e_i(t)\|] \quad (15)$$

By suitable choice of μ_i in (11) then $\sum_{i=1}^{n}\dot{V}_i(\sigma_i(x_i,t)) \le 0$.

To minimize of norms $\|\Psi_i\zeta_i(t)\|$ and $\|\Psi_iE_id_i(t)\|$ corresponding to the unmatched uncertainty and disturbances, respectively, the matrix G_i must be carefully chosen [13]. One choice is $G_i = B_i^TC_i^{+}$ which if substituted into (8) leads to the following:

(i) The term: $[I_i - B_i(B_i^{+}B_i)^{-1}B_i^{+}]B_i^{\perp}B_i^{\perp +}Z_i(t)$, with $B_i^TB_i^{\perp} = 0$, i.e.:

$$[I_i - B_i(B_i^{+}B_i)^{-1}B_i^{+}]B_i^{\perp}B_i^{\perp +}Z_i(t) = B_i^{\perp}B_i^{\perp +}Z_i(t) \quad (16)$$

(ii) The term:

$$[I_i - B_i(B_i^{+}B_i)^{-1}B_i^{+}]E_id_i(t) = [I_i - B_iB_i^{+}]E_id_i(t) \quad (17)$$

Substituting (16) & (17) into (8) yields the subsystem dynamics during sliding:

$$\dot{x}_i(t) = A_ix_i(t) + B_iu_i^{OBC}(t) + T_iZ_i(t) + H_id_i(t) - M_ie_i(t) \quad (18)$$

where $T_i = B_i^{\perp}B_i^{\perp +}$ and $H_i = [I_i - B_iB_i^{+}]E_i$.

From (18) it can be observed that the unknown unmatched uncertainties and disturbances $T_iZ_i(t), H_id_i(t)$ have not been minimized. Hence, another method must be found to minimize these terms and to limit their influence on the subsystem dynamics.

B. *LMI OBSERVER-BASED CONTROL DESIGN*

After designing the ISMC, the subsystem sliding dynamics are:

$$\dot{x}_i(t) = A_ix_i(t) + B_iu_i^{OBC}(t) + \Gamma_iJ_i(t) - M_ie_i(t) \quad (19)$$

where $\Gamma_i = [T_i \quad H_i]$ and $J_i(t) = \begin{bmatrix} Z_i(t) \\ d_i(t) \end{bmatrix}$

The aggregated system dynamics are given by:

$$\dot{X}(t) = A_dX(t) + B_dU^{OBC}(t) + \Gamma_dJ(t) - M_de(t) \quad (20)$$

where: $X(t) = [x_1, x_2, \ldots\ldots, x_n]$ $U^{OBC}(t) = [u_1, u_2, \ldots\ldots, u_n]$, $e(t) = [e_1, e_2, \ldots\ldots, e_n]$, $A_d = diag(A_i)$, $B_d = diag(B_i)$, $\Gamma_d = diag(\Gamma_i)$ and $J(t) = [J_1, J_2, \ldots\ldots, J_n]$, where "$diag$" represents the block diagonal matrix.

To develop a robust control law for the aggregate system consider a state estimate feedback of the form:

$$U^{OBC}(t) = K\hat{X}(t) = KX(t) - Ke(t) \quad (21)$$

where $K = diag(k_i)$ is the decentralized system gain that stabilizes the system under a specific performance objective. The design objective is to choose the gain K to minimize the effect of $J(t)$ on the system. Suppose further that $J(t)$ is the unknown input disturbance which satisfies the quadratic inequality:

$$J^T(t)J(t) \le \alpha^2X^T(t)X(t) \quad (22)$$

where $\alpha > 0$ a positive constant. A suitable observer can estimate the aggregate system state $\hat{X}(t)$ any suitable observer can be used. However, the observer subsystems are given by:

$$\dot{\hat{x}}_i(t) = A_i\hat{x}_i(t) + B_i u_i^{OBC}(t) + L_i(y_i(t) - C_i\hat{x}_i(t)) \quad (23)$$

where L_i is the subsystem observer gain. The aggregate observer dynamics are thus:

$$\dot{\hat{X}}(t) = A_d\hat{X}(t) + B_d U^{OBC}(t) + L_d(Y(t) - C_d\hat{X}(t)) \quad (24)$$

where $Y(t) = [y_1, y_2, \dots\dots, y_n]$, $L_d = diag(L_i)$ and $C_d = diag(C_i)$

Subtracting (20) from (24) yields the state estimation error:

$$\dot{e}(t) = (A_d - L_d C_d)e(t) + \Gamma_d J(t) - M_d e(t) \quad (25)$$

To check the stability of the observer-based closed-loop system the following candidate Lyapunov function is used:

$V(X,t) = X^T(t)PX(t) + e^T(t)Fe(t)$ where $P > 0$ & $F > 0$.

The time derivative of $V(X,t)$ is thus:

$$\dot{V}(X,t) = \dot{X}^T(t)PX(t) + X^T(t)P\dot{X}(t) + \dot{e}^T(t)Fe(t) + e^T(t)F\dot{e}(t) \quad (26)$$

Substituting (21) and (20) into (26) and substituting (25) into (26):

$$\dot{V}(X,t) = X^T(t)[{A_d}^T P + K^T {B_d}^T P + PB_d K]X(t) - e^T[K^T {B_d}^T P - {M_d}^T P]X(t) + J^T(t){\Gamma_d}^T PX(t) + -X^T(t)[PB_d K - (t)PM_d]e(t) + X^T(t)P\Gamma_d J(t) + e^T(t)[{A_d}^T F - {C_d}^T {L_d}^T F - {M_d}^T F + FA_d - FL_d C_d - FM_d]e(t) + J^T(t){\Gamma_d}^T Fe(t) + e^T(t)F\Gamma_d J(t) \quad (27)$$

The stability of subsystem (27) requires that $\dot{V}(X,t) < 0$ $\forall\, X(t) \neq 0$. Equation (27) can then be re-written as:

$$\mathcal{Z}^T\mathcal{D}\mathcal{Z} < 0 \quad (28)$$

where: $\mathcal{Z} = \begin{bmatrix} X(t) \\ e(t) \\ J(t) \end{bmatrix}$ and

$$\mathcal{D} = \begin{bmatrix} {A_d}^T P + PA_d + K^T {B_d}^T P + PB_d K & -PB_d K - PM_d & P\Gamma_d \\ -K^T {B_d}^T P - {M_d}^T P & {A_d}^T F + FA_d + {C_d}^T {L_d}^T F + FL_d C_d - FM_d - {M_d}^T F & F\Gamma_d \\ {\Gamma_d}^T P & {\Gamma_d}^T F & 0 \end{bmatrix} \quad (29)$$

To guarantee stability of the system (28) the matrix $\mathcal{D}$ must be negative-definite.

Furthermore, Eq (23) can be rewritten as:

$$\mathcal{Z}^{\mathrm{T}}\mathcal{O}\mathcal{Z} \leq 0 \quad (30)$$

where: $\mathcal{Z}_i = \begin{bmatrix} X(t) \\ e(t) \\ J(t) \end{bmatrix}$ and $\mathcal{O} = \begin{bmatrix} -\alpha^2 I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$.

To combine (28) & (30) into a single inequality the so-called S-procedure is now used [14].

If $\mathcal{D}$ and $\mathcal{O}$ can be considered as symmetric matrices then $\mathcal{Z}^T\mathcal{D}\mathcal{Z} < 0$ and $\mathcal{Z}^T\mathcal{O}\mathcal{Z} \leq 0$. Hence, there is a number $\tau > 0$ where $-\tau\mathcal{O} < 0$ and it follows that:

$$\mathcal{D} - \tau\mathcal{O} = \begin{bmatrix} {A_d}^T P + PA_d + K^T {B_d}^T P + PB_d K + \tau\alpha^2 I & -PB_d K - PM_d & P\Gamma_d \\ -K^T {B_d}^T P - {M_d}^T P & {A_d}^T F + FA_d + {C_d}^T {L_d}^T F + FL_d C_d - FM_d - {M_d}^T F & F\Gamma_d \\ {\Gamma_d}^T P & {\Gamma_d}^T F & -\tau I \end{bmatrix} < 0 \quad (31)$$

Substituting $\mathcal{Y} = \frac{P}{\tau}$ and $\Finv = \frac{F}{\tau}$ into (31) yields:

$$\Pi = \begin{bmatrix} {A_d}^T \mathcal{Y} + \mathcal{Y}A_d + K^T {B_d}^T \mathcal{Y} + \mathcal{Y}B_d K + \tau\alpha^2 I & -\mathcal{Y}B_d K - \mathcal{Y}M_d & \mathcal{Y}\Gamma_d \\ -K^T {B_d}^T \mathcal{Y} - {M_d}^T \mathcal{Y} & {A_d}^T \Finv + \Finv A_d + {C_d}^T {L_d}^T \Finv + \Finv L_d C_d - \Finv M_d - {M_d}^T \Finv & \Finv\Gamma_d \\ {\Gamma_d}^T \mathcal{Y} & {\Gamma_d}^T \Finv & -I \end{bmatrix} < 0 \quad (32)$$

The inequality (32) cannot be solved via an LMI since it includes the term $B_d K$ to overcome this problem both sides of (32) must be multiplied by the matrix $\mathcal{W} = \begin{bmatrix} \mathcal{Y}^{-1} & 0 \\ 0 & \mathcal{T} \end{bmatrix}$ where $\mathcal{T} = \begin{bmatrix} \mathcal{Y}^{-1} & 0 \\ 0 & S \end{bmatrix}$ where S is a design parameter.

Hence $\mathcal{W}\Pi\mathcal{W}^T = \begin{bmatrix} \mathcal{Y}^{-1} & 0 \\ 0 & \mathcal{T} \end{bmatrix} \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \begin{bmatrix} \mathcal{Y}^{-1} & 0 \\ 0 & \mathcal{T} \end{bmatrix} =$

$$\begin{bmatrix} \mathcal{Y}^{-1}\Pi_{11}\,\mathcal{Y}^{-1} & \mathcal{Y}^{-1}\Pi_{12}\mathcal{T} \\ \mathcal{T}\Pi_{21}\mathcal{Y}^{-1} & \mathcal{T}\Pi_{22}\mathcal{T} \end{bmatrix} \quad (33)$$

Also, by making a partition of Π_i into four parts:

$$\Pi_{11} = {A_d}^T \mathcal{Y} + \mathcal{Y}A_d + K^T {B_d}^T \mathcal{Y} + \mathcal{Y}B_d K + \tau\alpha^2 I \quad (34)$$

$$\Pi_{12} = [-\mathcal{Y}B_d K - \mathcal{Y}M_d \quad \mathcal{Y}\Gamma_d] \quad (35)$$

$$\Pi_{21} = \begin{bmatrix} -K^T {B_d}^T \mathcal{Y} - {M_d}^T \mathcal{Y} \\ {\Gamma_d}^T \mathcal{Y} \end{bmatrix} \quad (36)$$

$$\Pi_{22} = \begin{bmatrix} {A_d}^T \Finv + \Finv A_d + {C_d}^T {L_d}^T \Finv + \Finv L_d C_d - \Finv M_d - {M_d}^T \Finv & \Finv\Gamma_d \\ {\Gamma_d}^T \Finv & -I \end{bmatrix} \quad (37)$$

The term $\mathcal{T}\Pi_{22}\mathcal{T}$ can then be described using [17] as:

$$\mathcal{T}\Pi_{22}\mathcal{T} \leq -\lambda(\mathcal{T} + \mathcal{T}^T) - \lambda^2\Pi_{22}^{-1} \quad (38)$$

where $\lambda > 0$ is used for tuning to get an acceptable response.

$$\therefore \mathcal{W}\Pi\mathcal{W}^T = \begin{bmatrix} \mathcal{Y}^{-1}\Pi_{11}\,\mathcal{Y}^{-1} & \mathcal{Y}^{-1}\Pi_{12}\mathcal{T} \\ \mathcal{T}\Pi_{21}\mathcal{Y}^{-1} & -\lambda(\mathcal{T} + \mathcal{T}^T) - \lambda^2\Pi_{22}^{-1} \end{bmatrix} < 0 \quad (39)$$

By using the Schur complement (39) could rewritten as:

$$\begin{bmatrix} \mathcal{Y}^{-1}\Pi_{11}\,\mathcal{Y}^{-1} & \mathcal{Y}^{-1}\Pi_{12}\mathcal{T} & 0 \\ \mathcal{T}\Pi_{21}\mathcal{Y}^{-1} & -2\lambda\mathcal{T} & \lambda I \\ 0 & \lambda I & \Pi_{22} \end{bmatrix} < 0 \quad (40)$$

Substituting $\mathcal{P} = \mathcal{Y}^{-1}$ in (40) and also substituting (34), (35), (36) and (37) into (40) yields:

$$\begin{bmatrix} \mathbb{W} & -B_d K\mathcal{P} - M_d\mathcal{P} & \Gamma_d S & 0 & 0 \\ -\mathcal{P}K^T {B_d}^T - \mathcal{P}{M_d}^T & -2\lambda\mathcal{P} & 0 & \lambda I & 0 \\ S{\Gamma_d}^T & 0 & -2\lambda S & 0 & \lambda I \\ 0 & \lambda I & 0 & \mathbb{L} & F\Gamma_d \\ 0 & 0 & \lambda I & {\Gamma_d}^T \Finv & -I \end{bmatrix} < 0 \quad (41)$$

where $\mathbb{W} = \mathcal{P}{A_d}^T + A_d\mathcal{P} + \mathcal{P}K^T {B_d}^T + B_d K\mathcal{P} + \alpha^2\mathcal{P}\mathcal{P}$ and $\mathbb{L} = {A_d}^T \Finv + \Finv A_d + {C_d}^T {L_d}^T \Finv + \Finv L_d C_d - \Finv M_d - {M_d}^T \Finv$

Choose $= I$, and substitute $N = K\mathcal{P}$, $R = Ⅎ L_d$ and $\epsilon = \frac{1}{\alpha^2}$, by using the Schur complement (41) is re-written as:

$$\begin{bmatrix} \overline{\overline{\mathbb{W}}} & -B_d N - M_d\mathcal{P} & \Gamma_d S & 0 & 0 & \mathcal{P} \\ -N^T {B_d}^T - \mathcal{P}{M_d}^T & -2\lambda\mathcal{P} & 0 & \lambda I & 0 & 0 \\ S{\Gamma_d}^T & 0 & -2\lambda S & 0 & \lambda I & 0 \\ 0 & \lambda I & 0 & \overline{\overline{\mathbb{L}}} & F\Gamma_d & 0 \\ 0 & 0 & \lambda I & {\Gamma_d}^T Ⅎ & -I & 0 \\ \mathcal{P} & 0 & 0 & 0 & 0 & -\epsilon I \end{bmatrix} < 0 \quad (42)$$

where $\overline{\overline{\mathbb{W}}} = \mathcal{P}{A_d}^T + A_d\mathcal{P} + N^T{B_d}^T + B_d N$,
$\overline{\overline{\mathbb{L}}} = {A_d}^T Ⅎ + Ⅎ A_d + {C_d}^T R_i^T + RC_d - Ⅎ M_d - {M_d}^T Ⅎ$.

There are *two* approaches to solving the LMI (42)

Algorithm 1:

Minimize ϵ subject to $\mathcal{P} > 0$, $Ⅎ > 0$ and (42).

To minimize the gain magnitude the conditioning of the matrices N and R in terms of norm bounds $\|N\|_2 < K_N I$ and $\|R\|_2 < K_R I$ are used as further inequality conditions [18]:

where K_N and K_R are scalar variables, and by using the Schur complement inequalities (43) and (44) can be added to (42) as follows:

$$\begin{bmatrix} -K_N I & N^T \\ N & -I \end{bmatrix} < 0 \text{ and } \begin{bmatrix} -K_R I & R^T \\ R & -I \end{bmatrix} < 0 \quad (43)$$

Additional inequalities can be added to the matrices $\mathcal{P}$ and $Ⅎ$ [18].

$$\begin{bmatrix} \mathcal{P} & I \\ I & K_P I \end{bmatrix} > 0 \text{ and } \begin{bmatrix} Ⅎ & I \\ I & K_Ⅎ I \end{bmatrix} > 0 \quad (44)$$

where K_P and $K_Ⅎ$ are scalar variables.

Algorithm 2:

Minimize $(\epsilon + K_N + K_P + K_R + K_Ⅎ)$ subject to $\mathcal{P} > 0$,$Ⅎ > 0$, (42) , (43) and (44).

IV. NUMERICAL EXAMPLE

Consider a numerical example consisting of three inter-connected nonlinear systems.

Subsystem 1:

$A_1 = \begin{bmatrix} 0 & -6 \\ 6 & 1 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E_1 = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}$,

$z_1 = (\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_{31} \\ x_{32} \end{bmatrix})$

$W_1(x_1, t) = \begin{bmatrix} 0 \\ 4\cos(2t)x_{11} - 2\sin(t)x_{12} \end{bmatrix}$, $x_1(0) = \begin{bmatrix} 0.4 \\ -0.1 \end{bmatrix}$ and $x_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix}$

Subsystem 2:

$A_2 = \begin{bmatrix} 0 & -1 \\ -2 & -7 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}$,

$z_2 = (\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_{31} \\ x_{32} \end{bmatrix})$

$W_2(x_2, t) = \begin{bmatrix} 0 \\ 2\sin(t)x_{21} + 4\cos(2t)x_{22} \end{bmatrix}$, $x_2(0) = \begin{bmatrix} 0.3 \\ -0.2 \end{bmatrix}$ and $x_2(t) = \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix}$

Subsystem 3:

$A_3 = \begin{bmatrix} 0 & -1 \\ -4 & -5 \end{bmatrix}$, $B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}$,

$z_3 = (\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix})$

$W_3(x_3, t) = \begin{bmatrix} 0 \\ 6\sin(t)x_{31} + 2\cos(2t)x_{32} \end{bmatrix}$, $x_3(0) = \begin{bmatrix} -0.3 \\ -0.3 \end{bmatrix}$ and $x_3(t) = \begin{bmatrix} x_{31}(t) \\ x_{32}(t) \end{bmatrix}$

The systems without feedback are unstable.

Simulation Results

The continuous control $u_i^{OBC}(t)$ designed by LMI where the solution of Algorithm 2 yields the gains:

$K1 = [-4.9522 \quad -4.4306$, $K2 = [5.6854 \quad 3.6946]$ and $K3 = [7.6854 \quad 1.6946]$

$L1 = \begin{bmatrix} 5.2947 & -0.1504 \\ 0 & 0 \end{bmatrix}$, $L2 = \begin{bmatrix} 7.7790 & 0 \\ 0 & 0 \end{bmatrix}$, $L3 = \begin{bmatrix} 7.7790 & 0 \\ 0 & 0 \end{bmatrix}$

where the value of the aggregate system tuning design parameter is $\lambda = 1.86$, where $\mu_1 = \mu_2 = \mu_3 = 5$ and $\mathfrak{z}_1 = \mathfrak{z}_2 = \mathfrak{z}_3 = 0.2$

All three subsystems without controls are unstable. Fig.2 shows the response of all three subsystems using the output decentralized system (LMI+ISMC) with no faults. These results illustrate that the controllers give an acceptable response with stable subsystems.

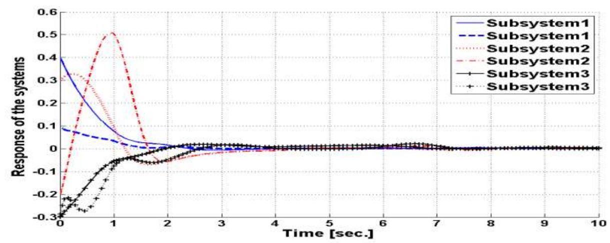

Figure.2 Three subsystems with controls and without faults

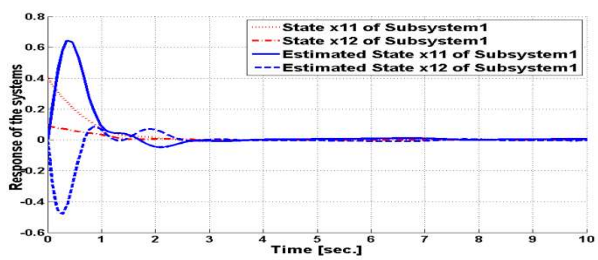

Figure.3 States of 1st subsystem and its estimated

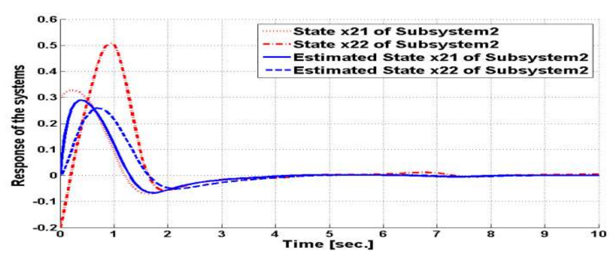

Figure.4 States of 2nd subsystem and it's estimated

Figs. 3, 4 & 5 illustrate the simulation results of the states of every subsystem and their estimates when there are no faults (actuators and sensors). In all subsystems from these results, the estimated and true state values are almost identical, with a little difference at the start of the simulation responses. These estimated states are used as feedback signals to control the subsystems.

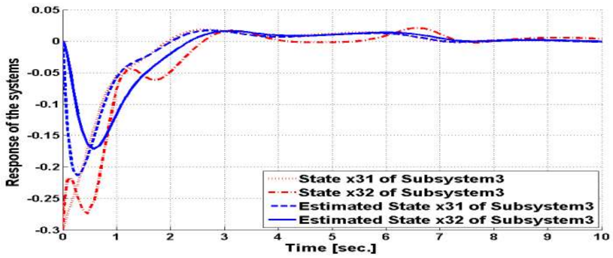

Figure.5 States of 3th subsystem and its estimated

For comparison with the results above, Fig.6 shows the response of all *three* independent linear subsystems using three separate observer-based controls to achieve output decentralized (LMI+ISMC) with no faults, no interconnections and no uncertainties. These results demonstrate very clearly the value of decentralized control approach described in the paper.

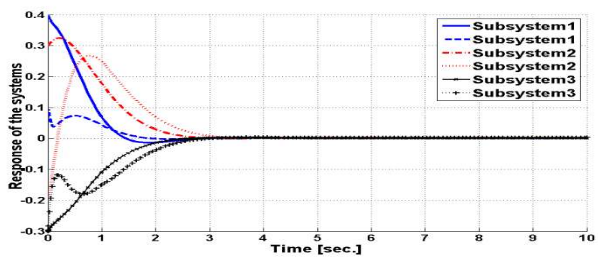

Figure. 6 Three linear subsystems with controls and without (faults + interconnections + uncertainties)

V. Conclusion

A major challenge of the control of uncertain inter-connected systems is to remove or compensate for the effects of uncertainly and disturbances acting in the subsystems so that an ideal decentralization can be achieved. In the ideal case, the resulting hitherto inter-connected system now becomes a truly decentralized structure in which the subsystems can be designed independently. This approach to control of complex systems has important consequences for security and fault-tolerance, e.g. if one subsystem fails then this failure does not influence the integrity of the remaining subsystems.

It is assumed in this work that the subsystem states are not available for control and hence the outputs are used together with the classical notion of state estimate feedback to develop a strategy for decentralization. Hence, the output decentralized control is achieved via ISMC together with linear observer design to give robust performance for both matched and unmatched uncertainty and disturbances. The design uses a single LMI to achieve stability of the aggregate system, minimization of matched/unmatched uncertainties and interactions and control performance specification. Whilst the design procedure is considerably complex the system implementation is nothing more than the ISMC and linear observers applied locally to each subsystem.

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