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Robust Design of a Switched PI Controller for an Uncertain Traffic Model

Antoine Lemarchand , Damien Koenig and John J. Martinez

Abstract— This paper deals with the robust local H_∞ regulation of a freeway section. The model used to describe the on-ramp neighborhood is a discrete-time linear switching model. We extended this model with parametric uncertainties. We propose to design a bank of robust PI controllers to ensure that the system follows the concentration references even in presence of parametric uncertainties and exogenous disturbances. These controllers are computed using LMI resolution.

I. INTRODUCTION

The amount of vehicles on roads increases every day, and causes a waste of time and money [13]. With the construction of new roads, traffic control via on-ramp metering is one of the most explored ways to deal with this problem. The on-ramp metering aims at the improving of traffic condition by controlling the inflow of freeways.

Most of the solution proposed are based on a hierarchical architecture [14],[7],[9]. This architecture is constituted by the high level (⇔optimization layer) and by the low level (⇔local controller level).

The high level layer computes the optimal references (concentrations) to be tracked by the system. This layer is usually based on linear programming algorithms [6],[7],[4]. These references are open loop calculation based on a nominal model. Due to parametric uncertainties and disturbances, the optimal solution computed by this algorithm can't be applied directly on the system.

Therefore, we use the low level layer to ensure that the system really tracks optimal references. In this layer, references are locally applied to on-ramps neighborhoods as shown in Figure 1. Several approaches have been explored to solve this issue such as ALINEA [5], and others such as [15],[16]. In this article, we focus on the local regulation layer. We assume that the optimal trajectories computed by the optimization layer are known. Since the parameters of the model are strongly uncertain, the Cell Transmission Model (CTM) is extended with a model of uncertainties. Thanks to this model, we design a bank of robust switched PI controller which guarantee a certain attenuation of the H_∞ norm of disturbance transfer function.

The paper is organised as follows. A brief presentation of the CTM is provided in Section II. This model is extended with parametric uncertainties and written as a discrete-time switching uncertain model (Section III). The design of robust switched PI controller via LMI formulation is presented in Section IV. Some simulation results are provided in Section V. In Section VI, some concluding remarks end the paper.
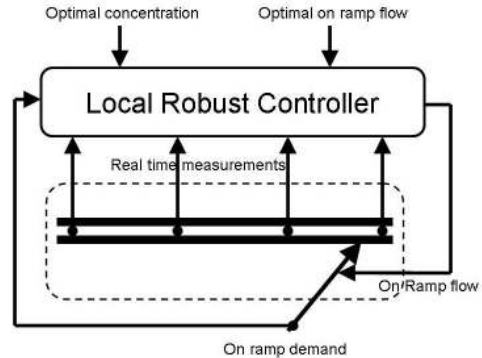

Fig. 1.   Local regulation.

II. CELL TRANSMISSION MODEL

The CTM is a first order linear discrete traffic model proposed in [2],[1] and used in many traffic issues. The CTM model is presented in three parts: Section II-A presents the elementary cells and Section II-B presents the junction model.

A. Elementary Cell

Consider the road section divided into elementary cells. Assuming that in each cell, the density, $\rho(k)$, of vehicles is homogenous. Then $\rho(k)$ can be related to $\phi_i(k)$ and $\phi_{i+1}(k)$ (respectively the flow entering and leaving cell i) by

$$\rho_i(k+1) = \rho_i(k) + \tfrac{T}{l_i}(\phi_i(k) - \phi_{i+1}(k)), \qquad (1)$$
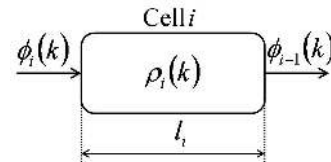
as depicted in Figure 2

Fig. 2.   Elementary cell.

where l_i is the length of cell i and T the period of discretized time. To guarantee numerical stability, T has to fit the condition $T < inf(\tfrac{l_i}{v_i}), \forall i = 1, ..., N$ (N, the number of cells). Notice that (1) is the conservation law of vehicles [8].

B. Junctions

Assuming that a cell can contain at most one on ramp at the beginning of the cell and one off ramp at the end as depicted in Figure 3. Denote u_i the flow entering cell i via on ramp, and ϕ_{out_i} the flow leaving cell $i-1$ via off ramp. We consider that off-ramps have infinite capacities (i.e. no congestion propagates in the freeway section via off-ramps)
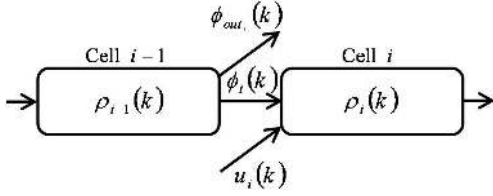

Fig. 3.   Junction.

Define $S_{\phi_i}(k)$ as the flow that upstream cell can supply, and $R_{\phi_i}(k)$ as the flow that downstream cell can receive. $S_{\phi_i}(k)$ and $R_{\phi_i}(k)$ are characterized by

$$S_{\phi_i}(k) = min((1-\beta_i).v_{i-1}.\rho_{i-1}(k), \phi_{M_i}), \quad (2)$$
$$R_{\phi_i}(k) = min(w_i.(\rho_{J_i} - \rho_i(k)) - u_i(k), \phi_{M_i}). \quad (3)$$

with $v_i, (km/h)$ the free flow speed, w_i the backward congestion propagation speed, $\rho_{J_i}, (veh/km)$ the jam density (i.e. maximal density), and $\phi_{M_i}, (veh/h)$ the maximum flow that can travel from upstream to downstream cell and β_i the split ratio (i.e. the part of the flow going out of cell $i-1$ that leave main road via off ramp) defined by

$$\beta_i = \frac{\phi_{out_i}(k)}{\phi_{out_i}(k) + \phi_i(k)}. \quad (4)$$

Obviously the flow moving from upstream to downstream cell is the minimum of (2) and (3), i.e.

$$\phi_i(k) = min(S_{\phi_i}(k), R_{\phi_i}(k)). \quad (5)$$

Junction modes:
From (2), (3), and (5) one can identify the three possible modes of the junction: A free mode (F) where the flow is proportional to the upstream cell concentration, a decoupled mode (D) where the flow is equal to the maximal flow, and a congested mode (C) where the flow is proportional to the remaining space in downstream cell. A graphical representation of (2), (3) and (5) is provided in Figure 4. It is called the fundamental diagram. This kind of diagram appears in every traffic issues using macroscopic model [3].

III. UNCERTAIN SWITCHED CTM

A. Uncertain Parameters

The nominal parameters of the fundamental diagram can be computed using the calibration methods described in [12] with experimental datas[1]. The obtain diagram contains

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[1]Real time measurements realized on D383 road (near Lyon, France) provided by DDE69

---

strong uncertainties. They can be modeled as the following parametric uncertainties:

$$
\begin{aligned}
v_{i-1}(k) &= v_{0_{i-1}} + \Delta v_{i-1}(k), \\
\phi_{M_i}(k) &= \phi_{M_{0_i}} + \Delta\phi_{M_i}(k), \\
w_i(k) &= w_{0_i} + \Delta w_i(k),
\end{aligned}
\quad (6)
$$

where v_0, ϕ_{M_0} and w_0 are respectively the nominal free flow speed, maximum flow, and backward congestion propagation speed computed thanks to [12], and Δv_0, $\Delta\phi_{M_0}$ and Δw_0 the corresponding uncertainties. The new uncertain fundamental diagram is depicted in Figure 4.
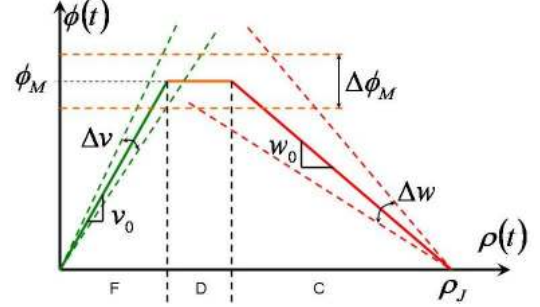

Fig. 4.   Fundamental diagram with parametric uncertainties.

B. Matrix formulation

Consider a freeway section, divided into N cells, $N+1$ junctions and N_{in} inputs. Define

$$\alpha(k) := [\alpha_1(k), \cdots, \alpha_{N+1}(k)] \quad (7)$$

where $\alpha_i(k) \in \{F, D, C\}$ represents the mode of junction i presented in Section II-B.

Proposition 3.1: For all $\alpha(k)$ the dynamics of a freeway section can be written as an uncertain dynamical system:

$$
\begin{aligned}
\rho(k+1) = &\ (A_{0_{\alpha(k)}} + F_{\alpha(k)}\Delta(k)G).\rho(k) \\
&+ B_{\alpha(k)}.u(k) + E_{\alpha(k)}.d(k)
\end{aligned}
\quad (8)
$$

where $\rho(k) = [\rho_1(k), ..., \rho_N(k)]$ is the state vector of vehicle densities for each cell, $u(k)$ the on ramp controlled flows and $d(k)$ a vector of external disturbances. $A_{0_{\alpha(k)}}, B_{\alpha(k)}, E_{\alpha(k)}, F_{\alpha(k)}, G$ are know matrices of appropriate dimensions. Each matrices and vectors are defined bellow. (see (11-14) and (16-17))

Proof: First, the conservation law (1) is rewritten as follows:

$$\rho(k+1) = \rho(k) + \Theta.\phi(k) + B.u(k) \quad (9)$$

where $\phi = [\phi_1, \cdots, \phi_{N+1}]$ is the vector of flows in each junctions, and

$$
\Theta = \begin{pmatrix} \frac{T}{l_1} & -\frac{T}{l_1} & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \frac{T}{l_N} & -\frac{T}{l_N} \end{pmatrix},
$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ \frac{T}{l_{i1}} & 0 & \vdots \\ 0 & \frac{T}{l_{ix}} & \vdots \\ \vdots & 0 & 0 \\ 0 & 0 & \frac{T}{l_{iN_{in}}} \end{pmatrix}.$$

Secondly, the vector $\phi(k)$ is reformulated thanks to the results of Section II-B with a matrix formulation as follows:

$$\phi(k) = A_{\phi_{\alpha(k)}}(k).\rho(k) + B_{\phi_{\alpha(k)}}.u(k) + E_{\phi_{\alpha(k)}}.d(k) \\ + a_{J_{\alpha(k)}}(k).\rho_J + a_{\phi_{M\alpha(k)}}.\phi_M(k), \quad (10)$$

where, $\rho_J := [\rho_{J_1}, \cdots, \rho_{J_{N+1}}]$ is the vector jam densities of each cell, $\phi_M(k) := [\phi_{M_1}(k), \cdots, \phi_{M_{N+1}}(k)]$ is the vector of maximal flows in each junction (see Figure 4), $d(k)$ the boundary conditions, and the matrices $A_{\phi_{\alpha(k)}}(k), B_{\phi_{\alpha(k)}}, E_{\phi_{\alpha(k)}}, a_{J_{\alpha(k)}}(k), a_{\phi_{M\alpha(k)}}$ are respectively described by:

$$A_{\phi_{\alpha(k)}}(k) = \begin{pmatrix} -w_1(k).e_{C_1} & 0 & 0 \\ v_1(k).e_{F_2} & \ddots & 0 \\ 0 & \ddots & -w_N(k).e_{C_N} \\ 0 & 0 & v_N(k).e_{F_{N+1}} \end{pmatrix},$$

$$B_{\phi_{\alpha(k)}} = \begin{pmatrix} 0 & 0 & 0 \\ -e_{C_1} & 0 & \vdots \\ 0 & -e_{C_x} & \vdots \\ \vdots & 0 & 0 \\ 0 & 0 & -e_{C_{Nin}} \end{pmatrix},$$

$$E_{\phi_{\alpha(k)}} = \begin{pmatrix} e_{F_1} & 0 \\ 0 & \vdots \\ \vdots & 0 \\ 0 & e_{C_{N+1}} \end{pmatrix},$$

$$a_{J_{\alpha(k)}}(k) = \begin{pmatrix} e_{C_1}w_1(k) & 0 \\ & \ddots & \\ 0 & e_{C_{N+1}}w_{N+1}(k) \end{pmatrix},$$

$$a_{\phi_{M\alpha(k)}} = \begin{pmatrix} e_{D_1} & 0 \\ & \ddots & \\ 0 & e_{D_N} \end{pmatrix},$$

with,

$$e_{F_i} = \begin{cases} 1 & \text{if junction } i \text{ is Free} \\ 0 & \text{otherwise} \end{cases}$$

$$e_{D_i} = \begin{cases} 1 & \text{if junction } i \text{ is Decoupled} \\ 0 & \text{otherwise} \end{cases}$$

$$e_{C_i} = \begin{cases} 1 & \text{if junction } i \text{ is Congested} \\ 0 & \text{otherwise} \end{cases}$$

Substituting (10) into (9) and using the relations (11-14) the formulation (15) is obtained.

$$A_{\alpha(k)}(k) = I + \Theta.A_{\phi_{\alpha(k)}}(k), \quad (11)$$
$$B_{\alpha(k)} = B + \Theta.B_{\phi_{\alpha(k)}}, \quad (12)$$
$$E_{d_{\alpha(k)}} = \Theta.E_{\phi_{\alpha(k)}}, \quad (13)$$
$$a_{\alpha(k)}(k) = \Theta.(a_{J_{\alpha(k)}}(k).\rho_J + a_{\phi_{M\alpha(k)}}.\phi_M(k)).(14)$$

$$\rho(k+1) = A_{\alpha(k)}(k).\rho(k) + B_{\alpha(k)}.u(k) \\ + E_{d_{\alpha(k)}}.d(k) + a_{\alpha(k)}(k). \quad (15)$$

Notice that (11) and (14) contains uncertain parameters. Let's separate the nominal parts and the uncertain parts of (11) and (14). From (6), the term (11) can be decomposed as a nominal and uncertain part:

$$A_{\alpha(k)}(k) = A_{0_{\alpha(k)}} + F_{\alpha(k)}.\Delta(k).G \quad (16)$$

with,

$$A_{0_{\alpha(k)}} = A_{\alpha(k)}(k)|_{v_i(k)=v_{0_i},w_i(k)=w_{0_i}},$$

$$F_{\alpha(k)} = \Theta. \begin{pmatrix} 0 & \cdots & 0 & e_{C_1}. & 0 \\ e_{F_2} & & 0 & & \ddots \\ & \ddots & & 0 & e_{C_N} \\ 0 & e_{F_{N+1}} & 0 & \cdots & 0 \end{pmatrix},$$

$$G = \begin{pmatrix} diag(||\Delta v_i||_\infty)_2^{N+1} & 0 \\ 0 & diag(||\Delta w_i||_\infty)_1^N \end{pmatrix}$$

and,

$$\Delta(k) = \begin{pmatrix} diag(\delta v_i(k))_2^{N+1} & 0 \\ 0 & diag(\delta w_i(k))_1^N \end{pmatrix}$$
$$\delta v_i(k) = \frac{\Delta v_i(k)}{||\Delta v_i||_\infty}$$
$$\delta w_i(k) = \frac{\Delta w_i(k)}{||\Delta w_i||_\infty}$$

so $\Delta(k)^T \Delta(k) < 1$.

From (6) the term (14) can be written as an external disturbance:

$$a_{\alpha(k)}(k) = E_{\phi_{M\alpha(k)}}.\phi_M(k) + E_{J_{\alpha(k)}}.\rho_J + E_{w_{\alpha(k)}}.w_\Delta(k) \quad (17)$$

with:

$$E_{\phi_{M\alpha(k)}} = \Theta.a_{\phi_{M\alpha(k)}} \quad (18)$$
$$E_{J_{\alpha(k)}} = \Theta.a_{J_{\alpha(k)}}(k)|_{w_i(k)=w_{0_i}} \quad (19)$$
$$E_{w_{\alpha(k)}} = \Theta.diag(e_{C_i})_1^{N+1} \quad (20)$$
$$E_\Delta(k) = \begin{pmatrix} \Delta w_1(k).\rho_{J_1} \\ \vdots \\ \Delta w_{N+1}(k).\rho_{J_{N+1}} \end{pmatrix} \quad (21)$$

Substituting (16) and (17) in (15), the formulation (8) of the proposition is obtained. ∎

IV. REGULATOR DESIGN

A. Study case

For our study, we consider that optimal trajectories ($\rho^*(k)$ for mainline concentration and $u^*(k)$ for on-ramp flow are known. Without loss of generality, we consider the neighborhood depicted in Figure 5.
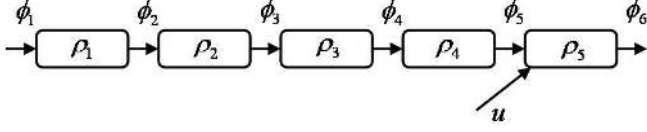

Fig. 5.   On-ramp neighborhood.

We consider here that in normal traffic condition (i.e. without accidents) the different mode of the section that can occurs are the following:

- case 1 : $\alpha(k) = [FFFFFF]$
- case 2 : $\alpha(k) = [FFFFFD]$
- case 3 : $\alpha(k) = [FFFFCD]$
- case 4 : $\alpha(k) = [FFFCCD]$
- case 5 : $\alpha(k) = [FFCCCD]$
- case 6 : $\alpha(k) = [FCCCCD]$

$\alpha(k)$ defined in 7. We notate Ω this set of case. In case 1, all the junctions of the section are in free mode (F). Just before congestion appears (case 2) junction 4 switches to decoupled mode (D). Then in case 3, 4, 5 and 6, the congestion propagates backward (i.e. junction 5 to 2 switch to congested mode (C)). We denote τ_{ij} the transition from case i to case j. So the set of possible transition is:

$$\Upsilon = \{\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}, \cdots, \tau_{55}, \tau_{65}, \tau_{66}\}. \quad (22)$$

B. Extended system

The problem now, is to design a switched PI controller in order to stabilize the system and attenuate the disturbances. For this purpose, we have to extend our system with an integrator. To keep the stabilizability property of the system, we can only extend one state of the system with an integrator. For the case 1 and 2, we choose to extend the system with an integrator on $\epsilon_5(k)$ (where $\epsilon_5(k) = \rho_5(k) - \rho_5^*(k)$) . For the case 3 to 6, we choose to extend the system with an integrator on the density of the cell where the congestion front stand (respectively $\epsilon_4(k)$ to $\epsilon_1(k)$). So, the new state vector becomes

$$X(k) = \begin{pmatrix} \rho(k) \\ z(k) \end{pmatrix} \quad (23)$$

with:

$$z(k+1) = z(k) + \begin{cases} \epsilon_5(k) & \text{, if case 1 or 2} \\ \epsilon_4(k) & \text{, if case 3} \\ \epsilon_3(k) & \text{, if case 4} \\ \epsilon_2(k) & \text{, if case 5} \\ \epsilon_1(k) & \text{, if case 6} \end{cases} \quad (24)$$

The extended system is described in a compact form as follows:

$$\begin{cases} X(k+1) = & (A_{a_{\alpha(k)}} + F_{a_{\alpha(k)}}.\Delta(k).G_a).X(k) \\ & +B_{a_{\alpha(k)}}.u(k) + E_{a_{\alpha(k)}}.w(k) \\ Z(k) = & C_{a_{\alpha(k)}}.X(k) \end{cases} \quad (25)$$

with

$$A_{a_{\alpha(k)}} = \begin{pmatrix} A_{0_{\alpha(k)}} & 0 \\ A_{z_{\alpha(k)}} & 1 \end{pmatrix},$$

$$F_{a_{\alpha(k)}} = \begin{pmatrix} F_{1_{\alpha(k)}} \\ 0 \end{pmatrix},$$

$$G_a = \begin{pmatrix} G & 0 \end{pmatrix},$$

$$B_{a_{\alpha(k)}} = \begin{pmatrix} B_{\alpha(k)} \\ 0 \end{pmatrix},$$

$$E_{a_{\alpha(k)}} = \begin{pmatrix} f_d.E_{\alpha(k)} & 0 & f_\phi.H_{\phi_{M_{\alpha(k)}}} \\ 0 & -f_z.A_{z_{\alpha(k)}} & 0 \\ f_J.H_{J_{\alpha(k)}} & f_w.H_{w_{\alpha(k)}} \\ 0 & 0 \end{pmatrix},$$

$$C_{a_{\alpha(k)}} = \begin{pmatrix} A_{z_{\alpha(k)}} & 0 \end{pmatrix},$$

$$\bar{d}(k) = \begin{pmatrix} d(k) & \rho^*(k) & \phi_M(k) & \rho_J & w_\Delta(k) \end{pmatrix}^T,$$

According to (24), we have

$$A_{z_{\alpha(k)}} = \begin{cases} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \text{, if case 1 or 2} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix} & \text{, if case 3} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix} & \text{, if case 4} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \end{pmatrix} & \text{, if case 5} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} & \text{, if case 6} \end{cases}$$

f_d, f_z, f_ϕ, f_J and f_w are added to the matrix $E_{a_{\alpha(k)}}$ to weight the influence of the different perturbations for the H_∞ attenuation criteria.

C. LMI formulation

Take the following notations, $\alpha(k) = i, \alpha(k+1) = j$. Assume that each cell density is measured. This seems to be a fair assumption since sensor becomes cheaper and cheaper, and since much roads are equipped of sensors.

Proposition 4.1: if $\exists\ Q_i = Q_i^T > 0, e_i > 0, U_i, i \in \Omega$ such that (26) holds for all transition $\tau_{ij} \in \Upsilon$,

Then:

- The extended system (25) is stable under state feedback $u(k) = -K_i X(k), K_i = U_i Q_i^{-1}$,
- The poles of the system are placed in a circle of center σ and radius r,
- The H_∞ norm of the transfer function between $w(k)$ and $Z(k)$ is bounded by γ (weighted by f_d, f_z, f_ϕ, f_J and f_w).

Proof: The core of our approach is the use of the following candidate Lyapunov function

$$V_i(k) = X(k)^T P_i X(k). \quad (27)$$

Where P_i is a positive definite matrix. Since $P_i = P_i^T > 0$. We can write the following H_∞ attenuation criteria:

$$\begin{aligned} X(k+1)^T P_j X(k+1) - X(k)^T P_i X(k) \\ + Z(k)^T Z(k) - \gamma^2 w(k)^T w(k) < 0 \end{aligned} \quad (28)$$

$$\begin{pmatrix} \epsilon_i F_{a_i} F_{a_i}^T - rQ_j & A_{a_i}Q_i + B_{a_i}U_i - \sigma Q_i & rE_{a_i} & 0 & 0 \\ * & -rQ_i & 0 & Q_iG_a^T & Q_iC_a^T \\ * & * & -r\gamma^2 I & 0 & 0 \\ * & * & * & -\epsilon_i I & 0 \\ * & * & * & * & -r^{-1}I \end{pmatrix} < 0 \qquad (26)$$

Substituting (25) in (28), the following condition is obtained:

$$((A_{a_i} + F_{a_i}.\Delta(k).G_a - B_{a_{\alpha(k)}}K_i).X(k) + E_{a_i}.\bar{d}(k))^T$$
$$.P_j((A_{a_i} + F_{a_i}.\Delta(k).G_a - B_{a_{\alpha(k)}}K_i).X(k)$$
$$+E_{a_i}.\bar{d}(k)) - X(k)^T P_i X(k) + (C_iX(k))^T(C_iX(k)) \qquad (29)$$
$$-\gamma^2 \bar{d}(k)^T \bar{d}(k) < 0,$$

Taking the notation $A_{a_i} - B_{a_i}K_i = \Gamma_i$ the previous inequality is equivalent to:

$$\begin{pmatrix} \Pi_{11}^1 & \Pi_{21}^1 \\ * & E_{a_i}^T E_{a_i} - \gamma^2 I \end{pmatrix} < 0 \qquad (30)$$

$$\Pi_{11}^1 = \quad (\Gamma_i + F_{a_i}.\Delta(k).G_a)^T P_j(\Gamma_i + F_{a_i}.\Delta(k).G_a)$$
$$-P_i + C_i^T C_i$$
$$\Pi_{21}^1 = \qquad (\Gamma_i + F_{a_i}.\Delta(k).G_a)^T P_j E_{a_i}$$

Applying Schur complement, we obtain the following inequality:

$$\begin{pmatrix} -P_i + C_i^T C_i & 0 & (\Gamma_i + F_{a_i}.\Delta(k).G_a)^T \\ * & -\gamma^2 I & E_{a_i} \\ * & * & -P_j^{-1} \end{pmatrix} < 0 \qquad (31)$$

Using majoration lemma, we can demonstrate that (31) holds if $\exists \epsilon_i > 0$ such that

$$\begin{pmatrix} -P_i + C_i^T C_i & & \\ +\epsilon_i^{-1}G_a^T G_a & 0 & \Gamma_i^T \\ * & -\gamma^2 I & E_{a_i} \\ * & * & -P_j^{-1} + \epsilon_i F_{a_i} F_{a_i}^T \end{pmatrix} < 0 \qquad (32)$$

Applying the Schur complement

$$\begin{pmatrix} -P_i + C_i^T C_i + \epsilon_i^{-1}G_a^T G_a & 0 \\ * & -\gamma^2 I \end{pmatrix}$$
$$-\begin{pmatrix} \Gamma_i^T \\ E_{a_i}^T \end{pmatrix} (P_j^{-1} - \epsilon_i F_{a_i}F_{a_i}^T)^{-1} \begin{pmatrix} \Gamma_i & E_{a_i} \end{pmatrix} < 0 \qquad (33)$$

Multiplying both side of (33) by $\begin{pmatrix} Pi^{-1} & 0 \\ 0 & I \end{pmatrix}$ and applying Schur complement, (34) is obtain.

$$\begin{pmatrix} -P_j^{-1} + \epsilon_i F_{a_i}F_{a_i}^T & \Gamma P_i^{-1} & E_{a_i} \\ * & \Pi_{22}^2 & 0 \\ * & * & -\gamma^2 I \end{pmatrix} < 0 \qquad (34)$$

$$\Pi_{22}^2 = -P_i^{-1} + P_i^{-1}C_i^T C_i P_i^{-1} + \epsilon_i^{-1}P_i^{-1}G_a^T G_a P_i^{-1}$$

Applying twice Schur lemma on term Π_{22}^2, (34) becomes:

$$\begin{pmatrix} -P_j^{-1} + \epsilon_i F_{a_i}F_{a_i}^T & \Gamma P_i^{-1} & E_{a_i} & 0 & 0 \\ * & -P_i^{-1} & 0 & P_i^{-1}G_a^T & P_i^{-1}C_i^T \\ * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & -\epsilon_i I & 0 \\ * & * & * & * & -I \end{pmatrix} < 0 \qquad (35)$$

It is stated in that In order to improve the performance, the poles of the closed loop system are placed in a particular region. like in [11], we place the poles in a circle of center σ and radius r by replacing $A_{a_i} - B_{a_i}K_i$ by $\frac{A_{a_i} - B_{a_i}K_i - \sigma I}{r}$. This particular region is depicted in Figure 6. With this change and taking $P_i^{-1} = Q_i$, $P_j^{-1} = Q_j$ and $U_i = K_iP_i^{-1}$ (26) is obtained. ∎
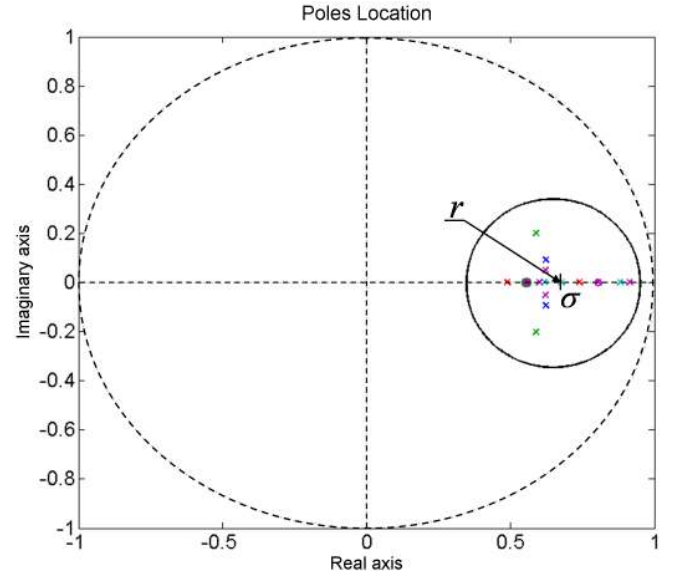

Fig. 6. Pole placement region.

V. SIMULATION RESULTS

For simulation, we take the on-ramp neighborhood depicted in Figure 5. We choose the following parameters (cells are homogenous):

- Cell length $l = 0.3 km$
- Free flow speed $v = 80 km/h \pm 5\%$
- Backward congestion speed $v = 35 km/h \pm 15\%$
- Maximum flow $\phi_M = 7000 km/h \pm 8\%$

These parameters values and uncertainties ranges are chosen according to experimental data.

We take the following weighting constants: $f_d = 1, f_z = 1/200, f_J = 1/22, f_p = 1, f_w = 1/4$. These constants are chosen with respect to the range of disturbance values.

To compute our controllers, we take $\sigma = 0.6$ and $r = 0.35$ for transition $\tau_{ij}, i = j$ and $\sigma = 0$ and $r = 1$ for transition $\tau_{ij}, i \neq j$, this means that we doesn't place pole for the switching between modes. This leads to reduce the conservatism of our approach. The result of pole placement for each of the 6 cases is depicted in Figure 6. We detail our results for case 4. We obtain the following

state feedback gain:

$$K_4 = \begin{pmatrix} 221.24 & 216.49 & 191.24 & 113.05 & 36.36 & 13.53 \end{pmatrix}$$

The sensitivity functions between disturbances and ρ_3 are given in Figure 7, we can see only 6 sensitivity function because the matrix E_{a_4} has several columns of zeros. The attenuation is good in low frequency, where disturbances occurs [10].
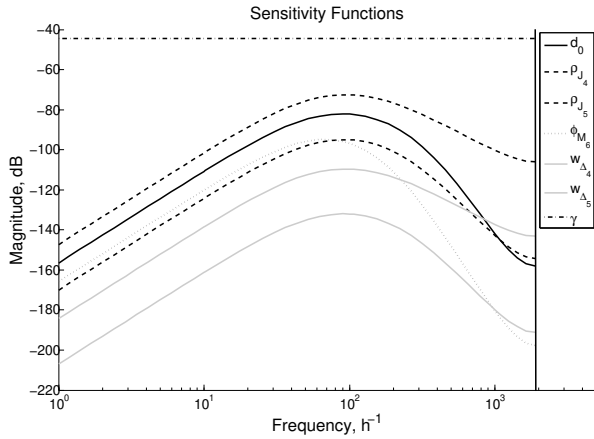

Fig. 7. Sensitivity functions.

In the scenario we have chosen, the system has to track a congestion front which propagates backward then forward. The system goes from case 1 to case 5 via all the intermediate cases then goes back to case 1. The result of the simulation is depicted in Figure 8. We can see that the system is following the congestion front despite parametric uncertainties and disturbance. The active controller (K_1 to K_5) is chosen with respect to traffic condition. The choice of this controller is based on the nominal parameters of the system and so the controller chosen may not be the appropriate one. This can explain the switching problem that occurs around $t = 1.7h$ when the system switches from case 3 to case 2.
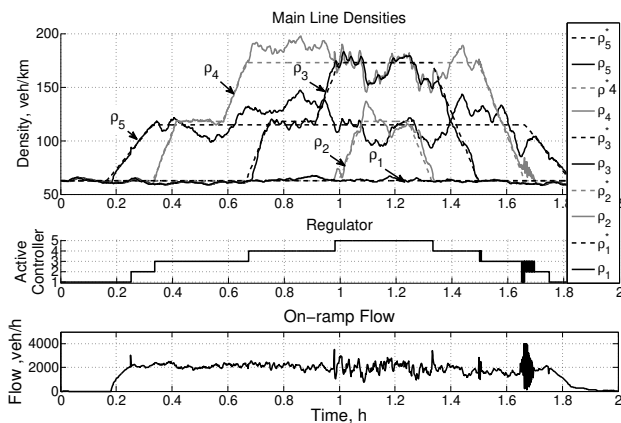

Fig. 8. Simulation results.

VI. Conclusions

In this paper, the CTM has been extended with parametric uncertainties. A new robust switched H_∞ PI controller has been designed, taking account of the uncertainty model. The simulation result shows that this solution is efficient to track congestion front, and shows a good attenuation of disturbance. Some improvements on switching conditions and queue regulation still have to be done.

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