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Robust H^∞ Control of an Uncertain System Via a Stable Output Feedback Controller

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***Abstract*—This technical note presents a new approach to the robust control of an uncertain system via a stable output feedback controller. The uncertain systems under consideration contain structured uncertainty described by integral quadratic constraints. The controller is designed to achieve absolute stabilization with a specified level of disturbance attenuation. The main result involves solving a state feedback version of the problem by solving an algebraic Riccati equation dependent on a set of scaling parameters. Then two further algebraic Riccati equations are solved, which depend on a further set of scaling parameters.**

***Index Terms*—Absolute stabilization, H^∞ control, integral quadratic constraints, strong stabilization.**

I. Introduction

This technical note considers the problem of robust H^∞ control via a stable output feedback controller. It is well known that the use of stable controllers is preferable to the use of unstable feedback controllers in many practical control problems; e.g., see [1]–[3]. Indeed, the use of unstable controllers can lead to problems with actuator and sensor failure, sensitivity to plant uncertainties and implementation problems. This has motivated a number researchers to consider problems of H^∞ control via the use of stable controllers; e.g., see [1]–[4].

In this technical note, we propose a new approach to the problem of robust H^∞ control via a stable output feedback controller. We consider a class of uncertain systems with structured uncertainty described by integral quadratic constraints (IQCs); e.g., see [5] and [6]. Indeed, our results build on the results of [5] which provide necessary and sufficient conditions for the absolute stabilization of such uncertain systems with a specified level of disturbance attenuation (but with no requirement that the output feedback controller is stable). The key idea behind our approach is to begin with an uncertain system of the type considered in [5] and then add an additional uncertainty to form a new uncertain system. This additional uncertainty has the property that for one specific value of the uncertainty, the new uncertain system reduces to the original uncertain system and thus any suitable controller for the new uncertain system will also solve the problem of absolute stabilization with a specified level of disturbance attenuation for the original system. Also, for a different value of the new uncertainty, the new uncertain system reduces to a certain open-loop system in such a way that the controller is forced to be stable. Because our approach involves the addition of new uncertainties, our results provide only sufficient conditions rather than necessary and sufficient conditions for absolute stabilization with a specified level of disturbance attenuation. However, because the new uncertainty is explicitly constructed, this can give some indication about the degree of conservatism introduced.

Our main result is obtained applying the results of [5] to the new uncertain system. This gives a stable output feedback controller solving a problem of absolute stabilization with a specified level of disturbance attenuation. This is achieved by solving a pair of algebraic Riccati equations dependent on a set of scaling parameters. The controller obtained is of the same order of the plant.

The remainder of the technical note proceeds as follows: In Section II of the technical note, we set up the problem of absolute stabilization with a specified level of disturbance attenuation via a stable output feedback controller. Section III introduces the new uncertain system for which we will apply the results of [5] in order to obtain a stable controller which guarantees absolute stabilization with a specified level of disturbance attenuation. The construction of this new uncertain system involves solving a state feedback version of the approach of [5] applied to the original uncertain system. This involves solving an algebraic Riccati equation of the H^∞ type which is dependent on a set of scaling parameters. This leads to our main result which is a procedure for constructing the required stable controller. This procedure involves solving a pair algebraic Riccati equations of the H^∞ type which are dependent on an additional set of scaling parameters. The final controller is constructed from the solutions to these Riccati equations. Section IV presents an example which illustrates the theory presented in the technical note. This example, which involves an H^∞ control problem for a linear time-invariant (LTI) system without uncertainty, is taken from [2]. We show that for this example, our approach is slightly less conservative than the approach of [2].

II. Problem Statement

We consider an output feedback H^∞ control problem for an uncertain system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) + \sum_{s=1}^{k} D_s \xi_s(t) \\ z(t) &= C_1 x(t) + D_{12} u(t) \\ \zeta_1(t) &= K_1 x(t) + G_1 u(t) \\ &\vdots \\ \zeta_k(t) &= K_k x(t) + G_k u(t) \\ y(t) &= C_2 x(t) + D_{21} w(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the *state*, $w(t) \in \mathbf{R}^p$ is the *disturbance input*, $u(t) \in \mathbf{R}^m$ is the *control input*, $z(t) \in \mathbf{R}^q$ is the *error output*, $\zeta_1(t) \in \mathbf{R}^{h_1}, \ldots, \zeta_k(t) \in \mathbf{R}^{h_k}$ are the *uncertainty outputs*, $\xi_1(t) \in \mathbf{R}^{r_1}, \ldots, \xi_k(t) \in \mathbf{R}^{r_k}$ are the *uncertainty inputs* , and $y(t) \in \mathbf{R}^l$ is the *measured output*. The uncertainty in this system is described by a set of equations of the form

$$\begin{aligned} \xi_1(t) &= \phi_1\left(t, \zeta_1(\cdot)|_0^t\right) \\ &\vdots \\ \xi_k(t) &= \phi_k\left(t, \zeta_k(\cdot)|_0^t\right) \end{aligned} \tag{2}$$

where the following IQC is satisfied.

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Definition 1: (IQC; see [5] and [6]) An uncertainty of the form (2) is an admissible uncertainty for the system (1) if the following conditions hold: given any locally square integrable control input $u(\cdot)$ and disturbance input $w(\cdot)$, and any corresponding solution to the system (1) and (2), let $(0, t_*)$ be the interval on which this solution exists. Then there exist constants $d_1 \geq 0, \ldots, d_k \geq 0$ and a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \rightarrow t_*, t_i \geq 0$ and

$$\int_0^{t_i} \|\xi_s(t)\|^2 dt \leq \int_0^{t_i} \|\zeta_s(t)\|^2 dt + d_s \quad \forall\, i \;\; \forall\, s = 1, \ldots, k. \quad (3)$$

Here, $\|\cdot\|$ denotes the standard Euclidean norm and $\mathbf{L}_2[0, \infty)$ denotes the Hilbert space of square integrable vector valued functions defined on $[0, \infty)$. Note that t_* may be equal to infinity. The class of all such admissible uncertainties $\xi(\cdot) = [\xi_1(\cdot), \ldots, \xi_k(\cdot)]$ is denoted Ξ.

For the uncertain system (1) and (3), we consider a problem of absolute stabilization with a specified level of disturbance attenuation. The class of controllers considered are stable output feedback controllers of the form

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c y(t) \\ u(t) &= C_c x_c(t) \end{aligned} \quad (4)$$

where A_c is a Hurwitz matrix.

Definition 2: The uncertain system (1) and (3) is said to be *absolutely stabilizable with disturbance attenuation* γ via stable output feedback control if there exists a stable output feedback controller (4) and constants $c_1 > 0$ and $c_2 > 0$ such that the following conditions hold:

1) For any initial condition $[x(0), x_c(0)]$, any admissible uncertainty inputs $\xi(\cdot)$ and any disturbance input $w(\cdot) \in \mathbf{L}_2[0, \infty)$, then

$$[x(\cdot), x_c(\cdot), u(\cdot), \xi_1(\cdot), \ldots, \xi_k(\cdot)] \in \mathbf{L}_2[0, \infty)$$

(hence, $t_* = \infty$) and

$$\begin{aligned} &\|x(\cdot)\|_2^2 + \|x_c(\cdot)\|_2^2 + \|u(\cdot)\|_2^2 + \sum_{s=1}^{k} \|\xi_s(\cdot)\|_2^2 \\ &\qquad \leq c_1 \left[\|x(0)\|^2 + \|x_c(0)\|^2 + \|w(\cdot)\|_2^2 + \sum_{s=1}^{k} d_s \right]. \end{aligned} \quad (5)$$

2) The following H^∞ norm bound condition is satisfied: If $x(0) = 0$ and $x_c(0) = 0$, then

$$J \triangleq \sup_{w(\cdot) \in \mathbf{L}_2[0,\infty)} \; \sup_{[\xi_1(\cdot),\ldots,\xi_k(\cdot)] \in \Xi} \frac{\|z(\cdot)\|_2^2 - c_2 \sum_{s=1}^{k} d_s}{\|w(\cdot)\|_2^2} < \gamma^2. \quad (6)$$

Here, $\|q(\cdot)\|_2$ denotes the $\mathbf{L}_2[0, \infty)$ norm of a function $q(\cdot)$. That is, $\|q(\cdot)\|_2^2 \triangleq \int_0^\infty \|q(t)\|^2 dt$.

Assumption 1: The uncertain system (1) and (3) will be assumed to satisfy the following conditions:

i) The pair (A, C_1) is observable.
ii) The pair (A, B_1) is controllable.

III. The Main Results

The key idea behind our main result is to introduce some extra uncertainty into the uncertain system (1) and (3). This is done in such a way so that the controller must not only achieve absolute stabilization with disturbance attenuation γ when applied to the original uncertain system (1) and (3) but also the controller must achieve internal stability when applied to a "null" system; i.e., the controller itself must be stable. We first consider a state feedback version of the problem considered in [5]. Using the results of [5], we can give a Riccati equation condition for the uncertain system (1) and (3) to be absolutely stabilizable with a specified level of disturbance attenuation via a state feedback controller. The Riccati equation under consideration is defined as follows: Let $\tau_1 > 0, \ldots, \tau_k > 0$ be given constants and consider the algebraic Riccati equation

$$\begin{aligned} &\left(A - B_2 E_1^{-1} \hat{D}_{12}' \hat{C}_1\right)' X + X \left(A - B_2 E_1^{-1} \hat{D}_{12}' \hat{C}_1\right) \\ &\quad + X \left(\hat{B}_1 \hat{B}_1' - B_2 E_1^{-1} B'_2\right) X \\ &\quad + \hat{C}_1' \left(I - \hat{D}_{12} E_1^{-1} \hat{D}_{12}'\right) \hat{C}_1 = 0 \end{aligned} \quad (7)$$

where

$$\begin{aligned} &\hat{C}_1 = \begin{bmatrix} C_1 \\ \sqrt{\tau_1} K_1 \\ \vdots \\ \sqrt{\tau_k} K_k \end{bmatrix}; \quad \hat{D}_{12} = \begin{bmatrix} D_{12} \\ \sqrt{\tau_1} G_1 \\ \vdots \\ \sqrt{\tau_k} G_k \end{bmatrix} \\ &E_1 = \hat{D}_{12}' \hat{D}_{12} \\ &\hat{B}_1 = [\, \gamma^{-1} B_1 \quad \sqrt{\tau_1}^{-1} D_1 \quad \ldots \quad \sqrt{\tau_k}^{-1} D_k \,]. \end{aligned} \quad (8)$$

Assumption 2: The uncertain system (1) and (3) will be assumed to be such that $E_1 > 0$ for any $\tau_1 > 0, \ldots, \tau_k > 0$.

We now present a result which follows directly from [5].

Lemma 1: Suppose the uncertain system (1) and (3) satisfies Assumptions 1 and 2 and is absolutely stabilizable with disturbance attenuation γ via a controller of the form (4) (but which is not necessarily stable). Then, there exist constants $\tau_1 > 0, \ldots, \tau_k > 0$ such that the Riccati (7) has a solution $X > 0$. Furthermore, the uncertain system (1) and (3) is absolutely stabilizable with disturbance attenuation γ via the state feedback controller

$$u(t) = Kx(t) \quad (9)$$

where

$$K = -E_1^{-1}(B_2' X + \hat{D}_{12}' \hat{C}_1). \quad (10)$$

Proof: If the system (1) and (3) satisfies Assumptions 1 and 2 and is absolutely stabilizable with disturbance attenuation γ via a controller of the form (4), it follows from the Proof in [5, Theor. 4.1] that there exist constants $\tau_1 > 0, \ldots, \tau_k > 0$ such that the controller (4) solves the H^∞ control problem defined by the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \hat{B}_1 \hat{w}(t) + B_2 u(t) \\ \hat{z}(t) &= \hat{C}_1 x(t) + \hat{D}_{12} u(t) \\ y(t) &= C_2 x(t) + \hat{D}_{21} \hat{w}(t) \end{aligned} \quad (11)$$

and the H^∞ norm bound condition

$$\hat{J} \triangleq \sup_{\hat{w}(\cdot) \in \mathbf{L}_2[0,\infty), x(0)=0, x_c(0)=0} \frac{\|\hat{z}(\cdot)\|_2^2}{\|\hat{w}(\cdot)\|_2^2} < 1. \quad (12)$$

Here

$$\begin{aligned} \hat{w}(\cdot) &= [\, \gamma w(\cdot)' \quad \sqrt{\tau_1} \xi_1(\cdot)' \quad \ldots \quad \sqrt{\tau_k} \xi_k(\cdot)' \,]', \\ \hat{z}(\cdot) &= [\, z(\cdot)' \quad \sqrt{\tau_1} \zeta_1(\cdot)' \quad \ldots \quad \sqrt{\tau_k} \zeta_k(\cdot)' \,]' \end{aligned}$$

and the matrix coefficients $\hat{B}_1$, $\hat{C}_1$, and $\hat{D}_{12}$ are defined by (8) and

$$\hat{D}_{21} = [\, \gamma^{-1} D_{21} \quad 0_{l \times r_1} \quad \ldots \quad 0_{l \times r_k} \,]. \quad (13)$$

Then, it follows from a standard result on H^∞ control (e.g., see [7, Theor. 3.3]) that there exists a state feedback control law $u = Kx$ which stabilizes the system (11) and leads to the satisfaction of the H^∞ condition (12). Furthermore, it also follows from standard H^∞ control theory (e.g., see [7, Coroll. 3.1] or [8, Theor. 4.8 and Sec. 4.5.1]) that the Riccati equation (7) has a solution $X > 0$ and that the corresponding state feedback controller (9) and (10) stabilizes the system (11) and leads to the satisfaction of (12). It now follows using the same argument that is used in the [5, proof of Theorem 4.1] that the state feedback controller (9) and (10) absolutely stabilizes the uncertain system (1) and (3) with disturbance attenuation γ. □

We now suppose that constants $\tau_1 > 0, \ldots, \tau_k > 0$ have been found such that the Riccati (7) has a solution $X > 0$ and we will use the corresponding state feedback gain matrix K defined in (10) to define a new uncertain system as follows:

$$\begin{aligned}
\dot{x}(t) &= \tilde{A}x(t) + B_1 w(t) + \tilde{B}_2 u(t) + \sum_{s=1}^{k+1} D_s \xi_s(t)\\
z(t) &= \tilde{C}_1 x(t) + J\xi_{k+1} + \tilde{D}_{12} u(t)\\
\zeta_1(t) &= \tilde{K}_1 x(t) + F_1 \xi_{k+1} + \tilde{G}_1 u(t)\\
&\vdots\\
\zeta_k(t) &= \tilde{K}_k x(t) + F_k \xi_{k+1} + \tilde{G}_k u(t)\\
\zeta_{k+1}(t) &= \tilde{K}_{k+1} x(t) + \tilde{G}_{k+1} u(t)\\
y(t) &= C_2 x(t) + D_{21} w(t)
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
&\tilde{A} = A + \frac{1}{2}B_2 K; \quad \tilde{B}_2 = \frac{1}{2}B_2; \quad D_{k+1} = B_2\\
&\tilde{C}_1 = C_1 + \frac{1}{2}D_{12}K; \quad J = D_{12}; \quad \tilde{D}_{12} = \frac{1}{2}D_{12}\\
&\tilde{K}_1 = K_1 + \frac{1}{2}G_1 K; \quad F_1 = G_1; \quad \tilde{G}_1 = \frac{1}{2}G_1\\
&\vdots\\
&\tilde{K}_k = K_k + \frac{1}{2}G_k K; \quad F_k = G_k; \quad \tilde{G}_k = \frac{1}{2}G_k\\
&\tilde{K}_{k+1} = \frac{1}{2}K; \quad \tilde{G}_{k+1} = -\frac{1}{2}I_{m\times m}.
\end{aligned} \tag{15}$$

Also, we extend the IQC (3) to include the additional uncertainty input ξ_{k+1}

$$\int_0^{t_i} \|\xi_s(t)\|^2 dt \le \int_0^{t_i} \|\zeta_s(t)\|^2 dt + d_s \quad \forall i \quad \forall \quad s = 1, \ldots, k+1. \tag{16}$$

Here, d_{k+1} is any positive constant. We consider two special cases of the uncertainty input ξ_{k+1}.

1) Case 1: $\xi_{k+1}(t) \equiv \zeta_{k+1}(t) = (1/2)Kx(t) - (1/2)u(t)$. In this case, it is clear that this uncertainty input satisfies the IQC (16). Also, it is straightforward to verify that with this value of $\xi_{k+1}(t)$ the system (14) becomes

$$\begin{aligned}
\dot{x}(t) &= (A + B_2 K)\, x(t) + B_1 w(t) + \sum_{s=1}^{k} D_s \xi_s(t)\\
z(t) &= (C_1 + D_{12}K)\, x(t)\\
\zeta_1(t) &= (K_1 + G_1 K)\, x(t)\\
&\vdots\\
\zeta_k(t) &= (K_k + G_k K)\, x(t)\\
y(t) &= C_2 x(t) + D_{21} w(t)
\end{aligned} \tag{17}$$

where the IQC (3) is satisfied. However, the uncertain system (17) and (3) is the closed-loop uncertain system obtained when the state feedback control law (9) and (10) is applied to the original uncertain system (1) and (3). Thus, according to the construction of K and Lemma 1, this uncertain system will be absolutely stable with disturbance attenuation γ. Also note that for the system (17), the control input $u(t)$ (which is the output of the controller) does not affect the system.

2) Case 2: $\xi_{k+1}(t) \equiv -\zeta_{k+1}(t) = -(1/2)Kx(t) + (1/2)u(t)$. In this case, it is clear that this uncertainty input satisfies the IQC (16).

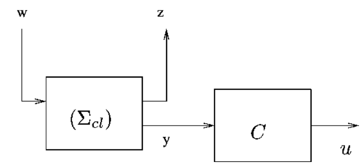

Fig. 1. Block diagram corresponding to Case 1.

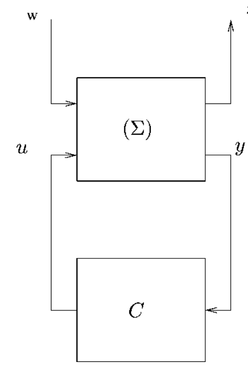

Fig. 2. Block diagram corresponding to Case 2.

Also, it is straightforward to verify that with this value of $\xi_{k+1}(t)$ the system (14) reduces to the original system (1).

In order to obtain our main result, we will apply the results of [5] to the uncertain system (14) and (16). Indeed, if the uncertain system (14) and (16) is absolutely stabilizable with disturbance attenuation γ via an output feedback controller of the form (4) (not necessarily stable) then it follows from Case 1 above that for the corresponding value of the additional uncertainty, this is equivalent to the open-loop situation illustrated in Fig. 1. In this block diagram the block (Σ_{cl}) refers to the closed-loop uncertain system defined by (17) and (3) and the block C refers to the output feedback controller of the form (4). Since definition of absolute stabilizability with disturbance attenuation γ requires the stability of the entire closed-loop system, it follows that the controller must be stable.

It follows from Case 2 above that for the corresponding value of additional uncertainty, when the controller (4) is applied to the uncertain system (14) and (16), this is equivalent to the situation shown in Fig. 2. In this block diagram, the block (Σ) refers to the original uncertain system defined by (1) and (3) and the block C refers to the output feedback controller of the form (4). From this, we can conclude that the output feedback controller (4) solves the original problem of absolute stabilizability with disturbance attenuation γ.

Combining the conclusions from both cases, we can conclude that the output feedback controller of the form (4) obtained by applying results of [5] to the uncertain system (14) and (16) is in fact a stable output feedback controller which solves the problem absolute stabilizability with disturbance attenuation γ for the original uncertain system (1) and (3). This leads us to the main result of this technical note which is stated in terms of a pair of algebraic Riccati equations. The Riccati equations under consideration are defined as follows: let $\tilde{\tau}_1 > 0, \ldots, \tilde{\tau}_{k+1} > 0$ be given constants and consider the Riccati equations

$$\begin{aligned}
&\left(\check{A} - \check{B}_2 \check{E}_1^{-1} \check{D}_{12}' \check{C}_1\right)' \check{X} + \check{X}\left(\check{A} - \check{B}_2 \check{E}_1^{-1} \check{D}_{12}' \check{C}_1\right)\\
&\quad + \check{X}\left(\check{B}_1 \check{B}_1' - \check{B}_2 \check{E}_1^{-1} \check{B}_2'\right)\check{X}\\
&\quad + \check{C}_1'\left(I - \check{D}_{12} \check{E}_1^{-1} \check{D}_{12}'\right)\check{C}_1 = 0
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\left(\check{A} - \check{B}_1 \check{D}_{21}' \check{E}_2^{-1} \check{C}_2\right)\check{Y} + \check{Y}\left(\check{A} - \check{B}_1 \check{D}_{21}' \check{E}_2^{-1} \check{C}_2\right)'\\
&\quad + \check{Y}\left(\check{C}_1' \check{C}_1 - \check{C}_2' \check{E}_2^{-1} \check{C}_2\right)\check{Y}\\
&\quad + \check{B}_1\left(I - \check{D}_{21}' \check{E}_2^{-1} \check{D}_{21}\right)\check{B}_1' = 0
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
\check{A} &= \tilde{A} + \bar{B}_1\bar{D}_{11}'\left(I_{\tilde{q}\times\tilde{q}} - \bar{D}_{11}\bar{D}_{11}'\right)^{-1}\bar{C}_1\\
\check{B}_2 &= \tilde{B}_2 + \bar{B}_1\bar{D}_{11}'\left(I_{\tilde{q}\times\tilde{q}} - \bar{D}_{11}\bar{D}_{11}'\right)^{-1}\bar{D}_{12}\\
\check{C}_2 &= C_2 + \bar{D}_{21}\bar{D}_{11}'\left(I_{\tilde{q}\times\tilde{q}} - \bar{D}_{11}\bar{D}_{11}'\right)^{-1}\bar{C}_1\\
\check{D}_{22} &= \bar{D}_{21}\bar{D}_{11}'\left(I_{\tilde{q}\times\tilde{q}} - \bar{D}_{11}\bar{D}_{11}'\right)^{-1}\bar{D}_{12}\\
\check{B}_1 &= \bar{B}_1\left(I_{\tilde{p}\times\tilde{p}} - \bar{D}_{11}'\bar{D}_{11}\right)^{-(1/2)}\\
\check{D}_{21} &= \bar{D}_{21}\left(I_{\tilde{p}\times\tilde{p}} - \bar{D}_{11}'\bar{D}_{11}\right)^{-(1/2)}\\
\check{C}_1 &= \left(I_{\tilde{q}\times\tilde{q}} - \bar{D}_{11}\bar{D}_{11}'\right)^{-(1/2)}\bar{C}_1\\
\check{D}_{12} &= \left(I_{\tilde{q}\times\tilde{q}} - \bar{D}_{11}\bar{D}_{11}'\right)^{-(1/2)}\bar{D}_{12}\\
\check{E}_1 &= \check{D}_{12}'\check{D}_{12}; \quad \check{E}_2 = \check{D}_{21}\check{D}_{21}'\\
\bar{B}_1 &= \left[\gamma^{-1}B_1 \quad \sqrt{\tilde{\tau}_1}^{-1}D_1 \quad \ldots \quad \sqrt{\tilde{\tau}_{k+1}}^{-1}D_{k+1}\right]\\
\bar{C}_1 &= \begin{bmatrix}\tilde{C}_1\\ \sqrt{\tilde{\tau}_1}\tilde{K}_1\\ \vdots\\ \sqrt{\tilde{\tau}_{k+1}}\tilde{K}_{k+1}\end{bmatrix}\\
\bar{D}_{11} &= \begin{bmatrix}0_{q\times p} & 0_{q\times r_1} & \ldots & 0_{q\times r_k} & \frac{1}{\sqrt{\tilde{\tau}_{k+1}}}J\\
0_{h_1\times p} & 0_{h_1\times r_1} & \ldots & 0_{h_1\times r_k} & \frac{\sqrt{\tilde{\tau}_1}}{\sqrt{\tilde{\tau}_{k+1}}}F_1\\
\vdots & & & & \\
0_{h_k\times p} & 0_{h_k\times r_1} & \ldots & 0_{h_k\times r_k} & \frac{\sqrt{\tilde{\tau}_k}}{\sqrt{\tilde{\tau}_{k+1}}}F_k\\
0_{m\times p} & 0_{m\times r_1} & \ldots & 0_{m\times r_k} & 0_{m\times m}\end{bmatrix}\\
\bar{D}_{12} &= \begin{bmatrix}\tilde{D}_{12}\\ \sqrt{\tilde{\tau}_1}\tilde{G}_1\\ \vdots\\ \sqrt{\tilde{\tau}_{k+1}}\tilde{G}_{k+1}\end{bmatrix}\\
\bar{D}_{21} &= \left[\gamma^{-1}D_{21} \quad 0_{l\times r_1} \quad \ldots \quad 0_{l\times r_k} \quad 0_{l\times m}\right].
\end{aligned} \tag{20}$$

Here, $\tilde{q} = q + h_1 \cdots + h_k + m$ and $\tilde{p} = p + r_1 \cdots + r_k + m$.

Assumption 3: The uncertain system (1) and (3) will be assumed to satisfy the following additional assumptions for any $\tilde{\tau}_1 > 0, \ldots, \tilde{\tau}_{k+1} > 0$: i) $\check{E}_1 > 0$; ii) $\check{E}_2 > 0$; iii) $\bar{D}_{11}\bar{D}_{11}' < I$.

Theorem 1: Suppose that the uncertain system (1) and (3) satisfies Assumptions 1–3 and that there exist constants $\tau_1 > 0, \ldots, \tau_k > 0$ such that the Riccati (7) has a solution $X > 0$ and let

$$K = -E_1^{-1}(B'_2X + \hat{D}_{12}'\hat{C}_1). \tag{21}$$

Furthermore, suppose there exist constants $\tilde{\tau}_1 > 0, \ldots, \tilde{\tau}_{k+1} > 0$ such that the Riccati (18) and (19) have solutions $\check{X} > 0$ and $\check{Y} > 0$ and such that the spectral radius of their product satisfies $\rho(\check{X}\check{Y}) < 1$. Then the uncertain system (1) and (3) is absolutely stabilizable with disturbance attenuation γ via a stable linear controller of the form (4) where

$$\begin{aligned}
A_c &= \check{A}_c - B_c\check{D}_{22}C_c\\
\check{A}_c &= \check{A} + \check{B}_2C_c - B_c\check{C}_2 + (\check{B}_1 - B_c\check{D}_{21})\check{B}_1'\check{X}\\
B_c &= (I - \check{Y}\check{X})^{-1}(\check{Y}\check{C}_2' + \check{B}_1\check{D}_{21}')\check{E}_2^{-1}\\
C_c &= -\check{E}_1^{-1}(\check{B}_2'\check{X} + \check{D}_{12}'\check{C}_1).
\end{aligned} \tag{22}$$

Proof: It follows via a similar argument to Proof in [5, Theor. 4.1] that the uncertain system (14) and (16) is absolutely stabilizable with disturbance attenuation γ via a controller of the form (4) if and only if there exist constants $\tilde{\tau}_1 > 0, \ldots, \tilde{\tau}_{k+1} > 0$ such that the controller (4) solves the H^∞ control problem defined by the system

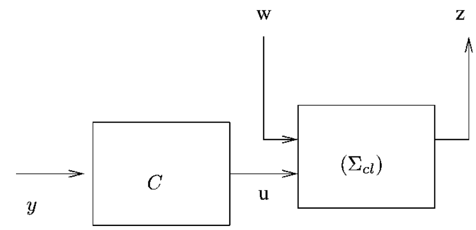

Fig. 3. Open-loop block diagram arising using an output injection approach.

$$\begin{aligned}
\dot{x}(t) &= \tilde{A}x(t) + \bar{B}_1\bar{w}(t) + \tilde{B}_2u(t)\\
\bar{z}(t) &= \bar{C}_1x(t) + \bar{D}_{11}\bar{w}(t) + \bar{D}_{12}u(t)\\
y(t) &= C_2x(t) + \bar{D}_{21}\bar{w}(t)
\end{aligned} \tag{23}$$

and the H^∞ norm bound condition

$$\bar{J} \triangleq \sup_{\bar{w}(\cdot)\in\mathbf{L}_2[0,\infty),x(0)=0,x_c(0)=0} \frac{\|\bar{z}(\cdot)\|_2^2}{\|\bar{w}(\cdot)\|_2^2} < 1. \tag{24}$$

Here

$$\begin{aligned}
\bar{w}(\cdot) &= \left[\gamma w(\cdot)' \quad \sqrt{\tilde{\tau}_1}\xi_1(\cdot)' \quad \ldots \quad \sqrt{\tilde{\tau}_{k+1}}\xi_{k+1}(\cdot)'\right]'\\
\bar{z}(\cdot) &= \left[z(\cdot)' \quad \sqrt{\tilde{\tau}_1}\zeta_1(\cdot)' \quad \ldots \quad \sqrt{\tilde{\tau}_{k+1}}\zeta_{k+1}(\cdot)'\right]'
\end{aligned}$$

and the matrix coefficients $\bar{B}_1$, $\bar{C}_1$, $\bar{D}_{11}$, $\bar{D}_{12}$, and $\bar{D}_{21}$ are defined by (20). Furthermore, it follows from standard loop shifting arguments in H^∞ control theory (e.g., see [8, Sec. 4.5.1 and 5.5.1] and [9, Sec. 17.2]) that the H^∞ control problem (23) and (24) has a solution if and only if the Riccati equations (18) and (19) have solutions $\check{X} > 0$ and $\check{Y} > 0$ and such that the spectral radius of their product satisfies $\rho(\check{X}\check{Y}) < 1$. Furthermore in this case, a controller of the form (4) which solves the H^∞ control problem (23) and (24) is defined by the (22).

Hence, if the conditions of the theorem are satisfied, then the controller (4) and (22) is absolutely stabilizing with disturbance attenuation γ for the uncertain system (14) and (16). Then, using the arguments given above, it follows that the controller (4) and (22) is stable and is absolutely stabilizing with disturbance attenuation γ for the uncertain system (1) and (3). □

Remark 1: The basis of our approach is to use a state feedback controller to define a new uncertain system which leads to the open-loop situation shown in Fig. 1 for a particular value of the additional uncertainty. However, one could equally apply a dual approach in which an output injection (e.g., see [10] and [9, Sec. 16.5]) is used to define a new uncertain system which leads to the open-loop situation shown in Fig. 3.

In practice, one could try both approaches and then chose the one which gave the smallest value of the disturbance attenuation parameter γ.

Remark 2: The key idea of this technical note involves introducing a new uncertainty. This approach will inherently lead to some conservatism. However, the new uncertainty is a constant but unknown uncertain parameter. Hence, the conservatism of our approach could be reduced by introducing multipliers to exploit the structure of this additional uncertainty such as Popov type multipliers (e.g., see [11]) or Zames-Falb type dynamic multipliers (e.g., see [12]). However, the use of dynamic multipliers would lead to a higher order controller.

IV. Illustrative Example

In this section, we consider an example originally presented in [2]. This example illustrates the theory developed above and also enables us to compare the conservatism of the approach of this technical note with that of the approach of [2]. Note that [2] considers only problems of H^∞ control for LTI systems with a stable controller whereas our results allow consideration of problems of robust H^∞ control for uncertain systems with an IQC uncertainty description. However in this example, we limit our attention to an H^∞ control problem for an LTI system without uncertainty.

The example in [2] is a mixed sensitivity H^∞ control problem corresponding to a plant transfer function

$$P(s) = \frac{(s+5)(s-1)(s-5)}{(s+2+j)(s+2-j)(s-20)(s-30)}$$

and weighting transfer functions

$$W_1(s) = \frac{1}{s+1}, \quad W_2(s) = 0.2.$$

The H^∞ optimal control problem under consideration is to find

$$\gamma_{opt} = \inf_{K \text{ stabilizing } P} \left\| \begin{bmatrix} W_1(1+PK)^{-1} \\ W_2K(1+PK)^{-1} \end{bmatrix} \right\|_\infty .$$

This H^∞ problem leads to a system of the form (1) described by the following state equations:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 46 & -12.656 & -8.398 & -2.930 & 0 \\ 32 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ -1 & 0.031 & 0.098 & -0.024 & -1 \end{bmatrix} x(t) \\ &\quad + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \\ z(t) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} u(t) \\ y(t) &= [1 \quad 0.0313 \quad 0.0977 \quad -0.0244 \quad 0]\, x(t) \\ &\quad + w(t). \end{aligned} \tag{25}$$

It is shown in [2] that the standard H^∞ central controller for this system is unstable for any value of the disturbance attenuation parameter $\gamma \geq \gamma_{opt} = 34.24$. We now apply the approach outlined in our main result Theorem 1 to this system. For $\gamma = 42.4$ and $\tilde{\tau}_1 = 0.34$, we find that the conditions of Theorem 1 are satisfied and we construct the corresponding controller (4) where

$$A_c = 10^3 \times \begin{bmatrix} -0.1816 & -0.0209 & 0.0023 & -0.0061 & 0.0000 \\ 1.7686 & -0.0543 & -0.1696 & 0.0424 & -0.0005 \\ 1.1174 & -0.0269 & -0.1091 & 0.0273 & -0.0003 \\ -0.6218 & 0.0194 & 0.0647 & -0.0152 & 0.0002 \\ 0 & 0 & 0 & 0 & -0.0010 \end{bmatrix}$$

$$B_c = 10^3 \times \begin{bmatrix} -0.1294 \\ 1.7366 \\ 1.1174 \\ -0.6218 \\ 0.0010 \end{bmatrix}$$

$$C_c = [-100.4857 \quad -12.5633 \quad -1.9669 \quad -0.0025 \quad 0].$$

This controller is stable and has poles at $s = -177.64 \pm 175.91j$, $s = -0.28833$, $s = -4.6508$, $s = -1.0000$. Furthermore, when this controller is applied to the system (25), the resulting closed-loop system has a maximum singular value plot as shown in Fig. 4. From this, we

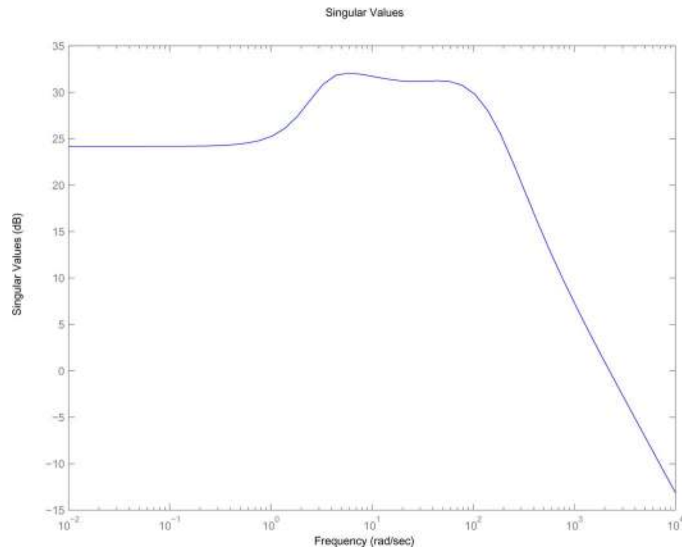

Fig. 4. Closed-loop maximum singular value plot with stable controller.

can see that this stable controller does indeed solve the H^∞ strong stabilization problem under consideration with a maximum closed-loop singular value of 32 dB. This corresponds to a closed-loop H^∞ norm of 40.11. This compares to closed-loop H^∞ norm of 42.51 which was obtained using the method of [2]. Thus, we can conclude that for this example, the approach is this technical note is slightly less conservative than the approach of [2].

V. Conclusions

In this technical note, we have presented a new approach to the problem of absolute stabilization with a specified level of disturbance attenuation via the use of a stable output feedback controller. The key idea of our approach is to add an additional uncertain parameter to the original uncertain system. For one value of this additional uncertain parameter, the new uncertain system reduces to the original uncertain system and for another value of the additional uncertain parameter, the system reduces to a system in which the control input has no effect and so the controller is effectively in open loop. This forces the controller to be stable.

A number of possible areas for future research are motivated by the results of this technical note. One would be to reduce the conservatism of the approach by introducing dynamic multipliers to exploit the fact that the additional uncertain parameter is really only required to be constant but unknown. The use of such dynamic multipliers would result in the synthesis of a controller which was of higher order than the original plant. Also, it would be useful to carry out an investigation into the use of optimization tools in order to construct the parameters on which the main result depends. Furthermore, one could consider extensions to the main results which require the controller to have additional properties other than stability such as satisfying a H^∞ norm bound or satisfying a positive real condition.

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Comments on "On Optimal Control of Spatially Distributed Systems"

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I. INTRODUCTION

Following in the footsteps of the paper by Bamieh *et al.*[1] on spatially invariant systems, the authors of the above paper [4] aim at developing a theory of spatial decaying operators for spatially varying systems that are continuous in time, but discrete in space. This falls clearly into the area of infinite-dimensional systems theory [2] and the authors do quote results from this text. The theory is developed in great generality, but even for the simplest class of spatially invariant systems, the main result in Theorem 6 proves to be false. To explain this I analyse the claim for the very special class of spatially invariant systems with scalar entries.

Consider the system

$$\dot{z}_r(t) = \sum_{l=-\infty}^{\infty} a_{r-l} z_r(t) + \sum_{l=-\infty}^{\infty} b_{r-l} u_l(t),$$
$$y_r(t) = \sum_{l=-\infty}^{\infty} c_{r-l} z_r(t) + \sum_{l=-\infty}^{\infty} d_{r-l} u_r(t), \quad r \in \mathbb{Z} \tag{1}$$

where $a_r, b_r, c_r, d_r \in \mathbb{C}$ and $z_r(t), u_r(t), y_r(t) \in \mathbb{C}$ are the state, the input and the output vectors, respectively, at time $t \geq 0$ and spatial point r. This can be formulated as a linear system

$$\dot{z}(t) = (Az)(t) + (Bu)(t) \tag{2}$$
$$y(t) = (Cz)(t) + (Du)(t), \quad t \geq 0 \tag{3}$$

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where the spatially invariant operators A, B, C, D are convolution operators of the type $T : \ell_2 \to \ell_2$ given by

$$(Tz)_r = \sum_{l=-\infty}^{\infty} t_{r-l} z_r.$$

T is bounded, i.e., is in $\mathcal{L}(\ell_2)$ if and only if

$$\|T\|_\infty = \operatorname{esssup}_{0 \leq \theta \leq 2\pi} \left| \sum_{r=-\infty}^{\infty} t_r e^{-\imath r \theta} \right| < \infty.$$

As in [1] the observation is made that if the Fourier transform $\check{T}(\theta) = \sum_{r=-\infty}^{\infty} t_r e^{-\imath r\theta}$ has an analytic continuation to an annulus around the unit circle $\Omega = \{z \in \mathbb{C} : e^{-\tau} < |z| < e^{\tau}, \tau > 0\}$, then there exists a constant μ such that

$$|t_r| \leq \mu e^{-\alpha |r|} \text{ for all } 0 < \alpha < \tau.$$

This motivates the introduction of so-called spatially decaying operators and weighted norms such as the following very special case

$$|||T|||_\tau := \sup_{0 \leq \alpha < \tau} \sum_{r \in \mathbb{Z}} |t_r| e^{\alpha |r|}. \tag{4}$$

This norm induces a $*$-Banach algebra $\mathcal{S}_\tau^\infty(\mathcal{C})$ (but not a B^*-algebra as the authors state-see later) as a subalgebra of $\mathcal{L}(\ell_2)$. Specialized to this class of spatially invariant operators Theorem 6 reduces to the following.

Theorem 6 for Scalar Spatially Invariant Systems: Consider the $*$-Banach algebra $\mathcal{S}_\tau^\infty(\mathcal{C})$ and the spatially invariant system (1) with $A, B, C \in \mathcal{S}_\tau^\infty(\mathcal{C})$ and $D = 0$. If the control Riccati equation

$$A^*P + PA - PBB^*P + CC^* = 0$$

has a unique nonnegative solution $P \in \mathcal{L}(\ell_2)$, then $P \in \mathcal{S}_\tau^\infty(\mathcal{C})$.

II. COUNTEREXAMPLE

Consider the system with the Fourier transforms $\check{A} = 0$, $\check{B} = 10 - e^{-\imath\theta} - e^{\imath\theta}$, $\check{C} = 1$, $\check{D} = 0$. The solution to the Fourier transformed Riccati equation is

$$\check{Q}(\theta) = \frac{1}{10 - 2\cos\theta} = \sum_{-\infty}^{\infty} q_k e^{-\imath\theta k}.$$

The Fourier coefficients for $k \geq 0$ can be calculated from the formula

$$\begin{aligned} q_k &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{10 - e^{-\imath\theta} - e^{\imath\theta}} e^{\imath k\theta} \mathrm{d}\theta \\ &= \frac{1}{2\pi\imath} \int_C \frac{z^k}{10z - z^2 - 1} \mathrm{d}z \\ &= \frac{1}{4\sqrt{6}} (5 - \sqrt{24})^k = \frac{1}{4\sqrt{6}} e^{-\delta k} \end{aligned}$$

where C denotes the unit circle and $\delta = -\ln(5 - \sqrt{24})$ is a small positive number. Since $\check{Q}$ is self adjoint we obtain the solution

$$\check{Q}(\theta) = \frac{1}{4\sqrt{6}} + \frac{1}{2\sqrt{6}} \sum_{k=1}^{\infty} e^{-\delta k} \cos k\theta.$$

Note that it is readily verified by direct substitution that this satisfies the Riccati equation.

Now A, B, C are trivially in $\in \mathcal{S}_\tau^\infty(\mathcal{C})$ for arbitrary $\tau > 0$, but this is not true for Q. Although $Q \in \mathcal{S}_\tau^\infty(\mathcal{C})$ for very small $\tau < \delta$, this does not hold for $\tau \geq \delta$. This shows that Theorem 6 is false.