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# Robust $H_2$ control of Markovian jump systems with uncertain switching probabilities

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This article deals with the robust $H_2$ control problem for a class of Markovian jump linear systems with uncertain switching probabilities. The uncertainties under consideration appear both in the system parameters and in the mode transition rates. First, a new criterion based on linear matrix inequalities is established for checking the robust $H_2$ performance of the uncertain system. Then, a sufficient condition for the existence of the state-feedback controllers is established such that the closed-loop system is quadratically mean square stable and has a certain level of robust $H_2$ performance in terms of linear matrix inequalities with equality constraints. A globally convergent algorithm is also presented to construct such controllers effectively. Finally, an illustrative numerical example is used to demonstrate the developed theory.

**Keywords:** linear matrix inequalities (LMIs); Markovian parameters; robust $H_2$ control; uncertainties



## 1. Introduction

The objective of the robust $H_2$ control problem is to design a controller such that the resulting closed-loop system achieves a certain level of $H_2$ performance in spite of the system model uncertainties. The $H_2$ performance of a system may be regarded as a measure of the average response energy over impulsive inputs (Dullerud and Paganini 2000) and hence can be used to character the transient response performance of the system. However, a system with an extremely good $H_2$ performance for the nominal operation model could be very sensitive to the parameter uncertainties (Doyle 1978). On the other hand, a very robust controller may also tend to make the $H_2$ performance poor generally (Zhou, Doyle and Glover 1996). Thus, it is very natural to keep both the required $H_2$ performance and the desired robustness of the system in mind when designing controllers.

On the other hand, a great deal of attention has recently been devoted to the study of Markovian jump linear systems (MJLSs). This class of systems can model dynamic systems subject to random abrupt variations in their structures and have many applications (Mariton 1990; Mahmoud and Shi 2003). From a mathematical point of view, MJLSs are a special class of stochastic systems with system parameters changed randomly at discrete time points governed by a Markov process. A great number of control issues concerning the nominal systems have been investigated, such as stabilisation (Ji and Chizeck 1990; Feng et al. 1992; Yuan and Mao 2004), $H_2$ control (Costa, do Val and Geromel 1999; de Farias et al. 2000; do Val, Geromel and Goncalves 2002), $H_\infty$ control (de Farias et al. 2000; Cao, Lam and Hu 2003) and model reduction (Zhang, Huang and Lam 2003). As for MJLSs with uncertainties only in the system matrices, the issues of robust stabilisation (El Ghaoui and Rami 1996; Boukas, Shi and Benjelloun 1999), robust Kalman filtering (Shi, Boukas and Agarwal 1999) and robust $H_\infty$ control (Shi and Boukas 1997; Cao and Lam 2000) have also been well studied.

Moreover, the study of MJLSs with uncertain switching probabilities is of its own interest because these uncertainties can destabilise MJLSs or degrade their performance as the uncertainties in system matrices do (Xiong, Lam, Gao and Ho 2005). In the literature, two descriptions concerning the uncertain switching probabilities have been proposed. The first is the polytopic model (El Ghaoui and Rami 1996; Costa, do Val and Geromel 1999), where the mode transition rate matrix is assumed to be in a convex hull with known vertices. However, this approach often leads to too many linear matrix inequalities (LMIs) (Xiong et al. 2005). The other is the element-wise description (Shi and Boukas 1997; Boukas, Shi and Benjelloun 1999; Mahmoud and Shi 2003), where bounded uncertainties can appear in all the elements of the mode transition rate matrix. Recently, a modified element-wise description is addressed in Xiong et al. (2005), where the robust stability and robust

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stabilisation problems were investigated. In the current article, we study the robust $H_2$ control problem for MJLSs and adopt an improved bounding technique for the matrix inequalities which gives less conservative results than those in Shi and Boukas (1997); Boukas, Shi and Benjelloun (1999); Mahmoud and Shi (2003); and Xiong et al. (2005).

In this article, we consider the robust $H_2$ control problem for uncertain continuous-time MJLSs. The uncertainties are assumed to be norm-bounded in the system matrices and to be element-wise bounded in the mode transition rate matrix. We aim at designing a linear state-feedback controller such that, over all admissible uncertainties, the closed-loop system is quadratically mean square stable and the $H_2$ norm of the operator from the disturbance inputs to the regulated outputs is no more than a prescribed upper bound. The solution to the addressed problem is related to a set of coupled linear matrix inequalities with equality constraints and an effective algorithm (El Ghaoui, Oustry and Rami 1997; Leibfritz 2001) is suggested to construct the controller. Finally, a numerical example is offered to illustrate the usefulness of the proposed approach.



**Notation:** The notations in this article are standard. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the $n$-dimensional Euclidean space and the set of all $m\times n$ real matrices, respectively. $\mathbb{R}^+$ refers to the set of all strictly positive real numbers. $\mathbb{S}^{n\times n}$ is the set of all $n\times n$ real symmetric positive definite matrices and the notation $X\geq Y$ (respectively, $X>Y$) where $X$ and $Y$ are real symmetric matrices, means that $X-Y$ is positive semi-definite (respectively, positive definite). $I$ denotes the identity matrix with compatible dimensions. The superscript '$T$' stands for the transpose and trace$(\cdot)$ is the trace of a square matrix. $\|\cdot\|_2$ refers to the Euclidean norm for vectors and induced two-norm for matrices. Moreover, let $(\Omega,\mathcal{F},\mathrm{P})$ be a complete probability space. $\mathrm{E}(\cdot)$ stands for the mathematical expectation operator.

## 2. Problem formulation

Consider the following class of MJLSs with uncertain switching probabilities defined on a complete probability space $(\Omega,\mathcal{F},\mathrm{P})$:

$$\begin{cases} \dot{x}(t)=\hat{A}(\hat{r}(t))x(t)+\hat{B}(\hat{r}(t))u(t)+\hat{B}_w(\hat{r}(t))w(t) \\ z(t)=\hat{C}(\hat{r}(t))x(t)+\hat{D}(\hat{r}(t))u(t), \quad t\geq 0 \end{cases}, \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the system state, $u(t)\in\mathbb{R}^{n_u}$ is the control input and $w(t)\in\mathbb{R}^{n_w}$ is the disturbance input and $z(t)\in\mathbb{R}^{n_z}$ is the regulated output. The mode jumping process $\{\hat{r}(t)\colon t\geq 0\}$ is a continuous-time, discrete-state homogeneous Markov process on the probability space, takes values in a finite state space $\mathcal{S}\triangleq\{1,2,\ldots,s\}$ and has the mode transition probabilities

$$Pr(\hat{r}(t+\delta t)=j\mid r(t)=i)=\begin{cases}\hat{\pi}_{ij}\delta t+o(\delta t) & \text{if } j\neq i\\ 1+\hat{\pi}_{ii}\delta t+o(\delta t) & \text{if } j=i\end{cases},$$

where $\delta t>0$ and $\lim_{\delta t\to 0}(o(\delta t)/\delta t)=0$, $\hat{\pi}_{ij}\geq 0$, $(i,j\in\mathcal{S}, j\neq i)$, denotes the switching rate from mode $i$ to mode $j$ and $\hat{\pi}_{ii}\triangleq-\sum_{j=1,j\neq i}^{s}\hat{\pi}_{ij}$ for all $i\in\mathcal{S}$. The initial condition of the system state is $x_0\triangleq x(0)$ and the initial probability distribution of $\hat{r}_0\triangleq\hat{r}(0)$ is given by $\mu\triangleq(\mu_1,\ldots,\mu_s)$ in such a way that $Pr(\hat{r}_0=i)=\mu_i$ with $\mu_i\geq 0$, $i\in\mathcal{S}$ and $\sum_{i=1}^{s}\mu_i=1$. The matrices $\hat{A}_i\triangleq\hat{A}(\hat{r}(t)=i)$, $\hat{B}_i\triangleq\hat{B}(\hat{r}(t)=i)$, $\hat{B}_{wi}\triangleq\hat{B}_w(\hat{r}(t)=i)$, $\hat{C}_i\triangleq\hat{C}(\hat{r}(t)=i)$ and $\hat{D}_i\triangleq\hat{D}(\hat{r}(t)=i)$, $i\in\mathcal{S}$, are appropriately dimensioned constant real matrices for each operation mode $i\in\mathcal{S}$ and it is supposed that the system matrices $\hat{A}_i$, $\hat{B}_i$, $\hat{B}_{wi}$, $\hat{C}_i$, $\hat{D}_i$, $i\in\mathcal{S}$ and the mode transition rate matrix $\hat{\Pi}\triangleq(\hat{\pi}_{ij})\in\mathbb{R}^{s\times s}$ are not precisely known *a priori*, but belong to the following uncertainty domains, respectively:

$$\mathcal{D}_a\triangleq\{\hat{A}_i=A_i+E_{ai}F_{ai}H_{ai}\colon F_{ai}^TF_{ai}\leq I,\quad \text{for all } i\in\mathcal{S}\} \tag{2a}$$

$$\mathcal{D}_b\triangleq\{\hat{B}_i=B_i+E_{ai}F_{ai}H_{bi}\colon F_{ai}^TF_{ai}\leq I,\quad \text{for all } i\in\mathcal{S}\} \tag{2b}$$

$$\mathcal{D}_{bw}\triangleq\{\hat{B}_{wi}=B_{wi}+E_{bwi}F_{bwi}H_{bwi}\colon F_{bwi}^TF_{bwi}\leq I, \text{ for all } i\in\mathcal{S}\} \tag{2c}$$

$$\mathcal{D}_c\triangleq\{\hat{C}_i=C_i+E_{ci}F_{ci}H_{ci}\colon F_{ci}^TF_{ci}\leq I,\quad \text{for all } i\in\mathcal{S}\} \tag{2d}$$

$$\mathcal{D}_d\triangleq\{\hat{D}_i=D_i+E_{ci}F_{ci}H_{di}\colon F_{ci}^TF_{ci}\leq I,\quad \text{for all } i\in\mathcal{S}\} \tag{2e}$$

$$\mathcal{D}_\pi\triangleq\{\hat{\Pi}=\Pi+\Delta\Pi\colon|\Delta\pi_{ij}|\leq 2\varepsilon_{ij},\ \varepsilon_{ij}\geq 0, \text{ for all } i,j\in\mathcal{S}, \text{ if } j\neq i\} \tag{2f}$$

where matrices $A_i$, $B_i$, $B_{wi}$, $C_i$, $D_i$, $E_{ai}$, $H_{ai}$, $H_{bi}$, $E_{bwi}$, $H_{bwi}$, $E_{ci}$, $H_{ci}$, $H_{di}$, $(i\in\mathcal{S})$ and $\Pi\triangleq q(\pi_{ij})$ are known constant real matrices of appropriate dimensions. The matrices $F_{ai}$, $F_{bwi}$, $F_{ci}$ and $\Delta\Pi\triangleq(\Delta\pi_{ij})$ denote the uncertainties in the system matrices and the mode transition rate matrix, respectively. Moreover, $\pi_{ij}\,(\geq 0)$ denotes the estimated value of $\hat{\pi}_{ij}$ and $\Delta\pi_{ij}\triangleq\hat{\pi}_{ij}-\pi_{ij}$ is referred to as switching probability uncertainty and can take any value in $[-2\varepsilon_{ij},2\varepsilon_{ij}]$ for all $i,j\in\mathcal{S}$, $j\neq i$. For all $i\in\mathcal{S}$, we have $\pi_{ii}\triangleq-\sum_{j=1,j\neq i}^{s}\pi_{ij}$ and $\Delta\pi_{ii}\triangleq-\sum_{j=1,j\neq i}^{s}\Delta\pi_{ij}$.

Let $x(t;x_0,\hat{r}_0)$ be the trajectory of the system state of (1) from the initial system state $x_0\in\mathbb{R}^n$ and the initial operation mode $\hat{r}_0\in\mathcal{S}$, we have the following definition and result on the stochastic stability for the nominal Markovian jump system of (1).

**Definition 1** (de Farias et al. 2000)**:** The nominal Markovian jump system of (1) with $u(t) \equiv 0$ and $w(t) \equiv 0$ is said to be mean square stable if

$$\lim_{t\to\infty} \mathrm{E}(\|x(t; x_0, \hat{r}_0)\|_2^2) = 0$$

for any initial conditions $x_0 \in \mathbb{R}^n$ and initial distribution for $\hat{r}_0 \in \mathcal{S}$.

**Proposition 1** (de Farias et al. 2000)**:** *The nominal Markovian jump system of* (1) *with* $u(t) \equiv 0$ *and* $w(t) \equiv 0$ *is mean square stable if, and only if, the coupled linear matrix inequalities*

$$A_i^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_j < 0, \quad \text{for all } i \in \mathcal{S} \tag{3}$$

*are feasible for matrices* $P_i \in \mathbb{S}^{n\times n}$, $i \in \mathcal{S}$.

The next definition generalises the $H_2$-norm concept from continuous-time deterministic systems to the stochastic Markovian jump case.

**Definition 2** (Costa, do Val and Geromel 1999)**:** Consider nominal Markovian jump system of (1) with $u(t) \equiv 0$, let $G_{zw}$ denote the operator from $w(t)$ to $z(t)$, the $H_2$-norm of the operator $G_{zw}$ is defined as

$$\|G_{zw}\|_2^2 \triangleq \sum_{k=1}^{n_w} \sum_{i=1}^{s} \mu_i \|z_{k,i}\|_2^2,$$

where $z_{k,i}$ represents the output given by (1) when

(a) $w(t) = e_k \delta(t)$, $\delta(t)$ is the unit impulse and $e_k$ is the $n_w$-dimensional unit vector formed by 1 at the $k$th position and zeros elsewhere and
(b) $x_0 = 0$ and $\hat{r}_0 = i \in \mathcal{S}$ with probability distribution $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$.

The following proposition shows that the $H_2$ performance of the nominal system of (1) can be calculated precisely in terms of a set of coupled linear matrix equations.

**Proposition 2** (Costa, do Val and Geromel 1999)**:** *The nominal Markovian jump system of* (1) *with* $u(t) \equiv 0$ *is mean square stable and has* $H_2$ *performance*

$$\|G_{zw}\|_2^2 = \sum_{i=1}^{s} \mu_i \,\mathrm{trace}(B_{wi}^T P_i B_{wi}) \tag{4}$$

*if the coupled linear matrix equations*

$$A_i^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_j + C_i^T C_i = 0, \quad \textit{for all } i \in \mathcal{S} \tag{5}$$

*have a unique solution* $P_i \in \mathbb{S}^{n\times n}$, $i \in \mathcal{S}$.

Based on Proposition 2, we introduce the following definition for uncertain system (1).

**Definition 3:** For a prescribed scalar $\gamma_{H_2} \in \mathbb{R}^+$, uncertain MJLS (1) with $u(t) \equiv 0$ is said to be quadratically mean square stable and has robust $H_2$ performance $\|G_{zw}\|_2 < \gamma_{H_2}$ if there exist matrices $P_i \in \mathbb{S}^{n\times n}$, $i \in \mathcal{S}$, such that the coupled linear matrix inequalities

$$\sum_{i=1}^{s} \mu_i \,\mathrm{trace}(\hat{B}_{wi}^T P_i \hat{B}_{wi}) < \gamma_{H_2}^2 \tag{6}$$

$$\hat{A}_i^T P_i + P_i \hat{A}_i + \sum_{j=1}^{s} \hat{\pi}_{ij} P_j + \hat{C}_i^T \hat{C}_i < 0, \quad \text{for all } i \in \mathcal{S} \tag{7}$$

hold over all admissible uncertainty domains (2).

Now, consider the state-feedback control law

$$u(t) = K(\hat{r}(t))x(t) \tag{8}$$

where $K_i \triangleq K(\hat{r}(t) = i) \in \mathbb{R}^{n_u \times n}$ $(i \in \mathcal{S})$ is the controller to be designed. Substituting the state-feedback controller (8) into system (1) yields the corresponding closed-loop system

$$\begin{cases} \dot{x}(t) = \hat{A}_{\mathrm{cl}}(\hat{r}(t))x(t) + \hat{B}_w(\hat{r}(t))w(t) \\ z(t) = \hat{C}_{\mathrm{cl}}(\hat{r}(t))x(t), \quad t \geq 0 \end{cases} \tag{9}$$

where $\hat{A}_{\mathrm{cl}i} = (A_i + B_i K_i) + E_{ai} F_{ai} (H_{ai} + H_{bi} K_i)$ and $\hat{C}_{\mathrm{cl}i} = (C_i + D_i K_i) + E_{ci} F_{ci} (H_{ci} + H_{di} K_i)$, $i \in \mathcal{S}$.

The problems of robust $H_2$ performance analysis and synthesis for uncertain Markovian jump system (1) will be explored based on linear matrix inequality machinery.

To obtain the main results of this article, the following lemmas will be used.

**Lemma 1** (Xie 1996)**:** *Given real matrices* $Q$, $E$ *and* $H$ *of appropriate dimensions with* $Q = Q^T$, *then*

$$Q + EFH + (EFH)^T < 0$$

*for all* $F$ *satisfying* $F^T F \leq I$ *if, and only if, there exists some real number* $\lambda \in \mathbb{R}^+$ *such that*

$$Q + \lambda H^T H + \frac{1}{\lambda} E E^T < 0.$$

**Lemma 2:** *Given any real number* $\varepsilon \in \mathbb{R}$ *and any square matrix* $Q \in \mathbb{R}^{n\times n}$, *the matrix inequality*

$$\varepsilon(Q + Q^T) \leq \varepsilon^2 T + Q T^{-1} Q^T$$

*holds for any matrix* $T \in \mathbb{S}^{n\times n}$.

**Proof:** The proof follows from the inequality

$$\begin{aligned} 0 &\leq (\varepsilon T^{(1/2)} - Q T^{-(1/2)})(\varepsilon T^{(1/2)} - Q T^{-(1/2)})^T \\ &= \varepsilon^2 T + Q T^{-1} Q^T - \varepsilon(Q + Q^T) \end{aligned}$$

immediately. □

In order to simplify the proof of the main results, we present the following lemmas.

**Lemma 3:** *Given real matrices $Q$, $P$, $D$, $E$ and $H$ of appropriate dimensions with $Q = Q^T < 0$ and $P = P^T > 0$, then*

$$Q + (D + EFH)^T P(D + EFH) < 0 \tag{10}$$

*holds for all $F$ satisfying $F^T F \leq I$ if, and only if, one of the following conditions holds*:

(a) *there exists some real number $\lambda \in \mathbb{R}^+$ such that*

$$\begin{bmatrix} Q + \lambda H^T H + D^T PD & D^T PE \\ E^T PD & -\lambda I + E^T PE \end{bmatrix} < 0; \tag{11}$$

(b) *there exists some real number $\alpha \in \mathbb{R}^+$ such that*

$$\begin{bmatrix} Q & D^T & H^T \\ D & -P^{-1} + \alpha EE^T & 0 \\ H & 0 & -\alpha I \end{bmatrix} < 0. \tag{12}$$

**Proof:** We first prove part (a), in view of Schur complement equivalence, inequality (10) is equivalent to

$$\begin{bmatrix} Q & (D + EFH)^T \\ D + EFH & -P^{-1} \end{bmatrix} < 0$$

which can be rewritten as

$$\begin{bmatrix} Q & D^T \\ D & -P^{-1} \end{bmatrix} + \begin{bmatrix} 0 \\ E \end{bmatrix} F \begin{bmatrix} H & 0 \end{bmatrix} + \begin{bmatrix} H^T \\ 0 \end{bmatrix} F^T \begin{bmatrix} 0 & E^T \end{bmatrix} < 0.$$

Using Lemma 1, the above inequality holds for all $F$ satisfying $F^T F \leq I$ if, and only if, there exists a real number $\lambda \in \mathbb{R}^+$ such that

$$\begin{bmatrix} Q + \lambda H^T H & D^T \\ D & -P^{-1} + \frac{1}{\lambda} EE^T \end{bmatrix} < 0.$$

By applying Schur complement equivalence again, we conclude that the above inequality is equivalent to

$$\begin{bmatrix} Q + \lambda H^T H & D^T & 0 \\ D & -P^{-1} & E \\ 0 & E^T & -\lambda I \end{bmatrix} < 0$$

Pre- and post-multiply both sides of the above inequality by

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}$$

we have

$$\begin{bmatrix} Q + \lambda H^T H & 0 & D^T \\ 0 & -\lambda I & E^T \\ D & E & -P^{-1} \end{bmatrix} < 0$$

which is equivalent to (11) in view of Schur complement equivalence. This completes the proof of part (a). To prove part (b), define $\alpha \triangleq (1/\lambda)$; we have inequality (11) is equivalent to inequality (12) by Schur complement equivalence. This completes the proof. □

## 3. Robust $H_2$ control

In the section, the robust $H_2$ performance analysis problem is addressed first in terms of coupled linear matrix inequalities, then the associated synthesis problem is dealt with in terms of the solvability of a set of coupled linear matrix inequalities with equality constraints, which can be solved using the sequential linear programming method developed in Leibfritz (2001).

### 3.1. *Robust $H_2$ performance analysis*

The goal of this section is to develop a criterion for testing the robust $H_2$ performance of the uncertain Markovian jump system (1) over the uncertainty domains in (2). This criterion is stated in the following theorem in terms of coupled linear matrix inequalities.

**Theorem 1:** *For a prescribed scalar $\gamma_{H_2} \in \mathbb{R}^+$, uncertain Markovian jump system (1) with $u(t) \equiv 0$ is quadratically mean square stable and satisfies $\|G_{zw}\|_2 < \gamma_{H_2}$ over all the uncertainty domains in (2) if there exist matrices $P_i \in \mathbb{S}^{n \times n}$, $T_{ij} \in \mathbb{S}^{n \times n}$, $W_i \in \mathbb{S}^{n_w \times n_w}$ and scalars $\lambda_{ai} \in \mathbb{R}^+$, $\lambda_{bwi} \in \mathbb{R}^+$, $\lambda_{ci} \in \mathbb{R}^+$, $i, j \in \mathcal{S}$, $j \neq i$, such that the coupled linear matrix inequalities*

$$\sum_{i=1}^{s} \mu_i \operatorname{trace}(W_i) < \gamma_{H_2}^2 \tag{13}$$

$$\begin{bmatrix} -W_i + \lambda_{bwi} H_{bwi}^T H_{bwi} + B_{wi}^T P_i B_{wi} & B_{wi}^T P_i E_{bwi} \\ E_{bwi}^T P_i B_{wi} & -\lambda_{bwi} I + E_{bwi}^T P_i E_{bwi} \end{bmatrix} < 0, \quad \textit{for all } i \in \mathcal{S} \tag{14}$$

$$\begin{bmatrix} Q_{1i} & C_i^T E_{ci} & P_i E_{ai} & M_{1i} \\ E_{ci}^T C_i & -\lambda_{ci} I + E_{ci}^T E_{ci} & 0 & 0 \\ E_{ai}^T P_i & 0 & -\lambda_{ai} I & 0 \\ M_{1i}^T & 0 & 0 & -\Lambda_{1i} \end{bmatrix} < 0, \quad \textit{for all } i \in \mathcal{S} \tag{15}$$

*hold, where*

$$Q_{1i} = A_i^T P_i + P_i A_i + C_i^T C_i + \sum_{j=1}^{s} \pi_{ij} P_j + \sum_{j=1, j\neq i}^{s} \varepsilon_{ij}^2 T_{ij} + \lambda_{ai} H_{ai}^T H_{ai} + \lambda_{ci} H_{ci}^T H_{ci}$$

$$M_{1i} = \begin{bmatrix} P_i - P_1 & P_i - P_2 \cdots P_i - P_{i-1} \\ & P_i - P_{i+1} \cdots P_i - P_s \end{bmatrix}$$

$$\Lambda_{1i} = \mathrm{diag}(T_{i1}, T_{i2}, \ldots, T_{i(i-1)}, T_{i(i+1)}, \ldots, T_{is}).$$

**Proof:** According to Definition 3, inequality (6) holds if and only if there exist matrices $W_i \in \mathbb{S}^{n_w \times n_w}$, $i \in \mathcal{S}$, such that (13) and

$$\hat{B}_{wi}^T P_i \hat{B}_{wi} < W_i$$

hold. Note that $\hat{B}_{wi} = B_{wi} + E_{bwi} F_{bwi} H_{bwi}$, the above inequality is

$$-W_i + (B_{wi} + E_{bwi} F_{bwi} H_{bwi})^T P_i (B_{wi} + E_{bwi} F_{bwi} H_{bwi}) < 0. \quad (16)$$

Applying part (a) of Lemma 3, the above inequality holds for all $F_{bwi}$ satisfying $F_{bwi}^T F_{bwi} \leq I$ if and only if, there exists a real number $\lambda_{bwi} \in \mathbb{R}^+$ such that (14) holds.

On the other hand, because of $\hat{\pi}_{ij} = \pi_{ij} + \Delta\pi_{ij}$ and $\Delta\pi_{ii} = -\sum_{j=1, j\neq i}^{s} \Delta\pi_{ij}$, in view of Lemma 2, we have

$$\begin{aligned} \sum_{j=1}^{s} \Delta\pi_{ij} P_j &= \sum_{j=1, j\neq i}^{s} \Delta\pi_{ij}(P_j - P_i) \\ &= \sum_{j=1, j\neq i}^{s} \left[\frac{1}{2}\Delta\pi_{ij}(P_j - P_i) + \frac{1}{2}\Delta\pi_{ij}(P_j - P_i)\right] \\ &\leq \sum_{j=1, j\neq i}^{s} \left[(\frac{1}{2}\Delta\pi_{ij})^2 T_{ij} + (P_i - P_j) T_{ij}^{-1} (P_i - P_j)\right] \\ &\leq \sum_{j=1, j\neq i}^{s} \left[\varepsilon_{ij}^2 T_{ij} + (P_i - P_j) T_{ij}^{-1} (P_i - P_j)\right] \end{aligned}$$

holds for any matrix $T_{ij} \in \mathbb{S}^{n\times n}$, $i, j \in \mathcal{S}$, $j \neq i$. Hence, inequality (7) holds if

$$\hat{A}_i^T P_i + P_i \hat{A}_i + \sum_{j=1}^{s} \pi_{ij} P_j + \sum_{j=1, j\neq i}^{s} \left[\varepsilon_{ij}^2 T_{ij} + (P_i - P_j) T_{ij}^{-1} (P_i - P_j)\right] + \hat{C}_i^T \hat{C}_i < 0.$$

Note that $\hat{A}_i = A_i + E_{ai} F_{ai} H_{ai}$ and $\hat{C}_i = C_i + E_{ci} F_{ci} H_{ci}$, according to Lemma 1, the above inequality holds for all $F_{ai}$ satisfying $F_{ai}^T F_{ai} \leq I$ if and only if there exists a real number $\lambda_{ai} \in \mathbb{R}^+$, such that

$$L_{1i} + (C_i + E_{ci} F_{ci} H_{ci})^T (C_i + E_{ci} F_{ci} H_{ci}) < 0 \quad (17)$$

where

$$L_{1i} = A_i^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_j + \lambda_{ai} H_{ai}^T H_{ai} + \frac{1}{\lambda_{ai}} P_i E_{ai} E_{ai}^T P_i + \sum_{j=1, j\neq i}^{s} \left[\varepsilon_{ij}^2 T_{ij} + (P_i - P_j) T_{ij}^{-1} (P_i - P_j)\right].$$

In view of part (a) of Lemma 3 again, we would conclude that inequality (17) holds for all $F_{ci}$ satisfying $F_{ai}^T F_{ai} \leq I$ if and only if there exists a real number $\lambda_{ci} \in \mathbb{R}^+$ such that

$$\begin{bmatrix} L_{1i} + \lambda_{ci} H_{ci}^T H_{ci} + C_i^T C_i & C_i^T E_{ci} \\ E_{ci}^T C_i & -\lambda_{ci} I + E_{ci}^T E_{ci} \end{bmatrix} < 0$$

which is equivalent to (15) by Schur complement equivalence. This completes the proof. □

In the following remarks, we provide a comparison of the results in (Shi and Boukas 1997; Boukas, Shi and Benjelloun 1999; Mahmoud and Shi 2003) and the current article.

**Remark 1:** The model of the uncertain mode transition rate matrix considered in Shi and Boukas (1997); Boukas, Shi and Benjelloun (1999); Mahmoud and Shi (2003) is of the form

$$\mathcal{D}'_\pi \triangleq \{\hat{\Pi} = \Pi + \Delta\Pi : |\Delta\pi_{ij}| \leq 2\varepsilon_{ij}, \quad \varepsilon_{ij} \geq 0, \text{ for all } i, j \in \mathcal{S}\} \quad (18)$$

A crucial difference between (18) and (2f) is that $\varepsilon_{ii}$ is undefined in (2f) for all $i \in \mathcal{S}$ because we have considered the probability constraint $\sum_{j=1}^{s} \Delta\pi_{ij} = 0$ to ensure $\sum_{j=1}^{s}(\pi_{ij} + \Delta\pi_{ij}) = 0$, which implies $\varepsilon_{ii} = \sum_{j=1, j\neq i}^{s} \varepsilon_{ij}$, for all $i \in \mathcal{S}$.

Based upon Remark 1, we can prove that our technique adopted in Theorem 1 gives less conservative results than those in Shi and Boukas (1997); Boukas, Shi and Benjelloun (1999); Mahmoud and Shi (2003) to deal with the element-wise uncertainties.

**Remark 2:** Suppose there do exist uncertainties, that is, at least one $\varepsilon_{ij} > 0, j \neq i$. The bounding technique for the matrix inequalities used in Shi and Boukas (1997); Boukas, Shi and Benjelloun (1999); Mahmoud and Shi (2003) is

$$\sum_{j=1}^{s} \Delta\pi_{ij} P_j \leq \sum_{j=1}^{s} 2\varepsilon_{ij} P_j = \sum_{j=1, j\neq i}^{s} 2\varepsilon_{ij}(P_i + P_j).$$

The bounding technique used in this article is

$$\sum_{j=1}^{s} \Delta\pi_{ij} P_j = \sum_{j=1, j\neq i}^{s} \Delta\pi_{ij}(P_j - P_i) \leq \sum_{j=1, j\neq i}^{s} \left[\varepsilon_{ij}^2 T_{ij} + (P_i - P_j) T_{ij}^{-1} (P_i - P_j)\right]$$

for any $T_{ij}\in\mathbb{S}^{n\times n}$. Then for those $\varepsilon_{ij}>0$, $(j\neq i)$, we choose $T_{ij}=(1/\varepsilon_{ij})(P_i+P_j)$ and have

$$\begin{aligned}
&\varepsilon_{ij}^2T_{ij}+(P_i-P_j)T_{ij}^{-1}(P_i-P_j)\\
&=\varepsilon_{ij}(P_i+P_j)+\varepsilon_{ij}(P_i+P_j-2P_j)(P_i+P_j)^{-1}(P_i+P_j-2P_j)\\
&=\varepsilon_{ij}(P_i+P_j)+\varepsilon_{ij}[P_i+P_j+4P_j(P_i+P_j)^{-1}P_j-4P_j]\\
&=2\varepsilon_{ij}(P_i+P_j)+4\varepsilon_{ij}P_j[(P_i+P_j)^{-1}-P_j^{-1}]P_j\\
&<2\varepsilon_{ij}(P_i+P_j)
\end{aligned}$$

For those $\varepsilon_{ij}=0$, $(j\neq i)$, we choose $T_{ij}=(1/\alpha)I$ with $\alpha\in\mathbb{R}^+$ sufficiently small, such that

$$\sum_{j=1,j\neq i}^{s}\left[\varepsilon_{ij}^2T_{ij}+(P_i-P_j)T_{ij}^{-1}(P_i-P_j)\right]<\sum_{j=1,j\neq i}^{s}2\varepsilon_{ij}(P_i+P_j)=\sum_{j=1}^{s}2\varepsilon_{ij}P_j.$$

That is, our result is less conservative than the one in Shi and Boukas (1997); Boukas, Shi and Benjelloun (1999); Mahmoud and Shi (2003) as long as there exist uncertainties.

### 3.2. *Robust $H_2$ controller synthesis*

This section aims at designing a state-feedback controller (8) such that the closed-loop system (9) is quadratically mean square stable and satisfies a prescribed level of $H_2$ performance. The following result provides a solution to the *robust $H_2$ control problem* ($RH_2P$) for the uncertain system (1) with uncertain switching probabilities in terms of coupled linear matrix inequalities and equality constraints.

**Theorem 2:** *Consider uncertain Markovian jump system* (1), *for a prescribed scalar $\gamma_{H_2}\in\mathbb{R}^+$, there exists a state-feedback controller* (8) *such that the closed-loop system* (9) *is quadratically mean square stable and has robust $H_2$ performance $\|G_{zw}\|_2<\gamma_{H_2}$ over all the uncertainty domains in* (2) *if there exist matrices $P_i\in\mathbb{S}^{n\times n}$, $X_i\in\mathbb{S}^{n\times n}$, $V_i\in\mathbb{S}^{n\times n}$, $Z_i\in\mathbb{S}^{n\times n}$, $T_{ij}\in\mathbb{S}^{n\times n}$, $W_i\in\mathbb{S}^{n_w\times n_w}$, $Y_i\in\mathbb{R}^{n_u\times n}$ and scalars $\alpha_{ai}\in\mathbb{R}^+$, $\alpha_{bwi}\in\mathbb{R}^+$, $\alpha_{ci}\in\mathbb{R}^+$, $i,\ j\in\mathcal{S}$, $j\neq i$, such that the coupled linear matrix inequalities*

$$\sum_{i=1}^{s}\mu_i\,\mathrm{trace}(W_i)<\gamma_{H_2}^2 \tag{19}$$

$$\begin{bmatrix}-W_i & B_{wi}^T & H_{bwi}^T\\ B_{wi} & -X_i+\alpha_{bwi}E_{bwi}E_{bwi}^T & 0\\ H_{bwi} & 0 & -\alpha_{bwi}I\end{bmatrix}<0 \tag{20}$$

$$\begin{bmatrix}Q_{2i} & (C_iX_i+D_iY_i)^T & (H_{ci}X_i+H_{di}Y_i)^T & (H_{ai}X_i+H_{bi}Y_i)^T & X_i\\ C_iX_i+D_iY_i & -I+\alpha_{ci}E_{ci}^TE_{ci} & 0 & 0 & 0\\ H_{ci}X_i+H_{di}Y_i & 0 & -\alpha_{ci}I & 0 & 0\\ H_{ai}X_i+H_{bi}Y_i & 0 & 0 & -\alpha_{ai}I & 0\\ X_i & 0 & 0 & 0 & -Z_i\end{bmatrix}<0 \tag{21}$$

$$\begin{bmatrix}Q_{3i} & M_{1i}\\ M_{1i}^T & -\Lambda_{1i}\end{bmatrix}\leq 0 \tag{22}$$

*with equality constraints*

$$P_iX_i=I,\quad V_iZ_i=I \tag{23}$$

*hold for all $i\in\mathcal{S}$, where*

$$Q_{2i}=(A_iX_i+B_iY_i)+(A_iX_i+B_iY_i)^T+\alpha_{ai}E_{ai}E_{ai}^T$$

$$Q_{3i}=-V_i+\sum_{j=1}^{s}\pi_{ij}P_j+\sum_{j=1,j\neq i}^{s}\varepsilon_{ij}^2T_{ij}$$

*and $M_{1i}$ and $\Lambda_{1i}$ are given in Theorem* 1. *In this case, a controller* (8) *is given by $K_i=Y_iP_i$, $i\in\mathcal{S}$.*

**Proof:** Firstly, in view of Lemma 3, we have that LMIs (13) and (14) are equivalent to LMIs (19) and (20) with $X_i\triangleq P_i^{-1}$ and $\alpha_{bwi}\triangleq(1/\lambda_{bwi})$, respectively. Next, consider the closed-loop system (9), let $\bar{A}_i\triangleq A_i+B_iK_i$, $\bar{C}_i\triangleq C_i+D_iK_i$, $\bar{H}_{ai}\triangleq H_{ai}+H_{bi}K_i$ and $\bar{H}_{ci}\triangleq H_{ci}+H_{di}K_i$, then replacing matrices $A_i$, $C_i$, $H_{ai}$, $H_{ci}$ in inequality (17) with matrices $\bar{A}_i$, $\bar{C}_i$, $\bar{H}_{ai}$, $\bar{H}_{ci}$, respectively, one has

$$\begin{aligned}
&\bar{A}_i^TP_i+P_i\bar{A}_i+\alpha_{ai}P_iE_{ai}E_{ai}^TP_i\\
&+\frac{1}{\alpha_{ai}}\bar{H}_{ai}^T\bar{H}_{ai}+(\bar{C}_i+E_{ci}F_{ci}\bar{H}_{ci})^T(\bar{C}_i+E_{ci}F_{ci}\bar{H}_{ci})\\
&+\sum_{j=1}^{s}\pi_{ij}P_j+\sum_{j=1,j\neq i}^{s}[\varepsilon_{ij}^2T_{ij}+(P_i-P_j)T_{ij}^{-1}(P_i-P_j)]<0
\end{aligned} \tag{24}$$

where $\alpha_{ai}\triangleq(1/\lambda_{ai})$. Now let $V_i\in\mathbb{S}^{n\times n}$ such that

$$\sum_{j=1}^{s}\pi_{ij}P_j+\sum_{j=1,j\neq i}^{s}\left[\varepsilon_{ij}^2T_{ij}+(P_i-P_j)T_{ij}^{-1}(P_i-P_j)\right]\leq V_i$$

which is equivalent to (22) in view of Schur complement equivalence and inequality (24) is equivalent to

$$\bar{A}_i^T P_i + P_i \bar{A}_i + V_i + \alpha_{ai} P_i E_{ai} E_{ai}^T P_i + \frac{1}{\alpha_{ai}} \bar{H}_{ai}^T \bar{H}_{ai} + (\bar{C}_i + E_{ci} F_{ci} \bar{H}_{ci})^T (\bar{C}_i + E_{ci} F_{ci} \bar{H}_{ci}) < 0.$$

Now, pre- and post-multiply both sides of the above inequality by $X_i$ and apply the changes of variables $Z_i \triangleq V_i^{-1}$ and $Y_i \triangleq K_i X_i$, and one obtains

$$L_{2i} + [(C_i X_i + D_i Y_i) + E_{ci} F_{ci} (H_{ci} X_i + H_{di} Y_i)]^T [(C_i X_i + D_i Y_i) + E_{ci} F_{ci} (H_{ci} X_i + H_{di} Y_i)] < 0$$

where

$$L_{2i} = (A_i X_i + B_i Y_i) + (A_i X_i + B_i Y_i)^T + X_i Z_i^{-1} X_i + \alpha_{ai} E_{ai} E_{ai}^T + \frac{1}{\alpha_{ai}} (H_{ai} X_i + H_{bi} Y_i)^T (H_{ai} X_i + H_{bi} Y_i).$$

According to part (b) of Lemma 3, the above inequality holds for all $F_{ci}$ satisfying $F_{ci}^T F_{ci} \leq I$ if and only if there exists a real number $\alpha_{ci} \in \mathbb{R}^+$ such that

$$\begin{bmatrix} L_{2i} & (C_i X_i + D_i Y_i)^T & (H_{ci} X_i + H_{di} Y_i)^T \\ C_i X_i + D_i Y_i & -I + \alpha_{ci} E_{ci} E_{ci}^T & 0 \\ H_{ci} X_i + H_{di} Y_i & 0 & -\alpha_{ci} I \end{bmatrix} < 0$$

which is equivalent to (21) in view of Schur complement equivalence. This completes the proof. □

In the case when the mode transition rate matrix is known exactly, we do not need to introduce the additional variables $V_i$, $Z_i$, $i \in \mathcal{S}$ and the equality constraints (23). The corresponding result is stated in the following corollary in terms of coupled linear matrix inequalities and can be proved similarly to that of Theorem 2. It should be noticed that the condition is necessary and sufficient since Lemma 2 is no longer needed in the proof.

**Corollary 1:** *Consider uncertain Markovian jump system* (1) *with mode transition rate matrix known exactly, for a prescribed scalar* $\gamma_{H_2} \in \mathbb{R}^+$, *there exists a state-feedback controller* (8) *such that the closed-loop system* (9) *is quadratically mean square stable and has robust* $H_2$ *performance* $\|G_{zw}\|_2 < \gamma_{H_2}$ *over all the uncertainty domains in* (2a)–(2e) *if, and only if, there exist matrices* $X_i \in \mathbb{S}^{n \times n}$, $W_i \in \mathbb{S}^{n_w \times n_w}$, $Y_i \in \mathbb{R}^{n_u \times n}$ *and scalars* $\alpha_{ai} \in \mathbb{R}^+$, $\alpha_{bwi} \in \mathbb{R}^+$, $\alpha_{ci} \in \mathbb{R}^+$, $i \in \mathcal{S}$, *such that the coupled linear matrix inequalities*

$$\sum_{i=1}^{s} \mu_i \operatorname{trace}(W_i) < \gamma_{H_2}^2$$

$$\begin{bmatrix} -W_i & B_{wi}^T & H_{bwi}^T \\ B_{wi} & -X_i + \alpha_{bwi} E_{bwi} E_{bwi}^T & 0 \\ H_{bwi} & 0 & -\alpha_{bwi} I \end{bmatrix} < 0$$

$$\begin{bmatrix} Q_{4i} & (C_i X_i + D_i Y_i)^T & (H_{ci} X_i + H_{di} Y_i)^T & (H_{ai} X_i + H_{bi} Y_i)^T & M_{2i} \\ C_i Y_i + D_i Y_i & -I + \alpha_{ci} E_{ci} E_{ci}^T & 0 & 0 & 0 \\ H_{ci} X_i + H_{di} Y_i & 0 & -\alpha_{ci} I & 0 & 0 \\ H_{ai} X_i + H_{bi} Y_i & 0 & 0 & -\alpha_{ai} I & 0 \\ M_{2i}^T & 0 & 0 & 0 & -\Lambda_{2i} \end{bmatrix} < 0$$

*hold for all* $i \in \mathcal{S}$, *where*

$$Q_{4i} = (A_i X_i + B_i Y_i) + (A_i X_i + B_i Y_i)^T + \pi_{ii} X_i + \alpha_{ai} E_{ai} E_{ai}^T$$

$$M_{2i} = \begin{bmatrix} \sqrt{\pi_{i1}} X_i & \sqrt{\pi_{i2}} X_i & \cdots & \sqrt{\pi_{i(i-1)}} X_i & \sqrt{\pi_{i(i+1)}} X_i & \cdots & \sqrt{\pi_{is}} X_i \end{bmatrix}$$

$$\Lambda_{2i} = \operatorname{diag}(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_s)$$

*In this case, controller* (8) *is given by* $K_i = Y_i X_i^{-1}$, $i \in \mathcal{S}$.

It is observed that the solution set to Theorem 2 is not convex due to the equality constraints (23). Now, let the equality constraints (23) be weakened to the following semi-definite programming relaxations:

$$\begin{bmatrix} P_i & I \\ I & X_i \end{bmatrix} \geq 0, \quad \begin{bmatrix} V_i & I \\ I & Z_i \end{bmatrix} \geq 0 \tag{25}$$

and for a sufficiently small number $\beta \in \mathbb{R}^+$, let the strict inequalities (19), (20), (21) be replaced by

$$\sum_{i=1}^{s} \mu_i \operatorname{trace}(W_i) + \beta \leq \gamma_{H_2}^2 \tag{26}$$

$$\begin{bmatrix} -W_i + \beta I & B_{wi}^T & H_{bwi}^T \\ B_{wi} & -X_i + \alpha_{bwi} E_{bwi} E_{bwi}^T & 0 \\ H_{bwi} & 0 & -\alpha_{bwi} I \end{bmatrix} \leq 0 \tag{27}$$

and

$$\begin{bmatrix} Q_{3i}+\beta I & (C_iX_i+D_iY_i)^T & (H_{ci}X_i+H_{di}Y_i)^T & (H_{ai}X_i+H_{bi}Y_i)^T & X_i \\ C_iY_i+D_iY_i & -I+\alpha_{ci}E_{ci}^TE_{ci} & 0 & 0 & 0 \\ H_{ci}X_i+H_{di}Y_i & 0 & -\alpha_{ci}I & 0 & 0 \\ H_{ai}X_i+H_{bi}Y_i & 0 & 0 & -\alpha_{ai}I & 0 \\ X_i^T & 0 & 0 & 0 & -Z_i \end{bmatrix} \le 0, \quad (28)$$

respectively, then the sequential linear programming method (Leibfritz 2001) can be employed to find a solution of Theorem 2. The solution of $RH_2P$ is summarised below.

**Algorithm $RH_2P$:** For a given precision $\delta\in\mathbb{R}^+$, let $N$ be the maximum number of iterations and a sufficiently small number $\beta\in\mathbb{R}^+$ be given.

(1) Determine $P_i^0$, $X_i^0$, $V_i^0$, $Z_i^0$, $T_{ij}^0$, $W_i^0$, $Y_i^0$, $\alpha_{ai}^0$, $\alpha_{bwi}^0$, $\alpha_{ci}^0$, $i,j\in\mathcal{S}$, $j\neq i$, satisfying (22) and (25)–(28). Let $k:=0$.

(2) Solve the following convex optimisation problem for the variables $P_i$, $X_i$, $V_i$, $Z_i$, $T_{ij}$, $W_i$, $Y_i$, $\alpha_{ai}$, $\alpha_{bwi}$, $\alpha_{ci}$, $i,j\in\mathcal{S}$, $j\neq i$:

$$\min\sum_{i=1}^{s}\operatorname{trace}(P_iX_i^k+P_i^kX_i+V_iZ_i^k+V_i^kZ_i)$$

subject to (22) and (25)−(28) for all $i\in\mathcal{S}$.

(3) Let $T_i^k:=P_i$, $L_i^k:=X_i$, $U_i^k:=V_i$ and $R_i^k:=Z_i$ for all $i\in\mathcal{S}$.

(4) If

$$\left|\sum_{i=1}^{s}\operatorname{trace}(T_i^kX_i^k+P_i^kL_i^k+U_i^kZ_i^k+V_i^kR_i^k) - 2\sum_{i=1}^{s}\operatorname{trace}(P_i^kX_i^k+V_i^kZ_i^k)\right|<\delta$$

then go to step (7), else go to step (5).

(5) Compute $\theta^*\in[0,1]$ by solving

$$\min_{\theta\in[0,1]}\sum_{i=1}^{s}\operatorname{trace}([P_i^k+\theta(T_i^k-P_i^k)][X_i^k+\theta(L_i^k-X_i^k)] + [V_i^k+\theta(U_i^k-V_i^k)][Z_i^k+\theta(R_i^k-Z_i^k)])$$

(6) Let

$$\begin{aligned} P_i^{k+1}&:=P_i^k+\theta^*(T_i^k-P_i^k),\\ X_i^{k+1}&:=X_i^k+\theta^*(L_i^k-X_i^k),\\ V_i^{k+1}&:=V_i^k+\theta^*(U_i^k-V_i^k),\\ Z_i^{k+1}&:=Z_i^k+\theta^*(R_i^k-Z_i^k),\end{aligned}$$

for all $i\in\mathcal{S}$, and $k:=k+1$, if $k<N$, then go to step (2), else go to step (7).

(7) Stop. If $\sum_{i=1}^{s}\operatorname{trace}(P_i^kX_i^k+V_i^kZ_i^k)=2sn$, then a solution is found successfully, else a solution cannot be found.

**Remark 3:** As explained in (Leibfritz 2001), Algorithm $RH_2P$ always generates a strictly decreasing sequence of the values of the objective function

$$f(k)\triangleq\sum_{i=1}^{s}\operatorname{trace}(P_i^kX_i^k+V_i^kZ_i^k).$$

Thus, $\{f(k)\}$ always converges to some $f^*\ge 2sn$ and if $f^*=2sn$, then the corresponding optimal values $P_i^*$, $X_i^*$, $V_i^*$, $Z_i^*$, $T_{ij}^*$, $W_i^*$, $Y_i^*$, $\alpha_{ai}^*$, $\alpha_{bwi}^*$ and $\alpha_{ci}^*$, $(i,j\in\mathcal{S},j\neq i)$, are a solution of Theorem 2. Moreover, the sequence $\{(P_i^k,X_i^k,V_i^k,Z_i^k,T_{ij}^k,W_i^k,Y_i^k,\alpha_{ai}^k,\alpha_{bwi}^k,\alpha_{ci}^k)\}$ generated by Algorithm $RH_2P$ is bounded for all $i\in\mathcal{S}$.

## 4. Numerical example

In this section, in order to illustrate the usefulness and flexibility of the theory developed in this article, we present a numerical example. Attention is focused on designing a robust $H_2$ controller such that the closed-loop system has guaranteed $H_2$ performance with respect to the uncertain switching probabilities. It is assumed that the system under consideration has two switching modes with uncertainties only in the mode transition rate matrix. The system data of (1) are as follows:

$$A_1=\begin{bmatrix}0&0.1\\0&1\end{bmatrix},\quad A_2=\begin{bmatrix}-1&0.1\\0&-1\end{bmatrix},$$
$$B_1=\begin{bmatrix}0.9\\-1\end{bmatrix},\quad B_2=\begin{bmatrix}0.1\\1\end{bmatrix}\quad C_1=\begin{bmatrix}1&-0.1\\0&1\end{bmatrix},$$
$$C_2=\begin{bmatrix}1&0.1\\0&1\end{bmatrix},\quad D_1=\begin{bmatrix}0\\0\end{bmatrix},\quad D_2=\begin{bmatrix}0\\0\end{bmatrix}$$
$$\Pi=\begin{bmatrix}-1.9&1.9\\10&-10\end{bmatrix},\quad B_{w1}=\begin{bmatrix}1\\0\end{bmatrix},\quad B_{w2}=\begin{bmatrix}0.1\\1\end{bmatrix},$$
$$x_0=\begin{bmatrix}0\\0\end{bmatrix}\quad \varepsilon_{12}=0.9,\quad \varepsilon_{21}=4,\quad \mu_1=0.5,\quad \mu_2=0.5.$$

The nominal system of the uncertain system given above is not mean square stable. Suppose that a controller (8) is desired such that the closed-loop system (9) is robustly mean square stable and has robust $H_2$ performance $\|G_{zw}\|_2 < \gamma_{H_2}$ with $\gamma_{H_2}=2$ over all the uncertainties $\Delta\pi_{12}\in[-1.8, 1.8]$ and $\Delta\pi_{21}\in[-8, 8]$. One controller can be obtained based on Corollary 1 by ignoring the effect of the uncertainties and one solution is as follows:

$$X_1 = \begin{bmatrix} 2.7721 & -2.1463 \\ -2.1463 & 2.3428 \end{bmatrix},$$
$$X_2 = \begin{bmatrix} 1.8839 & -0.7832 \\ -0.7832 & 2.0859 \end{bmatrix}, \quad W_1 = 2.9163,$$
$$Y_1 = \begin{bmatrix} -14.7217 & 15.2169 \end{bmatrix},$$
$$Y_2 = \begin{bmatrix} -12.4819 & -15.0692 \end{bmatrix}, \quad W_2 = 2.7685,$$
$$K_1 = \begin{bmatrix} -0.9693 & 5.6072 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -11.4095 & -11.5081 \end{bmatrix}.$$

Applying this controller, the resulting nominal closed-loop system becomes mean square stable and has the $H_2$ performance $\gamma^*_{H_2} = 0.6022$ (according to Proposition 2) with associated Gramian matrices

$$P_1 = \begin{bmatrix} 0.9028 & 0.7742 \\ 0.7742 & 0.8526 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.4156 & 0.1794 \\ 0.1794 & 0.2616 \end{bmatrix}.$$

However, this controller cannot guarantee the $H_2$ performance, even the stability of the closed-loop system, over the admissible uncertainties. Let us consider the case $\Delta\pi_{12}=-1.3$ and $\Delta\pi_{21}=6$; the closed-loop system remains mean square stable but has a largely degraded $H_2$ performance as $\gamma^*_{H_2} = 12.4895$ with associated Gramian matrices given by

$$P_1 = \begin{bmatrix} 16.5255 & 17.6342 \\ 17.6342 & 19.0324 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 5.9375 & 6.3536 \\ 6.3536 & 7.1234 \end{bmatrix}.$$

Moreover, in the case $\Delta\pi_{12}=-1.4$ and $\Delta\pi_{21}=6$, the closed-loop system becomes mean square unstable.

Fortunately, Algorithm RH$_2$P can be employed here to construct a more powerful controller such that the closed-loop system is robustly mean square stable and preserves the desired $H_2$ performance over all the admissible uncertainties in the switching probabilities. To compute with Algorithm RH$_2$P for this example, it is chosen that $\delta=10^{-10}$, $N=100$ and $\beta=0.01$. One set of solutions is

$$P_1 = \begin{bmatrix} 3.1248 & 3.1564 \\ 3.1564 & 4.0721 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.0913 & 3.0874 \\ 3.0874 & 4.2268 \end{bmatrix},$$
$$V_1 = \begin{bmatrix} 0.1470 & 0.2766 \\ 0.2766 & 1.6164 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0.7002 & 0.6272 \\ 0.6272 & 1.0958 \end{bmatrix},$$
$$X_1 = \begin{bmatrix} 1.4744 & -1.1428 \\ -1.1428 & 1.1314 \end{bmatrix},$$
$$X_2 = \begin{bmatrix} 1.1959 & -0.8735 \\ -0.8735 & 0.8746 \end{bmatrix},$$
$$Z_1 = \begin{bmatrix} 10.0361 & -1.7172 \\ -1.7172 & 0.9125 \end{bmatrix},$$
$$Z_2 = \begin{bmatrix} 2.9306 & -1.6773 \\ -1.6773 & 1.8726 \end{bmatrix},$$
$$T_{12} = \begin{bmatrix} 0.1300 & 0.2516 \\ 0.2516 & 0.8163 \end{bmatrix},$$
$$T_{21} = \begin{bmatrix} 0.0114 & -0.0020 \\ -0.0020 & 0.0826 \end{bmatrix},$$
$$W_1 = 3.1248, \ W_2 = 4.8752,$$
$$Y_1 = \begin{bmatrix} -4.8662 & 5.5920 \end{bmatrix},$$
$$Y_2 = \begin{bmatrix} -0.3370 & -5.7670 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 2.4441 & 7.4113 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -18.8467 & -25.4163 \end{bmatrix}.$$

It can be verified that $\|P_1X_1-I\|_2=2.1292\times10^{-12}$, $\|P_2X_2-I\|_2=2.1270\times10^{-12}$, $\|V_1Z_1-I\|_2=2.1220\times10^{-12}$, $\|V_2Z_2-I\|_2=2.1324\times10^{-12}$. Therefore, the equality constraints (23) are satisfied. By applying this controller, the resulting nominal closed-loop system is mean square stable and has the $H_2$ performance $\gamma^*_{H_2} = 0.8113$ with associated Gramian matrices

$$P_1 = \begin{bmatrix} 1.3998 & 1.1331 \\ 1.1331 & 1.1344 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.5475 & 0.1689 \\ 0.1689 & 0.1835 \end{bmatrix}.$$

To contrast with the previous controller, let us consider the same case $\Delta\pi_{12}=-1.3$ and $\Delta\pi_{21}=6$; the closed-loop system remains mean square stable and achieves the guaranteed $H_2$ performance $\gamma^*_{H_2} = 1.2813$ with associated Gramian matrices

$$P_1 = \begin{bmatrix} 2.0227 & 1.8720 \\ 1.8720 & 1.9836 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.7768 & 0.4365 \\ 0.4365 & 0.4448 \end{bmatrix}.$$

In the case $\Delta\pi_{12}=-1.4$ and $\Delta\pi_{21}=6$; the closed-loop system remains mean square stable as well as having guaranteed $H_2$ performance $\gamma^*_{H_2} = 1.3296$ with associated Gramian matrices

$$P_1 = \begin{bmatrix} 2.0938 & 1.9561 \\ 1.9561 & 2.0814 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.7947 & 0.4563 \\ 0.4563 & 0.4661 \end{bmatrix}.$$

Even in the extreme case $\Delta\pi_{12}=-1.8$ and $\Delta\pi_{21}=8$, the closed-loop system is still mean square stable and has guaranteed $H_2$ performance $\gamma^*_{H_2} = 1.6052$ with associated Gramian matrices

$$P_1 = \begin{bmatrix} 2.4519 & 2.3802 \\ 2.3802 & 2.5770 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.9316 & 0.6099 \\ 0.6099 & 0.6271 \end{bmatrix}.$$

## 5. Conclusions

This article discussed the robust $H_2$ control problem for MJLSs with uncertain switching probabilities. Attention was focussed on the design of a robust controller such that the closed-loop system is quadratically mean square stable and guarantees a desired robust $H_2$ performance over all the admissible uncertainties both in the system matrices and in the switching probabilities. It led to a non-linear problem consisting of a set of coupled linear matrix inequalities and a set of equality constraints. An algorithm involving convex optimisation was addressed to solve such a problem. The developed theory was illustrated by a numerical example and presented powerful utility and flexibility.


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## Notes on contributors

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