Robust H∞ Control of Markov Jump Linear Systems with Uncertain Switching Probabilities

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This paper the problem of robust H∞ control for Markov jump systems with uncertain transition rates is investigated. A robust H∞ performance criterion is established for a given Markov jump system. The robust H∞ control performance analysis in terms of coupled linear matrix inequalities is proposed, then convex optimization problem is solved with constraints defined in terms of the solvability of the linear matrix inequalities. Based on the solution of the optimization problem, the condition of robust stochastic stability for closed-loop systems is found, which minimizes disturbance attenuation level. Depending on the developed performance criterion, the H∞ state-feedback controller is designed too, which warranties the robust H∞ control of the closed-loop system. All the conditions are linear matrix inequalities, and therefore they can be solved by any linear matrix inequalities solver. Finally, a numerical example is given to show the effectiveness of the method of robust H∞ control for Markov jump systems.

Introduction

Many dynamic systems undergo sudden random changes, which may be caused by random component failures and repairs, sudden environmental changes, changes in the interconnectedness of subsystems, etc. Usually most conventional dynamic systems are powerless to overcome with these abrupt random changes. Markov jump systems (MJSs) are special class of stochastic hybrid systems (dynamical systems that exhibits both continuous and discrete dynamic behavior). MJSs have applied in many fields, uncrewed aerial vehicle [1], solar power stations [2], communication protocols [3], control of power systems [4], economic systems, [5].

MJSs have been investigated extensively and many beneficial results have been obtained, such as the stabilizability, and continuous-time MJL quadratic control [6], controller design for MJL [7–9] and robust linear filtering for MJL [10-12]. The nonlinearity in systems may lead to unstable behavior of the systems, robust stabilization and H∞ control for nonlinear systems with Markovian jump [13–15].

The transition rates are essential to set the MJSs. So, the main investigation on MJSs is to assume that the transition rates are well known. In application, the estimated values of transition rates are only available, and estimation errors, i.e., in the transition rates, the uncertainties may be given instability or deterioration of a system. There have been some works regarding control of this type of system [16], the robust stabilization and control problems are considered for MJS with uncertain switching probabilities by using restrictive Young inequality. In [17] results by using general Young inequality less conservative than those of [16] are proposed. Because of the use of Young inequality, the proposed controller design methods in [16, 17] need to solve a set of nonlinear matrix inequalities (NLMIs). It is still not possible to fully resolve these NLMIs yet. The H∞ control problem for nonlinear MJSs with uncertain transition rates has not been completely scrupulous [18]. It remains important and hard.

This study is interested with the robust H∞ control for MJLS with uncertain transition rates. First, the robust H∞ performance criterion is found. Therewith, the method for designing the H∞ controller based on the proposed performance criterion is presented. We assume an improved bounding for the uncertain terms instead of using the traditional Young inequality. As an advantage, the design method of obtained controller only needs to solve a set of linear matrix inequalities (LMIs) instead of NLMIs, we can easily solve by any LMI solution. At last, a numerical example is given to confirm the efficacy of the proposed methods.

Notations: Let R^n is n-dimensional Euclidean space and by $B(R^n, R^m)$ is space of all $n \times m$ norm bounded linear matrices. For a matrix $A \in B(R^n)$, A^T is the transpose of A. $A \geq 0$ (resp., $A \leq 0$): will mean that the symmetric (resp., semi-negative definite) matrix $A = B(R^n)$ is posi-

tive semidefinite, and $A > 0$ (resp., $A < 0$): that is positive definite (resp., negative definite) matrix. I is the $n \times n$ identity matrix. $trace\{A\}$ is trace of a matrix. $\mathbf{E}(\cdot)$ is the expectation operator.

Problem statement

Let the following MJLS, defined on a complete probability space, are described as:

$$\begin{cases} \dot{x}(t) = A(\theta(t))x(t) + B(\theta(t))u(t) \\ y(t) = C(\theta(t))x(t) + D(\theta(t))u(t) \end{cases} \quad (1)$$

where $x(t) \in \mathrm{R}^n$ is standing for the state variable of the system, $u(t) \in \mathrm{R}^n$ is the control variable. We assume the set $S = \{1, 2, \ldots, N\}$, $\{\theta(t), \, t \geq 0\}$ is a continuous-time Markov chain on the probability space, with transition rate matrix $\Pi = (\pi_{ij})_{N \times N}$ given by [19]

$$\mathrm{P}(\theta(t+h) = j \,|\, \theta(t) = i) = $$
$$= \begin{cases} \pi_{ij}h + o(h), \, i \neq j, \\ 1 + \pi_{ii}h + o(h), \, i = j. \end{cases} \quad (2)$$

The function $o(h)$ satisfies $h > 0$, $\lim_{h \to 0} \dfrac{o(h)}{h} = 0$ and $\pi_{ij} \geq 0$ $(i, j \in S, i \neq j)$, are the transition rate from i to j, such that $\pi_{ii} = -\sum_{j \neq i} \pi_{ij}$ for all $i \in S$. For $\Pi = (\pi_{ij})_{N \times N}$ the error between them is $\Delta \pi_{ij}$ which take any value in interval $[-\varepsilon_{ij}, \varepsilon_{ij}]$.

The linear state-feedback control law is:

$$u(t) = K(\theta(t))x(t), \quad (3)$$

where the controller gain matrices $K_i = K(\theta(t) = i) \in R^{m \times n}$ to be design. The closed-loop system is

$$\begin{cases} \dot{x}(t) = \{A(\theta(t)) + B(\theta(t))K(\theta(t))\}x(t) \\ y(t) = \{C(\theta(t)) + D(\theta(t))K(\theta(t))\}x(t). \end{cases} \quad (4)$$

Now, we introduce the following definitions [20].

Definition 1. The MJLS (1) is robustly stochastically stable if $\lim_{t \to \infty} \mathbf{E}(x(t)^2) = 0$.

Lemma 1. Let the $Q_j^a(t)$ is solution of (1) with initial condition $Q_j(0) = \alpha Q_j^a(0)$ the solution of (1) is $Q_j(t) = \alpha Q_j^a(t), \forall t > 0$

Lemma 2. If, at a certain time instant τ, $\dot{Q}_j(t) < 0$; $\forall j \in S$ then $\dot{Q}_j(t) < 0$; $\forall t > \tau$, $j \in S$.

Lemma 3. Let $Q_j^a(t)$ and $Q_j^b(t)$ be the solution of (1) the $Q_j^a(0)$ and $Q_j^b(0)$ is initial condition respectively. If $Q_j^a(0) < Q_j^b(0), \forall j \in S$ then $Q_j^a(t) < Q_j^b(t), \forall j \in S$.

Definition 2. The MJLS (1) is exponentially mean square stable (EMS-stable) if there exist α and β are positive real scalar such that $E(x(t)^2) < \alpha e^{-\beta t}$.

Proposition 1. The MJLS of (1) is MS-stable if the LMIs problems

$$A_j Q_j(t) + Q_j(t) A_j^T + \sum_{i=1}^{N} \pi_{ij} Q_i(t) \quad (5)$$

are possible for a matrices $\{Q_i; Q_i \in S^{n \times n}\}$

Lemma 4. (Schur Complement) Given constant matrices $\Omega_1, \Omega_2, \Omega_3$, with $\Omega_1 = \Omega_1^T$ then $\Omega_3^T \Omega_2^{-1} \Omega_3 - \Omega_1 < 0$ if and only if

$$\begin{bmatrix} -\Omega_1 & \Omega_3^T \\ * & \Omega_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ * & -\Omega_1 \end{bmatrix} < 0.$$

Robust Stochastic H$_\infty$ Performance Analysis

In this section, analysis performance the problem H$_\infty$ of system is considered in terms of LMI, and then dealt with in terms of the solvability of a set of LMI with equality constraints

The following theorem [20] gives a robust stochastic H$_\infty$ criterion for MJLS of (1).

Theorem 1. Let the MJS (1) with uncertain transition rates. The controller gains $K_i, i \in S$, the closed-loop system (4) is robustly stochastically stable if there exist matrices $P_i > 0, M_{ij} \geq 0$, $(i, j \in S, i \neq j)$ such that for $\forall i \in S$ the following LMIs are feasible:

$$\begin{bmatrix} \Theta_i & P_i B_i & C_i^T \\ * & -\gamma^2 I & D_i^T \\ * & * & -I \end{bmatrix} < 0, \quad (6)$$

$$P_i - \alpha_i I \leq 0, \quad (7)$$

$$P_j - P_i - M_{ij} \le 0; \left(\forall j \in S, i \ne j\right), \qquad (8)$$

where $\Theta_i = A_i^T P_i + P_i A_i + \sum\limits_{j=1, j \ne i}^{N} \left\{ \hat{\pi}_{ij}(P_j - P_i) + 2\varepsilon_{ij} M_{ij} \right\},$

$$\hat{\pi}_{ij} = \pi_{ij} - \varepsilon_{ij}.$$

Robust Stochastic H$_\infty$ Controller Design

In this section, design problem the H$_\infty$ controller of system (1) is studied. The following theorem has been proposed for designing the robust stochastic H$_\infty$ controller for system (1) in the form (3).

Theorem 2. Let the MJLS (1). The closed-loop system is robustly stochastically stable with disturbance attenuation level γ if there are matrices $Q_i > 0, N_{ij} \ge 0, Y_i \quad (i, j \in S, i \ne j)$ and scalars $\beta_i, i \in S$, $F_i, i \in S$, is a known constant matrix such that for $\forall i \in S$ the LMIs have solutions

$$\begin{bmatrix} \Phi_i & B_i & Q_i C_i^T + Y_i^T D_i^T & Q_i F_i^T & E_i \\ * & -\gamma^2 I & D_i^T & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\beta_i I & 0 \\ * & * & * & * & -\Lambda_i \end{bmatrix} < 0, \qquad (9)$$

$$\begin{bmatrix} -\beta_i I & \beta_i I \\ * & -Q_i \end{bmatrix} \le 0, \qquad (10)$$

$$\begin{bmatrix} -Q_i - N_{ij} & Q_i \\ * & -Q_j \end{bmatrix} \le 0, \forall j \in S, j \ne i, \qquad (11)$$

where

$$\Phi_i = \left(A_i Q_i + B_i Y_i\right)^T +$$
$$+ \left(A_i Q_i + B_i Y_i\right) + \sum\limits_{j=1, j \ne i}^{s} \left\{ 2\varepsilon_{ij} N_{ij} - \pi_{ij} Q_i \right\},$$

$$E_i = \left[\sqrt{\hat{\pi}_{i1}} Q_i \dots \sqrt{\hat{\pi}_{i(i-1)}} Q_i \ \sqrt{\hat{\pi}_{i(i+1)}} Q_i \dots \sqrt{\hat{\pi}_{is}} Q_i \right],$$

$$\Lambda_i = \text{diag}\left\{ Q_1, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_s \right\}.$$

Furthermore, a controller gain is given by

$$K_i = Y_i Q_i^{-1}. \qquad (12)$$

Proof. From Theorem 1, we have that system (4) with uncertain transition rates is robustly stochastically stable if inequalities (6)-(8) holds with disturbance attenuation level γ. By applay the

Schur complement and noting (6)-(8) are equivalent to the relations (13)–(15).

$$\begin{bmatrix} \bar{\Phi}_i & P_i B_{vi} & C_i^T + K_i^T D_i^T & F_i^T & \bar{E}_i \\ * & -\gamma^2 I & D_{vi}^T & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\alpha_i^{-1} & 0 \\ * & * & * & * & -\bar{\Lambda}_i \end{bmatrix} < 0, \qquad (13)$$

$$\begin{bmatrix} -\alpha_i I & I \\ * & -P_i^{-1} \end{bmatrix} \le 0, \qquad (14)$$

$$\begin{bmatrix} -P_i - M_{ij} & I \\ * & -P_j^{-1} \end{bmatrix} \le 0, \forall j \in S, j \ne i, \qquad (15)$$

where

$$\bar{\Phi}_i = \left(A_i + B_i K_i\right)^T P_i +$$
$$+ P_i \left(A_i + B_i K_i\right) + \sum\limits_{j=1, j \ne i}^{N} \left\{ 2\varepsilon_{ij} M_{ij} - \pi_{ij} P_i \right\},$$

$$\bar{E}_i = \left[\sqrt{\hat{\pi}_{i1}} I \dots \sqrt{\hat{\pi}_{i,i-1}} I \ \sqrt{\hat{\pi}_{i,i+1}} I \dots \sqrt{\hat{\pi}_{is}} I \right],$$

$$\bar{\Lambda}_i = \text{diag}\left\{ P_1^{-1}, \dots, P_{i-1}^{-1}, P_{i+1}^{-1}, \dots, P_s^{-1} \right\}.$$

Consider $Q_i = P_i^{-1}$, for transformation congruence to the inequality in (13) by $diag\{Q_i, I, \dots, I\}$ and setting the change of variable $N_{ij} = Q_i M_{ij} Q_i$, $Y_i = K_{ii} Q_i$, and $\beta_i = \alpha_i^{-1}$, we can write the inequality in (9). Transformation the congruence to the inequality in (14) by $diag\{\beta_i I, I\}$, we find the inequality in (10). and transformation the congruence to the inequality in (15) by $diag\{Q_i, I\}$, we can get the inequality in (11). In addition, we have $Y_i = K_i Q_i$, the gain of wanted controller is given by $K_i = Y_i Q_i^{-1}$. This finished the proof.

Remark. From Theorem 2, the H$_\infty$ control problem for MJLS with uncertain transition rates can be solved in terms of the LMIs in (9)– (11). The inequalities in (9)– (11) are not only linear with the variables $Q_i, N_{ij}, Y_i, \beta_i$ but also linear with regard to the scalar γ^2. Then, can be readily found the robust H$_\infty$ control with minimum guaranteed cost by solving the convex optimization problem:

$$\varphi_0 = \min\left\{ \varphi = \gamma^2 \ \middle|\ Q_i > 0, N_{ij} \ge 0, Y_i, \beta_i \ \forall i, j \in S, \text{subject to (9)-(11)} \right\}.$$

Such that, the minimum cost is given by $\gamma_0 = \sqrt{\phi_0}$.

Numerical example

We give an example of simulation to illustrate the usefulness and of the theory developed in this work. We find the design of a robust H$_\infty$ controller to the MJLS.

Switch between two dynamic subsystems with Markov jump parameters $S = \{1, 2\}$:

$$A_1 = \begin{bmatrix} 0.1769 & 0.7843 \\ 0.9266 & 0.1363 \end{bmatrix}, B_1 = \begin{bmatrix} 0.2995 \\ 0.4471 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -0.3 & 0 \end{bmatrix}, D_1 = -0.5,$$

$$A_2 = \begin{bmatrix} 0.5478 & 0.1279 \\ 0.6160 & 0.9657 \end{bmatrix}, B_1 = \begin{bmatrix} 0.7417 \\ 0.7957 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.4 & 0.2 \end{bmatrix}, D_2 = 0.76.$$

The uncertain transition rates given by:

$$\Pi = \begin{bmatrix} -0.5 & 0.5 \\ 0.3 & -0.3 \end{bmatrix}, \left| \Delta\pi_{ij} \right| \le 0.8\pi_{ij},$$

$$\forall i, j \in S, j \ne i.$$

The robust H$_\infty$ controller is designed such that the closed-loop system is robustly stochastically stable with γ over all the and uncertain transition rates. We obtain minimum disturbance attenuation level is $\gamma_0 = 0.753$ by Theorem 2 with the corresponding controller gain matrices

$$K_1 = \begin{bmatrix} -1.9328 & -3.3166 \end{bmatrix},$$
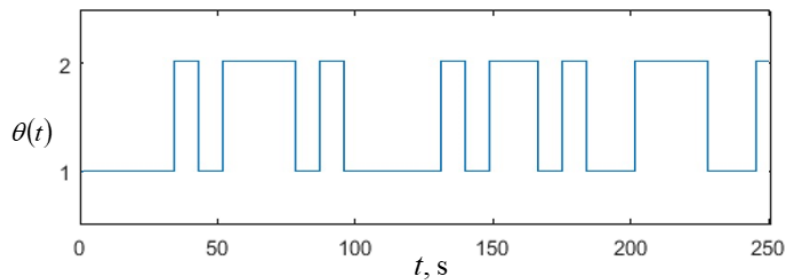$$K_2 = \begin{bmatrix} -1.5559 & -3.4950 \end{bmatrix}.$$

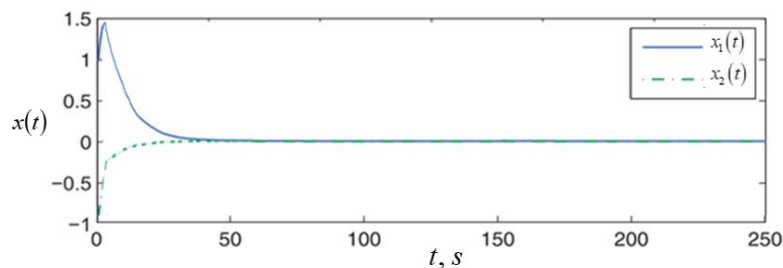*Fig. 1.* Switch between two dynamic subsystems with Markov jump

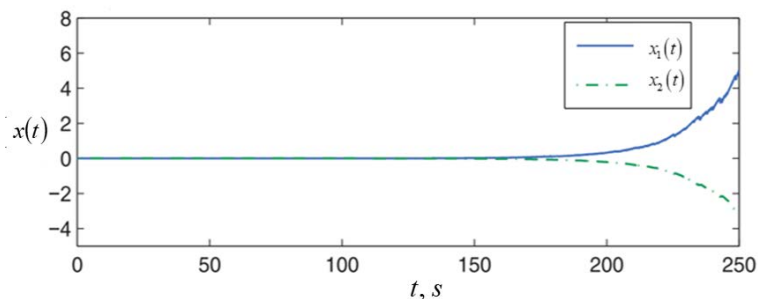*Fig. 2.* State of closed-loop MJLS with robust stabilizing control

Fig. 3. State of closed-loop MJLS without stabilizing control

In Fig. 1, the effect of weak noise with an abrupt change in intensity leads to the emergence of unstable modes and an increase in the "stopping distance".

The state response of the resulting closed-loop system with uncertain switching probability is given in Fig. 2, which are switching two models and $x_0 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, and the switching signal is

shown in Fig. 1, obviously, the disturbance observers are effect to handle the random switching disturbances. Based on the above analysis, it can be asserted that the desired controller has good robust performance. On Fig. 3 we seen that the closed-loop system without stabilizing control is unstable.

Conclusion

In this study, the problem H_∞ control is discussed for MJSs with uncertain transition rates. And the controller is designed such that the closed-loop system is robustly stochastically stable and guarantees a desired robust H_∞ performance over the transition rates. The disturbances are considered as switching systems under Markov jump parameters. The method for design the controller is in linear matrix inequalities, which can be solved as LMI problem. The numerical example shows the effectiveness of the method.

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Робастное H_∞ управление линейными системами с марковскими скачками и неопределенными вероятностями переключения

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В статье исследуется проблема робастного управления H_∞ для систем с марковскими скачками и неопределенными матрицами интенсивностей перехода. Для рассматриваемой системы предлагается исполь-

зовать робастный критерий качества управления H_∞. На первом этапе проводится анализ характеристик робастного управления H_∞ на основе сопряженных линейных матричных неравенств, далее решается задача выпуклой оптимизации с ограничениями, определенными в терминах разрешимости линейных матричных неравенств. На основе решения задачи оптимизации определено условие робастной стохастической устойчивости для систем с обратной связью, которое минимизирует уровень затухания возмущений. На основе рассматриваемого критерия спроектирован контроллер, который гарантирует робастное управление H_∞ системой с обратной связью. Все условия выражаются в терминах линейных матричных неравенств, и, следовательно, они могут быть определены решателем линейных матричных неравенств. Представлен численный пример решения задачи, демонстрирующий эффективность предложенного метода робастного управления стохастической системы с марковскими скачкам.

Ключевые слова: линейные системы с марковскими скачками, робастное управление H_∞, линейные матричные неравенства, неопределенные марковские параметры.