Robust Graph Ideals

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Abstract

Let I be a toric ideal. We say I is *robust* if its universal Gröbner basis is a minimal generating set. We show that any robust toric ideal arising from a graph G is minimally generated by its Graver basis. We then completely characterize all graphs which give rise to robust ideals. Our characterization shows that robustness can be determined solely in terms of graph-theoretic conditions on the set of circuits of G.

Motivation

- Many interesting and fundamental examples in algebraic geometry are furnished by *determinental ideals*.
- For the ideal of maximal minors of a generic matrix, many of its homological invariants can be studied in the context of its robustness (defined below) [Boo11].
- In the $2 \times n$ case, these ideals arise from a simple family of graphs.

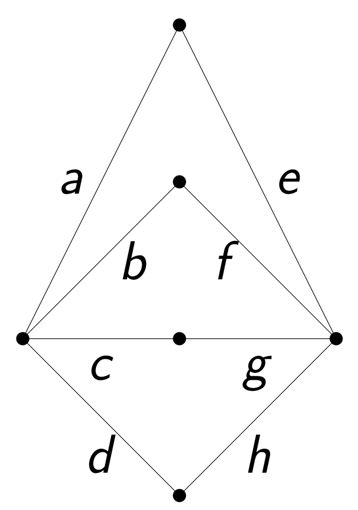


Figure: A simple graph gives rise to the ideal of maximal minors for the 2×4 matrix of indeterminates $\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$.

Toric Ideals and Gröbner Basics

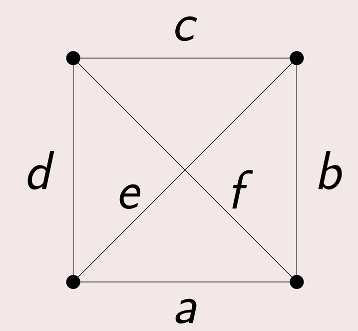
- Let $m_1, \ldots, m_d$ be monomials in $k[x_1, \ldots, x_n]$. The kernel of the map
$$\varphi : k[x_1, \ldots, x_d] \to k[x_1, \ldots, x_n]$$
$$x_i \mapsto m_i$$
is a prime ideal generated by differences of monomials. Geometrically, it is the affine closure of a (nonnormal) toric variety, so we call $\ker \varphi$ a *toric ideal*.
- Let I be a toric ideal. The *reduced Gröbner basis* of I (for a fixed term order $<$ on $k[x_1, \ldots, x_k]$) is generating set whose initial terms minimally generate the monomial ideal $\mathrm{in}_< \ker I$. A generating set which satisfies this form all term orders simulateneously is a *universal Gröbner basis*.
- The *support* of a binomial $x^a - x^b$ written in multindex notation is the set of all indices i for which a_i or b_i is nonzero. An irreducible binomial in I of minimal support is called a *circuit*. I is generated up to radical by its circuits.
- $x^a - x^b \in I$ is said to be *primitive* if there exists no binomial $x^c - c^d \in I$ such that x^c divides x^a and x^d divides x^b. The set of all primitive binomials in I is called its *Graver basis*.
- Every circuit of a toric ideal is contained in its universal Gröbner basis, which in turns sits inside of the Graver basis; the reverse inclusions are all generally strict [Stu96].
- An ideal is said to be *robust* if it is minimally generated by its universal Gröbner basis.

Theorem [BR]

All robust ideals generated by quadratic binomials are essentially determinental. The picture is considerably more complicated for ideals generated in degree 3 and higher.

Graph Ideals

- Let $G = (V, E)$ be a simple, connected graph. We define the *toric ideal of* G as the kernel of the following monomial map
$$\varphi_G : k[E] \to k[V]$$
$$(u, v) \mapsto uv$$
We denote the ideal by I_G.
- I_G is generated by binomials corresponding to closed walks of even length in G.
- (Example) The Graver basis of the graph below consists of three binomials.



$$B_1 = ac - bd$$
$$B_2 = ac - ef$$
$$B_3 = bd - ef$$

All three binomials are circuits, thus also elements of the universal Gröbner basis. However, only two are needed to generate the ideal. This graph is not robust.

First Result

- Our result shows that robustness for graph ideals is equivalent to a seemingly much stronger condition.

Theorem

The ideal I_G is robust if and only if it is minimally generated by its Graver basis.

- This result greatly propelled the study of robust graph ideals, because in general, the Graver basis is easier to compute than the universal Gröbner basis [Stu96].

Second Result

- In light of the above result, we sought to characterize all robust graph ideals using graph theoretic language.
- We focused on conditions of walks which give rise to primitive binomials and those which correspond to circuits. This became the following result.

Theorem

A graph G is robust if and only if no walk corresponding to a circuit binomial has a non-odd chord, an effective chrossing, and if no two such walks share exactly one edge belonging to a cyclic block of both.

- This is interesting because the set circuits is often a strict subset of the Graver basis.

Methods

- Our two theorems were proven using graph theoretic techniques. This is made possible by results of [RTT12] and [TT11] about translating the algebraic definitions above into graph theoretic conditions on closed walks in the graph, such as the theorem below, due to [Vil95].

Theorem

A closed walk corresponds to a circuit binomial if and only if the walk is an even cycle, two odd cycles joined at a vertex, or two vertex-disjoint odd cycles joined by a simple path.

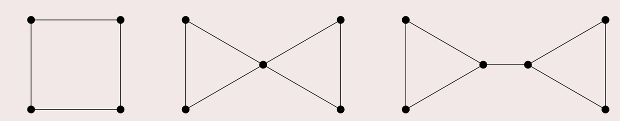


Figure: The three types of walks corresponding to circuit binomials.

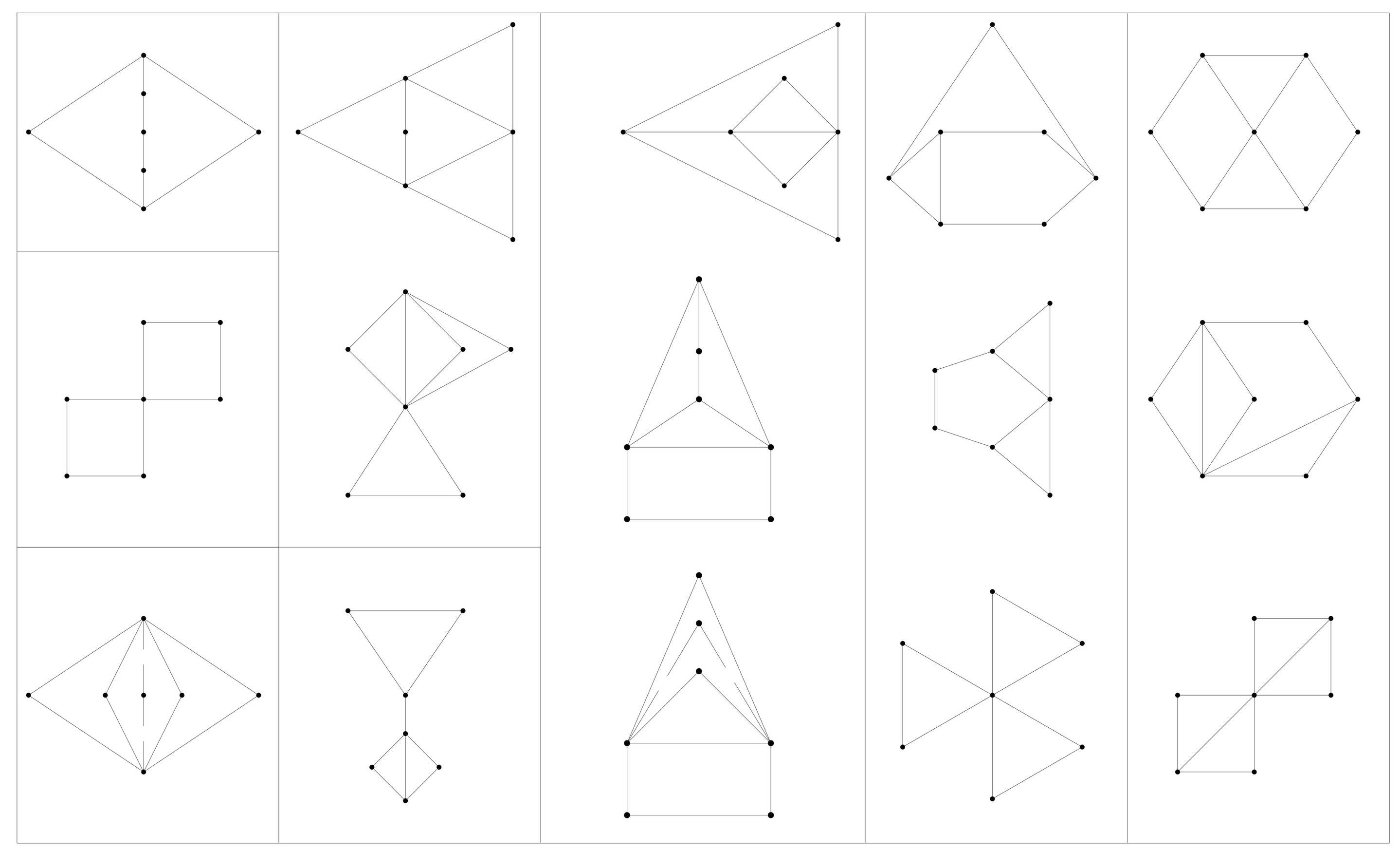


Figure: All connected robust graphs G on 7 vertices such that the ideal I_G has full support in its edge ring, divided up into isomorphism classes of I_G.

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