

Robust H_∞ Control for Uncertain Discrete-Time-Delay Fuzzy Systems Via Output Feedback Controllers

Shengyuan Xu and James Lam

Abstract—This paper investigates the problem of robust output feedback H_∞ control for a class of uncertain discrete-time fuzzy systems with time delays. The state-space Takagi–Sugeno fuzzy model with time delays and norm-bounded parameter uncertainties is adopted. The purpose is the design of a full-order fuzzy dynamic output feedback controller which ensures the robust asymptotic stability of the closed-loop system and guarantees an H_∞ norm bound constraint on disturbance attenuation for all admissible uncertainties. In terms of linear matrix inequalities (LMIs), a sufficient condition for the solvability of this problem is presented. Explicit expressions of a desired output feedback controller are proposed when the given LMIs are feasible. The effectiveness and the applicability of the proposed design approach are demonstrated by applying this to the problem of robust H_∞ control for a class of uncertain nonlinear discrete delay systems.

Index Terms—Discrete systems, linear matrix inequality (LMI), output feedback, robust H_∞ control, Takagi–Sugeno (T–S) fuzzy models, time-delay systems, uncertain systems.

I. INTRODUCTION

AS AN alternative method to conventional control approach for complex control systems, fuzzy logic control has received much attention in the past decades. It has been shown that fuzzy logic control is one of the most useful techniques for utilizing the qualitative knowledge of a system to design controllers. A great number of industrial applications via fuzzy logic control have been reported [14], [15], [28]. Among various model-based fuzzy control approaches, the method based on the Takagi–Sugeno (T–S) fuzzy model has become popular today, which gives a simple and effective way to control complex nonlinear systems. The main features of this approach are as follows: first, a nonlinear system is represented by a T–S fuzzy model, in which local dynamics in different state space regions are represented by linear models. Then, the overall model of the system is achieved by a fuzzy “blending” of these fuzzy models. Based on this, the control design can be carried out by

the so-called parallel distributed compensation (PDC) scheme. Applications of such a fuzzy control scheme can be found in [10], [21], and [23].

Recently, stability analysis of T–S fuzzy control systems has been investigated, and several stability criteria have been proposed; see, e.g., [18] and [19]. When parameter uncertainties appear, the problems of robust stability analysis and robust stabilization for fuzzy systems have been studied. For example, by a linear matrix inequality (LMI) approach, some robust stability results were presented in [17] in the continuous case; based on these, robust fuzzy stabilizing controllers were constructed via the PDC scheme. The corresponding results for discrete case can be found in [11]. Very recently, the robust H_∞ control problem for fuzzy systems described by T–S fuzzy model has been addressed. By the LMI approach, sufficient conditions for the solvability of this issue were proposed in [4] and [12] for the discrete and continuous cases, respectively. It should be pointed out that in both [4] and [12], state feedback controllers were designed under the assumption that all state variables are available.

On the other hand, it is well-known that time delay arises quite naturally in propagation phenomena, population dynamics or engineering systems such as chemical processes, long transmission lines in pneumatic systems [8]. Many results on estimation and control issues related to time-delay systems have been proposed [8], [9], [25], [26]. Recently, fuzzy systems with time delays have been introduced in [3] and [5], where several stability analysis results were given via different approaches, and stabilizing controllers were also designed. When delays are time-varying, the stability results for fuzzy delay systems were given in [27]. It is noted that in [3], [5], and [27], no parameter uncertainties were taken into account. In the case when parameter uncertainties arise and not all of the states are available directly, the robust output feedback H_∞ control problem for fuzzy systems with time delays was discussed in [13]. In terms of solutions to a certain LMI, an output feedback controller was designed in [13]. These results were further applied to a class of nonlinear delay systems. It is worth noting that the results in [13] were obtained in the context of continuous T–S fuzzy systems. For discrete fuzzy systems with time delays and parameter uncertainties subject to that all state variables are not available, however, the problem of robust H_∞ control via output feedback controllers is still open and remains unsolved, which motivates the present study.

Manuscript received February 10, 2003; revised August 30, 2003 and March 31, 2004. This work was supported by RGC HKU 7103/01P, the Foundation for the Author of National Excellent Doctoral Dissertation of China under Grant 200240, the National Natural Science Foundation of China under Grant 60304001, and the Fok Ying Tung Education Foundation under Grant 91061.

S. Xu is with the Department of Automation, Nanjing University of Science and Technology, Nanjing 210094, China.

J. Lam is with the Department of Mechanical Engineering, University of Hong Kong, Hong Kong.

Digital Object Identifier 10.1109/TFUZZ.2004.839661

In this paper, we are concerned with the problem of robust output feedback H_∞ control for a class of discrete fuzzy systems with parameter uncertainties and time delays. The T–S fuzzy model is adopted for fuzzy modeling of a discrete uncertain nonlinear systems with time delays. The parameter uncertainties are assumed to be time varying but norm bounded. The purpose is the design of a full-order fuzzy dynamic output feedback controller such that the resulting closed-loop system is robustly asymptotically stable while satisfying a prescribed H_∞ performance level irrespective of the parameter uncertainties. Sufficient conditions for the solvability of this problem are obtained in terms of LMIs. A desired output feedback controller can be constructed by using standard numerical algorithms to solve these given LMIs [2], and no tuning of parameters is required.

Notation: Throughout this paper, for real symmetric matrices X and Y , the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive–semidefinite (respectively, positive–definite). I is an identity matrix with appropriate dimension. \mathbb{N} is the set of natural numbers. $l_2[0, \infty)$ refers to the space of square summable infinite vector sequences. $\|\cdot\|_2$ stands for the usual $l_2[0, \infty)$ norm. The notation M^T represents the transpose of the matrix M . Matrices, if not explicitly stated, are assumed to have compatible dimensions.

II. PROBLEM FORMULATION

The T–S fuzzy dynamic model is described by fuzzy IF–THEN rules, which locally represent linear input–output relations of nonlinear systems. Similar to [16], a discrete-time T–S fuzzy model with time delays and parameter uncertainties can be described by

Plant Rule i : IF $s_1(k)$ is μ_{i1} and $s_2(k)$ is μ_{i2} and \dots
and $s_g(k)$ is μ_{ig} , THEN

$$x(k+1) = [A_i + \Delta A_i(k)]x(k) + [A_{di} + \Delta A_{di}(k)] \times x(k-\tau) + [B_i + \Delta B_i(k)]u(k) + D_{1i}\omega(k) \quad (1)$$

$$y(k) = C_i x(k) + C_{di} x(k-\tau) + D_{2i} \omega(k) \quad (2)$$

$$z(k) = E_i x(k) + E_{di} x(k-\tau) + G_i u(k) \quad (3)$$

$$x(k) = \phi(k) \quad \forall k \in [-\tau, 0], \quad i = 1, 2, \dots, r \quad (4)$$

where μ_{ij} is the fuzzy set and r is the number of IF–THEN rules; $x(k) \in \mathbb{R}^n$ is the state; $u(k) \in \mathbb{R}^m$ is the control input; $y(k) \in \mathbb{R}^s$ is the measured output; $z(k) \in \mathbb{R}^q$ is the controlled output; $\omega(k) \in \mathbb{R}^p$ is the disturbance input which is assumed to belong to $l_2[0, \infty)$; $\tau > 0$ is an integer representing the time delay of the fuzzy system; $s_1(k), s_2(k), \dots, s_g(k)$ are the premise variables. Throughout this paper, it is assumed that the premise variables do not depend on the input variables $u(k)$ explicitly. $A_i, A_{di}, B_i, C_i, C_{di}, D_{1i}, D_{2i}, E_i,$ and G_i are known real constant matrices; $\Delta A_i(k), \Delta A_{di}(k)$ and $\Delta B_i(k)$ are real-valued unknown matrices representing time-varying parameter uncertainties, and are assumed to be of the form

$$\begin{bmatrix} \Delta A_i(k) & \Delta A_{di}(k) & \Delta B_i(k) \end{bmatrix} = M_i F_i(k) \begin{bmatrix} N_{1i} & N_{2i} & N_{3i} \end{bmatrix}, \quad i = 1, 2, \dots, r \quad (5)$$

where $M_i, N_{1i}, N_{2i},$ and N_{3i} are known real constant matrices and $F_i(\cdot) : \mathbb{N} \rightarrow \mathbb{R}^{l_1 \times l_2}$ are unknown time-varying matrix function satisfying

$$F_i(k)^T F_i(k) \leq I \quad \forall k. \quad (6)$$

The parameter uncertainties $\Delta A_i(k), \Delta A_{di}(k)$ and $\Delta B_i(k)$ are said to be admissible if both (5) and (6) hold. It is worth mentioning that interval bounded parameters can also be used to describe uncertain systems. In the discrete-time case, interval model control and applications can be found in [1], [29], and the references therein.

Given a pair $(x(k), u(k))$, the final output of the fuzzy system is inferred as follows:

$$x(k+1) = \sum_{i=1}^r h_i(s(k)) \{ [A_i + \Delta A_i(k)]x(k) + [A_{di} + \Delta A_{di}(k)]x(k-\tau) + [B_i + \Delta B_i(k)]u(k) + D_{1i}\omega(k) \} \quad (7)$$

$$y(k) = \sum_{i=1}^r h_i(s(k)) [C_i x(k) + C_{di} x(k-\tau) + D_{2i} \omega(k)] \quad (8)$$

$$z(k) = \sum_{i=1}^r h_i(s(k)) [E_i x(k) + E_{di} x(k-\tau) + G_i u(k)] \quad (9)$$

where

$$h_i(s(k)) = \frac{\varpi_i(s(k))}{\sum_{j=1}^r \varpi_j(s(k))} \quad \varpi_i(s(k)) = \prod_{j=1}^g \mu_{ij}(s_j(k))$$

$$s(k) = [s_1(k) \quad s_2(k) \quad \dots \quad s_g(k)]$$

in which $\mu_{ij}(s_j(k))$ is the grade of membership of $s_j(k)$ in μ_{ij} . Then, it can be seen that

$$\varpi_i(s(k)) \geq 0, \quad i = 1, 2, \dots, r$$

$$\sum_{j=1}^r \varpi_j(s(k)) > 0$$

for all k . Therefore, for all k

$$h_i(s(k)) \geq 0, \quad i = 1, 2, \dots, r \quad (10)$$

$$\sum_{j=1}^r h_j(s(k)) = 1. \quad (11)$$

Now, by the parallel distributed compensation (PDC) technique, we consider the following full-order fuzzy dynamic output feedback controller for the fuzzy system (7)–(9):

Control Rule i : IF $s_1(k)$ is μ_{i1} and $s_2(k)$ is μ_{i2} and \dots and $s_g(k)$ is μ_{ig} , THEN

$$\hat{x}(k+1) = A_{Ki} \hat{x}(k) + B_{Ki} y(k) \quad (12)$$

$$u(k) = C_{Ki} \hat{x}(k), \quad i = 1, 2, \dots, r \quad (13)$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the controller state, $A_{Ki}, B_{Ki},$ and C_{Ki} are matrices to be determined later. Then, the overall fuzzy output feedback controller is given by

$$\hat{x}(k+1) = \sum_{i=1}^r h_i(s(k)) [A_{Ki} \hat{x}(k) + B_{Ki} y(k)] \quad (14)$$

$$u(k) = \sum_{i=1}^r h_i(s(k)) C_{Ki} \hat{x}(k). \quad (15)$$

From (7)–(9), (14), and (15), the closed-loop system can be obtained as

$$\begin{aligned} \xi(k+1) &= \sum_{i=1}^r \sum_{j=1}^r h_i(s(k))h_j(s(k))[A_{cij}(k)\xi(k) \\ &\quad + A_{cdij}(k)H\xi(k-\tau) + D_{cij}\omega(k)] \end{aligned} \quad (16)$$

$$\begin{aligned} z(k) &= \sum_{i=1}^r \sum_{j=1}^r h_i(s(k))h_j(s(k))[E_{cij}\xi(k) \\ &\quad + E_{cdi}H\xi(k-\tau)] \end{aligned} \quad (17)$$

where

$$\xi(k) = \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}$$

and

$$\begin{aligned} A_{cij}(k) &= A_{cij} + \Delta A_{cij}(k) \\ A_{cdij}(k) &= A_{cdij} + \Delta A_{cdij}(k) \end{aligned} \quad (18)$$

$$\begin{aligned} A_{cij} &= \begin{bmatrix} A_i & B_i C_{Kj} \\ B_{Kj} C_i & A_{Kj} \end{bmatrix} \\ \Delta A_{cij}(k) &= \begin{bmatrix} \Delta A_i(k) & \Delta B_i(k) C_{Kj} \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (19)$$

$$A_{cdij} = \begin{bmatrix} A_{di} \\ B_{Kj} C_{di} \end{bmatrix} \quad \Delta A_{cdij}(k) = \begin{bmatrix} \Delta A_{di}(k) \\ 0 \end{bmatrix}$$

$$D_{cij} = \begin{bmatrix} D_{1i} \\ B_{Kj} D_{2i} \end{bmatrix} \quad (20)$$

$$E_{cij} = [E_i \quad G_i C_{Kj}] \quad E_{cdi} = E_{di} \quad H = [I \quad 0]. \quad (21)$$

Then, the robust fuzzy H_∞ control problem to be addressed in this paper can be formulated as follows: given an uncertain fuzzy system (7)–(9) and a scalar $\gamma > 0$, determine an output feedback fuzzy controller in the form of (12) and (13) such that

- R1) The closed-loop system in (16) and (17) is robustly asymptotically stable when $\omega(k) = 0$.
- R2) Under zero-initial condition, the controlled output z satisfies

$$\|z\|_2 < \gamma \|\omega\|_2 \quad (22)$$

for any nonzero $\omega \in l_2$ and all admissible uncertainties.

III. MAIN RESULTS

In this section, an LMI approach will be developed to solve the problem of robust output feedback H_∞ control of uncertain discrete delay fuzzy systems formulated in the previous section. We first give the following results which will be used in the proof of our main results.

Lemma 1 [24]: Given any matrices \mathcal{X} , \mathcal{Y} , and \mathcal{Z} with appropriate dimensions such that $\mathcal{Y} > 0$. Then, we have

$$\mathcal{X}^T \mathcal{Z} + \mathcal{Z}^T \mathcal{X} \leq \mathcal{X}^T \mathcal{Y} \mathcal{X} + \mathcal{Z}^T \mathcal{Y}^{-1} \mathcal{Z}.$$

Lemma 2 [22]: Let A , D , S , \mathcal{W} , and F be real matrices of appropriate dimensions such that $\mathcal{W} > 0$ and $F^T F \leq I$. Then, for any scalar $\epsilon > 0$ such that $\mathcal{W} - \epsilon D D^T > 0$, we have

$$\begin{aligned} (A + D F S)^T \mathcal{W}^{-1} (A + D F S) \\ \leq \mathcal{A}^T (\mathcal{W} - \epsilon D D^T)^{-1} \mathcal{A} + \epsilon^{-1} S^T S. \end{aligned}$$

Theorem 1: The uncertain system in (16) and (17) is robustly asymptotically stable and (22) is satisfied if there exist matrices $P > 0$, and $Q > 0$, and scalars $\epsilon_{ij} > 0$, $1 \leq i \leq j \leq r$, such that the matrix inequalities shown in (23) and (24) at the bottom of the page, hold, where

$$\begin{aligned} \tilde{N}_{ij} &= [N_{1i} \quad N_{3i} C_{Kj}] \quad \tilde{M}_i = [M_i^T \quad 0]^T \\ \tilde{M}_{ij} &= [\tilde{M}_i \quad \tilde{M}_j]. \end{aligned} \quad (25)$$

Proof: Under the conditions of the theorem, we first establish the robust asymptotic stability of the system in (16). To this end, we consider (16) with $\omega(k) = 0$; that is

$$\begin{aligned} \xi(k+1) &= \sum_{i=1}^r \sum_{j=1}^r h_i(s(k))h_j(s(k)) \\ &\quad \times [A_{cij}(k)\xi(k) + A_{cdij}(k)H\xi(k-\tau)]. \end{aligned} \quad (26)$$

Choose a Lyapunov function candidate for the system in (26) as follows:

$$V(\xi(k)) = \xi(k)^T P \xi(k) + \sum_{i=k-\tau}^{k-1} \xi(i)^T H^T Q H \xi(i). \quad (27)$$

$$\begin{bmatrix} H^T Q H - P & 0 & 0 & A_{cii}^T & \tilde{N}_{ii}^T & E_{cii}^T \\ 0 & -Q & 0 & A_{cdii}^T & N_{2i}^T & E_{cdii}^T \\ 0 & 0 & -\gamma^2 I & D_{cii}^T & 0 & 0 \\ A_{cii} & A_{cdii} & D_{cii} & \epsilon_{ii} \tilde{M}_i \tilde{M}_i^T - P^{-1} & 0 & 0 \\ \tilde{N}_{ii} & N_{2i} & 0 & 0 & -\epsilon_{ii} I & 0 \\ E_{cii} & E_{cdii} & 0 & 0 & 0 & -I \end{bmatrix} < 0, \quad 1 \leq i \leq r \quad (23)$$

and

$$\begin{bmatrix} 4H^T Q H - 4P & 0 & 0 & A_{cij}^T + A_{cji}^T & \tilde{N}_{ij}^T & \tilde{N}_{ji}^T & E_{cij}^T + E_{cji}^T \\ 0 & -4Q & 0 & A_{cdij}^T + A_{cdji}^T & N_{2i}^T & N_{2j}^T & E_{cdij}^T + E_{cdji}^T \\ 0 & 0 & -4\gamma^2 I & D_{cij}^T + D_{cji}^T & 0 & 0 & 0 \\ A_{cij} + A_{cji} & A_{cdij} + A_{cdji} & D_{cij} + D_{cji} & \epsilon_{ij} \tilde{M}_{ij} \tilde{M}_{ij}^T - P^{-1} & 0 & 0 & 0 \\ \tilde{N}_{ij} & N_{2i} & 0 & 0 & -\epsilon_{ij} I & 0 & 0 \\ \tilde{N}_{ji} & N_{2j} & 0 & 0 & 0 & -\epsilon_{ij} I & 0 \\ E_{cij} + E_{cji} & E_{cdij} + E_{cdji} & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0, \quad 1 \leq i < j \leq r \quad (24)$$

Then it can be verified that

$$\begin{aligned}
\Delta V(\xi(k)) &= V(\xi(k+1)) - V(\xi(k)) \\
&= \sum_{i=1}^r \sum_{j=1}^r \sum_{u=1}^r \sum_{v=1}^r h_i(s(k)) h_j(s(k)) h_u(s(k)) h_v(s(k)) \\
&\quad \times [A_{cij}(k)\xi(k) + A_{cdij}(k)H\xi(k-\tau)]^T P \\
&\quad \times [A_{cuv}(k)\xi(k) + A_{cdvu}(k)H\xi(k-\tau)] \\
&\quad + \xi(k)^T (H^T QH - P)\xi(k) \\
&\quad - \xi(k-\tau)^T H^T QH\xi(k-\tau) \\
&= \frac{1}{4} \sum_{i=1}^r \sum_{j=1}^r \sum_{u=1}^r \sum_{v=1}^r h_i(s(k)) h_j(s(k)) h_u(s(k)) h_v(s(k)) \\
&\quad \times \{[A_{cij}(k) + A_{cji}(k)]\xi(k) \\
&\quad + [A_{cdij}(k) + A_{cdji}(k)]H\xi(k-\tau)\}^T \\
&\quad \times P\{[A_{cuv}(k) + A_{cvu}(k)]\xi(k) \\
&\quad + [A_{cdvu}(k) + A_{cdvu}(k)]H\xi(k-\tau)\} \\
&\quad + \xi(k)^T (H^T QH - P)\xi(k) \\
&\quad - \xi(k-\tau)^T H^T QH\xi(k-\tau). \tag{28}
\end{aligned}$$

Using Lemma 1 and noting (11), we have

$$\begin{aligned}
&\sum_{i=1}^r \sum_{j=1}^r \sum_{u=1}^r \sum_{v=1}^r h_i(s(k)) h_j(s(k)) h_u(s(k)) h_v(s(k)) \\
&\quad \times \{[A_{cij}(k) + A_{cji}(k)]\xi(k) \\
&\quad + [A_{cdij}(k) + A_{cdji}(k)]H\xi(k-\tau)\}^T \\
&\quad \times P\{[A_{cuv}(k) + A_{cvu}(k)]\xi(k) \\
&\quad + [A_{cdvu}(k) + A_{cdvu}(k)]H\xi(k-\tau)\} \\
&\leq \sum_{i=1}^r \sum_{j=1}^r h_i(s(k)) h_j(s(k)) \{[A_{cij}(k) + A_{cji}(k)]\xi(k) \\
&\quad + [A_{cdij}(k) + A_{cdji}(k)]H\xi(k-\tau)\}^T \\
&\quad \times P\{[A_{cij}(k) + A_{cji}(k)]\xi(k) \\
&\quad + [A_{cdij}(k) + A_{cdji}(k)]H\xi(k-\tau)\}.
\end{aligned}$$

This together with (28) and the relationship

$$x(k-\tau) = H\xi(k-\tau) \tag{29}$$

implies

$$\begin{aligned}
\Delta V(\xi(k)) &\leq \frac{1}{4} \sum_{i=1}^r \sum_{j=1}^r h_i(s(k)) h_j(s(k)) \alpha(k)^T [\mathcal{M}_{cij}(k) \\
&\quad + \mathcal{M}_{cji}(k)]^T P [\mathcal{M}_{cij}(k) + \mathcal{M}_{cji}(k)] \alpha(k) \\
&\quad + \xi(k)^T (H^T QH - P)\xi(k) \\
&\quad - x(k-\tau)^T Qx(k-\tau)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^r h_i^2(s(k)) \alpha(k)^T [\mathcal{M}_{cii}(k)]^T \\
&\quad \times P \mathcal{M}_{cii}(k) + Z_c \alpha(k) \\
&\quad + \frac{1}{2} \sum_{i,j=1, i<j}^r h_i(s(k)) h_j(s(k)) \\
&\quad \times \alpha(k)^T \{[\mathcal{M}_{cij}(k) + \mathcal{M}_{cji}(k)]^T \\
&\quad \times P[\mathcal{M}_{cij}(k) + \mathcal{M}_{cji}(k)] + 4Z_c\} \alpha(k) \tag{30}
\end{aligned}$$

where

$$\begin{aligned}
\alpha(k) &= [\xi(k)^T \quad x(k-\tau)^T]^T \\
\mathcal{M}_{cij}(k) &= [A_{cij}(k) \quad A_{cdij}(k)], \\
Z_c &= \begin{bmatrix} H^T QH - P & 0 \\ 0 & -Q \end{bmatrix}.
\end{aligned}$$

Now, from (24), it is easy to see that for $1 \leq i < j \leq r$, (31), as shown at the bottom of the page, holds. Set

$$\begin{aligned}
\mathcal{M}_{cij} &= [A_{cij} \quad A_{cdij}] \quad N_{cij} = [\tilde{N}_{ij} \quad N_{2i}] \\
\mathcal{N}_{ij} &= \begin{bmatrix} N_{cij} \\ N_{cji} \end{bmatrix}. \tag{32}
\end{aligned}$$

for $1 \leq i \leq j \leq r$. Then, applying the Schur complement to (31) gives

$$P^{-1} - \epsilon_{ij} \tilde{M}_{ij} \tilde{M}_{ij}^T > 0 \tag{33}$$

and

$$\begin{aligned}
&(\mathcal{M}_{cij} + \mathcal{M}_{cji})^T (P^{-1} - \epsilon_{ij} \tilde{M}_{ij} \tilde{M}_{ij}^T)^{-1} \\
&\quad \times (\mathcal{M}_{cij} + \mathcal{M}_{cji}) + \epsilon_{ij}^{-1} \mathcal{N}_{ij}^T \mathcal{N}_{ij} + 4Z_c < 0 \tag{34}
\end{aligned}$$

for $1 \leq i < j \leq r$. Considering (33) and using Lemma 2, we have that for $1 \leq i < j \leq r$

$$\begin{aligned}
&[\mathcal{M}_{cij}(k) + \mathcal{M}_{cji}(k)]^T P [\mathcal{M}_{cij}(k) + \mathcal{M}_{cji}(k)] \\
&= [\mathcal{M}_{cij} + \mathcal{M}_{cji} + \tilde{M}_{ij} F_{ij}(k) \mathcal{N}_{ij}]^T \\
&\quad \times P [\mathcal{M}_{cij} + \mathcal{M}_{cji} + \tilde{M}_{ij} F_{ij}(k) \mathcal{N}_{ij}] \\
&\leq (\mathcal{M}_{cij} + \mathcal{M}_{cji})^T (P^{-1} - \epsilon_{ij} \tilde{M}_{ij} \tilde{M}_{ij}^T)^{-1} \\
&\quad \times (\mathcal{M}_{cij} + \mathcal{M}_{cji}) + \epsilon_{ij}^{-1} \mathcal{N}_{ij}^T \mathcal{N}_{ij} \tag{35}
\end{aligned}$$

where

$$F_{ij}(k) = \begin{bmatrix} F_i(k) & 0 \\ 0 & F_j(k) \end{bmatrix}.$$

Then, it follows from (34) and (35) that for $1 \leq i < j \leq r$

$$\begin{aligned}
&[\mathcal{M}_{cij}(k) + \mathcal{M}_{cji}(k)]^T \\
&\quad \times P [\mathcal{M}_{cij}(k) + \mathcal{M}_{cji}(k)] + 4Z_c < 0. \tag{36}
\end{aligned}$$

$$\begin{bmatrix} 4H^T QH - 4P & 0 & A_{cij}^T + A_{cji}^T & \tilde{N}_{ij}^T & \tilde{N}_{ji}^T \\ 0 & -4Q & A_{cdij}^T + A_{cdji}^T & N_{2i}^T & N_{2j}^T \\ A_{cij} + A_{cji} & A_{cdij} + A_{cdji} & \epsilon_{ij} \tilde{M}_{ij} \tilde{M}_{ij}^T - P^{-1} & 0 & 0 \\ \tilde{N}_{ij} & N_{2i} & 0 & -\epsilon_{ij} I & 0 \\ \tilde{N}_{ji} & N_{2j} & 0 & 0 & -\epsilon_{ij} I \end{bmatrix} < 0. \tag{31}$$

On the other hand, from (23), we have

$$\begin{bmatrix} H^T Q H - P & 0 & A_{cii}^T & \tilde{N}_{1ii}^T \\ 0 & -Q & A_{cdii}^T & \tilde{N}_{2i}^T \\ A_{cii} & A_{cdii} & \epsilon_{ii} \tilde{M}_i \tilde{M}_i^T - P^{-1} & 0 \\ \tilde{N}_{1ii} & \tilde{N}_{2i} & 0 & -\epsilon_{ii} I \end{bmatrix} < 0, \quad 1 \leq i \leq r \quad (37)$$

which, by the Schur complement, implies

$$P^{-1} - \epsilon_{ii} \tilde{M}_i \tilde{M}_i^T > 0 \quad (38)$$

and

$$\begin{aligned} & \mathcal{M}_{cii}^T \left(P^{-1} - \epsilon_{ii} \tilde{M}_i \tilde{M}_i^T \right)^{-1} \mathcal{M}_{cii} \\ & + \epsilon_{ii}^{-1} N_{cii}^T N_{cii} + Z_c < 0 \end{aligned} \quad (39)$$

where \mathcal{M}_{cii} and N_{cii} are given in (32). Taking into account (38) and using Lemma 2 again, we have

$$\begin{aligned} & \mathcal{M}_{cii}(k)^T P \mathcal{M}_{cii}(k) \\ & = [\mathcal{M}_{cii} + \tilde{M}_i F_i(k) N_{cii}]^T P [\mathcal{M}_{cii} + \tilde{M}_i F_i(k) N_{cii}] \\ & \leq \mathcal{M}_{cii}^T \left(P^{-1} - \epsilon_{ii} \tilde{M}_i \tilde{M}_i^T \right)^{-1} \mathcal{M}_{cii} + \epsilon_{ii}^{-1} N_{cii}^T N_{cii} \end{aligned} \quad (40)$$

for $1 \leq i \leq r$. Then, by (39) and (40), it can be established that for $1 \leq i \leq r$

$$\mathcal{M}_{cii}(k)^T P \mathcal{M}_{cii}(k) + Z_c < 0. \quad (41)$$

Then, it follows from (30), (36), and (41) that

$$\Delta V(\xi(k)) < 0$$

for all $\alpha(k) \neq 0$. Hence, the uncertain system (16) with $\omega(k) = 0$ is robustly asymptotically stable.

Next, we show that for any nonzero $\omega \in l_2$ the uncertain system in (16) and (17) satisfies (22) under zero initial condition. To this end, we introduce

$$J_N = \sum_{k=0}^N [z(k)^T z(k) - \gamma^2 \omega(k)^T \omega(k)] \quad (42)$$

where the scalar $N \in \mathbb{N}$. Noting the zero initial condition and (16), (17), and (29), we can deduce

$$\begin{aligned} J_N &= \sum_{k=0}^N \{ z(k)^T z(k) - \gamma^2 \omega(k)^T \omega(k) \\ & + [V(\xi(k+1)) - V(\xi(k))] \} - V(\xi(N+1)) \\ &\leq \sum_{k=0}^N \{ z(k)^T z(k) - \gamma^2 \omega(k)^T \omega(k) \\ & + [V(\xi(k+1)) - V(\xi(k))] \} \\ &= \sum_{k=0}^N \left\{ \sum_{i=1}^r \sum_{j=1}^r \sum_{u=1}^r \sum_{v=1}^r \right. \\ & h_i(s(k)) h_j(s(k)) h_u(s(k)) h_v(s(k)) \\ & \times [(E_{cij} \xi(k) + E_{cdi} x(k-\tau))]^T \\ & \times [E_{cuv} \xi(k) + E_{cdv} x(k-\tau)] \\ & + [A_{cij}(k) \xi(k) + A_{cdij}(k) x(k-\tau) + D_{cij} \omega(k)]^T \\ & \times P [A_{cuv}(k) \xi(k) + A_{cdv}(k) x(k-\tau) + D_{cuv} \omega(k)] \\ & \left. + \xi(k)^T (H^T Q H - P) \xi(k) \right. \\ & \left. - x(k-\tau)^T Q x(k-\tau) - \gamma^2 \omega(k)^T \omega(k) \right\} \end{aligned} \quad (43)$$

where $V(\xi(k))$ is given in (27). By Lemma 1, we have

$$\begin{aligned} & \sum_{i=1}^r \sum_{j=1}^r \sum_{u=1}^r \sum_{v=1}^r h_i(s(k)) h_j(s(k)) h_u(s(k)) h_v(s(k)) \\ & \times [E_{cij} \xi(k) + E_{cdi} x(k-\tau)]^T [E_{cuv} \xi(k) + E_{cdv} x(k-\tau)] \\ & = \frac{1}{4} \sum_{i=1}^r \sum_{j=1}^r \sum_{u=1}^r \sum_{v=1}^r h_i(s(k)) h_j(s(k)) h_u(s(k)) h_v(s(k)) \\ & \times [(E_{cij} + E_{cji}) \xi(k) + (E_{cdi} + E_{cdj}) x(k-\tau)]^T \\ & \times [(E_{cuv} + E_{cvu}) \xi(k) + (E_{cdv} + E_{cdu}) x(k-\tau)] \\ & \leq \frac{1}{4} \sum_{i=1}^r \sum_{j=1}^r h_i(s(k)) h_j(s(k)) [(E_{cij} + E_{cji}) \xi(k) \\ & + (E_{cdi} + E_{cdj}) x(k-\tau)]^T \\ & \times [(E_{cij} + E_{cji}) \xi(k) + (E_{cdi} + E_{cdj}) x(k-\tau)] \end{aligned}$$

and

$$\begin{aligned} & \sum_{i=1}^r \sum_{j=1}^r \sum_{u=1}^r \sum_{v=1}^r h_i(s(k)) h_j(s(k)) h_u(s(k)) h_v(s(k)) \\ & \times [A_{cij}(k) \xi(k) + A_{cdij}(k) x(k-\tau) + D_{cij} \omega(k)]^T \\ & \times P [A_{cuv}(k) \xi(k) + A_{cdv}(k) x(k-\tau) + D_{cuv} \omega(k)] \\ & = \frac{1}{4} \sum_{i=1}^r \sum_{j=1}^r \sum_{u=1}^r \sum_{v=1}^r h_i(s(k)) h_j(s(k)) h_u(s(k)) h_v(s(k)) \\ & \times \{ [A_{cij}(k) + A_{cji}(k)] \xi(k) + [A_{cdij}(k) \\ & + A_{cdji}(k)] x(k-\tau) + [D_{cij} + D_{cji}] \omega(k) \}^T \\ & \times P \{ [A_{cuv}(k) + A_{cvu}(k)] \xi(k) + [A_{cdv}(k) \\ & + A_{cdvu}(k)] x(k-\tau) + [D_{cuv} + D_{cuv}] \omega(k) \} \\ & \leq \frac{1}{4} \sum_{i=1}^r \sum_{j=1}^r h_i(s(k)) h_j(s(k)) \\ & \times \{ [A_{cij}(k) + A_{cji}(k)] \xi(k) + [A_{cdij}(k) \\ & + A_{cdji}(k)] x(k-\tau) + [D_{cij} + D_{cji}] \omega(k) \}^T \\ & \times P \{ [A_{cij}(k) + A_{cji}(k)] \xi(k) + [A_{cdij}(k) \\ & + A_{cdji}(k)] x(k-\tau) + [D_{cij} + D_{cji}] \omega(k) \}. \end{aligned}$$

Therefore

$$\begin{aligned} J_N &\leq \sum_{k=0}^N \frac{1}{4} \left\{ \sum_{i=1}^r \sum_{j=1}^r h_i(s(k)) h_j(s(k)) \right. \\ & \times \{ [(E_{cij} + E_{cji}) \xi(k) + (E_{cdi} + E_{cdj}) x(k-\tau)]^T \\ & \times [(E_{cij} + E_{cji}) \xi(k) + (E_{cdi} + E_{cdj}) x(k-\tau)] \\ & + \{ [A_{cij}(k) + A_{cji}(k)] \xi(k) + [A_{cdij}(k) \\ & + A_{cdji}(k)] x(k-\tau) + [D_{cij} + D_{cji}] \omega(k) \} \\ & \times P \{ [A_{cij}(k) + A_{cji}(k)] \xi(k) + [A_{cdij}(k) \\ & + A_{cdji}(k)] x(k-\tau) + [D_{cij} + D_{cji}] \omega(k) \} \\ & \left. + \xi(k)^T (4H^T Q H - 4P) \xi(k) \right. \\ & \left. - 4x(k-\tau)^T Q x(k-\tau) - 4\gamma^2 \omega(k)^T \omega(k) \right\} \end{aligned}$$

where S and W are any nonsingular matrices satisfying

$$SW^T = I - XY. \quad (52)$$

Proof: Under the conditions of the theorem, we first show that there always exist nonsingular matrices S and W such that (52) is satisfied. To this end, we note that (48) implies

$$\begin{bmatrix} -Y & -I \\ -I & -X \end{bmatrix} < 0$$

which, by the Schur complement formula, gives that $X - Y^{-1} > 0$, therefore $I - XY$ is nonsingular. This ensures that there always exist nonsingular matrices S and W such that (52) is satisfied. Now, we introduce the following nonsingular matrices

$$\Pi_1 = \begin{bmatrix} Y & I \\ W^T & 0 \end{bmatrix} \quad \Pi_2 = \begin{bmatrix} I & X \\ 0 & S^T \end{bmatrix}. \quad (53)$$

Let

$$\hat{P} = \Pi_2 \Pi_1^{-1}. \quad (54)$$

Then

$$\hat{P} = \begin{bmatrix} X & S \\ S^T & \Xi \end{bmatrix}$$

where

$$\Xi = W^{-1}Y(X - Y^{-1})YW^{-T} > 0. \quad (55)$$

Observe that

$$\Xi - S^T X^{-1} S = S^T (YX - I)^{-1} \times (Y - X^{-1})(XY - I)^{-1} S > 0. \quad (56)$$

$$\begin{bmatrix} -4J_1 + H_{4ij}^T H_{4ij} & 0 & H_{1ij}^T + H_{1ji}^T & 0 & H_{2ij}^T & H_{2ji}^T & H_{3ij}^T + H_{3ji}^T & \tilde{H}^T \\ 0 & -4J_2 & H_{5ij}^T + H_{5ji}^T & 0 & 0 & 0 & 0 & 0 \\ H_{1ij} + H_{1ji} & H_{5ij} + H_{5ji} & -J_3 + H_{7ij} H_{7ij}^T & H_{6ij}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & H_{6ij} & -J_{4ij} & 0 & 0 & 0 & 0 \\ H_{2ij} & 0 & 0 & 0 & -J_{5ij} & 0 & 0 & 0 \\ H_{2ji} & 0 & 0 & 0 & 0 & -J_{5ij} & 0 & 0 \\ H_{3ij} + H_{3ji} & 0 & 0 & 0 & 0 & 0 & -I & 0 \\ \tilde{H} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4}J_6 \end{bmatrix} < 0. \quad (57)$$

$$\begin{aligned} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & (\Psi_j - \Psi_i)^T (B_i - B_j)^T X \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ X(B_i - B_j)(\Psi_j - \Psi_i) & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ & \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ X(B_i - B_j) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ X(B_i - B_j) \end{bmatrix}^T + \begin{bmatrix} (\Psi_j - \Psi_i)^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} (\Psi_j - \Psi_i)^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \end{aligned} \quad (58)$$

and

$$\begin{aligned} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & Y(C_i - C_j)^T (\Phi_j - \Phi_i)^T \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ (\Phi_j - \Phi_i)(C_i - C_j)Y & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ & \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \Phi_j - \Phi_i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \Phi_j - \Phi_i \end{bmatrix}^T + \begin{bmatrix} Y(C_i - C_j)^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} Y(C_i - C_j)^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T. \end{aligned} \quad (59)$$

numerical algorithms, which involves no tuning of parameters [2]. In the case when the parameters $Q > 0$ and $\epsilon_{ij} > 0, 1 \leq i \leq j \leq r$, are not fixed, it can be seen that (48) and (49) are not LMIs with respect to the parameters $Q > 0$ and $\epsilon_{ij} > 0, 1 \leq i \leq j \leq r$, since these parameters appear in (48) and (49) in a nonlinear fashion, which is sometimes encountered when dealing with the output feedback control problem for time-delay systems with or without parameter uncertainties; see, e.g., [6], [7], [9]. In order to cast the output H_∞ control problem in this paper into an LMI framework, we therefore fix the parameters $Q > 0$ and $\epsilon_{ij} > 0, 1 \leq i \leq j \leq r$; such an approach was also adopted in [6], [7], [9].

Remark 2: It is worth pointing out that the result in Theorem 2 can be readily extended to the case with multiple delays. It is also noted that the result in Theorem 2 are independent of the delay size; therefore, Theorem 2 can be applicable to the case when no *a priori* knowledge about the size of the time delay is available.

IV. NUMERICAL EXAMPLE

In this section, we will apply the proposed method to design a fuzzy dynamic controller for an uncertain nonlinear discrete delay system. The uncertain discrete nonlinear time-delay system is described as follows:

$$\begin{aligned} x_1(k+1) &= -x_1(k)^2 + 0.3x_2(k) + 0.1x_1(k-2)^2 \\ &\quad - 0.2x_1(k-2)x_2(k-2) \\ &\quad + u_1(k) + x_1(k)\omega(k) + 0.1c_1(k)x_1(k)x_2(k) \\ &\quad + 0.1c_2(k)x_2(k-2) \\ x_2(k+1) &= 0.1x_1(k) + x_2(k) + 0.1c_1(k)x_2(k) + 0.5u_2(k) \\ y(k) &= 0.6x_1(k) + 0.1\omega(k) \\ z(k) &= 0.1x_1(k)^2 + 0.2x_2(k-2) \end{aligned}$$

where $c_1(k)$ and $c_2(k)$ are uncertain parameters satisfying

$$c_1(k) \in [-0.2, 0.2] \quad c_2(k) \in [-0.1, 0.1].$$

$$\left[\begin{array}{cccccccccc} -4\hat{P} & 0 & 0 & A_{cij}^T + A_{cji}^T & 0 & 0 & \tilde{N}_{ij}^T & \tilde{N}_{ji}^T & E_{cij}^T + E_{cji}^T & H^T \\ 0 & -4Q & 0 & A_{cdij}^T + A_{cdji}^T & 0 & 0 & N_{2i}^T & N_{2j}^T & E_{cdij}^T + E_{cdji}^T & 0 \\ 0 & 0 & -4\gamma^2 I & D_{cij}^T + D_{cji}^T & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{cij} + A_{cji} & A_{cdij} + A_{cdji} & D_{cij} + D_{cji} & -\hat{P}^{-1} & \tilde{M}_i & \tilde{M}_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{M}_i^T & -\epsilon_{ij}^{-1} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{M}_j^T & 0 & -\epsilon_{ij}^{-1} I & 0 & 0 & 0 & 0 \\ \tilde{N}_{ij} & N_{2i} & 0 & 0 & 0 & 0 & -\epsilon_{ij} I & 0 & 0 & 0 \\ \tilde{N}_{ji} & N_{2j} & 0 & 0 & 0 & 0 & 0 & -\epsilon_{ij} I & 0 & 0 \\ E_{cij} + E_{cji} & E_{cdij} + E_{cdji} & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 \\ H & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4}Q^{-1} \end{array} \right] < 0$$

$$\left[\begin{array}{cccccccc} 4H^T Q H - 4\hat{P} & 0 & 0 & A_{cij}^T + A_{cji}^T & \tilde{N}_{ij}^T & \tilde{N}_{ji}^T & E_{cij}^T + E_{cji}^T \\ 0 & -4Q & 0 & A_{cdij}^T + A_{cdji}^T & N_{2i}^T & N_{2j}^T & E_{cdij}^T + E_{cdji}^T \\ 0 & 0 & -4\gamma^2 I & D_{cij}^T + D_{cji}^T & 0 & 0 & 0 \\ A_{cij} + A_{cji} & A_{cdij} + A_{cdji} & D_{cij} + D_{cji} & \epsilon_{ij} \tilde{M}_i \tilde{M}_j^T - \hat{P}^{-1} & 0 & 0 & 0 \\ \tilde{N}_{ij} & N_{2i} & 0 & 0 & -\epsilon_{ij} I & 0 & 0 \\ \tilde{N}_{ji} & N_{2j} & 0 & 0 & 0 & -\epsilon_{ij} I & 0 \\ E_{cij} + E_{cji} & E_{cdij} + E_{cdji} & 0 & 0 & 0 & 0 & -I \end{array} \right] < 0 \quad (61)$$

$$\left[\begin{array}{ccccccc} H^T Q H - \hat{P} & 0 & 0 & A_{cii}^T & \tilde{N}_{ii}^T & E_{cdii}^T \\ 0 & -Q & 0 & A_{cdii}^T & N_{2i}^T & E_{cdii}^T \\ 0 & 0 & -\gamma^2 I & D_{cii}^T & 0 & 0 \\ A_{cii} & A_{cdii} & D_{cii} & \epsilon_{ii} \tilde{M}_i \tilde{M}_i^T - \hat{P}^{-1} & 0 & 0 \\ \tilde{N}_{ii} & N_{2i} & 0 & 0 & -\epsilon_{ii} I & 0 \\ E_{cii} & E_{cdii} & 0 & 0 & 0 & -I \end{array} \right] < 0 \quad (62)$$

Similar to [20], we assume that $x_1(k) \in [-0.5, 0.5]$, and select the membership functions as

$$\mu_{11}(x_1(k)) = \frac{1}{2}(1 - 2x_1(k)) \quad \mu_{21}(x_1(k)) = \frac{1}{2}(1 + 2x_1(k)).$$

Then, the nonlinear time-delay system can be represented by the following uncertain time-delay T-S model:

Plant Rule 1: IF $x_1(k)$ is μ_{11} THEN

$$\begin{aligned} x_1(k+1) &= [A_1 + \Delta A_1(k)]x(k) \\ &\quad + A_{d1}x(k-2) + B_1u(k) \\ &\quad + D_{11}\omega(k) \\ y(k) &= C_1x(k) + D_{21}\omega(k) \\ z(k) &= E_1x(k) + E_{d1}x(k-2) \end{aligned}$$

and

Plant Rule 2: IF $x_1(k)$ is μ_{21} THEN

$$\begin{aligned} x_1(k+1) &= [A_2 + \Delta A_2(k)]x(k) \\ &\quad + A_{d2}x(k-2) + B_2u(k) \\ &\quad + D_{12}\omega(k) \\ y(k) &= C_2x(k) + D_{22}\omega(k) \\ z(k) &= E_2x(k) + E_{d2}x(k-2) \end{aligned}$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.5 & 0.3 \\ 0.1 & 1 \end{bmatrix} & A_{d1} &= \begin{bmatrix} -0.05 & 0.1 \\ 0 & 0 \end{bmatrix} \\ B_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} & D_{11} &= \begin{bmatrix} -0.5 \\ 0 \end{bmatrix} \\ C_1 &= [0.6 \quad 0] & D_{21} &= 0.1 \\ E_1 &= [-0.05 \quad 0] & E_{d1} &= [0 \quad 0.2] \\ A_2 &= \begin{bmatrix} -0.5 & 0.3 \\ 0.1 & 1 \end{bmatrix} & A_{d2} &= \begin{bmatrix} 0.05 & -0.1 \\ 0 & 0 \end{bmatrix} \\ B_2 &= B_1 & D_{12} &= \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \\ C_2 &= C_1 & D_{22} &= D_{21} & E_2 &= [0.05 \quad 0] & E_{d2} &= E_{d1} \end{aligned}$$

and $\Delta A_1(k)$ and $\Delta A_2(k)$ can be represented in the form of (5) and (6) with

$$\begin{aligned} M_1 &= \begin{bmatrix} 0.05 & 0.1 \\ 0.1 & 0 \end{bmatrix} & N_{11} &= \begin{bmatrix} 0 & 0.2 \\ 0 & 0 \end{bmatrix} \\ N_{21} &= \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix} \\ M_2 &= M_1 & N_{12} &= N_{11} & N_{22} &= N_{21}. \end{aligned}$$

In this example, we choose the H_∞ performance level $\gamma = 1.8$. In order to design a fuzzy H_∞ output feedback controller for the T-S model, we first choose

$$\epsilon_{11} = 10 \quad \epsilon_{12} = 0.2 \quad \epsilon_{22} = 1 \quad Q = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

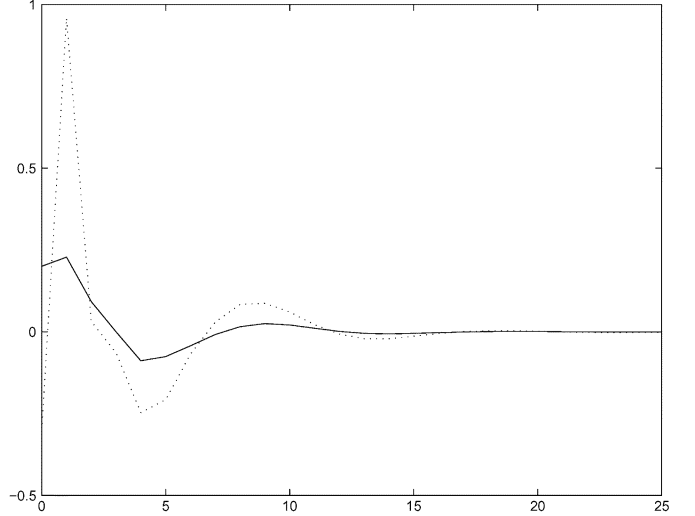


Fig. 1. State response of $x_1(k)$ (—) and $x_2(k)$ (···).

Then, using the Matlab LMI Control Toolbox to solve the LMIs in (48) and (49), we obtain the solution as follows:

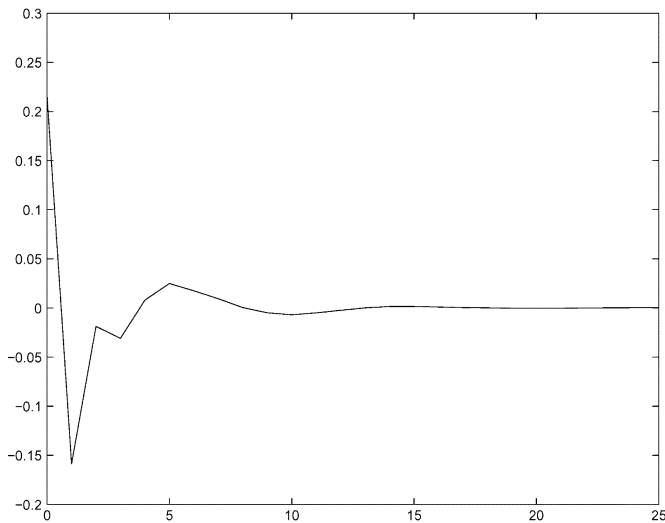
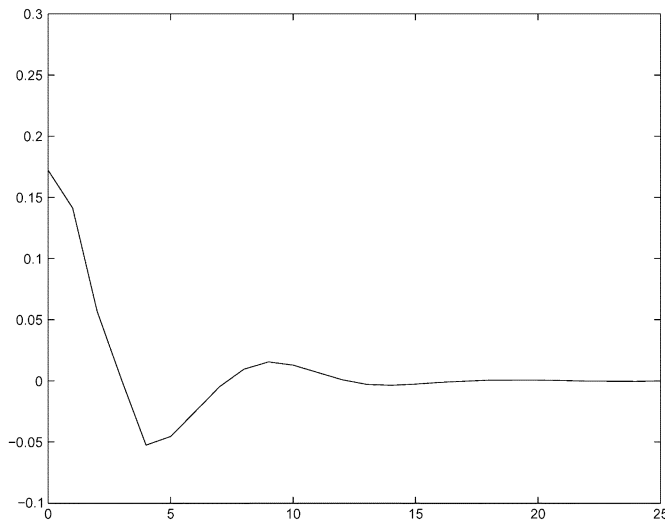
$$\begin{aligned} X &= \begin{bmatrix} 2.0126 & -0.7162 \\ -0.7162 & 1.2392 \end{bmatrix} & Y &= \begin{bmatrix} 3.2210 & 0.2170 \\ 0.2170 & 3.9002 \end{bmatrix} \\ \Omega_1 &= \begin{bmatrix} 0.2384 & 0.1137 \\ -0.0989 & 0.3758 \end{bmatrix} \\ \Omega_2 &= \begin{bmatrix} -0.0997 & -0.4315 \\ -0.0040 & 0.3465 \end{bmatrix} & \Phi_1 &= \begin{bmatrix} -0.2762 \\ -1.1165 \end{bmatrix} \\ \Phi_2 &= \begin{bmatrix} 0.8337 \\ -1.4828 \end{bmatrix} \\ \Psi_1 &= \begin{bmatrix} -0.6124 & -1.2324 \\ -0.9050 & -7.1605 \end{bmatrix} \\ \Psi_2 &= \begin{bmatrix} 0.6401 & -1.2887 \\ -0.8744 & -7.1387 \end{bmatrix}. \end{aligned}$$

Now, choose

$$S = \begin{bmatrix} 1 & 1 \\ 8 & -5 \end{bmatrix} \quad W = \begin{bmatrix} -1.8922 & -3.4350 \\ 0.6235 & 1.7331 \end{bmatrix}. \quad (63)$$

It is easy to verify that the matrices S and W in (63) satisfy the equality in (52); therefore, by Theorem 2, a desired fuzzy output feedback controller can be constructed as in (12) and (13) with

$$\begin{aligned} A_{K1} &= \begin{bmatrix} 0.0666 & 0.0487 \\ 1.0122 & -0.2629 \end{bmatrix} \\ A_{K2} &= \begin{bmatrix} -0.5355 & 0.2276 \\ -0.2573 & 0.1555 \end{bmatrix} \\ B_{K1} &= \begin{bmatrix} -0.1921 \\ -0.0841 \end{bmatrix} & B_{K2} &= \begin{bmatrix} 0.2066 \\ 0.6271 \end{bmatrix} \\ C_{K1} &= \begin{bmatrix} 4.6541 & -2.3854 \\ 22.9993 & -12.4055 \end{bmatrix} \\ C_{K2} &= \begin{bmatrix} 2.9160 & -1.7926 \\ 22.8867 & -12.3524 \end{bmatrix}. \end{aligned}$$

Fig. 2. Control input $u(k)$.Fig. 3. Measured output $y(k)$.

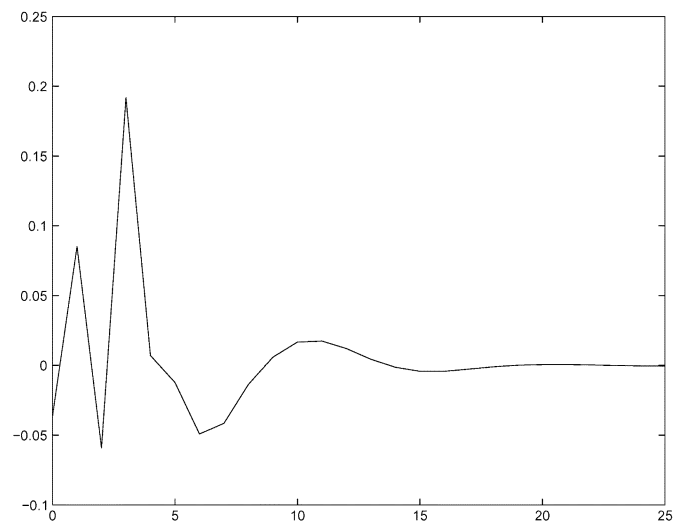
With the aforementioned fuzzy controller, the simulation results of the state response of the nonlinear system are given in Fig. 1, where the initial conditions are set as

$$\begin{aligned} x(-2) &= [0.5 \quad -0.2]^T & x(-1) &= [0.1 \quad 0.4]^T \\ x(0) &= [0.2 \quad -0.3]^T \end{aligned}$$

and the exogenous disturbance input $\omega(k) \in l_2[0, \infty)$ is defined as

$$\omega(k) = \frac{r}{1 + 15(k + 1)}$$

where r is a random number taken from a uniform distribution over $[0, 2]$, and $c_1(k) = 0.2 \cos(k)$ and $c_2(k) = 0.1 \sin(k)$. Fig. 2 shows the control input, while Figs. 3 and 4 present the corresponding measured output and the controlled output, respectively. From these simulation results, it can be seen the designed fuzzy output feedback controller ensures the robust asymptotic stability of the closed-loop system and guarantees a prescribed H_∞ performance level.

Fig. 4. Controlled output $z(k)$.

V. CONCLUSION

The problem of robust output feedback H_∞ control for uncertain discrete T-S fuzzy systems with time-varying norm-bounded parameter uncertainties and time delays has been studied. A sufficient condition for the existence of a full-order fuzzy dynamic output feedback controller, which robustly stabilizes the uncertain system and guarantees a prescribed level on disturbance attenuation, has been obtained. The design approach has been applied to the problem of robust H_∞ control of a class of nonlinear discrete delay systems, and the simulation results have showed the effectiveness of the proposed approach.

REFERENCES

- [1] C. Abdallah, P. Dorato, F. Pérez, and D. Docampo, "Controller synthesis for a class of interval plants," *Automatica*, vol. 31, pp. 341–343, 1995.
- [2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, ser. SIAM Studies in Applied Mathematics. Philadelphia, PA: SIAM, 1994.
- [3] Y.-Y. Cao and P. M. Frank, "Analysis and synthesis of nonlinear time-delay systems via fuzzy control approach," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 2, pp. 200–211, Apr. 2000.
- [4] —, "Robust H_∞ disturbance attenuation for a class of uncertain discrete-time fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 4, pp. 406–415, Aug. 2000.
- [5] —, "Stability analysis and synthesis of nonlinear time-delay systems via linear Takagi-Sugeno fuzzy models," *Fuzzy Sets Syst.*, vol. 124, pp. 213–229, 2001.
- [6] H. H. Choi and M. J. Chung, "An LMI approach to H_∞ controller design for linear time-delay systems," *Automatica*, vol. 33, pp. 737–739, 1997.
- [7] S. H. Eshfahani and I. R. Petersen, "An LMI approach to output-feedback-guaranteed cost control for uncertain time-delay systems," *Int. J. Robust Nonlinear Control*, vol. 10, pp. 157–174, 2000.
- [8] J. K. Hale, *Theory of Functional Differential Equations*. New York: Springer-Verlag, 1977.
- [9] E. T. Jeung, J. H. Kim, and H. B. Park, " H_∞ -output feedback controller design for linear systems with time-varying delayed state," *IEEE Trans. Autom. Control*, vol. 43, no. 7, pp. 971–974, Jul. 1998.
- [10] Y.-H. Joo, L.-S. Shieh, and G. Chen, "Hybrid state-space fuzzy model-based controller with dual-rate sampling for digital control of chaotic systems," *IEEE Trans. Fuzzy Syst.*, vol. 7, no. 4, pp. 394–408, Aug. 1999.
- [11] H. J. Lee, J. B. Park, and G. Chen, "Robust fuzzy control of nonlinear systems with parametric uncertainties," *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 2, pp. 369–379, Apr. 2001.

- [12] K. R. Lee, E. T. Jeung, and H. B. Park, "Robust fuzzy H_∞ control for uncertain nonlinear systems via state feedback: An LMI approach," *Fuzzy Sets Syst.*, vol. 120, pp. 123–134, 2001.
- [13] K. R. Lee, J. H. Kim, E. T. Jeung, and H. B. Park, "Output feedback robust H_∞ control of uncertain fuzzy dynamic systems with time-varying delay," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 6, pp. 657–664, Dec. 2000.
- [14] E. H. Mamdani and S. Assilian, "Applications of fuzzy algorithms for control of simple dynamic plant," *Proc. Inst. Elect. Eng. Control Theory Appl.*, vol. 121, pp. 1585–1588, 1974.
- [15] M. Sugeno and M. Nishida, "Fuzzy control of model car," *Fuzzy Sets Syst.*, vol. 16, pp. 103–113, 1985.
- [16] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-15, no. 1, pp. 116–132, Jan./Feb. 1985.
- [17] K. Tanaka, T. Ikeda, and H. O. Wang, "Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: Quadratic stabilizability, H_∞ control theory, and linear matrix inequalities," *IEEE Trans. Fuzzy Syst.*, vol. 4, no. 1, pp. 1–13, Feb. 1996.
- [18] —, "Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs," *IEEE Trans. Fuzzy Syst.*, vol. 6, no. 2, pp. 250–265, May 1998.
- [19] K. Tanaka and M. Sugeno, "Stability analysis and design of fuzzy control systems," *Fuzzy Sets Syst.*, vol. 45, pp. 135–156, 1992.
- [20] K. Tanaka and H. O. Wang, *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*. New York: Wiley, 2001.
- [21] H. O. Wang, K. Tanaka, and M. F. Griffin, "An approach to fuzzy control of nonlinear systems: Stability and design issues," *IEEE Trans. Fuzzy Syst.*, vol. 4, no. 1, pp. 14–23, Feb. 1996.
- [22] Y. Wang, L. Xie, and C. E. de Souza, "Robust control of a class of uncertain nonlinear systems," *Syst. Control Lett.*, vol. 19, pp. 139–149, 1992.
- [23] L. K. Wong, F. H. F. Leung, and P. K. S. Tam, "Fuzzy model-based controller for inverted pendulum," *Electron. Lett.*, vol. 32, pp. 1683–1685, 1996.
- [24] L. Xie and C. E. de Souza, "Robust H_∞ control for linear systems with norm-bounded time-varying uncertainty," *IEEE Trans. Autom. Control*, vol. 37, no. 8, pp. 1188–1191, Aug. 1992.
- [25] S. Xu, J. Lam, and C. Yang, " H_∞ and positive real control for linear neutral delay systems," *IEEE Trans. Autom. Control*, vol. 46, no. 8, pp. 1321–1326, Aug. 2001.
- [26] —, "Quadratic stability and stabilization of uncertain linear discrete-time systems with state delay," *Syst. Control Lett.*, vol. 43, pp. 77–84, 2001.
- [27] Z. Yi and P. A. Heng, "Stability of fuzzy control systems with bounded uncertain delays," *IEEE Trans. Fuzzy Syst.*, vol. 10, pp. 92–97, 2002.
- [28] Y. M. Zhang and R. Kovacevic, "Neurofuzzy model-based predictive control of weld fusion zone geometry," *IEEE Trans. Fuzzy Syst.*, vol. 6, pp. 389–401, 1998.
- [29] Y. M. Zhang, E. Liguio, and B. Walcott, "Robust control of pulsed gas metal arc welding," *ASME J. Dyna. Syst., Meas., Control*, vol. 124, pp. 281–289, 2002.



Shengyuan Xu received the B.Sc. degree from Hangzhou Normal University, Hangzhou, China, the M.Sc. degree from Qufu Normal University, Qufu, China, and the Ph.D. degree from Nanjing University of Science and Technology, Nanjing, China, in 1990, 1996, and 1999, respectively.

From 1999 to 2000, he was a Research Associate in the Department of Mechanical Engineering, University of Hong Kong, Hong Kong. From December 2000 to November 2001 and from December 2001 to September 2002, he was a Postdoctoral Researcher in CESAME at the Université Catholique de Louvain, Louvain-la-Neuve, Belgium, and the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB, Canada, respectively. From September 2002 to August 2003, he was a William Mong Young Researcher in the Department of Mechanical Engineering, University of Hong Kong, Hong Kong. He is a Professor and Ph.D. Supervisor in the Department of Automation, Nanjing University of Science and Technology, Nanjing, China, and an Honorary Associate Professor in the Department of Mechanical Engineering, University of Hong Kong, Hong Kong. His current research interests include robust filtering and control, singular systems, time-delay systems, multidimensional systems, and nonlinear systems. He has published more than 40 papers in international scientific journals.

Dr. Xu was a recipient of the 2002 National Excellent Doctoral Dissertation Award from the National Education Commission of China.



James Lam received a first-class B.Sc. degree in mechanical engineering from the University of Manchester, Manchester, U.K., in 1983, and the M.Phil. and Ph.D. degrees in the area of control engineering from the University of Cambridge, Cambridge, U.K., in 1985 and 1988, respectively. His postdoctoral research was carried out at the Australian National University between 1990 and 1992.

He has held faculty positions at the City University of Hong Kong, Hong Kong, and the University of Melbourne, Melbourne, Australia. He is now an Associate Professor in the Department of Mechanical Engineering, the University of Hong Kong, and is holding a Concurrent Professorship at the Northeastern University, Guest Professorship at the Huazhong University of Science and Technology, Consulting Professorship at the South China University of Technology, and Guest Professorship of Shandong University, China. He is a Chartered Mathematician, a Fellow of the Institute of Mathematics and Its Applications (U.K.), a Member of the Institution of Electrical Engineers (U.K.). He is an Honorary Editor of the *IEE Proceedings: Control Theory and Applications* (from 2005), and an Associate Editor of the *Asian Journal of Control*, the *International Journal of Applied Mathematics and Computer Science*, and the *International Journal of Systems Science*. His research interests include model reduction, delay systems, descriptor systems, stochastic systems, multidimensional systems, robust control and filtering, fault detection, and reliable control.

Dr. Lam was awarded the Ashbury Scholarship, the A. H. Gibson Prize and the H. Wright Baker Prize for his academic performance. He is a Scholar (1984) and Fellow (1990) of the Croucher Foundation. He is on the Conference Editorial Board of the IEEE Control Systems Society.