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Robust Finite-Time Control of Discrete-Time Switched Positive Time-Varying Delay Systems with Exogenous Disturbance and Their Application

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Abstract: Many practical systems can be modeled in terms of uncertainties, which refer to the differences or errors between actual data and mathematical simulations. However, systems including slight uncertainties and exogenous disturbances may lead to the instability of those systems. Besides, the behavior of systems is preferable to investigate within a prescribed bound over a fixed time interval. Therefore, in this paper, we study a robust finite-time control of discrete-time linear switched positive time-varying delay systems with interval uncertainties and exogenous disturbance. A distinctive feature of this research is that the considered systems consist of finite-time bounded subsystems and finite-time unbounded subsystems. A class of quasi-alternative switching signals is validly designed to analyze the mechanism and switching behaviors of the systems among their subsystems. By utilizing a copositive Lyapunov–Krasovskii functional method combined with the slow mode-dependent average dwell time and the fast mode-dependent average dwell time switching techniques, new sufficient conditions containing several symmetric negative-definite matrices are derived to guarantee robust finite-time control of the systems. These results are applied to a water-quality controllability model in streams to the standard level. Finally, the consistent results between the theoretical analysis and the corresponding numerical simulations are shown.

Keywords: robust finite-time control; discrete-time systems; switched positive systems; time-varying delay; exogenous disturbance

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1. Introduction

During the last decades, there has been a growing interest in the study of hybrid systems. An essential example of analysis is the switched system, which consists of a finite number of subsystems and a switching rule specifying the switching among the multiple subsystems [1–3]. Considerations of switching behaviors and their applications are generally investigated in the community of mathematicians, scientists, and engineers. Especially, switched positive systems (SPSs) comprising all individual positive subsystems have been successfully applied and solved in many real-world problems, for instance, formation flying [4], network communication [5], wireless power control [6], congestion control [7], compartmental model [8], water-quality model [9], and positive circuit model [10].

In many actual systems, there exists a class of dynamical systems in which the future evolution of the state variables relies not only on their present value but also on their past values. The systems under this characteristic are called time-delay systems [11,12]. Numerous physical and biological phenomena can be discovered in the area of time-delay systems, such as fluid and mechanical transmissions, metallurgical and chemical processes, networked communications, stochastic models, and reproduction in an organism. However, the dynamic performance of systems caused by the time delay may lead to chaos and instability of systems [13–15]. For these reasons, it is meaningful to investigate the stability and stabilization of systems, including time delay. In particular, stability problems of discrete-time systems, discrete-time systems with time delay, and discrete-time switched systems with time delay have been addressed in [16–18], respectively. Due to the complex dynamics of switched systems with time delay and the existence of slight uncertainties in the systems, the stability of these systems is not a trivial problem. Consequently, the robust stability and stabilization problems of the switched systems including uncertainties have been reported in several publications, for example, switched continuous-time systems [19], switched discrete-time systems [20,21], stochastic switched discrete-time systems [22], SPSs [5,23,24], and switched positive time-varying delay systems [25].

It is well known that stability analysis for dynamical systems in the concept of classical Lyapunov stability is the study of the behavior of trajectories of systems over an infinite time interval with small perturbations of initial conditions. A variety of the stability results of dynamical systems and differential equations has been achieved theoretically and numerically by many researchers [26–30]. Furthermore, stability theory and optimal control have been studied and there is interest in them [31,32]. Different from the Lyapunov stability concept, finite-time stability (FTS) requires that the state trajectories of the considered systems do not exceed a certain bound during a fixed finite-time interval under some given constraints of the initial condition [33–36]. Besides, the concept of FTS is extended to a finite-time boundedness (FTB) if exogenous disturbances or the influence of perturbing forces are taken into account together [37]. Therefore, both FTS and FTB problems on switched systems with time delay as well as time-varying delay have drawn significant attention in the field of control (see [38–42] and references therein).

A switching law plays a crucial role in the stability analysis of switched systems. Some relevant research is based on the various types of time-dependent switching, such as dwell time (DT), average dwell time (ADT), mode-dependent average dwell time (MDADT), which is composed of both slow mode-dependent average dwell time (SMDADT) and fast mode-dependent average dwell time (FMDADT), and so on. Accordingly, the switching behaviors and the performance of switched systems, especially SPSs, under employing appropriate switching rules and some constraints of operation time have been investigated and reviewed in the following. In [43], Zhao et al. dealt with the problem of stability for a class of SPSs with time delay. Meanwhile, the sufficient conditions which can guarantee the L_1 FTB of SPSs with time-varying delay were derived by Xiang et al. [44]. Later, the results obtained by [45] concerned with finite-time stabilization for a class of uncertain SPSs with time-varying delays. Next, Liu et al. [46] studied FTB, stabilization and L_2-gain for SPSs with multiple time delays. All the results mentioned above used the method of ADT switching. Depending on the MDADT approach, the static output-feedback L_1 finite-time control problem for SPSs with time-varying delay was investigated in [47]. The same method of MDADT switching was also utilized in the stability study of SPSs with time delay discussed by Liu et al. [48]. For discrete-time SPSs, in [49], Hernandez-Vargas et al. presented the results for the synthesis of stabilizing, guaranteed performance, and optimal control laws for the systems. Their results were applied to a simplified human immunodeficiency viral mutation model. In [50], Zhang et al. provided sufficient conditions to ensure the FTS and FTB of discrete-time SPSs by using the ADT strategy. In addition, they solved the problem of robust finite-time stabilization of nonautonomous systems with a weighted L_1-gain. Furthermore, the exponential stability issue for a class of discrete-time SPSs with time delay via the ADT approach was examined in [51].

On the other hand, all the research referred to above was mainly focused on only stable (bounded) subsystems. In practical applications, the switched systems, including the stable (bounded) subsystems and unstable (unbounded) subsystems, can be widely implemented (see [52–58]). Among them, in [52], Li et al. discussed the FTS and FTB issues of nonlinear switched systems with subsystems that are not finite-time stable or finite-time bounded by using ADT switching. Later, Tan et al. [53] dealt with the FTS and FTB problems of switched systems comprising both finite-time stable and unstable subsystems by utilizing MDADT switching. Recently, sufficient conditions in terms of multiple Lyapunov functions for FTS of a class of hybrid systems with unstable modes were presented by Garg and Panagou [54]. Nevertheless, both the time delay phenomena and the positivity of the systems were not taken into account in the three mentioned references. For the stability analysis, the result of nonlinear switched time-delay systems, including stable and unstable subsystems, was reported in [55]. Next, Pashaei and Hashemzadeh [56] solved the problem of FTS and FTB for linear switched delayed systems with finite-time unstable and unbounded subsystems by employing ADT switching. More recently, Mouktonglang and Yimnet [59] studied the FTB of linear uncertain switched positive time-varying delay systems with finite-time unbounded subsystems and exogenous disturbance by using MDADT switching. In addition to continuous-time switched systems, some fruitful results about discrete-time switched systems have been stated as follows. In [57], Gao et al. investigated the FTS and FTB for a class of discrete-time switched nonlinear systems with partial finite-time unstable subsystems via the MDADT approach. In addition, the stability problem for discrete-time nonlinear SPSs with unstable modes under different switching signals was studied by Zhang et al. [58]. However, to the best of our knowledge, there is no result on the FTB for a class of discrete-time SPSs, including time-varying delay, interval uncertainties, exogenous disturbance, and finite-time unbounded subsystems in the literature.

The motivation for this article comes from the above observation. The main innovative contributions of this study are proposed in the following:

(i) The stability of discrete-time SPSs with time-varying delay, interval uncertainties, and exogenous disturbance is analyzed in the sense of the FTB when their subsystems are both bounded and unbounded subsystems.

(ii) The new copositive Lyapunov–Krasovskii functional (CLKF) and a class of quasi-alternative switching signals (QASSs) are perfectly designed to stabilize the systems during the fixed finite-time interval.

(iii) Combining the SMDADT and FMDADT switching laws, novel delay-dependent sufficient criteria (DDSC) containing several symmetric negative-definite matrices for the FTB of the systems are derived. This is an efficient method to deal with the case that subsystems are not just bounded.

(iv) Different from the previous results in [43,46,50,52,55,56], both the SMDADT and FMDADT laws that are less conservative and more applicable in practice than the ADT switching rule are utilized for studying the FTB of the systems.

(v) The water-quality controllability model in streams studied in [2,3,9] is adopted to match the discrete-time switched positive time-varying delay system with interval uncertainties and exogenous disturbance. In addition, the problem of water-quality control to standard level in the finite-time interval is investigated by this system.

The paper is organized in the following manner. In the next section, the system descriptions and preliminaries are stated. Then, in Section 3, the main results are shown. Next, in Section 4, a numerical example is presented to support and validate our theoretical results. Lastly, the discussion and conclusions are given in Section 4.

Notations: The following notations are utilized throughout this article. The sets of non-negative and positive integers are denoted by $\mathbb{N}_0$ and $\mathbb{N}$, respectively. $\mathbb{R}^n$ and $\mathbb{R}^n_+$ refer to the vectors of n-tuples of real and positive real numbers, respectively. The set of all $m \times n$ real matrices is represented by $\mathbb{R}^{m\times n}$. I_n and A^T are the $n \times n$ dimensional identity matrix and the transpose of matrix A, respectively. For given vector $\nu \in \mathbb{R}^n$, ν_i $(1 \leq i \leq n)$ is the ith component of ν. The notation $\nu \succeq 0$ $(\nu \succ 0)$ stands for non-negative (positive) vector, namely, all components of vector ν are non-negative (positive). $\overline{\beta}(\nu)$ and $\underline{\beta}(\nu)$ denote the maximal and minimal elements of vector $\nu \in \mathbb{R}^n$, respectively. $\mathbf{1}_n = [1\ 1\ ...\ 1]^T \in \mathbb{R}^n$. The matrix A is called non-negative if all entries are non-negative and defined by $A \succeq 0$. In addition, $\|\nu\|_1 = \sum_{i=1}^{n} |\nu_i|$ and $\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|$ are the 1-norm of vector $\nu \in \mathbb{R}^n$ and the 1-norm of matrix $A \in \mathbb{R}^{m\times n}$, respectively.

2. System Descriptions and Preliminaries

A class of discrete-time linear switched time-varying delay system with interval uncertainties and exogenous disturbance can be stated as

$$\begin{cases} x(k+1) = A_{\sigma(k)}x(k) + D_{\sigma(k)}x(k-d(k)) + G_{\sigma(k)}\omega(k), \\ x(k_0+\theta) = \psi(\theta), \;\; \theta = -d_2, -d_2+1, ..., 0, \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $\sigma(k) : \mathbb{N}_0 \to \underline{N} = \{1, 2, ..., N\}$ is the switching signal, and $N > 1$ is the number of subsystems or modes of the switched system. Given the switching signal $\sigma(k)$, we denote the set of switching moments by $\{k_m : k_m \in \mathbb{N}_0\}$ where k_0 is the initial time and $k_m < k_{m+1}$ for $m \in \mathbb{N}_0$. For two successive switching moments k_m and k_{m+1}, we impose that $\sigma(k-1) = \sigma(k_m - 1) = j$ and $\sigma(k) = \sigma(k_m) = i$, where $j, i \in \underline{N}$, and the $\sigma(k_m)$th subsystem is activated when $k \in [k_m, k_{m+1})$. Based on the logical rule of $\sigma(k)$ at k_m, system (1) switches from the jth subsystem to the ith subsystem. For being the interval uncertain of system (1), A_i, D_i, and G_i satisfy

$$\underline{A}_i \preceq A_i \preceq \overline{A}_i,$$

$$\underline{D}_i \preceq D_i \preceq \overline{D}_i,$$

and

$$\underline{G}_i \preceq G_i \preceq \overline{G}_i,$$

where $\underline{A}_i$, $\underline{D}_i$, $\underline{G}_i$, $\overline{A}_i$, $\overline{D}_i$, *and* $\overline{G}_i$ are the given constant system matrices with appropriate dimensions for all $i \in \underline{N}$. The time-varying delay $d(k) : \mathbb{N}_0 \to \mathbb{N}$ satisfying

$$0 < d_1 \leq d(k) \leq d_2, \;\; \forall k \in \mathbb{N}_0,$$

where d_1 and d_2 are positive integers. As mentioned in [36,50], the exogenous disturbance $\omega(k) \in \mathbb{R}^w$ satisfying the condition

$$\exists \rho > 0 : \sum_{k=0}^{K_f} \|\omega(k)\|_1 \leq \rho, \tag{2}$$

with a given time constant $K_f \in \mathbb{N}$. Moreover, $\psi(\cdot) : \{-d_2, -d_2+1, ..., 0\} \to \mathbb{R}^n$ is a given discrete vector-valued initial state with the norm $\|\psi\|_d = \max_{\theta \in \{-d_2, -d_2+1, ..., 0\}} \|\psi(\theta)\|_1$.

Remark 1. *The discrete-time linear switched system with time-varying delay and exogenous disturbance (1) and its system descriptions were studied in [2,3,9]. However, the authors did not consider the interval uncertain of system (1).*

The following definitions, lemma, and assumption are useful to derive the main results of the work.

Definition 1 ([47])**.** *System (1) is said to be positive if for any initial condition* $\psi(\theta) \succeq 0$, $\theta = -d_2, -d_2+1, ..., 0$ *for any exogenous disturbance* $\omega(k) \succeq 0$, *and for any switching signal* $\sigma(k)$, *the corresponding trajectory* $x(k) \succeq 0$ *holds for all* $k \in \mathbb{N}_0$.

Lemma 1 ([47])**.** *System (1) is positive if and only if* $A_i \succeq 0$, $D_i \succeq 0$, *and* $G_i \succeq 0$ *hold for all* $i \in \underline{N}$.

Assumption 1 ([5,25])**.** *For each* A_i, D_i, *and* G_i *in system (1) there are the known constant matrices* $\underline{A}_i \succeq 0$, $\underline{D}_i \succeq 0$, *and* $\underline{G}_i \succeq 0$, *such that* $A_i \in [\underline{A}_i, \overline{A}_i]$, $D_i \in [\underline{D}_i, \overline{D}_i]$, *and* $G_i \in [\underline{G}_i, \overline{G}_i]$, *where* $\underline{A}_i$, $\underline{D}_i$, $\underline{G}_i$, $\overline{A}_i$, $\overline{D}_i$, *and* $\overline{G}_i$ *are the given constant system matrices with appropriate dimensions for all* $i \in \underline{N}$.

Definition 2 (Finite-Time Boundedness [46])**.** *Given two positive constants c_1, c_2 with $c_1 < c_2$, the time constant K_f, a positive vector l, and a switching signal $\sigma(k)$. System (1) is said to be finite-time bounded with respect to $(c_1, c_2, K_f, l, \rho, \sigma(k))$ if the solution $x(k)$ of the system satisfies the condition:*

$$\max_{\theta \in \{-d_2, -d_2+1, \ldots, 0\}} \psi^T(\theta) l \leq c_1 \Longrightarrow x^T(k) l < c_2, \quad \forall k = 1, 2, \ldots, K_f,$$

where $\omega(k)$ satisfies inequality (2).

The finite set $\underline{N}$ is divided into $\underline{B}$ and $\underline{U}$; namely, $\underline{N} = \underline{B} \cup \underline{U}$ where $\underline{B} = \{1, 2, \ldots, B\}$ represents the set of finite-time bounded subsystems with respect to the required parameters $(c_1, c_2, K_f, l, \rho, \sigma(k))$, and $\underline{U} = \{B+1, \ldots, N\}$ denotes the set of finite-time unbounded subsystems with respect to the same required parameters $(c_1, c_2, K_f, l, \rho, \sigma(k))$, respectively. In addition, let Ω be the set of switching signals which has only a finite number of switching for any finite-time interval.

The definitions of both the SMDADT and FMDADT switching laws are stated as follows:

Definition 3 ([53])**.** *For any $K \geq k \geq 0$ and a switching signal $\sigma(k) \in \Omega$, let $N_{\sigma p}(K, k)$ be the number of times the pth subsystem is activated and $T_p(K, k)$ be the total running time of the pth subsystem, $p \in \underline{B}$. If there exist constants $N_{0p} \geq 0$ and $\tau_{ap} > 0$ satisfying*

$$N_{\sigma p}(K, k) \leq N_{0p} + \frac{T_p(K, k)}{\tau_{ap}}, \quad \forall K \geq k \geq 0, \tag{3}$$

then τ_{ap} is called the SMDADT of the switching signal $\sigma(k)$.

Definition 4 ([53])**.** *For any $K \geq k \geq 0$ and a switching signal $\sigma(k) \in \Omega$, let $N_{\sigma q}(K, k)$ be the number of times the qth subsystem is activated and $T_q(K, k)$ be the total running time of the qth subsystem, $q \in \underline{U}$. If there exist constants $N_{0q} \geq 0$ and $\tau_{aq} > 0$ satisfying*

$$N_{\sigma q}(K, k) \geq N_{0q} + \frac{T_q(K, k)}{\tau_{aq}}, \quad \forall K \geq k \geq 0, \tag{4}$$

then τ_{aq} is called the FMDADT of the switching signal $\sigma(k)$.

3. Main Results

In this section, the problem of the FTB for the uncertain system (1) with exogenous disturbance and partial finite-time unbounded subsystems is studied. Inspired by the idea in [40,53,59], a class of QASSs for the uncertain system (1) is designed as follows:

(a) If $\sigma(k_m) \in \underline{B}$, then $\sigma(k_{m+1}) \in \underline{N}$;
(b) If $\sigma(k_m) \in \underline{U}$, then $\sigma(k_{m+1}) \in \underline{B}$.

The mechanism of the above switching is that the system can switch from a finite-time bounded subsystem to any other subsystems. However, it cannot switch from a finite-time unbounded subsystem to another finite-time unbounded subsystem. To compensate for the state divergence caused by the unbounded subsystems and stabilize the system in the finite-time interval, the slow switching for the bounded subsystems and the fast switching for the unbounded subsystems are designed and applied reasonably. Now, the new DDSC for the FTB of the uncertain system (1) with finite-time bounded and finite-time unbounded subsystems are derived by using QASSs above and combining the SMDADT and FMDADT switching laws in the following theorem.

Theorem 1. *Consider the uncertain system (1) with exogenous disturbance and partial finite-time unbounded subsystems satisfying Assumption 1. Let $\gamma_p > 1$, $\mu_p > 1$, $p \in \underline{B}$, $\gamma_q > 1$, $0 < \mu_q < 1$, and $q \in \underline{U}$ be the constants. For the given constants $c_2 > c_1 > 0$, the time constant $K_f > 0$*

and the vector $l \succ 0$. *Suppose that there exist positive vectors* $\nu_p \succ 0$ *and* $\nu_q \succ 0$ *and constants* $\xi_p > 0,\ \xi_q > 0,\ \beta_p > 0,$ *and* $\beta_q > 0$, *such that*

$$\left[\overline{A}_p^T + (d_2 - d_1 + 1)\widetilde{D}^T - \gamma_p I_n\right]\nu_p \prec 0, \quad (5)$$

$$\left[\overline{A}_q^T + (d_2 - d_1 + 1)\widetilde{D}^T - \gamma_q I_n\right]\nu_q \prec 0, \quad (6)$$

$$\overline{G}_p^T \nu_p - \xi_p \mathbf{1}_w \prec 0, \quad (7)$$

$$\overline{G}_q^T \nu_q - \xi_q \mathbf{1}_w \prec 0, \quad (8)$$

$$\beta_p l \preceq \nu_p \preceq \mu_p \nu_r, \quad (9)$$

$$\beta_q l \preceq \nu_q \preceq \mu_q \nu_p, \quad (10)$$

$$\Gamma^{K_f} < \frac{\beta c_2}{\eta c_1 + \xi\rho}, \quad (11)$$

hold for every $p \in \underline{B},\ q \in \underline{U},\ r \in \underline{N},$ *and* $p \neq r$. *Then, the uncertain system (1) is positive and finite-time bounded with respect to* $(c_1, c_2, K_f, l, \rho, \sigma(k))$ *under the switching signals with SMDADT satisfying*

$$\tau_{ap} \geq \tau_{ap}^* = \frac{K_f \ln \mu_p}{\ln \frac{\beta c_2}{\eta c_1 + \xi\rho} - K_f \ln \gamma_p}, \quad \forall p \in \underline{B}, \quad (12)$$

and FMDADT satisfying

$$\tau_{aq} \leq \tau_{aq}^* = -\frac{\ln \mu_q}{\ln \gamma_q}, \quad \forall q \in \underline{U}, \quad (13)$$

where

$$\Gamma = \max_{p\in\underline{B}}\{\gamma_p\},\ \ \beta = \min_{p\in\underline{B},q\in\underline{U}}\{\beta_p, \beta_q\},\ \ \xi = \max_{p\in\underline{B},q\in\underline{U}}\{\xi_p, \xi_q\},\ \ \eta = \frac{\overline{\beta}(\nu_{\sigma(0)})}{\underline{\beta}(l)}\Big[1 + d_2(d_2 - d_1 + 1)\|\widetilde{D}^T\|_1\Big],$$

and

$$\widetilde{D} = \left(\overline{d}_{kl}\right) \in \mathbb{R}^{n\times n},\ \ \overline{d}_{kl} = \max_{i\in\underline{N}}\left\{\overline{D}_i^{(kl)}\right\},$$

$\overline{D}_i^{(kl)}$ *is the kth row and lth column entry of system matrices* $\overline{D}_i,\ i \in \underline{N}$.

Proof. We divide the proof process into the following two steps.

Step 1. We prove that the uncertain system (1) is positive.

Using Assumption 1, we obtain that $A_i \succeq 0,\ D_i \succeq 0,$ and $G_i \succeq 0$ for all $i \in \underline{N}$. According to Lemma 1, we can conclude that the uncertain system (1) is positive.

Step 2. We prove the FTB for the uncertain system (1) under the switching signals with SMDADT satisfying condition (12) and FMDADT satisfying condition (13).

For any $K_f > 0$, we denote $k_1, k_2, ..., k_m, k_{m+1}, ..., k_{N_\sigma(K_f,0)}$ the switching moments over the interval $[0, K_f]$ where $k_{N_\sigma(K_f,0)} = k_{\sum_{p\in\underline{B}} N_{\sigma p}(K_f,0) + \sum_{q\in\underline{U}} N_{\sigma q}(K_f,0)}$ and $K_f = \sum_{p\in\underline{B}} T_p(K_f,0) + \sum_{q\in\underline{U}} T_q(K_f,0)$. For $k \in [k_m, k_{m+1}),\ m \in \mathbb{N}_0$, we construct the following CLKF candidate for system (1):

$$V_{\sigma(k)}(k) = x^T(k)\nu_{\sigma(k)} + \sum_{s=-d_2}^{-d_1}\sum_{h=k+s}^{k-1} x^T(h)\widetilde{D}^T\nu_{\sigma(k)}, \quad (14)$$

where $\nu_{\sigma(k)} \succ 0$, $\sigma(k) \in \Omega$. Along the trajectory of the uncertain system (1), we have

$$\begin{aligned} V_{\sigma(k_m)}(k+1) &= x^T(k)A^T_{\sigma(k_m)}\nu_{\sigma(k_m)} + x^T(k-d(k))D^T_{\sigma(k_m)}\nu_{\sigma(k_m)} + \omega^T(k)G^T_{\sigma(k_m)}\nu_{\sigma(k_m)} \\ &\quad + \sum_{s=-d_2}^{-d_1}\sum_{h=k+1+s}^{k} x^T(h)\widetilde{D}^T\nu_{\sigma(k_m)} \\ &\le x^T(k)\overline{A}^T_{\sigma(k_m)}\nu_{\sigma(k_m)} + x^T(k-d(k))\overline{D}^T_{\sigma(k_m)}\nu_{\sigma(k_m)} + \omega^T(k)\overline{G}^T_{\sigma(k_m)}\nu_{\sigma(k_m)} \\ &\quad + \sum_{s=-d_2}^{-d_1}\sum_{h=k+1+s}^{k} x^T(h)\widetilde{D}^T\nu_{\sigma(k_m)}, \end{aligned}$$

for $k \in [k_m, k_{m+1})$, $m \in \mathbb{N}_0$. We observe that

$$\begin{aligned} &V_{\sigma(k_m)}(k+1) - \gamma_{\sigma(k_m)}V_{\sigma(k_m)}(k) \\ &\le x^T(k)\overline{A}^T_{\sigma(k_m)}\nu_{\sigma(k_m)} + x^T(k-d(k))\overline{D}^T_{\sigma(k_m)}\nu_{\sigma(k_m)} + \omega^T(k)\overline{G}^T_{\sigma(k_m)}\nu_{\sigma(k_m)} \\ &\quad + \sum_{s=-d_2}^{-d_1}\left(\sum_{h=k+1+s}^{k-1} x^T(h)\widetilde{D}^T\nu_{\sigma(k_m)} + x^T(k)\widetilde{D}^T\nu_{\sigma(k_m)}\right) \\ &\quad - \gamma_{\sigma(k_m)}x^T(k)\nu_{\sigma(k_m)} - \gamma_{\sigma(k_m)}\sum_{s=-d_2}^{-d_1}\sum_{h=k+s}^{k-1} x^T(h)\widetilde{D}^T\nu_{\sigma(k_m)} \\ &< x^T(k)\overline{A}^T_{\sigma(k_m)}\nu_{\sigma(k_m)} + x^T(k-d(k))\overline{D}^T_{\sigma(k_m)}\nu_{\sigma(k_m)} + \omega^T(k)\overline{G}^T_{\sigma(k_m)}\nu_{\sigma(k_m)} \\ &\quad + \sum_{s=-d_2}^{-d_1}\left(\sum_{h=k+1+s}^{k-1} x^T(h)\widetilde{D}^T\nu_{\sigma(k_m)} + x^T(k)\widetilde{D}^T\nu_{\sigma(k_m)}\right) \\ &\quad - \gamma_{\sigma(k_m)}x^T(k)\nu_{\sigma(k_m)} - \sum_{s=-d_2}^{-d_1}\left(x^T(k+s)\widetilde{D}^T\nu_{\sigma(k_m)} + \sum_{h=k+1+s}^{k-1} x^T(h)\widetilde{D}^T\nu_{\sigma(k_m)}\right) \\ &= x^T(k)\overline{A}^T_{\sigma(k_m)}\nu_{\sigma(k_m)} + x^T(k-d(k))\overline{D}^T_{\sigma(k_m)}\nu_{\sigma(k_m)} + \omega^T(k)\overline{G}^T_{\sigma(k_m)}\nu_{\sigma(k_m)} \\ &\quad + (d_2-d_1+1)x^T(k)\widetilde{D}^T\nu_{\sigma(k_m)} - \gamma_{\sigma(k_m)}x^T(k)\nu_{\sigma(k_m)} - \sum_{s=-d_2}^{-d_1} x^T(k+s)\widetilde{D}^T\nu_{\sigma(k_m)}, \end{aligned}$$

for $k \in [k_m, k_{m+1})$, $m \in \mathbb{N}_0$. From $\overline{D}_{\sigma(t_m)} \preceq \widetilde{D}$ for all $\sigma(t_m) \in \underline{N}$, one has

$$\begin{aligned} V_{\sigma(k_m)}(k+1) - \gamma_{\sigma(k_m)}V_{\sigma(k_m)}(k) &\le x^T(k)\left[\overline{A}^T_{\sigma(k_m)}\nu_{\sigma(k_m)} + (d_2-d_1+1)\widetilde{D}^T\nu_{\sigma(k_m)} - \gamma_{\sigma(k_m)}\nu_{\sigma(k_m)}\right] \\ &\quad + \omega^T(k)\overline{G}^T_{\sigma(k_m)}\nu_{\sigma(k_m)}, \end{aligned}$$

for $k \in [k_m, k_{m+1})$, $m \in \mathbb{N}_0$. According to the conditions (5) and (6), we obtain

$$V_{\sigma(k_m)}(k+1) - \gamma_{\sigma(k_m)}V_{\sigma(k_m)}(k) \le \omega^T(k)\overline{G}^T_{\sigma(k_m)}\nu_{\sigma(k_m)}, \tag{15}$$

for $k \in [k_m, k_{m+1})$, $m \in \mathbb{N}_0$. Substituting the conditions (7) and (8) into the inequality (15), it yields

$$V_{\sigma(k_m)}(k+1) - \gamma_{\sigma(k_m)}V_{\sigma(k_m)}(k) \le \xi\omega^T(k)\mathbf{1}_w, \tag{16}$$

for $k \in [k_m, k_{m+1})$, $m \in \mathbb{N}_0$, and $\xi = \max_{p\in\underline{B}, q\in\underline{U}}\{\xi_p, \xi_q\}$.

Considering the change in the value of the CLKF (14) at the switching moments and the positivity of $x(k)$ in the uncertain system (1). According to the condition (9), we get

$$\begin{aligned}V_p(k_m) &= x^T(k_m)\nu_p + \sum_{s=-d_2}^{-d_1}\sum_{h=k_m+s}^{k_m-1} x^T(h)\widetilde{D}^T\nu_p \\ &\le x^T(k_m^-)\mu_p\nu_r + \sum_{s=-d_2}^{-d_1}\sum_{h=k_m^-+s}^{k_m^--1} x^T(h)\widetilde{D}^T\mu_p\nu_r \\ &= \mu_p V_r(k_m^-), \end{aligned} \tag{17}$$

for all $p \in \underline{B}$, $r \in \underline{N}$, and $p \neq r$. Similarly, using the condition (10), we have

$$V_q(k_m) \le \mu_q V_p(k_m^-), \tag{18}$$

for all $q \in \underline{U}$, $p \in \underline{B}$. Suppose a switching time sequence $0 \le k_0 < k_1 < k_2 < \cdots$ and for every $K_f \in \mathbb{N}$ there exists a constant $m \in \mathbb{N}_0$ such that $K_f \in [k_m, k_{m+1})$. From the inequality (16), we have

$$V_{\sigma(k_m)}(K_f) \le \gamma_{\sigma(k_m)} V_{\sigma(k_m)}(K_f-1) + \xi\omega^T(K_f-1)\mathbf{1}_w.$$

That is,

$$V_{\sigma(k_m)}(K_f) \le \gamma_{\sigma(k_m)} V_{\sigma(k_m)}(K_f-1) + \xi\|\omega(K_f-1)\|_1.$$

Furthermore, it follows that

$$\begin{aligned}V_{\sigma(k_m)}(K_f) &\le \gamma^2_{\sigma(k_m)} V_{\sigma(k_m)}(K_f-2) + \xi\gamma_{\sigma(k_m)}\|\omega(K_f-2)\|_1 + \xi\|\omega(K_f-1)\|_1 \\ &\le \cdots \\ &\le \gamma_{\sigma(k_m)}^{K_f-k_m} V_{\sigma(k_m)}(k_m) + \xi\sum_{s=k_m}^{K_f-1}\gamma_{\sigma(k_m)}^{K_f-1-s}\|\omega(s)\|_1. \end{aligned} \tag{19}$$

At the switching moment k_m, according to (17) and (18), we can obtain

$$V_{\sigma(k_m)}(K_f) \le \gamma_{\sigma(k_m)}^{K_f-k_m}\mu_{\sigma(k_m)} V_{\sigma(k_m-1)}(k_m^-) + \xi\sum_{s=k_m}^{K_f-1}\gamma_{\sigma(k_m)}^{K_f-1-s}\|\omega(s)\|_1. \tag{20}$$

In fact, $\sigma(k_m-1) = \sigma(k_{m-1})$. then,

$$V_{\sigma(k_m)}(K_f) \le \mu_{\sigma(k_m)}\gamma_{\sigma(k_m)}^{K_f-k_m} V_{\sigma(k_{m-1})}(k_m^-) + \xi\sum_{s=k_m}^{K_f-1}\gamma_{\sigma(k_m)}^{K_f-1-s}\|\omega(s)\|_1. \tag{21}$$

Together with the inequalities (19) and (21), we obtain

$$\begin{aligned}V_{\sigma(k_m)}(K_f) \le{}& \mu_{\sigma(k_m)}\gamma_{\sigma(k_m)}^{K_f-k_m}\left[\gamma_{\sigma(k_{m-1})}^{k_m-k_{m-1}} V_{\sigma(k_{m-1})}(k_{m-1}) + \xi\sum_{s=k_{m-1}}^{k_m-1}\gamma_{\sigma(k_{m-1})}^{k_m-1-s}\|\omega(s)\|_1\right] \\ &+ \xi\sum_{s=k_m}^{K_f-1}\gamma_{\sigma(k_m)}^{K_f-1-s}\|\omega(s)\|_1. \end{aligned} \tag{22}$$

Based on the relationship among the inequalities (19)–(22), we can derive

$$
\begin{aligned}
&V_{\sigma(k_m)}(K_f)\\
&\leq \mu_{\sigma(k_m)}\mu_{\sigma(k_{m-1})}\gamma_{\sigma(k_m)}^{K_f-k_m}\gamma_{\sigma(k_{m-1})}^{k_m-k_{m-1}}\gamma_{\sigma(k_{m-2})}^{k_{m-1}-k_{m-2}}V_{\sigma(k_{m-2})}(k_{m-2})\\
&\quad+\mu_{\sigma(k_m)}\mu_{\sigma(k_{m-1})}\gamma_{\sigma(k_m)}^{K_f-k_m}\gamma_{\sigma(k_{m-1})}^{k_m-k_{m-1}}\xi\sum_{s=k_{m-2}}^{k_{m-1}-1}\gamma_{\sigma(k_{m-2})}^{k_{m-1}-1-s}\|\omega(s)\|_1\\
&\quad+\mu_{\sigma(k_m)}\gamma_{\sigma(k_m)}^{K_f-k_m}\xi\sum_{s=k_{m-1}}^{k_m-1}\gamma_{\sigma(k_{m-1})}^{k_m-1-s}\|\omega(s)\|_1+\xi\sum_{s=k_m}^{K_f-1}\gamma_{\sigma(k_m)}^{K_f-1-s}\|\omega(s)\|_1\\
&\leq\cdots\\
&\leq \mu_{\sigma(k_m)}\mu_{\sigma(k_{m-1})}\cdots\mu_{\sigma(k_1)}\gamma_{\sigma(k_m)}^{K_f-k_m}\gamma_{\sigma(k_{m-1})}^{k_m-k_{m-1}}\gamma_{\sigma(k_{m-2})}^{k_{m-1}-k_{m-2}}\cdots\gamma_{\sigma(k_0)}^{k_1-k_0}V_{\sigma(k_0)}(k_0)\\
&\quad+\mu_{\sigma(k_m)}\mu_{\sigma(k_{m-1})}\cdots\mu_{\sigma(k_1)}\gamma_{\sigma(k_m)}^{K_f-k_m}\gamma_{\sigma(k_{m-1})}^{k_m-k_{m-1}}\cdots\gamma_{\sigma(k_1)}^{k_2-k_1}\xi\sum_{s=k_0}^{k_1-1}\gamma_{\sigma(k_0)}^{k_1-1-s}\|\omega(s)\|_1\\
&\quad+\mu_{\sigma(k_m)}\cdots\mu_{\sigma(k_2)}\gamma_{\sigma(k_m)}^{K_f-k_m}\cdots\gamma_{\sigma(k_2)}^{k_3-k_2}\xi\sum_{s=k_1}^{k_2-1}\gamma_{\sigma(k_1)}^{k_2-1-s}\|\omega(s)\|_1+\xi\sum_{s=k_2}^{K_f-1}\gamma_{\sigma(k_2)}^{K_f-1-s}\|\omega(s)\|_1\\
&\leq \mu_{\sigma(k_m)}\mu_{\sigma(k_{m-1})}\cdots\mu_{\sigma(k_1)}\gamma_{\sigma(k_m)}^{K_f-k_m}\gamma_{\sigma(k_{m-1})}^{k_m-k_{m-1}}\gamma_{\sigma(k_{m-2})}^{k_{m-1}-k_{m-2}}\cdots\gamma_{\sigma(k_0)}^{k_1-k_0}V_{\sigma(k_0)}(k_0)\\
&\quad+\mu_{\sigma(k_m)}\mu_{\sigma(k_{m-1})}\cdots\mu_{\sigma(k_1)}\gamma_{\sigma(k_m)}^{K_f-k_m}\gamma_{\sigma(k_{m-1})}^{k_m-k_{m-1}}\cdots\gamma_{\sigma(k_1)}^{k_2-k_1}\xi\gamma_{\sigma(k_0)}^{k_1-k_0}\sum_{s=k_0}^{k_1-1}\|\omega(s)\|_1\\
&\quad+\mu_{\sigma(k_m)}\cdots\mu_{\sigma(k_2)}\gamma_{\sigma(k_m)}^{K_f-k_m}\cdots\gamma_{\sigma(k_2)}^{k_3-k_2}\xi\gamma_{\sigma(k_1)}^{k_2-k_1}\sum_{s=k_1}^{k_2-1}\|\omega(s)\|_1+\xi\gamma_{\sigma(k_2)}^{K_f-k_2}\sum_{s=k_2}^{K_f-1}\|\omega(s)\|_1.
\end{aligned}
$$

Without loss of generality, we impose that $k_0=0$. It yields that

$$
\begin{aligned}
&V_{\sigma(k_m)}(K_f)\\
&\leq\prod_{p\in\underline{B}}\mu_p^{N_{\sigma p}(K_f,0)}\prod_{q\in\underline{U}}\mu_q^{N_{\sigma q}(K_f,0)}\mathrm{e}^{(K_f-k_m)\ln\gamma_{\sigma(k_m)}+\sum_{s=1}^{m}(k_s-k_{s-1})\ln\gamma_{\sigma(k_{s-1})}}V_{\sigma(0)}(0)\\
&\quad+\prod_{p\in\underline{B}}\mu_p^{N_{\sigma p}(K_f,0)}\prod_{q\in\underline{U}}\mu_q^{N_{\sigma q}(K_f,0)}\mathrm{e}^{(K_f-k_m)\ln\gamma_{\sigma(k_m)}+\sum_{s=1}^{m}(k_s-k_{s-1})\ln\gamma_{\sigma(k_{s-1})}}\xi\sum_{s=0}^{k_1-1}\|\omega(s)\|_1\\
&\quad+\prod_{p\in\underline{B}}\mu_p^{N_{\sigma p}(K_f,0)}\prod_{q\in\underline{U}}\mu_q^{N_{\sigma q}(K_f,0)}\mathrm{e}^{(K_f-k_m)\ln\gamma_{\sigma(k_m)}+\sum_{s=2}^{m}(k_s-k_{s-1})\ln\gamma_{\sigma(k_{s-1})}}\xi\sum_{s=k_1}^{k_2-1}\|\omega(s)\|_1\\
&\quad+\prod_{p\in\underline{B}}\mu_p^{N_{\sigma p}(K_f,0)}\prod_{q\in\underline{U}}\mu_q^{N_{\sigma q}(K_f,0)}\mathrm{e}^{(K_f-k_2)\ln\gamma_{\sigma(k_2)}}\xi\sum_{s=k_2}^{K_f-1}\|\omega(s)\|_1\\
&\leq\prod_{p\in\underline{B}}\mu_p^{N_{\sigma p}(K_f,0)}\prod_{q\in\underline{U}}\mu_q^{N_{\sigma q}(K_f,0)}\mathrm{e}^{\left[\sum_{p\in\underline{B}}T_p(K_f,0)\ln\gamma_p+\sum_{q\in\underline{U}}T_q(K_f,0)\ln\gamma_q\right]}V_{\sigma(0)}(0)\\
&\quad+\prod_{p\in\underline{B}}\mu_p^{N_{\sigma p}(K_f,0)}\prod_{q\in\underline{U}}\mu_q^{N_{\sigma q}(K_f,0)}\mathrm{e}^{\left[\sum_{p\in\underline{B}}T_p(K_f,0)\ln\gamma_p+\sum_{q\in\underline{U}}T_q(K_f,0)\ln\gamma_q\right]}\xi\sum_{s=0}^{K_f-1}\|\omega(s)\|_1.
\end{aligned}
$$

According to a property of the exogenous disturbance in (2), it is immediate that

$$
\begin{aligned}
V_{\sigma(k_m)}(K_f)\leq&\prod_{p\in\underline{B}}\mu_p^{N_{\sigma p}(K_f,0)}\prod_{q\in\underline{U}}\mu_q^{N_{\sigma q}(K_f,0)}\mathrm{e}^{\left[\sum_{p\in\underline{B}}T_p(K_f,0)\ln\gamma_p+\sum_{q\in\underline{U}}T_q(K_f,0)\ln\gamma_q\right]}V_{\sigma(0)}(0)\\
&+\prod_{p\in\underline{B}}\mu_p^{N_{\sigma p}(K_f,0)}\prod_{q\in\underline{U}}\mu_q^{N_{\sigma q}(K_f,0)}\mathrm{e}^{\left[\sum_{p\in\underline{B}}T_p(K_f,0)\ln\gamma_p+\sum_{q\in\underline{U}}T_q(K_f,0)\ln\gamma_q\right]}\xi\rho.
\end{aligned}
$$

Without loss of generality, we can select the constants in (3) and (4) satisfying $N_{0p} = 0$ and $N_{0q} = 0$, for all $p \in \underline{B}$, $q \in \underline{U}$. Thus, we have

$$\begin{aligned} V_{\sigma(k_m)}(K_f) \leq & \prod_{p\in\underline{B}} \mu_p^{\frac{T_p(K_f,0)}{\tau_{ap}}} \prod_{q\in\underline{U}} \mu_q^{\frac{T_q(K_f,0)}{\tau_{aq}}} \mathrm{e}^{\left[\sum_{p\in\underline{B}} T_p(K_f,0)\ln\gamma_p + \sum_{q\in\underline{U}} T_q(K_f,0)\ln\gamma_q\right]} V_{\sigma(0)}(0) \\ & + \prod_{p\in\underline{B}} \mu_p^{\frac{T_p(K_f,0)}{\tau_{ap}}} \prod_{q\in\underline{U}} \mu_q^{\frac{T_q(K_f,0)}{\tau_{aq}}} \mathrm{e}^{\left[\sum_{p\in\underline{B}} T_p(K_f,0)\ln\gamma_p + \sum_{q\in\underline{U}} T_q(K_f,0)\ln\gamma_q\right]} \xi\rho \\ = & \, \mathrm{e}^{\sum_{p\in\underline{B}}\left(\frac{\ln\mu_p}{\tau_{ap}} + \ln\gamma_p\right)T_p(K_f,0)} \mathrm{e}^{\sum_{q\in\underline{U}}\left(\frac{\ln\mu_q}{\tau_{aq}} + \ln\gamma_q\right)T_q(K_f,0)} V_{\sigma(0)}(0) \\ & + \mathrm{e}^{\sum_{p\in\underline{B}}\left(\frac{\ln\mu_p}{\tau_{ap}} + \ln\gamma_p\right)T_p(K_f,0)} \mathrm{e}^{\sum_{q\in\underline{U}}\left(\frac{\ln\mu_q}{\tau_{aq}} + \ln\gamma_q\right)T_q(K_f,0)} \xi\rho. \end{aligned}$$

It can be derived from the inequality (13) that

$$\mathrm{e}^{\sum_{q\in\underline{U}}\left(\frac{\ln\mu_q}{\tau_{aq}} + \ln\gamma_q\right)T_q(K_f,0)} \leq 1,$$

for all $q \in \underline{U}$. Thus, we obtain

$$V_{\sigma(k_m)}(K_f) \leq \mathrm{e}^{\sum_{p\in\underline{B}}\left(\frac{\ln\mu_p}{\tau_{ap}} + \ln\gamma_p\right)T_p(K_f,0)} \left(V_{\sigma(0)}(0) + \xi\rho\right). \tag{23}$$

Using CLKF (14), the conditions (9), (10), and Definition 2 for the following estimations:

$$\begin{aligned} V_{\sigma(k_m)}(K_f) &\geq x^T(K_f)\nu_{\sigma(k_m)} \\ &\geq x^T(K_f)\beta l, \end{aligned}$$

where $\beta = \min_{p\in\underline{B},q\in\underline{U}}\{\beta_p, \beta_q\}$, and

$$\begin{aligned} V_{\sigma(0)}(0) &= x^T(0)\nu_{\sigma(0)} + \sum_{s=-d_2}^{-d_1}\sum_{h=s}^{-1} x^T(h)\widetilde{D}^T\nu_{\sigma(0)} \\ &\leq \frac{\overline{\beta}(\nu_{\sigma(0)})}{\underline{\beta}(l)} x^T(0)l + d_2(d_2-d_1+1)\frac{\overline{\beta}(\nu_{\sigma(0)})}{\underline{\beta}(l)}\|\widetilde{D}^T\|_1 \max_{s\in\{-d_2,-d_2+1,\ldots,0\}} x^T(s)l \\ &\leq \eta c_1, \end{aligned}$$

where $\eta = \frac{\overline{\beta}(\nu_{\sigma(0)})}{\underline{\beta}(l)}\left[1+d_2(d_2-d_1+1)\|\widetilde{D}^T\|_1\right]$. From the inequality (23), we obtain

$$x^T(K_f)l \leq \left(\frac{\eta c_1 + \xi\rho}{\beta}\right)\mathrm{e}^{\sum_{p\in\underline{B}}\left(\frac{\ln\mu_p}{\tau_{ap}} + \ln\gamma_p\right)T_p(K_f,0)}. \tag{24}$$

Let $\epsilon = \max_{p\in\underline{B}}\left(\frac{\ln\mu_p}{\tau_{ap}} + \ln\gamma_p\right)$. Hence,

$$x^T(K_f)l \leq \left(\frac{\eta c_1 + \xi\rho}{\beta}\right)\mathrm{e}^{\epsilon K_f}. \tag{25}$$

It follows from the condition (11) and the inequality (12) that

$$\begin{aligned} \epsilon K_f &= \max_{p\in\underline{B}}\left(\frac{K_f\ln\mu_p}{\tau_{ap}} + K_f\ln\gamma_p\right) \\ &< \ln\left(\frac{\beta c_2}{\eta c_1 + \xi\rho}\right). \end{aligned} \tag{26}$$

Utilizing the inequalities (25) and (26), we arrive at

$$x^T(K_f)l < c_2.$$

Similarly, we can have

$$x^T(k)l < c_2,$$

for all $k = 1, 2, ..., K_f$. It can be concluded by Definition 2 that the uncertain system (1) is finite-time bounded with respect to $(c_1, c_2, K_f, l, \rho, \sigma(k))$ for the switching signal $\sigma(k)$ with SMDADT (12) and FMDADT (13). □

Remark 2. *In Theorem 1, our switching scheme not only gives the lower bounds that the bounded subsystems should dwell on but also provides the upper bounds for the unbounded subsystems.*

Remark 3. *In order to obtain the feasible positive vectors ν_p, ν_q, positive constants $\xi_p, \xi_q, \beta_p, \beta_q$, the SMDADT τ_{ap}, and the FMDADT τ_{aq} guaranteeing the FTB of the uncertain system (1), an algorithm is provided as follows:*
Step 1: *Determine whether $\underline{A}_i \succeq 0$, $\underline{D}_i \succeq 0$, and $\underline{G}_i \succeq 0$ for all $i \in \underline{N}$, where these constant matrices are defined in Assumption 1, to ensure the positivity of the uncertain system (1);*
Step 2: *Choose the positive integers d_1 and d_2 from the given time-varying delay $d(k)$ and set the initial condition $\psi(\theta) \succeq 0$, where $\theta = -d_2, -d_2 + 1, ..., 0$;*
Step 3: *Choose the time constant K_f and compute the positive constant ρ from the condition of the exogenous disturbance defined as in (2);*
Step 4: *Construct the matrix $\widetilde{D}$ defined in Theorem 1;*
Step 5: *Choose the parameters $\gamma_p > 1$, $\mu_p > 1$, $p \in \underline{B}$, $\gamma_q > 1$, $0 < \mu_q < 1$, and $q \in \underline{U}$, the positive constants $c_1 < c_2$, and the positive vector l satisfying $\max_{\theta \in \{-d_2, -d_2+1, ..., 0\}} \psi^T(\theta)l \leq c_1$;*
Step 6: *Determine a feasibility of (5)–(10) by using some available algorithms such as the LP toolbox in Matlab. Then, we can obtain the feasible positive vectors ν_p, ν_q and positive constants $\xi_p, \xi_q, \beta_p, \beta_q$ for every $p \in \underline{B}$, $q \in \underline{U}$;*
Step 7: *Calculate the positive constants β, ξ, Γ, and η;*
Step 8: *Determine whether (11) is satisfied or not. If it is satisfied, then the values τ^*_{ap} and τ^*_{aq} can be obtained by (12) and (13), respectively. Thus, we can estimate the values τ_{ap} and τ_{aq}, which ensure that switching system (1) is the FTB. Otherwise, we repeat **Step 5** to obtain a proper value of c_2. Then, we determine whether (11) is satisfied or not.*

Next, another FTB result of system (1) without its interval uncertainty is presented as follows:

Corollary 1. *Consider system (1) with exogenous disturbance. Let $\gamma_p > 1$, $\mu_p > 1$, $p \in \underline{B}$, $\gamma_q > 1$, $0 < \mu_q < 1$, and $q \in \underline{U}$ be the constants. For the given constants $c_2 > c_1 > 0$, the time constant $K_f > 0$, and the vector $l \succ 0$. Suppose that there exist positive vectors $\nu_p \succ 0$, $\nu_q \succ 0$ and constants $\xi_p > 0$, $\xi_q > 0$, $\beta_p > 0$, and $\beta_q > 0$, such that*

$$\left[A_p^T + (d_2 - d_1 + 1)D^T - \gamma_p I_n\right]\nu_p \prec 0,$$

$$\left[A_q^T + (d_2 - d_1 + 1)D^T - \gamma_q I_n\right]\nu_q \prec 0,$$

$$G_p^T \nu_p - \xi_p \mathbf{1}_w \prec 0,$$

$$G_q^T \nu_q - \xi_q \mathbf{1}_w \prec 0,$$

$$\beta_p l \preceq \nu_p \preceq \mu_p \nu_r,$$

$$\beta_q l \preceq \nu_q \preceq \mu_q \nu_p,$$

$$\Gamma^{K_f} < \frac{\beta c_2}{\eta c_1 + \xi \rho},$$

hold for every $p \in \underline{B}$, $q \in \underline{U}$, $r \in \underline{N}$, $p \neq r$. Then system (1) is positive and finite-time bounded with respect to $(c_1, c_2, K_f, l, \rho, \sigma(k))$ under the switching signals with SMDADT satisfying

$$\tau_{ap} \geq \tau_{ap}^* = \frac{K_f \ln \mu_p}{\ln \frac{\beta c_2}{\eta c_1 + \xi \rho} - K_f \ln \gamma_p}, \quad \forall p \in \underline{B},$$

and FMDADT satisfying

$$\tau_{aq} \leq \tau_{aq}^* = -\frac{\ln \mu_q}{\ln \gamma_q}, \quad \forall q \in \underline{U},$$

where

$$\Gamma = \max_{p \in \underline{B}} \{\gamma_p\}, \ \beta = \min_{p \in \underline{B}, q \in \underline{U}} \{\beta_p, \beta_q\}, \ \xi = \max_{p \in \underline{B}, q \in \underline{U}} \{\xi_p, \xi_q\}, \ \eta = \frac{\overline{\beta}(\nu_{\sigma(0)})}{\underline{\beta}(l)} \Big[1 + d_2(d_2 - d_1 + 1) \|D^T\|_1\Big],$$

and

$$D = (d_{kl}) \in \mathbb{R}^{n \times n}, \ d_{kl} = \max_{i \in \underline{N}} \Big\{ D_i^{(kl)} \Big\},$$

$D_i^{(kl)}$ is the kth row and lth column entry of system matrices D_i, $i \in \underline{N}$.

Proof. With the same symbols in Theorem 1, this corollary can be proved by utilizing the following CLKF candidate for system (1):

$$V_{\sigma(k)}(k) = x^T(k)\nu_{\sigma(k)} + \sum_{s=-d_2}^{-d_1} \sum_{h=k+s}^{k-1} x^T(h) D^T \nu_{\sigma(k)},$$

where $\nu_{\sigma(k)} \succ 0$, $\sigma(k) \in \Omega$. The remainder of the proof is similar to that of Theorem 1. Hence, the detail is omitted. □

4. Numerical Simulations

In this section, we provide a numerical example together with the simulation results to demonstrate the correctness and effectiveness of our theoretical analysis presented in the previous section.

Example 1. *In [2,3,9], the preservation of water-quality standards in streams is discussed. References used the mathematical methods and computer control techniques to simulate the growth and management of water-quality constituents to the standard level in the form of a linearized model. The model can be described by a discrete-time switched positive time-varying delay system with interval uncertainties and exogenous disturbance, which is consistent with the uncertain system (1). For terminology in this model, $x(k)$ represents the state vector of water-quality constituents such as oxygen, algae, and ammonia nitrogen. The physical meaning and more details of the time-varying delay, the delayed factor, and the disturbance were exhibited in those references. Furthermore, the existence of interval uncertainties in the constant system matrices reflects the fact of the studied system. Namely, the process of simulation may have slight errors. Therefore, the problem of water-quality control to standard level in the finite-time interval has been intensively studied by the uncertain system (1) comprising two subsystems in this example. The system data are given as follows:*

$$\underline{A}_1 = \begin{bmatrix} 0.249 & 0.319 \\ 0.470 & 0.155 \end{bmatrix}, \ \overline{A}_1 = \begin{bmatrix} 0.251 & 0.321 \\ 0.490 & 0.165 \end{bmatrix},$$

$$\underline{D}_1 = \begin{bmatrix} 0.019 & 0.079 \\ 0.018 & 0.061 \end{bmatrix}, \ \overline{D}_1 = \begin{bmatrix} 0.021 & 0.081 \\ 0.022 & 0.063 \end{bmatrix},$$

$$\underline{G}_1 = \begin{bmatrix} 0.009 \\ 0.010 \end{bmatrix}, \ \overline{G}_1 = \begin{bmatrix} 0.011 \\ 0.010 \end{bmatrix},$$

$$\underline{A}_2 = \begin{bmatrix} 0.829 & 0.350 \\ 0.505 & 0.285 \end{bmatrix}, \ \overline{A}_2 = \begin{bmatrix} 0.831 & 0.370 \\ 0.515 & 0.285 \end{bmatrix},$$

$$\underline{D}_2 = \begin{bmatrix} 0.009 & 0.045 \\ 0.018 & 0.029 \end{bmatrix}, \ \overline{D}_2 = \begin{bmatrix} 0.011 & 0.049 \\ 0.022 & 0.031 \end{bmatrix},$$

$$\underline{G}_2 = \begin{bmatrix} 0.048 \\ 0.049 \end{bmatrix}, \ \overline{G}_2 = \begin{bmatrix} 0.052 \\ 0.051 \end{bmatrix},$$

$$d(k) = 1 + \sin^2\left(\frac{k\pi}{2}\right), \ \textit{and} \ \ \omega(k) = 0.01e^{0.04k}\sin(0.05k\pi).$$

Under the given time-varying delay above, we select $d_1 = 1$ *and* $d_2 = 2$*. It is obvious that* $\underline{A}_1 \succeq 0$, $\underline{A}_2 \succeq 0$, $\underline{D}_1 \succeq 0$, $\underline{D}_2 \succeq 0$, $\underline{G}_1 \succeq 0$, *and* $\underline{G}_2 \succeq 0$*. According to Assumption 1 and Lemma 1, the studied system is positive. For the numerical simulations, we set the initial condition as* $\psi(\theta) = [5 \ \ 8]^T$, $\theta = -d_2, -d_2+1, ..., 0$, *and let the system matrices be*

$$A_1 = \frac{\underline{A}_1 + \overline{A}_1}{2} = \begin{bmatrix} 0.250 & 0.320 \\ 0.480 & 0.160 \end{bmatrix}, \ D_1 = \frac{\underline{D}_1 + \overline{D}_1}{2} = \begin{bmatrix} 0.020 & 0.080 \\ 0.020 & 0.062 \end{bmatrix}, \ G_1 = \frac{\underline{G}_1 + \overline{G}_1}{2} = \begin{bmatrix} 0.010 \\ 0.010 \end{bmatrix},$$

and

$$A_2 = \frac{\underline{A}_2 + \overline{A}_2}{2} = \begin{bmatrix} 0.830 & 0.360 \\ 0.510 & 0.285 \end{bmatrix}, \ D_2 = \frac{\underline{D}_2 + \overline{D}_2}{2} = \begin{bmatrix} 0.010 & 0.047 \\ 0.020 & 0.030 \end{bmatrix}, \ G_2 = \frac{\underline{G}_2 + \overline{G}_2}{2} = \begin{bmatrix} 0.050 \\ 0.050 \end{bmatrix}.$$

Given the positive vector $l = [1 \ \ 1]^T$*, then, we assign the positive constants* $c_1 = 13$, $c_2 = 130$ *and the time constant* $K_f = 30$*, which satisfies* $\max_{\theta \in \{-d_2, -d_2+1, ..., 0\}} \psi^T(\theta)l \leq c_1$*. From the condition of the exogenous disturbance defined as in (2), it is obvious that* $\rho = 0.602$*. The state response and the value of* $x^T(k)l$, $k \in \{0, 1, 2, ..., K_f\}$ *for subsystem 1 and subsystem 2 are shown, respectively, in Figures 1 and 2. From two subsystems of the simulations, it is verified that the first subsystem is finite-time bounded with respect to* $(13, 130, 30, [1 \ \ 1]^T, 0.602, \sigma(k))$*, and the second one is finite-time unbounded with respect to the same required parameters.*

Then, it is obvious that

$$\widetilde{D} = \begin{bmatrix} 0.021 & 0.081 \\ 0.022 & 0.063 \end{bmatrix}.$$

For given sclars $\gamma_1 = 1.005$, $\mu_1 = 2.010$, $\gamma_2 = 1.400$*, and* $\mu_2 = 0.500$ *we can obtain a set of feasible solution for Theorem 1:*

$$\nu_1 = [2.7571 \ \ 2.5917]^T, \ \ \nu_2 = [1.3783 \ \ 1.2958]^T, \ \ \xi_1 = 0.1, \ \ \xi_2 = 0.2, \ \ \beta_1 = 2.1, \ \ \beta_2 = 1.2.$$

Thus, this system is finite-time bounded with respect to $(13, 130, 30, [1 \ \ 1]^T, 0.602, \sigma(k))$ *under the switching signal* $\sigma(k)$ *with SMDADT* $\tau^*_{a1} = 12.5602$ *and FMDADT* $\tau^*_{a2} = 2.0600$*, which satisfy the conditions specified by (12) and (13), respectively. Without loss of generality, we can select* $\tau_{a1} = 13 > 12.5602$ *and* $\tau_{a2} = 2 < 2.0600$ *for subsystem 1 and subsystem 2, respectively. The designed switching signal* $\sigma(k)$ *of this system is depicted in Figure 3. The state response and the value of* $x^T(k)l$ *of this system under the corresponding switching signal* $\sigma(k)$ *are shown at the left-hand side and the right-hand side of Figure 4, respectively. The plot indicates that the value of* $x^T(k)l$ *at* $K_f = 30$ *does not exceed* $c_2 = 130$*. From Figure 4, it can be seen that although the uncertain system (1) representing the water-quality controllability model in streams includes both the bounded subsystem and the unbounded subsystem, it is finite-time bounded with respect to* $(13, 130, 30, [1 \ \ 1]^T, 0.602, \sigma(k))$ *under the designed switching signals. These numerical results show that the state vectors representing water-quality constituents from two subsystems can be managed and restored to the standard level in the fixed finite-time interval by designing switching signals based on Theorem 8. Consequently, the correctness and effectiveness of our theoretical analysis and numerical simulations are illustrated.*

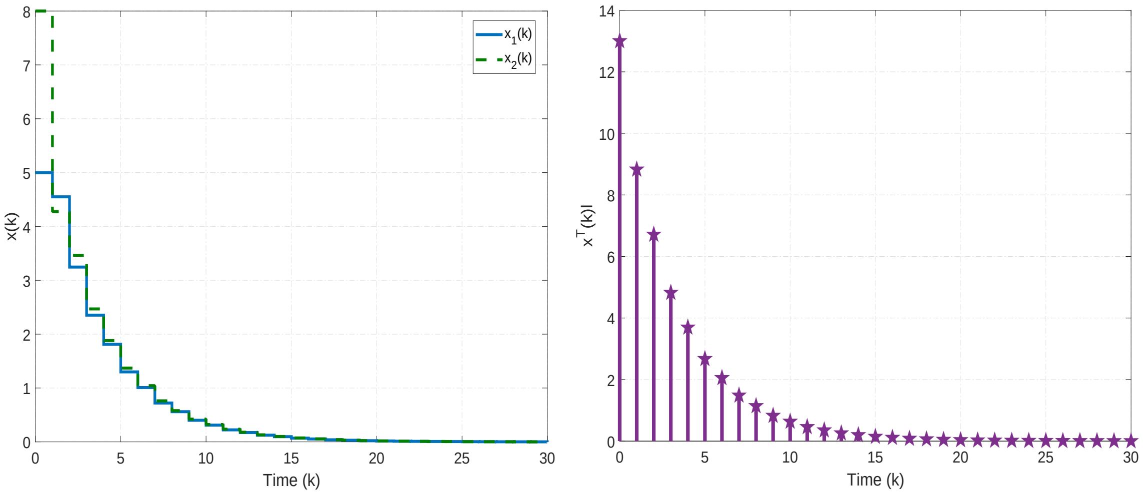

Figure 1. State response of the first subsystem.

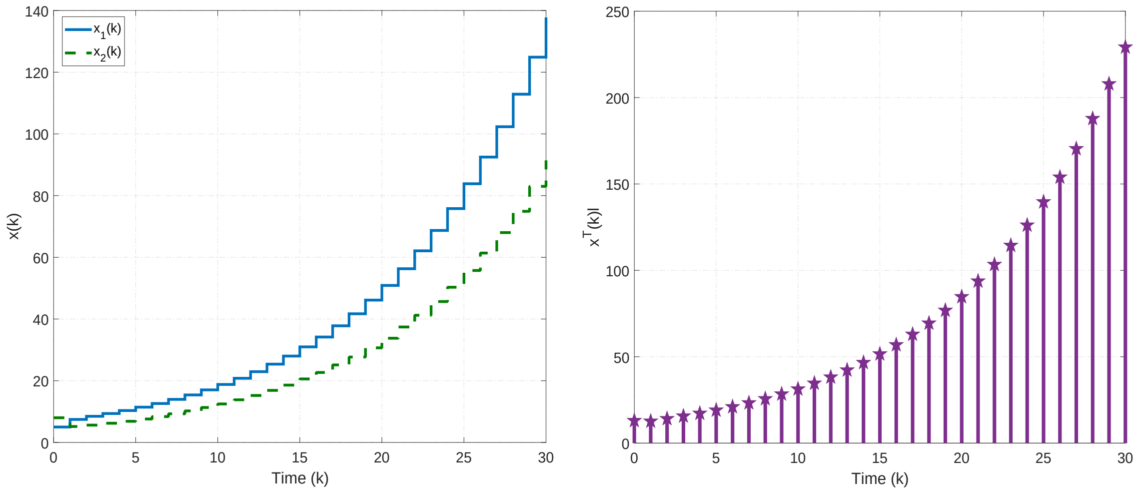

Figure 2. State response of the second subsystem.

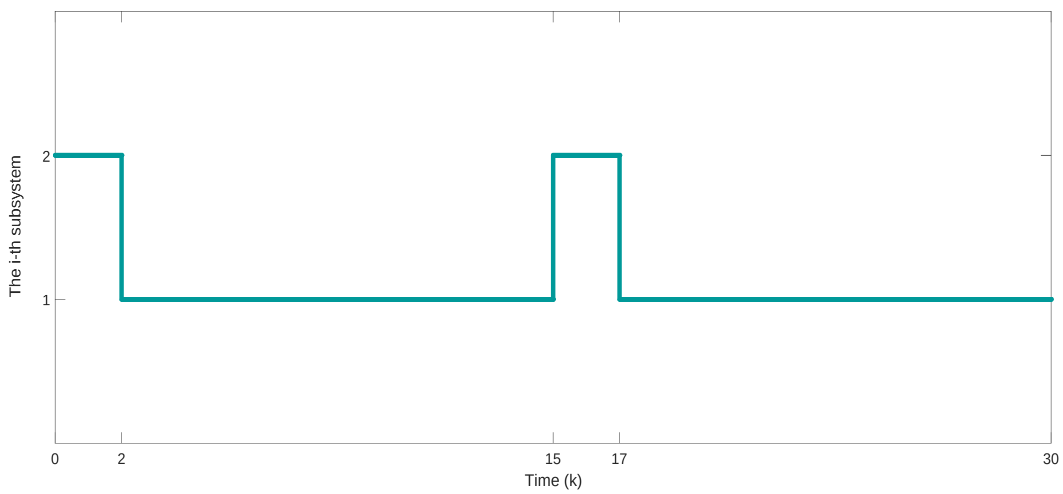

Figure 3. The designed switching signal $\sigma(k)$.

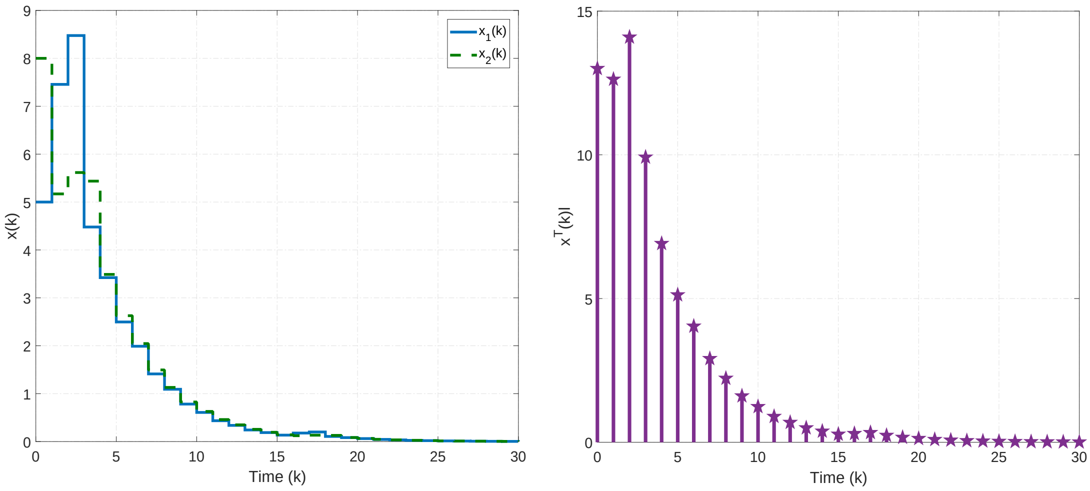

Figure 4. State response of the uncertain system (1) under the corresponding switching signal $\sigma(k)$.

Remark 4. *From sufficient conditions (5) and (6) in Theorem 1, the part of symmetric negative-definite matrices* $-\gamma_p I_n$ *and* $-\gamma_q I_n$*, where* $\gamma_p > 1,\ p \in \underline{B},\ \gamma_q > 1,\ q \in \underline{U}$*, is important for proving to be the robust finite-time control of the uncertain system (1). From the above example, it is obvious that*

$$-\gamma_1 I_2 = \begin{bmatrix} -1.005 & 0 \\ 0 & -1.005 \end{bmatrix} \text{ and } -\gamma_2 I_2 = \begin{bmatrix} -1.400 & 0 \\ 0 & -1.400 \end{bmatrix}.$$

5. Discussion and Conclusions

In this article, the FTB problem for a class of discrete-time SPSs with time-varying delay, interval uncertainties, exogenous disturbance, and finite-time unbounded subsystems was investigated. A class of QASSs for the systems with bounded and unbounded subsystems was designed to analyze the switching behaviors of the systems. Namely, the systems can switch from a finite-time bounded subsystem to any other subsystems. However, they cannot switch from a finite-time unbounded subsystem to another finite-time unbounded subsystem.

Since the advantage of the MDADT switching law is that the ADT of each subsystem can be calculated using the crucial characteristics of individual subsystems, the MDADT switching law is more general and less conservative than the ADT switching law used in [43,46,50,52,55,56]. Our switching law used in this work is MDADT, which consists of the SMDADT and FMDADT switching laws. It is utilized to deal with the problem of the FTB for the systems, including both the finite-time bounded and the finite-time unbounded subsystems under designing a class of QASSs. Namely, we use the SMDADT switching law with the finite-time bounded subsystems and use the FMDADT switching law with the finite-time unbounded subsystems to compensate for the divergent behavior of the state trajectory from the finite-time unbounded subsystems.

By the positivity of the studied systems together with the SMDADT and FMDADT switching laws, the suitable CLKF was established, and the systems can be ensured for being FTB by the sufficient conditions obtained in Theorem 1. In addition, a new DDSC of systems without the interval uncertainties was also derived in Corollary 1. These strategies are applied to the water-quality controllability model in streams. A numerical example was shown to illustrate the effectiveness of our analytical results. However, to the best of our knowledge, there is no result on the FTB for a class of discrete-time SPSs, including time-varying delay, interval uncertainties, exogenous disturbance, and finite-time unbounded subsystems in the literature. Therefore, this research is the first work for studying the FTB of the systems and cannot be compared directly with other results.

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