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Robust envelope-constrained filter with orthonormal bases and semi-definite and semi-infinite programming

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Abstract In this paper, the equivalence relation between a semi-infinite quadratically constrained convex quadratic programming problem and a combined semi-definite and semi-infinite programming problem is considered. Then, an efficient and reliable discretization algorithm for solving a general class of combined semi-definite and semi-infinite programming problems is developed. Both the continuous-time envelope-constrained optimal equalization filter and the corresponding robust envelope-constrained filter for a communication channel are solved by using the proposed algorithm.

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1 Introduction

In signal processing, the response of a filter with a linear time invariant finite impulse response $u(t)$ to a given input signal $s(t)$ can be corrupted by random noise. The filter output response should consist of two components $\psi(t)$ and $\zeta(t)$ that are due to the signal and noise, respectively. The continuous-time envelope-constrained (EC) filter design problem is to design a filter which minimizes the output noise power (Evans et al. 1977) while the noiseless output response $\psi(t)$ fits into an output pulse shape envelope defined by the lower and upper boundaries $\xi^-(t)$ and $\xi^+(t)$, respectively. Due to the fact that the output noise power is proportional to the squared L^2 norm of the filter when the input noise is white, the EC filter problem is to design a filter such that its squared L^2 norm is minimized, whereas $\psi(t)$ fits into an output pulse shape envelope. The EC filter problem was first posed in the early 1970*s* (see Evans et al. 1977, Fortmann and Evans 1974). Since then, it has become an active field. This filter design problem can be cast as a semi-infinite optimization problem involving a strictly quadratic cost and continuous linear constraints. Various methods for solving this problem have been obtained in the literature (see, for example, Evans et al. 1977, Tseng et al. 1999 and Vo et al. 1995). In Tseng et al. (1999), it is shown by using Carathéodory's dimensional theorem that the continuous-time EC filtering problem is equivalent to a finite optimization problem. The optimal solution obtained using these methods are such that the noiseless output response of the optimum filter to the prescribed input signal touches the output boundaries at some points. Thus, any perturbation of the prescribed input signal or error in the implementation of the optimal filter will result in the envelope constraints being violated. Clearly, it is of practical importance to improve the tolerance of the optimal filter to perturbations on the input signal and implementation errors. One approach is to maximize the minimum distance between the output response and the output envelope constraints, subject to a specified allowable increase in the optimal noise power gain. This formulation was called a robust envelope-constrained filter problem. It was first proposed in (Cantoni 1998) and (Zang et al. 1996) as a semi-infinite constrained optimization problem involving a linear cost, continuous linear constraints and a quadratic constraint. In Tseng et al. (2000), the robust EC filter problem is converted into an equivalent strictly convex constrained optimization problem with integral cost. Its solution can be obtained by solving a sequence of strictly convex optimization problems. This method is computationally rather expensive. There are the following two reasons. The first reason is the requirement of the numerical integration of the cost function. To explain the second reason, we need to point out that the solution method used in solving the sequence of strictly convex optimization problems with integral cost is the sequential quadratic programming technique with active set strategy (see, for example, Chap. 3 of Teo 1991 for details), where the search direction at each iteration is determined by the solution of a quadratic programming problem involving a quadratic cost and a set of linear constraints. The quadratic cost is obtained by taking the quadratic approximation of the cost function, while the linear constraints are from the linear approximation of the constraints. Thus, the sequential quadratic programming technique may not be computationally efficient for problems involving quadratic constraint such as the case of the robust EC filtering problem. In this paper, we aim to develop a more efficient method. For this, we first show that both the continuous-time EC filter and robust EC filter problems can be converted into respective equivalent combined semi-definite and semi-infinite programming problems. Then, an efficient and practical method is developed for solving a general combined semi-definite and semi-infinite programming (SDSIP) problem, which includes those mentioned above as special cases. The algorithm obtained is allowed to have an infeasible starting point. It is then used to solve the continuous-time EC filter and robust EC filter problems. Numerical results obtained in Sect. 5 clearly indicate the effectiveness of the proposed algorithm.

The rest of the paper is organized as follows. In Sect. 2, the continuous-time EC filtering problem and robust EC filtering problem are formulated. In Sect. 3, the relationship between a general semi-infinite quadratically constrained convex quadratic programming problem and a (SDSIP) problem is established. In Sect. 4, the Lagrangian dual problem is introduced for the (SDSIP) problem. An efficient and practical algorithm allowing an infeasible starting point is developed for solving the (SDSIP) problem. General convergence of the algorithm is established in the paper. In Sect. 5, the continuous-time EC filter and the robust envelope-constrained filter problems are solved by the proposed algorithm.

2 Problem formulations

In this section, we shall first review the formulations of the continuous-time EC filter and robust EC filter problems with orthonormal basis (Zang et al. 1996; Tseng et al. 2000). These problems can be written as respective semi-infinite constrained convex quadratic programming problems. Then, a general semi-infinite quadratically constrained convex quadratic programming problem, including those mentioned above as special cases, is introduced.

2.1 EC filter with orthonormal basis of $L^2([0,\infty))$

Let $L^2([0,\infty))$ denote the Hilbert space consisting of all real-valued Lebesgue measurable and square integrable functions on the semi-infinite interval $[0,\infty)$ with inner product

$$\langle f, g\rangle = \int_0^\infty f(t)g(t)dt, \quad \forall f, g \in L^2([0,\infty)).$$

The norm of $f \in L^2([0,\infty))$ is defined by

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_0^\infty f(t)^2 dt}.$$

Let $\{\varphi_j\}_{j=0}^{\infty}$ be a complete orthonormal basis of $L^2([0,\infty))$ space. Then, any $f \in L^2([0,\infty))$ can be expressed as

$$f(t) = \sum_{j=0}^{\infty} x_j \varphi_j(t) \quad \text{and} \quad x_j = \langle f, \varphi_j \rangle, \quad j = 0, 1, \ldots,$$

where

$$\langle \varphi_i, \varphi_j \rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

Let $u(t) \in L^2([0,\infty])$ be the impulse response of a linear-invariant filter and $\psi(t)$ be the continuous noiseless output response with support in $[0,\infty)$. Then, the output response $\psi(t)$ can be expressed as

$$\psi(t) = \int_0^{\infty} u(\tau) s(t-\tau) d\tau,$$

where $s(t)$ is the continuous input signal with support in $[0,\infty)$. It follows from (Evans et al. 1977; Tseng et al. 1999; Tseng et al. 2000; Vo et al. 1995) that the continuous-time EC filtering problem may be posed as follows:

$$\begin{aligned} &\min \quad \|u\|^2 = u^T u, \quad x \in \mathcal{R}^N \\ &\text{s.t.} \quad \xi^-(t) \le \psi(x,t) \le \xi^+(t), \quad \forall t \in [0,\infty), \end{aligned} \tag{1}$$

where $\xi^+(t)$ and $\xi^-(t)$ denote the continuous upper and lower mask boundaries, Since the function $u(t) \in L^2([0,\infty))$, $u(t)$ can be expressed as

$$u(t) = \sum_{j=0}^{\infty} x_j \varphi_j(t), \tag{2}$$

where $x_j = \langle u, \varphi_j \rangle$, $j = 0, 1, \ldots$ are the filter coefficients.

Consider only those filters $u_N(t)$ whose impulse responses are represented by a finite expansion on the orthonormal basis:

$$u_N(t) = \sum_{j=0}^{N-1} x_j \varphi_j(t). \tag{3}$$

The corresponding output response is

$$\psi_N(t) = \int_0^T u_N(\tau) s(t-\tau) d\tau = \Theta^T(t) x, \tag{4}$$

where the vector of filter coefficients $x = [x_0, x_1, \ldots, x_{N-1}]^T \in \mathcal{R}^N$, and the input signal $\Theta(t) = [\theta_0(t), \theta_1(t), \ldots, \theta_{N-1}]^T$ with

$$\theta_j(t) = \int_0^T \varphi_j(\tau) s(t-\tau) d\tau, \quad j = 0, \ldots, N-1, \tag{5}$$

where $\theta_j \in C([0,T])$, $j = 0, 1, \ldots, N-1$, and $C([0,T])$ denotes the Banach space of all real-valued continuous functions on $[0,T]$ with the norm defined by

$$\|\theta\|_{C[0,T]} = \max_{0\le t\le T} |\theta(t)|.$$

The norm $\|u_N\|$ of the filter can be written as:

$$\|u_N\| = \left(\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} x_i x_j \langle \varphi_i, \varphi_j\rangle\right)^{\frac{1}{2}} = (x^T x)^{\frac{1}{2}} = \|x\|. \tag{6}$$

From (3–6), the continuous-time EC filter problem with orthonormal basis can be written as the following semi-infinite programming problem (P):

$$\begin{aligned} \min \quad & \|x\|^2 = x^T x, \quad \in \mathcal{R}^N \\ \text{s.t.} \quad & \xi^-(t) \le \psi_N(x,t) \le \xi^+(t), \quad \forall t \in [0,T]. \end{aligned} \tag{7}$$

2.2 Robust EC filter formulation

For a given filter coefficient x, we define

$$\begin{aligned} [\phi^+(x)](t) &= \xi^+(t) - \psi_N(x,t), \\ [\phi^-(x)](t) &= \psi_N(x,t) - \xi^-(t). \end{aligned}$$

To quantify the notion of robustness, we define the constraint robustness margin as

$$\sigma(x) = \min\left\{ \min_t [\phi^+(x)](t), \min_t [\phi^-(x)](t) \right\}.$$

In practice, it may be required to have a larger constraint robustness margin over certain intervals. In this case, a weighting function β can be used to achieve the purpose. More specifically, we define the weighted constraint robustness margin as follows:

$$\sigma_\beta(x) = \min\left\{ \min_t \frac{[\phi^+(x)](t)}{\beta(t)}, \min_t \frac{[\phi^-(x)](t)}{\beta(t)} \right\}, \tag{8}$$

where β is a positive continuous weighting function that is normalized so that it attains a minimum of unity. Note that if x^* is the optimal solution of problem (P) and at least one of the constraints is active at the solution x^*, then $\sigma_\beta(x^*) = 0$.

The EC filtering problem with the constraint robustness is formulated (see Tseng et al. 2000) as the following semi-infinite programming problem (Q):

$$\begin{aligned} \max \quad & \sigma_\beta \\ \text{s.t.} \quad & \xi^-(t) + \beta(t)\sigma_\beta \le \psi_N(x,t) \le \xi^+(t) - \beta(t)\sigma_\beta, \quad \forall t \in [0,T], \\ & \|x\|^2 \le (1+\delta)\|x^*\|^2, \end{aligned} \tag{9}$$

where $\delta > 0$ is a constant that specifies the allowable amount of increase of the output noise power gain and x^* is the prior solution of (7).

The problem (Q) is much more expensive to solve than the problem (P) because of the additional quadratic constraint.

2.3 Semi-infinite quadratically constrained convex quadratic programming

Let

$$B(t)=\begin{bmatrix}\Theta^T(t)\\ -\Theta^T(t)\end{bmatrix}_{2\times N},\qquad b(t)=\begin{bmatrix}\xi^+(t)\\ \xi^-(t)\end{bmatrix}_{2\times 1},\qquad C=\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}_{N\times N}.$$

The EC filtering problem (P) can be cast as the following programming problem.

$$\begin{aligned}\min\quad & x^T C x\\ \text{s.t.}\quad & B(t)x-b(t)\le 0_2,\quad \forall t\in[0,T].\end{aligned}\tag{10}$$

On the other hand, by letting

$$B_1(t)=\begin{bmatrix}\Theta^T(t) & \beta(t)\\ -\Theta^T(t) & \beta(t)\end{bmatrix}_{2\times(N+1)},\qquad b(t)=\begin{bmatrix}\xi^+(t)\\ \xi^-(t)\end{bmatrix}_{2\times 1},$$

$$B_2=\begin{bmatrix}1 & & & 0\\ & \ddots & & \\ & & 1 & \\ 0 & & & 0\end{bmatrix}_{(N+1)\times(N+1)},$$

where $B_3=c=(0,\dots,0,-1)^T$ and $y=(x^T,\sigma_\beta)^T$ are $N+1$ dimensional vectors, the robust EC filtering problem (Q) can be written as the following programming problem.

$$\begin{aligned}\min\quad & c^T y\\ \text{s.t.}\quad & B_1(t)y-b(t)\le 0_2,\quad t\in[0,T],\\ & y^T B_2 y-(1+\delta)\|x^*\|^2\le 0_1,\\ & B_3^T y\le 0_1.\end{aligned}\tag{11}$$

Let us now consider a general semi-infinite quadratically constrained convex quadratic programming problem (Q^2P):

$$\begin{aligned}\min\quad & x^T Q_0 x+b_0^T x+c_0\\ \text{s.t.}\quad & x^T Q(t)x+b(t)^T x+c(t)\le 0,\quad t\in B.\end{aligned}\tag{12}$$

Here B is a compact set, Q_0 and $Q(t), t\in B$, are positive semi-definite matrices and x is an n dimensional vector in R^n. Clearly, the problems (10) and (11) are two special cases of the problem (Q^2P).

When the parameter set B is finite, the corresponding version of the problem (Q^2P) reduces to a special case considered in (Wolkowicz et al. 2000).

To avoid the trivial solution $u_N(t)=0$, i.e., $x=0_N$, we impose the following assumption.

Assumption 1 There exists at least one point in the output mask at which the upper and lower mask boundaries have the same sign, i.e., there exists at least one $t_0 \in [0, T]$ such that $\xi^+(t_0)\xi^-(t_0) > 0$.

3 Property of the optimal solution

In this section, we shall show that the problem (Q^2P) can be transformed into a combined semi-definite and semi-infinite programming (SDSIP) problem such that the solution of (Q^2P) can be obtained by solving a corresponding (SDSIP) problem to be defined in this section.

From Sect. 2, we see that the EC filtering problem and the robust EC filtering problem are special cases of the semi-infinite quadratically constrained convex quadratic programming problem (Q^2P). Thus, by virtue of the relationship between the problem (Q^2P) and the (SDSIP) problem, we can obtain the solutions to the EC filtering and robust filtering problems by solving their corresponding (SDSIP) problems.

Let S^n denote the set of real symmetric $n \times n$ matrices. The standard inner product on S^n is

$$A \bullet B = \mathrm{tr}\{AB\} = \sum_{i,j} a_{ij} b_{ij}.$$

By $X \succeq 0$, where $X \in S^n$, we mean that the matrix X is positive semidefinite. S^n_+ denotes the set of all positive semidefinite matrices in S^n.

Consider the problem (Q^2P). Suppose that

$$P_0 = \begin{pmatrix} c_0 & b_0^T/2 \\ b_0/2 & Q_0 \end{pmatrix}, \qquad P(t) = \begin{pmatrix} c(t) & b(t)^T/2 \\ b(t)/2 & Q(t) \end{pmatrix}, \quad t \in B,$$

and

$$y = (y_0, x^T)^T \in R^{n+1}.$$

Then, the problem (Q^2P) can be re-formulated equivalently as $(Q^2P)_y$ given below.

$$\begin{aligned} \min \quad & y^T P_0 y \\ \text{s.t.} \quad & y \in F, \end{aligned} \tag{13}$$

where

$$F = \{y \in R^{n+1} \mid y^T P(t) y \le 0, \ \forall t \in B, y_0 = 1\}.$$

Define

$$\Phi = \{Y \in S^{n+1}_+ \mid Y_{00} = 1, \ P(t) \bullet Y \le 0, \ \forall t \in B\}, \tag{14}$$

where Y_{00} denotes the element of the first row and the first column of the matrix Y.

We introduce a combined semi-definite and semi-infinite programming problem:

$$\begin{aligned} \min \quad & P_0 \bullet Y \\ \text{s.t.} \quad & Y \in \Phi. \end{aligned} \tag{15}$$

Let

$$\hat{F} = \{Ye_1 \mid Y \in \Phi\},$$

where e_1 is an $(n+1)$-dimensional unit vector with its first component being 1 and other components zero.

We introduce another quadratic programming problem

$$\begin{aligned} \min \quad & y^T P_0 y \\ \text{s.t.} \quad & y \in \hat{F}. \end{aligned} \tag{16}$$

It follows readily that Φ and $\hat{F}$ are convex subsets in S^{n+1} and $\mathcal{R}^{n+1}$, respectively.

Lemma 3.1 *Suppose that $Q(t)$, $t \in B$, is a positive semi-definite matrix. Then,*

$$y^T P(t) y \le 0, \quad t \in B \quad \textit{and} \quad y_0 = 1, \quad \forall y \in \hat{F}.$$

Proof Take any $y \in \hat{F}$. Then, there exists a $Y \in \Phi$ such that

$$y = Ye_1.$$

It follows from $Y \in \Phi$ and $y = Ye_1$ that

$$y_0 = 1 \quad \text{and} \quad P(t) \bullet Y \le 0, \quad t \in B.$$

Let

$$Y = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}.$$

Then,

$$y = Ye_1 = \begin{pmatrix} 1 \\ x \end{pmatrix},$$

and, for any $t \in B$,

$$\begin{aligned} y^T P(t) y &= x^T Q(t) x + b^T(t) x + c(t) \\ &= Q(t) \bullet X + b^T(t) x + c(t) - Q(t) \bullet (X - xx^T) \\ &\le -Q(t) \bullet (X - xx^T). \end{aligned}$$

Since Y is a positive semi-definite matrix, we have

$$X - xx^T \in S^n_+.$$

It follows from $Q(t) \in S^n_+$, $t \in B$, that

$$Q(t) \bullet (X - xx^T) \ge 0, \quad t \in B.$$

Thus,

$$y^T P(t) y \le 0, \quad t \in B.$$

This completes the proof. □

The following result shows that the optimal costs of the problem (15) and the problem (16) are equivalent.

Theorem 3.1 *Suppose that* Q_0 *and* $Q(t)$, $t \in B$, *are positive semi-definite matrices. Then*,

(i) y *is a feasible solution of the problem* (16) *if and only if* $y = Ye_1$, *where* Y *is some feasible solution of the problem* (15).
(ii) $\inf\{P_0 \bullet Y \mid Y \in \Phi\} = \inf\{y^T P_0 y \mid y \in \hat{F}\}$.

Proof The assertion (i) holds from the construction of Φ and $\hat{F}$. To prove the assertion (ii), take any $\bar{y} \in \hat{F}$. By Lemma 3.1, we have

$$\bar{y}_0 = 1 \quad \text{and} \quad \bar{y}^T P(t)\bar{y} \le 0, \quad \forall t \in B.$$

Then,

$$\bar{Y} = \bar{y}\bar{y}^T \in \Phi,$$

and

$$\inf\{P_0 \bullet Y \mid Y \in \Phi\} \le P_0 \bullet \bar{Y} = \bar{y}^T P_0 \bar{y}, \quad \forall \bar{y} \in \hat{F}.$$

Thus,

$$\inf\{P_0 \bullet Y \mid Y \in \Phi\} \le \inf\{y^T P_0 y \mid y \in \hat{F}\}. \tag{17}$$

Conversely, taking any $Y \in \Phi$, we have $y = Ye_1 \in \hat{F}$ and $y_0 = 1$. Suppose that

$$Y = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}.$$

Then,

$$y = Ye_1 = \begin{pmatrix} 1 \\ x \end{pmatrix},$$

and

$$\begin{aligned} P_0 \bullet Y &= P_0 \bullet Y - P_0 \bullet (yy^T) + y^T P_0 y \\ &= Q_0 \bullet (X - xx^T) + y^T P_0 y. \end{aligned} \tag{18}$$

Since Y and Q_0 are positive semi-definite matrices, we have

$$Q_0 \bullet (X - xx^T) \ge 0.$$

It follows that

$$P_0 \bullet Y \ge y^T P_0 y \ge \inf\{y^T P_0 y \mid y \in \hat{F}\}.$$

So,

$$\inf\{P_0 \bullet Y \mid Y \in \Phi\} \ge \inf\{y^T P_0 y \mid y \in \hat{F}\}. \tag{19}$$

Thus, by (17) and (19), the assertion (ii) holds. The proof is complete. □

Corollary 3.1 *Suppose that Y^* is an optimal solution of the problem* (15). *Then,* $y^* = Y^* e_1$ *is an optimal solution of the problem* (16).

Proof By examining the proof of Theorem 3.1 given in Appendix, we see that the result holds. □

Theorem 3.2 *If $Q(t)$, $t \in B$, is a positive semi-definite matrix, then,*

$$F = \hat{F}.$$

Proof Suppose that $y \in F$. Then,

$$y^T P(t) y \leq 0, \quad t \in B \quad \text{and} \quad y_0 = 1.$$

Let $Y = yy^T$. We have $Y_{00} = 1$ and

$$P(t) \bullet Y \leq 0, \quad t \in B.$$

Thus, $y = Ye_1 \in \hat{F}$, i.e.,

$$F \subset \hat{F}. \tag{20}$$

Conversely, suppose that $y \in \hat{F}$. By Lemma 3.1, we have

$$\hat{F} \subset F. \tag{21}$$

Thus, by (20) and (21), the result holds. □

Remark 1 It is well-known that the problem (15) is only a relaxation of the general semi-infinite quadratically constrained quadratic programming problem. In general, we cannot obtain the exact solution of the problem (Q^2P) from the solution of the problem (15). In fact, we cannot even get an approximate solution of the problem (Q^2P) satisfying a suitable accuracy from the solution of the problem (15). However, for the problem consider in this paper, we have shown in Corollary 3.1 and Theorem 3.2 that an exact solution of the semi-infinite quadratically constrained convex quadratic programming problem can be constructed from the solution of the problem (15). More specifically, if Y^* is an optimal solution of the problem (15), then, $y^* = Y^* e_1$ is a solution of the problem $(Q^2P)_y$.

4 Discretization algorithm

In this section, we shall develop an algorithm for solving a special (SDSIP) problem. In order to solve problems (10) and (11), consider the following combined semi-definite and semi-infinite linear programming problem (SDSIP):

$$\begin{aligned}
\inf \quad & C \bullet X \\
\text{s.t.} \quad & A_i \bullet X = a_i, \quad i = 1, 2, \ldots, l, \\
& B_q(t) \bullet X \leq b_q(t), \quad t \in B,\ q = 1, 2, \ldots, p, \\
& X \succeq 0.
\end{aligned}$$

Here, B is a compact set in R, C, A_i, $i = 1, 2, \dots, l$, and $B_q(t)$, $t \in B$, $q = 1, 2, \dots, p$, are all fixed matrices in S^n. Let a_i, $i = 1, 2, \dots, l$, and $b_q(t) \in R$, $t \in B$, $q = 1, 2, \dots, p$, be fixed real numbers, and let $X \in S^n$ be a decision matrix to be optimized upon.

Let us first introduce some notation. For a compact set B, let $R^B = \prod_B R$ denote the product space equipped with the product topology, which is a locally convex Hausdorff topological vector space; see (Kelley and Namtoka 1963). Then, the topological dual space of R^B is the generalized finite sequence space consisting of all functions $g : B \to R$ with a finite support. The set $R_+^B = \prod_B R_+$ denotes the convex cone of all nonnegative functions on B. Then, the dual cone of R_+^B is defined by

$$\Lambda_B = \{y = \{y(t)\}_{t \in B} \mid (\exists \text{a finite set } F \subseteq B)(\forall t \in B \backslash F)\ y(t) = 0 \\ \text{and } (\forall t \in F)\ y(t) \geq 0\}.$$

For this result, see (Jeyakumar and Gwinner 1991).

For the combined semi-definite and semi-infinite programming problem (SDSIP), we introduce the Lagrangian dual problem (DSDSIP) as follows:

$$\begin{aligned} \sup \quad & a^T z - \sum_{t \in B} b(t)^T y(t) \\ \text{s.t.} \quad & \sum_{i=1}^{l} z_i A_i - \sum_{q=1}^{p} \sum_{t \in B} B_q(t) y_q(t) + Z = C, \\ & Z \succeq 0, \quad y_q \in \Lambda_B,\ q = 1, 2, \dots, p, \end{aligned}$$

where $a = (a_1, a_2, \dots, a_l)^T$, $z = (z_1, z_2, \dots, z_l)^T$, $b(t) = (b_1(t), b_2(t), \dots, b_p)^T$ and $y(t) = (y_1(t), y_2(t), \dots, y_p(t))^T$.

We assume that the problem (SDSIP) and its dual problem (DSDSIP) have optimal solutions and their optimal values are equal.

When the parameter set B is finite, (SDSIP) and (DSDSIP) become a pair of primal and dual semi-definite programming problems. See (Vandenberghe and Boyd 1996) and (Wolkowicz et al. 2000) for relevant references. For detailed discussion, see (Jeyakumar and Wolkowicz 1990; Reemtsen 1994; Teo et al. 2000; Yang and Teo 2001).

Assume that $B = [T_1, T_2]$. We obtain a special class of combined semi-definite and semi-infinite programming problems (SDSIP) as follows:

$$\begin{aligned} (\mathrm{P}_0) \quad \inf \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = a_i, \quad i = 1, 2, \dots, l, \\ & B_q(t) \bullet X \leq b_q(t), \quad t \in [T_1, T_2],\ q = 1, 2, \dots, p, \\ & X \succeq 0. \end{aligned}$$

The Lagrangian dual problem of the problem (P_0) is:

$$(\mathrm{D}_0)\quad \sup \quad a^T z - \sum_{t\in[T_1,T_2]} b(t)^T y(t)$$
$$\text{s.t.}\quad \sum_{i=1}^{l} z_i A_i - \sum_{q=1}^{p} \sum_{t\in[T_1,T_2]} B_q(t) y_q(t) + Z = C,$$
$$Z \succeq 0,\ y_q \in \Lambda_{[T_1,T_2]},\ q = 1, 2, \ldots, p.$$

In this section, we develop a discretization method with an adaptive scheme for solving the problem (P_0). A sequence of discretized subproblems is obtained, and each (SDP) subproblem is solved by an infeasible interior point method (Potra and Sheng 1998).

The feasible set of the problem (P_0) is denoted by

$$\mathcal{F} = \{X \in S^n_+ : A_i \bullet X = a_i,\ i = 1, 2, \ldots, l,$$
$$B_q(t) \bullet X \le b_q(t),\ t \in [T_1, T_2], \quad q = 1, 2, \ldots, p\}.$$

We consider the following discretization scheme: given an integer $\bar{N} > 0$, let

$$\Omega_{\bar{N}} = \left\{ t_i = T_1 + \frac{i(T_2 - T_1)}{2^{\bar{N}}} : i = 0, 1, \ldots, 2^{\bar{N}} \right\}.$$

We introduce the following discretized problem $(\bar{P}_{\bar{N}})$:

$$\begin{aligned} \inf \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = a_i, \quad i = 1, 2, \ldots, l, \\ & B_q(t_j) \bullet X \le b_q(t_j), \quad t_j \in \Omega_{\bar{N}},\ q = 1, 2, \ldots, p, \\ & X \succeq 0. \end{aligned}$$

The feasible set of $(\bar{P}_{\bar{N}})$ is denoted by

$$\mathcal{F}_{\bar{N}} = \{X \in S^n_+ : A_i \bullet X = a_i,\ i = 1, 2, \ldots, l,$$
$$B_q(t_j) \bullet X \le b_q(t_j), \quad t_j \in \Omega_{\bar{N}},\ q = 1, 2, \ldots, p\}.$$

We have the following lemma.

Lemma 4.1 *Consider the problems* (P_0) *and* $(\bar{P}_{\bar{N}})$. *Then,*

$$\mathcal{F} \subset \mathcal{F}_{\bar{N}}.$$

A direct method for solving the problem (P_0) is to solve a sequence of discretized problems $(\bar{P}_N)$. The solution X_N of $(\bar{P}_N)$ is used as an approximate solution of the problem (P_0). However, the discretized problem $(\bar{P}_N)$ is a good approximation of the original (P_0) only if the integer N is large enough. Obviously, such a simple approximation of $[T_1, T_2]$ by the discretized subset Ω_N with a large number N leads to the problem $(\bar{P}_N)$ with a large number of inequality constraints. In order to overcome the problem of solving discretized problem $(\bar{P}_N)$ with a large number of inequality constraints, we introduce an adaptive scheme strategy. More specifically, at each iteration, we add only an additional constraint.

Discretization algorithm

Let $\{N_m\}$ be a strictly monotone increasing integer sequence with $N_m \to \infty$ (as $m \to \infty$). Given the integer $\bar{N} > 0$.

Step 1. $E_1 = \Omega_1, M_1 = \mathcal{F}_1, k = \bar{k} = 1, m = 1.$

Step 2. Find a solution $X_k \in M_k$ of the following semi-definite programming problem:

$$(P_k): \quad \sup \quad C \bullet X \qquad \text{s.t.} \quad X \in M_k.$$

Increase $\bar{k}$ to $\bar{k}+1$ and construct $\Omega_{\bar{k}+1}$. Go to Step 3.

Step 3. If $N_m > \bar{N}$, stop the algorithm. Otherwise, go to Step 4.

Step 4. Find a t_k and $1 \le q_k \le p$ such that

$$B_{q_k}(t_k) \bullet X_k - b_{q_k}(t_k) = \max_{\substack{t \in \Omega_{\bar{k}+1} \\ q=1,\dots,p}} (B_q(t) \bullet X_k - b_q(t)).$$

If $B_{q_k}(t_k) \bullet X_k - b_{q_k}(t_k) > 0$, go to Step 6.
If $B_{q_k}(t_k) \bullet X_k - b_{q_k}(t_k) \le 0$ and $\bar{k} < N_m$, go to Step 5.
If $B_{q_k}(t_k) \bullet X_k - b_{q_k}(t_k) \le 0$ and $\bar{k} \ge N_m$, increase m to $m+1$ and give N_{m+1}. Go to Step 8.

Step 5. Set $\bar{k} =: \bar{k}+1$. Increase $\bar{k}$ to $\bar{k}+1$ and construct $\Omega_{\bar{k}+1}$. Go to Step 4.

Step 6. Set

$$M_{k+1} = \{X \in S_+^n \mid A_i \bullet X = a_i,\ i = 1, 2, \dots, l,$$
$$B_q(t) \bullet X \le b_q(t),\ q = 1, \dots, p,\ t \in E_{k+1}\},$$

where $E_{k+1} = E_k \bigcup \{t_k\}$.

Increase m to $m+1$ and give N_{m+1}. Go to Step 7.

Step 7. Set $k =: k+1, \bar{k} =: \bar{k}+1, m =: m+1$ and go to Step 2.

Step 8. Set $k =: k+1, \bar{k} =: \bar{k}+1, m =: m+1$. Increase $\bar{k}$ to $\bar{k}+1$ and construct $\Omega_{\bar{k}+1}$. Go to Step 3.

For practical implementation, we will include a stopping criterion: we choose an integer $\bar{N}$, and we will terminate the algorithm when $N_m \ge \bar{N}$. For example, we can take $\bar{N} = 11$.

Theorem 4.1 *Suppose that $\mathcal{F}_1$ is a compact set. Then, any accumulation point of the sequence $\{X_k\}$ generated by the algorithm is an optimal solution of (P_0).*

Proof By the compactness of $\mathcal{F}_1$ and $X_k \in M_k$, the sequence $\{X_k\}$ has at least an accumulation point. Let $\bar{X}$ be an accumulation point of the sequence $\{X_k\}$. Then, there exists a subsequence $\{X_{k_j}\}$ of $\{X_k\}$ such that $\{X_{k_j}\}$ converges to the point $\bar{X}$. Suppose that X^* is an optimal solution of (P_0). It follows that $X^* \in M_k$. We have,

$$C \bullet X^* \geq C \bullet X_k, \quad \forall k.$$

Thus,

$$C \bullet X^* \geq C \bullet X_{k_j}, \quad \forall j.$$

As $j \to \infty$, we have

$$C \bullet X^* \geq C \bullet \bar{X}. \tag{22}$$

Now we prove that $C \bullet X^* \leq C \bullet \bar{X}$. There are two cases to be considered.

Case 1: There exists a subsequence $\{X_{k_{l_j}}\}$ of $\{X_{k_j}\}$ such that $B_{q_{k_{l_j}}}(t_{k_{l_j}}) \bullet X_{k_{l_j}} - b_{q_{k_{l_j}}}(t_{k_{l_j}}) > 0$, i.e., the algorithm goes to Step 6 from Step 4 in an infinite number of iterations. Since q_k takes an integer between 1 and p, we assume, without loss of generality, that $q_{k_{l_j}} = q_0$, $\forall j$. Suppose that the algorithm goes to Step 6 at $\bar{k} + 1 = \bar{k}_j$ (as $j \to \infty$). It follows from the convergence of $\Omega_{\bar{k}_j}$ that, for each $\xi \in B$, we can find an $\xi_{\bar{k}_j} \in \Omega_{\bar{k}_j}$ with $\xi_{\bar{k}_j} \to \xi$ (as $j \to \infty$). Thus,

$$B_{q_0}(t_{k_{l_j}}) \bullet X_{k_{l_j}} - b_{q_0}(t_{k_{l_j}}) \geq B_q(\xi_{\bar{k}_j}) \bullet X_{k_{l_j}} - b_q(\xi_{\bar{k}_j}), \quad q = 1, \ldots, p.$$

By the compactness of B, we can assume, without loss of generality, that the sequence $\{t_{k_{l_j}}\}$ is a convergent one with the limiting point $\bar{t}$. Therefore, we obtain

$$B_{q_0}(\bar{t}) \bullet \bar{X} - b_{q_0}(\bar{t}) \geq B_q(\xi) \bullet \bar{X} - b_q(\xi), \quad q = 1, \ldots, p.$$

By the construction of $E_{k_{j+1}}$ and $M_{k_{j+1}}$, we have

$$B_{q_0}(t_{k_j}) \bullet X_{k_{j+1}} - b_{q_0}(t_{k_j}) \leq 0.$$

So

$$B_{q_0}(\bar{t}) \bullet \bar{X} - b_{q_0}(\bar{t}) \leq 0,$$

and

$$B_q(\xi) \bullet \bar{X} - b_q(\xi) \leq 0, \quad q = 1, \ldots, p.$$

By $X_{k_j} \in M_{k_j}$, we have

$$A_i \bullet X_{k_j} = a_i, \quad \text{for } i = 1, 2, \ldots, l.$$

It follows that

$$A_i \bullet \bar{X} = a_i, \quad \text{for } i = 1, 2, \ldots, l,$$

$$\bar{X} \in \mathcal{F},$$

and

$$C \bullet X^* \le C \bullet \bar{X}. \tag{23}$$

Case 2: There does not exist any subsequence $\{X_{k_{l_j}}\}$ of $\{X_{k_j}\}$ such that

$$B_{q_{k_{l_j}}}(t_{k_{l_j}}) \bullet X_{k_{l_j}} - b_{q_{k_{l_j}}}(t_{k_{l_j}}) > 0.$$

Then, by the algorithm and the convergence of X_{k_j}, there exists a subsequence $\{X_{k_{r_j}}\}$ such that $B_{q_{k_{r_j}}}(t_{k_{r_j}}) \bullet X_{k_{r_j}} - b_{q_{k_{r_j}}}(t_{k_{r_j}}) \le 0$, i.e., the algorithm goes to Step 8 from Step 4 in an infinite number of iterations. Since q_k takes an integer between 1 and p, we assume, without loss of generality, that $q_{k_{r_j}} = q_0$, $\forall j$. Suppose that the algorithm goes to Step 8 at $\bar{k} + 1 = \bar{k}_j$. It follows from the convergence of $\Omega_{\bar{k}_j}$ that, for each $\xi \in B$, we can find $\xi_{\bar{k}_j} \in \Omega_{\bar{k}_j}$ with $\xi_{\bar{k}_j} \to \xi$ (as $j \to \infty$). Thus,

$$B_{q_0}(t_{k_{r_j}}) \bullet X_{k_{r_j}} - b_{q_0}(t_{k_{r_j}}) \ge B_q(\xi_{\bar{k}_j}) \bullet X_{k_{r_j}} - b_q(\xi_{\bar{k}_j}), \quad q = 1, \dots, p.$$

By the compactness of B, we can assume, without loss of generality, that the sequence $\{t_{k_{r_j}}\}$ is a convergent one with the limiting point $\tilde{t}$. Therefore, we obtain

$$B_{q_0}(\bar{t}) \bullet \bar{X} - b_{q_0}(\bar{t}) \ge B_q(\xi) \bullet \bar{X} - b_q(\xi), \quad q = 1, \dots, p.$$

By the condition of Case 2, we have

$$B_{q_0}(t_{k_{r_j}}) \bullet X_{k_{r_j}} - b_{q_0}(t_{k_{r_j}}) \le 0$$

So

$$B_{q_0}(\bar{t}) \bullet \bar{X} - b_{q_0}(\bar{t}) \le 0,$$

and

$$B_q(\xi) \bullet \bar{X} - b_q(\xi) \le 0, \quad q = 1, \dots, p.$$

Similarly, we have

$$\bar{X} \in \mathcal{F}, \quad C \bullet X^* \le C \bullet \bar{X}. \tag{24}$$

It follows from (22), (23) and (24) that

$$C \bullet X^* = C \bullet \bar{X}.$$

Thus, the proof is complete. □

In the algorithm, for each subproblem (P_k), we shall use the infeasible predictor corrector method (Potra and Sheng 1998) to solve for the exact solution X_k of (P_k).

Remark 2 It follows from (Lobo et al. 1998) that second-order-cone (SOC) problem is much simpler to solve than its SDP counterpart. Thus, one normally converts the convex quadratically constrained quadratic programming ($QCQP$) problem into an equivalent (SOC) problem, but not a SDP problem, and then apply any publicly available software like SDPT3 and SeDuMi for solving the resulted (SOC) problem. Therefore, we may use this method to solve the (Q^2P) problem. We believe, as shown in (Lobo et al. 1998), that this method should be better than our algorithm, which uses an equivalent SDSIP reformulation. The aim of this paper is to give an alternative method for solving the continuous-time EC filter and robust EC filter problems, which is shown to be more efficient than the methods used in (Tseng et al. 1999) and (Tseng et al. 2000) for solving these classes of problems respectively.

5 Numerical results with Laguerre basis

To illustrate the performance of the proposed algorithm derived in the previous section, we consider the Laguerre basis functions. We briefly introduce the orthonormal Laguerre basis and then apply them to a practical EC filter design example involving the channel equalization of a data communication (Kautz 1994).

5.1 Laguerre orthonormal basis of $L^2([0,\infty))$

Let $L_j^p(t)$ be the time-domain Laguerre function with an adjustable pole $p > 0$ defined by

$$L_j^p(t) = \sqrt{2p}e^{-pt}\ell_j(2pt), \quad j = 0, 1, \ldots,$$

where $\ell_j(t)$ is the classical Laguerre polynomial given by

$$\ell_j(t) = \frac{e^t}{j!}\frac{d^j}{dt^j}(e^{-t}t^j) = \sum_{i=0}^{j}\binom{j}{j-i}\frac{(-t)^i}{i!}, \quad j = 0, 1, \ldots .$$

It is known that the Laguerre sequence $\{L_j^p\}_{j=0}^{\infty}$ forms a uniformly bounded orthonormal basis for the Hilbert space $L^2([0,\infty))$. Thus, any $u(t) \in L^2([0,\infty))$ can be represented as

$$u(t) = \sum_{j=0}^{\infty} x_j L_j^p(t),$$

where $x_j = \langle u, L_j^p \rangle$, $j = 0, 1, \ldots$ are known as Laguerre Fourier coefficients.

Define

$$u_N(t) = \sum_{j=0}^{N-1} x_j L_j^p(t)$$

as a Laguerre filter of order N.

5.2 Numerical results

In the section, we consider the equalization of a digital transmission channel involving a coaxial cable operating at the DSX3 rate (44.736 Mb/s)(see G.707 1984). The design objective is to find an equalizer that takes the impulse response of a coaxial cable with a loss of 30 dB at 22 MHz and produces an output that lies within the DSX3 pulse template. The input signal $s(t)$ and the output pulse mask ($\xi^{+}(t)$ and $\xi^{-}(t)$) are given in the continuous-time domain (see Tseng et al. 1999).

For continuous-time EC filtering problem (10) with the Laguerre orthonormal basis, we choose the number of Laguerre coefficients $N = 8$, the scale factor in Laguerre filter $p = 14$ and the length of the interval time $T = 32$. Then, using our discretization algorithm and Corollary 3.1, the optimal cost value obtained is $\|x^*\|^2 = 56.08$. The simulation results are depicted in Fig. 1. It is clear that the output response fits into the output envelope mask.

For the robust envelope-constrained filtering problem (11) with the Laguerre orthonormal basis, we choose the weighting function $\beta(t) = (\xi(t)^{+} - \xi^{-}(t))/2$; $\beta(t)$ is a tolerance band about the desired pulse shape. For the improved robustness in problem (11), we are prepared to accept an additional 100% increase in the output noise power gain, i.e., $\delta = 1.0$. Then, using our discretization algorithm and Corollary 3.1, the weighted constraint robustness margin obtained is $\delta_\beta = 0.3950047$. The simula-

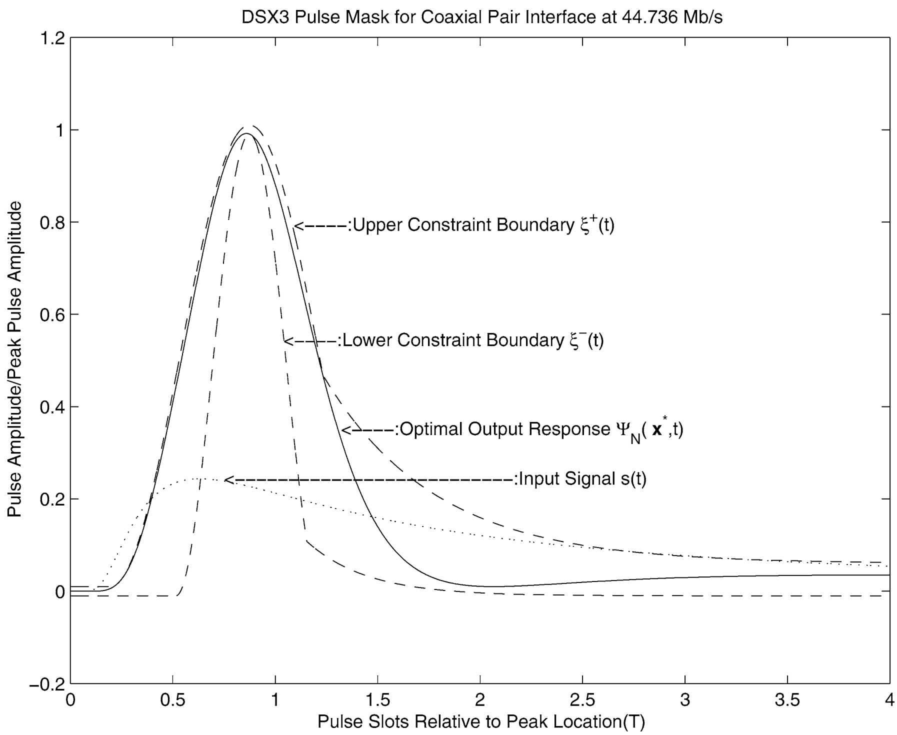

Fig. 1 DSX3 pulse template superimposed on coaxial cable response and filter output

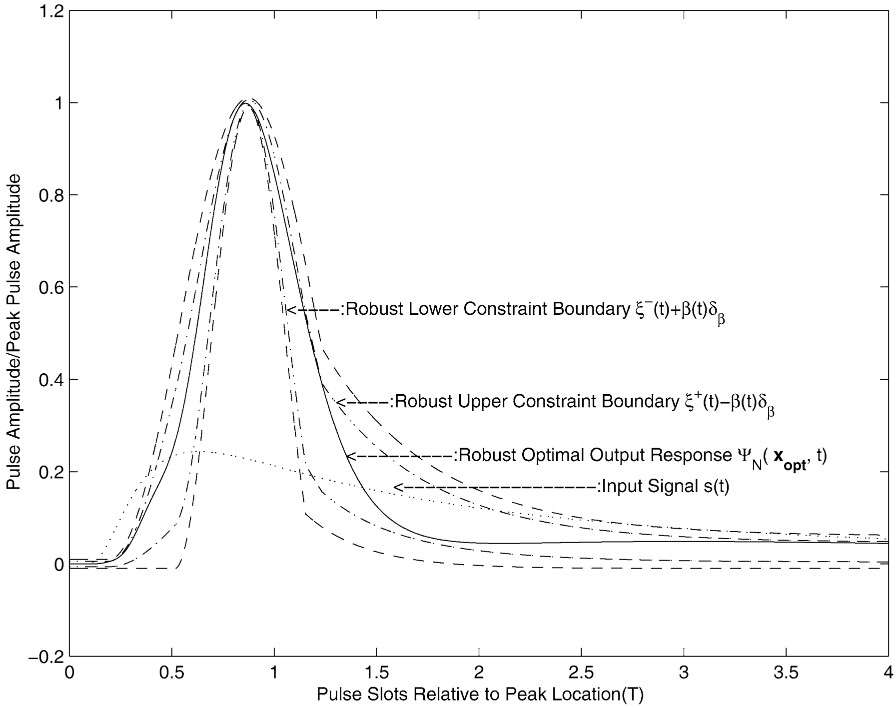

Fig. 2 The robust weighted EC filter output response

tion results are depicted in Fig. 2. It is clear that the output response fits into the robust output envelope mask. Figure 3 shows the comparison between the continuous-time EC optimal output response and the robust optimal output response. Clearly, the robust optimal output response is further away from the boundary of the output mask when compared with the continuous-time EC optimal output response.

In Tseng et al. (1999), the continuous-time EC filtering problem (10) was solved by a dual approach (cf. Algorithm 4.2 reported in Tseng et al. 1999). Now we present a comparison between our algorithm and the dual approach proposed in (Tseng et al. 1999) in Table 1:

- Computing time (seconds)—the required time in second to compute the continuous-time EC filtering problem (10).
- The optimal value—the optimal cost value $\|x^*\|^2$ of the continuous-time EC filtering problem (10).

In view of Table 1, we see that our algorithm and the dual approach are all efficient methods for solving the continuous-time EC filtering problem. The optimal values $\|x^*\|^2$ obtained by the two methods are almost the same. However, the dual approach of (Tseng et al. 1999) cannot be used to solve the robust envelope-constrained filtering problem (11). Moreover, the computing speed of our algorithm is a little faster than that of the dual approach proposed in (Tseng et al. 1999).

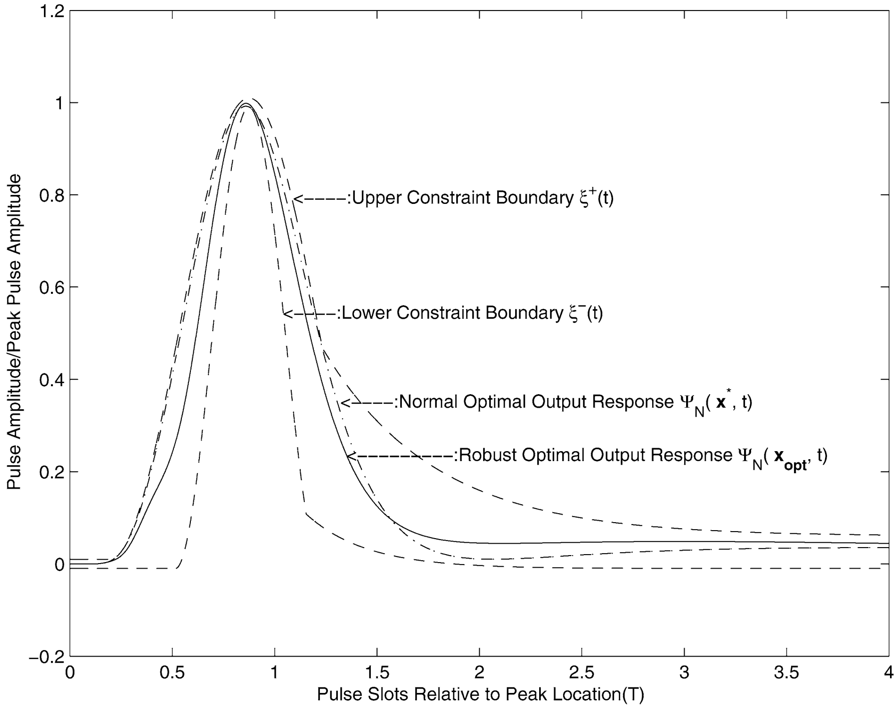

Fig. 3 Comparison between the normal EC optimal output response and the robust EC optimal output response while increasing an additional 100% noise power gain

Table 1 A comparison between our algorithm and the algorithm of (Tseng et al. 1999)

	Our algorithm	The dual approach given in (Tseng et al. 1999)
Computing time (seconds)	47.78	165.22
The optimal value	56.08	56.33

In Tseng et al. (2000), the solution of the robust envelope-constrained filtering problem (11) was obtained by solving a sequence of strictly convex optimization problems with integral cost (cf. Algorithm 3.2 given in (Tseng et al. 2000)). Now we present a comparison between our algorithm and the approach proposed in (Tseng et al. 2000) in Table 2:

- Computing time (seconds)—the required time in second to compute the robust EC filtering problem (11).
- δ_β—the weighted constraint robustness margin of the robust EC filtering problem (11).

From Table 2, we see that the weighted constraint robustness margins δ_β of the robust EC filtering problem (11) obtained by our algorithm and Algorithm 3.2 pro-

Table 2 A comparison between our algorithm and the algorithm of (Tseng et al. 2000)

	Our algorithm	The algorithm given in (Tseng et al. 2000)
Computing time (seconds)	57.12	1134.71
δ_β	0.3947	0.3957

posed in (Tseng et al. 2000) are almost the same. However, the computing speed of our algorithm is much faster than that of Algorithm 3.2 given in (Tseng et al. 2000).

6 Conclusion

In this paper, we have developed an algorithm for solving the continuous-time EC filtering and robust EC filtering problems. The numerical examples presented show that the algorithm is effective and highly efficient.

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