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Abstract

This paper investigates the investment and reinsurance problem in the presence of stochastic volatility for an ambiguity-averse insurer (AAI) with a general concave utility function. The AAI concerns about model uncertainty and seeks for an optimal robust decision. We consider a Brownian motion with drift for the surplus of the AAI who invests in a risky asset following a multiscale stochastic volatility (SV) model. We formulate the robust optimal investment and reinsurance problem for a general class of utility functions under a general SV model. Applying perturbation techniques to the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation associated with our problem, we derive an investment-reinsurance strategy that well approximates the optimal strategy of the robust optimization problem under a multiscale SV model. We also provide a practical strategy that requires no tracking of volatility factors. Numerical study is conducted to demonstrate the practical use of theoretical results and to draw economic interpretations from the robust decision rules.

Key words: Investment and reinsurance, Mixture of power utilities,

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Hamilton-Jacobi-Bellman-Isaacs equation, Multiscale stochastic volatility, Perturbation methods

1. Introduction

Many risk and insurance problems can be formulated as a stochastic control problem. For instance, the allocation of an insurer's reserves can be viewed as a stochastic control problem of maximizing an objective function which specifies the balance between profit and risk, see Yong and Zhou (1999). Two major financial activities for insurers to achieve this goal are: entering a reinsurance contract to transfer its risks to other firms, and investing in the risk-free and risky financial assets. Such problems were studied under different objectives such as minimizing the ruin probability (Promislow and Young, 2005); optimization with a VaR constraint (Chen et al., 2010); maximizing the utility with no-shorting constraint (Bai and Guo, 2008); investigating the Hamilton-Jacobi-Bellman (HJB) equation of related problems (Cao and Wan, 2009); incorporating general insurance (Liu and Ma, 2009); and considering the mean-variance objectives in Zeng and Li (2011) and Chen and Yam (2013).

Apart from a certain objective function and a known stochastic model, recent advances take the ambiguity aversion, uncertainty associated with the model and the risk aversion, into account as suggested by the Ellsberg paradox.Knight (1921) points out that ambiguity is distinct from the familiar notion of risk but subtle. Ambiguity and risk aversion are distinct factors in explaining the insurer's behaviors. We are interested in maximizing the anticipated utility of the terminal wealth of the insurer with the concerns of ambiguity and risk aversion, by adopting the notion of robust portfolio optimization. In fact, the robust control

theory has been applied to the investment-reinsurance (IR) problems. For example, Zhang and Siu (2009) study the robust IR problem via a max-min approach. Robust asset allocation problem via a penalty max-min approach is studied by Maenhout (2004) in accordance with the robust decision rules in Anderson et al. (1999). Yi et al. (2013) investigate the IR problem with model uncertainty under the Heston stochastic volatility (SV) model. However, these analysis only focus on a specific family of utility functions: power utility and exponential utility. We aim to extend existing results to a general class of concave utility functions under SV models, and offer implementable solution to the case of multi-scale SV model using asymptotic theory.

In this paper, we consider the surplus process of the insurer following a Brownian motion with drift by adopting the framework of Promislow and Young (2005). Investment can be made between a risky asset and a risk-free money market account, where the risky asset can be interpreted as the market index. This consideration facilitates the implementation because we use option data to infer the volatility surface to enhance investment decision. Usually, index option data are large enough for model calibration purpose. Although our analysis applies to different stochastic models for the risky asset price, we concentrate on a fixed surplus process for the insurer. Therefore, we formulate the robust IR problem for a general utility, analogously to Maenhout (2004), under a general SV model for the risky asset.

We derive an asymptotic solution to the robust IR problem under the multi-scale SV model, which contains a fast time scale factor and a slow time scale factor. These models and the related perturbation techniques are described in Fouque et al. (2011). The effect of multiscale SV on dynamic fund protection is studied by

Wong and Chan (2007). One advantage of using the multiscale SV model is that it allows us to infer the parameters by calibrating to market information. Moreover, the implementation is simple and accurate. In portfolio optimization, Fouque et al. (2013) obtain the asymptotic solution for the nonlinear Merton problem for the general utility. This motivates us to extend their results to the robust IR problem with general utility functions. In addition, the market information can be effectively incorporated into the optimal strategy under this framework. To make the presentation comprehensible, we illustrate the formulation and the derivation with the fast mean-reverting SV (FMRSV) model (i.e. only the fast time scale factor is considered) in the main context. The corresponding results to multiscale (or multifactor) SV model are collected in the appendix because the additional effort for the derivation is minimal.

A key advantage of our approach is its application to general utility functions. To show this advantage, we use the mixture of power utilities as an example as this utility function produces a nonlinear risk-tolerances and a non-constant relative risk aversion. The empirical studies in Brunnermeir and Nagel (2008) documents the relevancy of the time-varying relative risk aversion in practical decision making process.

The remainder of this paper is organized as follows. Section 2 presents the formulation of the robust IR problem with a general utility function under general SV models. We then asymptotically solve the robust IR problem under the FMRSV model in Section 3. A practical portfolio strategy is proposed in Section 4 so that no tracking of the instantaneous volatility is required. Section 5 uses numerical studies to examine the impact of SV factor in the robust IR problem and the performance of the strategy under the mixture of power utilities. We also

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address the implication of the robustness in our formulation. Section 6 concludes.

2. Problem formulation

2.1. The reference model

Consider the continuous-time surplus process with reinsurance and investment opportunities. The reference model is defined over the physical measure $\mathbb{P}$. Following Promislow and Young (2005), the claim process C of the insurer assumed as

$$dC(t) = adt - bdW_t^G,$$

where $a, b > 0$ are rate of the claim and the volatility of the claim process, respectively, and W_t^G is the standard $\mathbb{P}$-Brownian motion. To make the model more appealing, we further assume that the ratio a/b is large enough ($a/b > 3$) such that the probability of realizing a negative claim is small in any period of time. When the reinsurance strategy is absent in the analysis, the insurance premium rate is $\varsigma_0 = (1 + \tau)a$ with the safety loading $\tau > 0$ implies the surplus process G_0 as

$$dG_0(t) = \varsigma_0 dt - dC(t) = \tau adt + bdW_t^G.$$

When reinsurance is allowed, the insurer can divert a proportion of all premiums to another insurer (reinsurer) to manage the insurance risk. Let $1 - q(t)$ be the reinsurance fraction at time t where $q(t) \in [0, +\infty)$. The process $\{q(t)\}_{t \in [0,T]}$ is called a reinsurance strategy. When $q(t) > 1$, the underlying insurer itself offers reinsurance service to other insurers. When $q(t) \in [0, 1]$, the insurer makes proportional reinsurance. In this case, the fraction $1 - q(t)$ of each claim is paid by the counterparty reinsurer. When the reinsurance premium rate $\varsigma_1 = (1 + \eta)(1 -$

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$q(t))a$ with safety loading $\eta \geq \tau > 0$ is charged as the expense for reducing the potential risk, the surplus process G with the reinsurance strategy $q(t)$ becomes,

$$dG(t) = \varsigma_0 dt - q(t)dC(t) - \varsigma_1 dt = [\lambda + \eta q(t)]a dt + bq(t)dW_t^G, \qquad (1)$$

where $\lambda = \tau - \eta \leq 0$.

In addition, the underlying insurer can invest in a risky asset and a risk-free asset. We postulate the price of the risky asset S to follow an Itô process with SV driven by Y:

$$\begin{cases} dS_t &= \mu(Y_t)S_t dt + \sigma(Y_t)S_t dW_t^S, \\ dY_t &= m(Y_t)dt + \alpha(Y_t)[\rho dW_t^S + \bar{\rho}dW_t^Y], \end{cases} \qquad (2)$$

where $\bar{\rho} = \sqrt{1 - \rho^2}$, and W_t^S and W_t^Y are independent standard $\mathbb{P}$-Brownian motions, while W_t^G is independent of W_t^S and W_t^Y.

The insurer determines her wealth X allocation between the risky and risk-free assets and the reinsurance strategy. We use $l(t)$ to denote the amount of wealth in the risky asset at time t and the remaining amount in risk-free asset at rate r. The reinsurance strategy $q(t)$ and investment strategy $l(t)$ then form the IR strategy pair $\pi = (q, l)'$. The principle of continuous-time self-financing trading yields the following dynamics for the wealth process X:

$$\begin{aligned} dX_t &= dG_t + \frac{l}{S_t}dS_t + r(X_t - l)dt \\ &= [a\lambda + a\eta q + (\mu(Y_t) - r)l + rX_t]dt + bq dW_t^G + \sigma(Y_t)l dW_t^S. \quad (3) \end{aligned}$$

2.2. Robust stochastic control problem

Classical approaches aim at maximizing the anticipated utility of the terminal wealth with the fixed investment horizon $T < \infty$:

$$\sup_{\pi \in \Pi} \mathbb{E}^{\mathbb{P}}\left[U(X_T)\right], \qquad (4)$$

where Π is the set of admissible strategies π. But we are interested in incorporating the ambiguity aversion into the problem for an ambiguity-averse insurer.

Our approach stems on the belief that the insurer has some confidence in the reference measure $\mathbb{P}$ and is willing to consider a class of possible measures $\mathcal{Q}$, which are "similar" to $\mathbb{P}$. To clarify the meaning of "similar" there, we employ the concept of equivalent measures, analogous to Anderson et al. (1999). Specifically, alternative measures are induced by a class of probability measures equivalent to $\mathbb{P}$: $\mathcal{Q} := \{\mathbb{Q} \mid \mathbb{Q} \sim \mathbb{P}\}$. By the Girsanov theorem, for each $\mathbb{Q} \in \mathcal{Q}$, there is a stochastic process $\varphi^{\mathbb{Q}}(t) = (\varphi_G^{\mathbb{Q}}(t), \varphi_S^{\mathbb{Q}}(t), \varphi_Y^{\mathbb{Q}}(t))'$, which can be regarded as the model misspecification factors, such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \nu(t) = \exp\left(\int_0^t \varphi^{\mathbb{Q}}(s)'dW_s - \frac{1}{2}\int_0^t \varphi^{\mathbb{Q}}(s)'\varphi^{\mathbb{Q}}(s)ds\right),$$

where $W_t = (W_t^G, W_t^S, W_t^Y)'$. Moreover, if $\varphi^{\mathbb{Q}}(t)$ satisfies the Novikov condition,

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \varphi^{\mathbb{Q}}(s)'\varphi^{\mathbb{Q}}(s)ds\right)\right] < \infty,$$

then the process $\nu(t)$ is a positive $\mathbb{P}$-martingale and $\widetilde{W}_t := (\widetilde{W}_t^G, \widetilde{W}_t^S, \widetilde{W}_t^Y)$ becomes a $\mathbb{Q}$-Brownian motion in $\mathbb{R}^3$, where $d\widetilde{W}_t = dW_t - \varphi^{\mathbb{Q}}(t)dt$. An alternative measure $\mathbb{Q}$ is characterized by both $\varphi^{\mathbb{Q}}$ and the reference measure $\mathbb{P}$. In other words, choosing the measure $\mathbb{Q}$ is equivalent to determining the stochastic process $\varphi^{\mathbb{Q}}$ for a given $\mathbb{P}$.

Inspired by Maenhout (2004) and Yi et al. (2013), our objective function with the ambiguity aversion is defined through a penalty function:

$$\sup_{\pi \in \Pi} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\left[U(X_T) + \frac{1}{\xi}\int_0^T \frac{P(s)}{\phi(s)}ds\right], \tag{5}$$

where ξ is a measure of ambiguity aversion, $P(t) := \varphi^{\mathbb{Q}}(t)'\varphi^{\mathbb{Q}}(t)/2$ measures the relative entropy between $\mathbb{P}$ and $\mathbb{Q}$, and $\phi(t) \geq 0$ is the preference parameter related

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to the ambiguity aversion. As discussed in Maenhout (2004), $\int_0^T P(s)/\phi(s)ds$ acts as the penalty for the model choice in accordance with the preference parameter. The infimum in our problem (5) is to minimize the penalty and meanwhile consider the worst case scenario over feasible decisions, with respect to the choice of $\mathbb{Q}$. In particular, as $\xi \downarrow 0$, the insurer picks the reference measure $\mathbb{P}$ and the robust problem (5) is reduced to the problem (4), because $P \equiv 0$ in this case. If $\xi \uparrow \infty$, then the penalty function vanishes and all candidate measures are identical. Such a consideration is studied by Zhang and Siu (2009). The key step of the formulation is to find a suitable preference function ϕ. Similar construction is also considered by Maenhout (2004) for a power utility and by Yi et al. (2013) for an exponential utility. If one wants to consider a general utility, we propose a general form for ϕ in the subsequent context.

We first define the value function

$$V(t,x,y) = \sup_{\pi \in \Pi} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left[U(X_T) + \frac{1}{\xi} \int_t^T \frac{P(s)}{\phi(s)} ds \,\middle|\, X_t = x, Y_t = y \right], \quad (6)$$

where V is smooth on $[0,T] \times (0,\infty) \times \mathbb{R}$. Notice that $V(t,\cdot,y)$ happens to be the same functional form as the utility function $U(\cdot)$. Throughout this paper, we look for $V(t,x,y)$ which has a utility function characteristics, i.e. $V_x > 0, V_{xx} < 0$ (Fouque et al., 2013). We adopt the ϕ suggested by Maenhout (2004) for that it is economically meaningful and facilitates analytical tractability:

$$\phi(t,x,y) = \frac{1}{R(t,x,y)V_x(t,x,y)} = -\frac{V_{xx}(t,x,y)}{V_x^2(t,x,y)} \geq 0, \quad (7)$$

where $R(t,x,y) = -\frac{V_x}{V_{xx}} \geq 0$ is risk-tolerance function. This choice of ϕ is reasonable in the sense that ϕ is monotonically decreasing with respect to R (i.e. more risk aversion implies more robustness). Moreover, it implicitly imposes the homotheticity, that robustness will not wear off as the wealth rises. For power (or

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CRRA) utility and exponential (or CARA) utility functions, we summarize the corresponding ansatzes of V and the choices of ϕ in the Table 1. Essentially, our choice of ϕ matches those discussed in Maenhout (2004) and Yi et al. (2013) for specific utility functions.

Table 1: V and ϕ with the common utilities $(c, \gamma, \vartheta > 0, \gamma \neq 1)$

Utility functions	Ansatz of $V(t, x, y)$	$\phi(t, x, y)$
$U_{CRRA}(x) = c\frac{x^{1-\gamma}}{1-\gamma}$	$cK(t,y)\frac{x^{1-\gamma}}{1-\gamma}$	$\frac{\gamma}{(1-\gamma)V(t,x,y)}$
$U_{CARA}(x) = -\frac{c}{\vartheta}e^{-\vartheta x}$	$-\frac{c}{\vartheta}\exp\left\{-\vartheta[K_1(t,y)x + K_2(t,y)]\right\}$	$-\frac{1}{V(t,x,y)}$

With our problem formulation, the original maximin utility problem can be transformed into a utility maximization problem under an alternative equivalent measure. Thus, we need the dynamics in (3) under such a measure to derive the analytical solution. Specifically, the joint dynamics of the wealth process X and the SV factor Y under the measure $\mathbb{Q}$ (for any $\mathbb{Q} \in \mathcal{Q}$) is given by

$$
\begin{cases}
dX_t = \left\{a\lambda + [a\eta + b\varphi_G^{\mathbb{Q}}(t)]q + [\mu(Y_t) - r + \sigma(Y_t)\varphi_S^{\mathbb{Q}}(t)]l + rX_t\right\} dt \\
\qquad\quad + bqd\widetilde{W}_t^G + \sigma(Y_t)ld\widetilde{W}_t^S, \\
dY_t = \left\{m(Y_t) + \alpha(Y_t)[\rho\varphi_S^{\mathbb{Q}}(t) + \bar{\rho}\varphi_Y^{\mathbb{Q}}(t)]\right\} dt + \alpha(Y_t)[\rho d\widetilde{W}_t^S + \bar{\rho}d\widetilde{W}_t^Y],
\end{cases}
\tag{8}
$$

with $\mathbb{E}[d\widetilde{W}_t^S d\widetilde{W}_t^Y] = \mathbb{E}[d\widetilde{W}_t^G d\widetilde{W}_t^S] = \mathbb{E}[d\widetilde{W}_t^G d\widetilde{W}_t^Y] = 0$.

2.3. The Hamilton-Jacobi-Bellman framework

The associated Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation of the value function (6) is given by

$$
V_t + \mathcal{L}V + \sup_\pi \inf_{\varphi^{\mathbb{Q}}} \left\{ \left([a\eta + b\varphi_G^{\mathbb{Q}}(t)]q + [\mu(y) - r + \sigma(y)\varphi_S^{\mathbb{Q}}(t)]l \right) V_x \right.
$$

$$
+ \frac{1}{2}[b^2 q_t^2 + \sigma^2(y)l_t^2]V_{xx} + \alpha(y)[\rho\varphi_S^{\mathbb{Q}}(t) + \bar{\rho}\varphi_Y^{\mathbb{Q}}(t)]V_y + \rho l\sigma(y)\alpha(y)V_{xy}
$$

$$
\left. + [a\lambda + rx]V_x + \frac{1}{2\xi\phi}\left([\varphi_G^{\mathbb{Q}}(t)]^2 + [\varphi_S^{\mathbb{Q}}(t)]^2 + [\varphi_Y^{\mathbb{Q}}(t)]^2 \right) \right\} = 0,
$$

with the terminal condition $V(T, x, y) = U(x)$, where $\mathcal{L}$ is the infinitesimal generator of Y:

$$
\mathcal{L} = \frac{1}{2}\alpha^2(y)\frac{\partial^2}{\partial y^2} + m(y)\frac{\partial}{\partial y}.
$$

We refer to Yong and Zhou (1999) and Cao and Wan (2009) for the connection between the optimization problem and the HJB framework.

Minimizing the quadratic forms of $\varphi_G^{\mathbb{Q}}$, $\varphi_S^{\mathbb{Q}}$ and $\varphi_Y^{\mathbb{Q}}$ yields that

$$
\varphi_G^* = -\xi\phi qbV_x, \;\; \varphi_S^* = -\xi\phi[l\sigma(y)V_x + \alpha(y)\rho V_y], \;\; \varphi_Y^* = -\xi\phi\alpha(y)\bar{\rho}V_y. \quad (9)
$$

Substituting (9) and (7) into the HJBI equation, we obtain the HJB equation:

$$
V_t + [a\lambda + rx]V_x + \mathcal{L}V + \sup_\pi \left\{ [a\eta q + (\mu(y) - r)l] V_x + \frac{\xi}{2}\alpha^2(y)\frac{V_{xx}V_y^2}{V_x^2} \right. \quad (10)
$$

$$
\left. + \frac{\xi+1}{2}\left[b^2 q_t^2 + \sigma^2(y)l_t^2\right] V_{xx} + \rho l\sigma(y)\alpha(y)\left(V_{xy} + \xi\frac{V_{xx}V_y}{V_x}\right) \right\} = 0,
$$

with the terminal condition $V(T, x, y) = U(x)$.

Hence the robust stochastic control problem is transformed into a standard stochastic optimal control problem. This is because the value function (6) can be described by an HJB equation (10) and embed the ambiguity aversion coefficient ξ into an alternative stochastic model. The following theorem states this fact.

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Theorem 2.1. With the choice of ϕ in (7), the value function V defined in (6) and satisfying (10) can be written as

$$
\begin{aligned}
V(t,x,y) \;=\; \sup_{\pi\in\Pi}\mathbb{E}^{\mathbb{Q}^*}\Bigg[\xi\int_t^T \left(\frac{\alpha^2(y)}{2}\frac{V_{xx}V_y^2}{V_x^2} + \rho l\sigma(y)\alpha(y)\frac{V_{xx}V_y}{V_x} \right)(s)ds \\
+U(\widetilde{X}_T)\Big|\, \widetilde{X}_t = x, Y_t = y \Bigg],
\end{aligned} \tag{11}
$$

where $\widetilde{X}$ is the robust wealth process that evolves with Y as follows.

$$
\begin{cases}
d\widetilde{X}_t \;=\; \Big\{ a\lambda + a\eta q + (\mu(Y_t) - r)l + r\widetilde{X}_t \Big\} dt \\
\qquad\qquad + \sqrt{\xi+1}\,bq\,d\widetilde{W}_t^G + \sqrt{\xi+1}\,\sigma(Y_t)l\,d\widetilde{W}_t^S, \\
dY_t \;=\; m(Y_t)dt + \alpha(Y_t)[\rho_\xi d\widetilde{W}_t^S + \bar\rho_\xi d\widetilde{W}_t^Y],
\end{cases} \tag{12}
$$

with $\mathbb{E}[d\widetilde{W}_t^S d\widetilde{W}_t^Y] = \mathbb{E}[d\widetilde{W}_t^G d\widetilde{W}_t^S] = \mathbb{E}[d\widetilde{W}_t^G d\widetilde{W}_t^Y] = 0$, where $\widetilde{W}^G, \widetilde{W}^S, \widetilde{W}^Y$ are $\mathbb{Q}^*$-standard Brownian motions, $\rho_\xi = \frac{\rho}{\sqrt{\xi+1}}$ and $\bar\rho_\xi = \sqrt{1 - \rho_\xi^2}$.

Proof. The proof is based on the HJB framework and the previous discussion. $\square$

Remark: When $\xi = 0$, it is easy to see that the new HJB problem (11) is reduced to the ordinary IR problem.

Theorem 2.1 holds true for general SV models, including the multi-factor SV models because one can simply view the Y as a vector of stochastic factors. This theorem offers an interesting interpretation to the robust IR problem associated with our choice of the penalty function. It is seen that the maximin problem is transformed into a maximization problem in which the objective function is the classical utility function penalized by an integration related to the risk aversion of the insurer and the leverage effect while the stochastic volatility model is introduced. In addition, the corresponding wealth process has its volatility adjusted and the leverage effect is also diminished by a factor related to ξ.

Although we can solve the value function V from (10), the corresponding HJB equation is highly nonlinear and it is still hard to find an explicit solution. Fortunately, we will show shortly that this problem can be solved asymptotically under the multiscale SV model. The involved singular and regular perturbation techniques are described in Fouque et al. (2011) and the recent paper Fouque et al. (2013) considers the case of portfolio optimization.

3. Theoretical solution: The full feedback control

In this section, we present the asymptotic solution of (10) when the risky asset follows fast mean-reverting stochastic volatility (FMRSV) model. In particular, we postulate that $m(Y_t) = \frac{1}{\epsilon}\theta(Y_t)$ and $\alpha(Y_t) = \frac{1}{\sqrt{\epsilon}}\zeta(Y_t)$ in (8) and (10). Specifically, the risky asset evolves as follows.

$$\begin{cases} dS_t &=& \mu(Y_t)S_t dt + \sigma(Y_t)S_t dW_t^S, \\ dY_t &=& \frac{1}{\epsilon}\theta(Y_t)dt + \frac{1}{\sqrt{\epsilon}}\zeta(Y_t)[\rho dW_t^S + \bar{\rho}dW_t^Y], \end{cases}$$

where $0 < \epsilon \ll 1$. Let $Y_t = Y_{t/\epsilon}^{(1)}$ in distribution, where $Y^{(1)}$ is an ergodic diffusion process with the unique invariant distribution Φ. For detailed exposition of this model, we refer to Fouque et al. (2011). We denote $\langle \cdot \rangle$ as the invariant expectation with respect to Φ:

$$\langle g \rangle = \int g(y)\Phi(dy).$$

To execute the optimal strategy $\pi^*(t, X_t, Y_t)$ derived in this section, we need to observe or filter the volatility process Y_t, that is why we call the optimal strategy as the full feedback control. In practice, market practitioners may prefer to extract information from forward-looking volatility surfaces rather than filtering out the realized volatilities. It motivates us to consider a partial feedback control which is

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independent of Y_t, i.e. $\pi^* = \pi^*(t, X_t)$ in the next section. In Fouque et al. (2011) and Fouque et al. (2013), this latter type of control is called a practical strategy.

3.1. Asymptotic solution

Substituting $m(Y_t) = \frac{1}{\epsilon}\theta(Y_t)$ and $\alpha(Y_t) = \frac{1}{\sqrt{\epsilon}}\zeta(Y_t)$ and maximizing the quadratic forms of q and l in (10), the optimal trading strategy pair $\pi^* = (q^*, l^*)$ is given by

$$q^* = \frac{a\eta}{(\xi+1)b^2}R(t,x,y), \; l^* = \frac{\varpi(y)}{(\xi+1)\sigma(y)}R(t,x,y) - \frac{\rho\zeta(y)}{\sqrt{\epsilon}(\xi+1)\sigma(y)}\left(\frac{V_{xy}}{V_{xx}} + \xi\frac{V_y}{V_x}\right),$$
(13)

where $R(t,x,y) = -V_x/V_{xx}$ is risk-tolerance function and $\varpi(y) = \frac{\mu(y)-r}{\sigma(y)}$ is the Sharpe ratio. Generally speaking, from the expression of strategies (13), the more the ambiguity aversion (larger ξ) the larger the reinsurance proportion $(1-q)$ and the larger amount of money invested in risk-free asset. Under the fast mean-reverting SV model, the HJB equation (10) becomes

$$V_t + \frac{1}{\epsilon}\left[\mathcal{L}_0 V + \frac{\xi}{2}\zeta^2(y)\frac{V_{xx}V_y^2}{V_x^2}\right] - \frac{a^2\eta^2}{2b^2(\xi+1)}\frac{V_x^2}{V_{xx}}$$

$$+[a\lambda + rx]V_x - \frac{\left[\frac{1}{\sqrt{\epsilon}}\rho\zeta(y)\left(V_{xy} + \xi\frac{V_{xx}V_y}{V_x}\right) + \varpi(y)V_x\right]^2}{2(\xi+1)V_{xx}} = 0, \quad (14)$$

where

$$\mathcal{L}_0 = \frac{1}{2}\zeta^2(y)\frac{\partial^2}{\partial y^2} + \theta(y)\frac{\partial}{\partial y}.$$
(15)

Equation (14) is a nonlinear partial differential equation (PDE) which poses analytical and numerical challenges, especially for the general utility functions. We hence seek for an asymptotic solution of the value function with the form:

$$V(t,x,y) = V^{(0)}(t,x,y) + \sqrt{\epsilon}V^{(1)}(t,x,y) + \epsilon V^{(2)}(t,x,y) + \epsilon^{\frac{3}{2}}V^{(3)}(t,x,y) + \cdots.$$

Although we are satisfied with the first order approximation $(V^{(0)} + \sqrt{\epsilon}V^{(1)})$, $V^{(2)}$ and $V^{(3)}$ do provide us with information for solving $V^{(0)}$ and $V^{(1)}$. We then apply the singular perturbation with respect to this expansion form.

Inserted this expansion into (14), we collect the highest order ϵ^{-1} equation as

$$\mathcal{L}_0 V^{(0)} + \frac{\zeta^2(y)}{2} \left(\xi \frac{V_{xx}^{(0)} V_y^{(0)2}}{V_x^{(0)2}} - \frac{\rho^2}{\xi+1} \frac{\left(V_{xy}^{(0)} + \xi \frac{V_{xx}^{(0)} V_y^{(0)}}{V_x^{(0)}} \right)^2}{V_{xx}^{(0)}} \right) = 0.$$

Noting that $\mathcal{L}_0$ is taking derivative on y, this equation satisfied by selecting $V^{(0)}$ independent of y, i.e. $V^{(0)} = V^{(0)}(t,x)$. Successively, we collect the terms at the order $\epsilon^{-\frac{1}{2}}$: $\mathcal{L}_0 V^{(1)} = 0$ by recognizing the fact that $V_y^{(0)} = 0$. Similarly, we have $V^{(1)} = V^{(1)}(t,x)$ independent of y.

The order one terms in (14) are:

$$\mathcal{L}_0 V^{(2)} + V_t^{(0)} + [a\lambda + rx]V_x^{(0)} - \frac{1}{2} \frac{a^2\eta^2/b^2 + \varpi^2(y)}{(\xi+1)} \frac{V_x^{(0)2}}{V_{xx}^{(0)}} = 0. \qquad (16)$$

Analogous to Fouque et al. (2013), we introduce the risk-tolerance function at zeroth order and operators to ease the notational burden:

$$R^{(0)}(t,x) = -\frac{V_x^{(0)}(t,x)}{V_{xx}^{(0)}(t,x)}; \quad D_k = R^{(0)}(t,x)^k \frac{\partial^k}{\partial x^k};$$

$$\mathcal{L}_{t,x}(\psi^2) = \frac{\partial}{\partial t} + [a\lambda + rx]\frac{\partial}{\partial x} + \frac{1}{2}\psi^2 D_2 + \psi^2 D_1. \qquad (17)$$

Then the order one equation (16) can be rewritten as

$$\mathcal{L}_0 V^{(2)} + \mathcal{L}_{t,x}(\psi_\xi^2(y))V^{(0)} = 0, \qquad (18)$$

where $\psi_\xi^2(y) = \frac{a^2\eta^2/b^2 + \varpi^2(y)}{\xi+1}$ and this equation is followed by

$$-\frac{1}{2}\frac{V_x^{(0)2}}{V_{xx}^{(0)}} = \frac{1}{2}\left(\frac{V_x^{(0)}}{V_{xx}^{(0)}}\right)^2 V_{xx}^{(0)} + \left(-\frac{V_x^{(0)}}{V_{xx}^{(0)}}\right)V_x^{(0)} = \frac{1}{2}D_2 V^{(0)} + D_1 V^{(0)}.$$

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Notice that the equation (18) is a Poisson equation for $V^{(2)}$ whose solvability condition yields that $\langle \mathcal{L}_{t,x}(\psi_\xi^2(y))V^{(0)} \rangle = 0$ and $V^{(0)}$ is independent of y. Hence, $V^{(0)}$ is governed by the following PDE:

$$\mathcal{L}_{t,x}(\bar{\psi}_\xi^2)V^{(0)} = 0, \quad V^{(0)}(T, x) = U(x), \tag{19}$$

where $\bar{\psi}_\xi^2 = \langle \psi_\xi^2(y) \rangle$. It is easily seen that $V^{(0)}$ is exactly the solution of the IR problem under the Black-Scholes model. In order words, the availability of our asymptotic solution solely depends on the solvability of the problem under the Black-Scholes setting. For some specific utilities, the explicit formulas of $V^{(0)}$ can be written out. Table 2 illustrates with the power utility and exponential utility. In the case of power utility, we assume $\lambda = 0$.

Table 2: $V^{(0)}$ and $R^{(0)}$ with the common utilities ($c, \gamma, \vartheta > 0, \gamma \neq 1$)

$U(x)$	$V^{(0)}(t, x)$	$R^{(0)}(t, x)$
$c\frac{x^{1-\gamma}}{1-\gamma}$	$c\frac{x^{1-\gamma}}{1-\gamma} \exp\left\{ \left[r(1-\gamma) + \frac{\bar{\psi}_\xi^2(1-\gamma)}{2\gamma} \right](T-t) \right\}$	$\frac{x}{\gamma}$
$-\frac{c}{\vartheta}e^{-\vartheta x}$	$-\frac{c}{\vartheta} \exp\left\{ -\vartheta x e^{r(T-t)} + \frac{a\lambda\vartheta}{r}(1 - e^{r(T-t)}) - \frac{\bar{\psi}_\xi^2}{2}(T-t) \right\}$	$\frac{e^{-r(T-t)}}{\vartheta}$

Meanwhile, from the Poisson equation (18) and (19), we have

$$\mathcal{L}_0 V^{(2)} = -(\mathcal{L}_{t,x}(\psi_\xi^2(y)) - \mathcal{L}_{t,x}(\bar{\psi}_\xi^2))V^{(0)} = -(\psi_\xi^2(y) - \bar{\psi}_\xi^2)\left(\frac{1}{2}D_2 + D_1 \right)V^{(0)}.$$

Therefore,

$$V^{(2)} = -\frac{\chi(y)}{\xi + 1}\left(\frac{1}{2}D_2 + D_1 \right)V^{(0)} + K(t, x), \tag{20}$$

where $\chi(y)$ is the solution of the ODE: $\mathcal{L}_0\chi(y) = \varpi^2(y) - \langle \varpi^2(y) \rangle$.

To derive the explicit expression for $V^{(1)}$, we need the following two lemmas which are extensions of Lemma 2.1 and Lemma 2.3 in Fouque et al. (2013). The first lemma provides the PDE driving $R^{(0)}$.

Lemma 3.1. The risk-tolerance function at zeroth order $R^{(0)}(t, x)$ satisfies PDE:

$$R_t^{(0)} + \frac{1}{2}\bar{\psi}_\xi^2 (R^{(0)})^2 R_{xx}^{(0)} + (a\lambda + rx)R_x^{(0)} - rR^{(0)} = 0, \tag{21}$$

with the terminal condition $R^{(0)}(T, x) = -U'(x)/U''(x)$.

Proof. Differentiating (19) with respect to x yields

$$V_{tx}^{(0)} + rV_x^{(0)} + (a\lambda + rx)V_{xx}^{(0)} = -\frac{1}{2}\bar{\psi}_\xi^2 (R_x^{(0)} - 1)V_x^{(0)}, \tag{22}$$

where we have used the fact that $(R^{(0)})^2 V_{xxx}^{(0)} = (R_x^{(0)} + 1)V_x^{(0)}$ by the definition of $R^{(0)}$. Again, differentiating (22) with respect to x gives

$$V_{txx}^{(0)} + 2rV_{xx}^{(0)} + (a\lambda + rx)V_{xxx}^{(0)} = -\frac{1}{2}\bar{\psi}_\xi^2 (R_x^{(0)} - 1)V_{xx}^{(0)} - \frac{1}{2}\bar{\psi}_\xi^2 R_{xx}^{(0)} V_x^{(0)}. \tag{23}$$

Notice that from the definition of $R^{(0)}$, we have

$$R_t^{(0)} = -\frac{V_{tx}^{(0)}}{V_{xx}^{(0)}} + \frac{V_x^{(0)}}{(V_{xx}^{(0)})^2}V_{txx}^{(0)}.$$

Substituting $V_{tx}^{(0)}$ in (22) and $V_{txx}^{(0)}$ in (23) into the right hand side of the above equation induces the desirable result. $\qquad\square$

Suppose that we obtained the $R^{(0)}$ by solving (21), then we can infer $V_x^{(0)}$ and $V^{(0)}$ by taking integration and noting that

$$\frac{1}{R^{(0)}} = \frac{\partial}{\partial x}(-\ln V_x^{(0)}).$$

The second lemma asserts that the operators $\mathcal{L}_{t,x}(\bar{\psi}_\xi^2)$ and D_1 commute.

Lemma 3.2. $\mathcal{L}_{t,x}(\psi^2)D_1 = D_1\mathcal{L}_{t,x}(\psi^2)$.

Proof. We first rewrite $\mathcal{L}_{t,x}(\psi^2)$ as $M_x + \widetilde{\mathcal{L}}_{t,x}(\psi^2)$, where $M_x = [a\lambda + rx]\frac{\partial}{\partial x}$ and $\widetilde{\mathcal{L}}_{t,x}(\psi^2) = \mathcal{L}_{t,x}(\psi^2) - M_x$. For any smooth function $w(t,x)$, from the Lemma 2.3 in Fouque et al. (2013), we have

$$\widetilde{\mathcal{L}}_{t,x}(\psi^2)D_1 w = D_1\widetilde{\mathcal{L}}_{t,x}(\psi^2)w + \left(R_t^{(0)} + \frac{1}{2}\psi^2(R^{(0)})^2 R_{xx}^{(0)}\right) w_x.$$

On the other hand, we compute

$$M_x D_1 w = (a\lambda + rx)[R_x^{(0)}w_x + R^{(0)}w_{xx}];$$
$$D_1 M_x w = R^{(0)}rw_x + R^{(0)}(a\lambda + rx)w_{xx}.$$

Therefore, $M_x D_1 w = D_1 M_x w + [(a\lambda + rx)R_x^{(0)} - rR^{(0)}]w_x$.

To conclude the above analysis,

$$\mathcal{L}_{t,x}(\psi^2)D_1 w$$
$$= D_1\mathcal{L}_{t,x}(\psi^2)w + \left[R_t^{(0)} + \frac{1}{2}\psi^2(R^{(0)})^2 R_{xx}^{(0)} + (a\lambda + rx)R_x^{(0)} - rR^{(0)}\right] w_x$$
$$= D_1\mathcal{L}_{t,x}(\psi^2)w,$$

where the last equation is due to the equation (21). Hence, $\mathcal{L}_{t,x}(\psi^2)$ and D_1 commute. $\square$

We proceed to the order $\sqrt{\epsilon}$ terms in (14), which are collected as follows.

$$\mathcal{L}_0 V^{(3)} + \mathcal{L}_{t,x}(\psi_\xi^2(y))V^{(1)} - \frac{\rho\zeta(y)\varpi(y)}{2(\xi+1)^2}\chi_y(D_1^2 - \xi D_1)V^{(0)} = 0. \qquad (24)$$

The solvability condition of this Poisson equation for $V^{(3)}$ yields that

$$\mathcal{L}_{t,x}(\bar{\psi}_\xi^2)V^{(1)} = -\frac{1}{(\xi+1)^2}B(D_1^2 - \xi D_1)V^{(0)}, \text{ where } B = -\frac{\rho}{2}\langle\varpi(y)\zeta(y)\chi_y\rangle.$$

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Here, B is related to the slope of the volatility surface. Then, it is easy to verify that

$$V^{(1)}(t,x) = \frac{T-t}{(\xi+1)^2}B(D_1^2 - \xi D_1)V^{(0)}(t,x) = \frac{T-t}{(\xi+1)^2}B(R_x^{(0)} - 1 - \xi)R^{(0)}V_x^{(0)},$$

(25)

by noting that

$$\mathcal{L}_{t,x}(\bar{\psi}_\xi^2)\left(\frac{T-t}{(\xi+1)^2}B(D_1^2 - \xi D_1)V^{(0)}\right)$$

$$= \frac{-1}{(\xi+1)^2}B(D_1^2 - \xi D_1)V^{(0)} + \frac{T-t}{(\xi+1)^2}B(D_1^2 - \xi D_1)\mathcal{L}_{t,x}(\bar{\psi}_\xi^2)V^{(0)}$$

$$= \frac{-1}{(\xi+1)^2}B(D_1^2 - \xi D_1)V^{(0)}.$$

Thus, we get the corrected value function

$$V(t,x) \simeq \left[1 + \sqrt{\epsilon}\frac{T-t}{(\xi+1)^2}B(D_1^2 - \xi D_1)\right]V^{(0)}(t,x;\bar{\psi}_\xi).$$

(26)

It can been seen that the first order correction is contributed by the stochastic volatility. The introduction of the ambiguity aversion diminishes the effect of stochastic volatility. It makes the value function is less sensible to the choice of stochastic volatility model with the larger ξ.

The above analyses are based on FMRSV model. In fact, the extension to multiscale stochastic volatility model is straightforward but more tedious. The corresponding result is arranged in the Appendix A.

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3.2. Optimal strategy

From the expression of (26), we can expand $R(t, x, y) = -\frac{V_x}{V_{xx}}$ and $\frac{1}{\sqrt{\epsilon}}\left(\frac{V_{xy}}{V_{xx}} + \xi\frac{V_y}{V_x}\right)$ up to order $\sqrt{\epsilon}$ as

$$
\begin{aligned}
R(t, x) &= -\frac{V_x^{(0)} + \sqrt{\epsilon}V_x^{(1)} + \mathcal{O}(\epsilon)}{V_{xx}^{(0)} + \sqrt{\epsilon}V_{xx}^{(1)} + \mathcal{O}(\epsilon)} = R^{(0)} - \sqrt{\epsilon}\left[\frac{V_x^{(1)}}{V_{xx}^{(0)}} - \frac{V_x^{(0)}V_{xx}^{(1)}}{(V_{xx}^{(0)})^2}\right] + \mathcal{O}(\epsilon) \\
&= R^{(0)} - \sqrt{\epsilon}\frac{B(T-t)}{V_x^{(0)}}(D_1^2 D_2 - D_2 D_1^2)V^{(0)} + \mathcal{O}(\epsilon),
\end{aligned}
$$

$$
\frac{1}{\sqrt{\epsilon}}\left(\frac{V_{xy}}{V_{xx}} + \xi\frac{V_y}{V_x}\right) = \sqrt{\epsilon}\left(\frac{V_{xy}^{(2)}}{V_{xx}^{(0)}} + \xi\frac{V_y^{(2)}}{V_x^{(0)}}\right) + \mathcal{O}(\epsilon) = \sqrt{\epsilon}\frac{\chi_y(y)}{2V_x^{(0)}}(D_1^2 - \xi D_1)V^{(0)} + \mathcal{O}(\epsilon).
$$

Therefore, the opmtial strategy pair in (13) up to the order $\sqrt{\epsilon}$ is given by $\pi^* = (q^*, l^*)$:

$$
\begin{aligned}
q^*(t, x) &\simeq \frac{a\eta}{(\xi+1)b^2}\left[R^{(0)} - \sqrt{\epsilon}\frac{B(T-t)}{V_x^{(0)}}(D_1^2 D_2 - D_2 D_1^2)V^{(0)}\right], \\
l^*(t, x) &\simeq \frac{\varpi(y)}{(\xi+1)\sigma(y)}R^{(0)} - \frac{\sqrt{\epsilon}}{2(\xi+1)\sigma(y)V_x^{(0)}}\left[\rho\zeta(y)\chi_y(y)(D_1^2 - \xi D_1)V^{(0)}\right. \\
&\quad \left. + 2\varpi(y)B(T-t)(D_1^2 D_2 - D_2 D_1^2)V^{(0)}\right].
\end{aligned}
$$

Yet the optimal strategy depends on y even the value function is independent of y in terms of the first order approximation. In fact, the corrected term for the optimal strategy only takes effect on the value function at order ϵ. In other words, we can reproduce the value function up to the order $\sqrt{\epsilon}$ with the strategy at zeroth order. This fact is verified in the next subsection.

3.3. Using the zeroth order strategy

In this subsection, we demonstrate that using the zeroth order strategy of (13) results in the value function up to $\sqrt{\epsilon}$ (26). Define the zeroth order strategy pair

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$\pi^{(0)} = (q^{(0)}, l^{(0)})$ as follows.

$$q^{(0)} = \frac{a\eta}{(\xi+1)b^2}R^{(0)}, \quad l^{(0)} = \frac{\varpi(y)}{(\xi+1)\sigma(y)}R^{(0)}. \tag{27}$$

Trading with this strategy, the ambiguity-averse insurer's robust wealth process evolves as (12) with the substitution of $\pi^{(0)}$:

$$\begin{cases} d\widetilde{X}_t &= \left\{ a\lambda + a\eta q^{(0)} + (\mu(Y_t) - r)l^{(0)} + r\widetilde{X}_t \right\} dt \\ &\quad + \sqrt{\xi+1}bq^{(0)}d\widetilde{W}_t^G + \sqrt{\xi+1}\sigma(Y_t)l^{(0)}d\widetilde{W}_t^S, \\ dY_t &= \frac{1}{\epsilon}\theta(Y_t)dt + \frac{1}{\sqrt{\epsilon}}\zeta(Y_t)[\rho_\xi d\widetilde{W}_t^S + \bar{\rho}_\xi d\widetilde{W}_t^Y], \end{cases}$$

For comparison purpose, the value of this strategy, analogous to (11), is defined as

$$\widetilde{V}(t,x,y) = \mathbb{E}^{\mathbb{Q}^*}\left[U(\widetilde{X}_T) + \xi \int_t^T \left(\frac{\zeta^2(y)}{2\epsilon}\frac{\widetilde{V}_{xx}\widetilde{V}_y^2}{\widetilde{V}_x^2} + \rho l^{(0)}\sigma(y)\alpha(y)\frac{\widetilde{V}_{xx}\widetilde{V}_y}{\widetilde{V}_x} \right)(s)ds \,\middle|\, \widetilde{X}_t = x, Y_t = y \right],$$

By Feynman-Kac formula, $\widetilde{V}$ solves the PDE

$$\widetilde{V}_t + \frac{1}{\epsilon}\mathcal{L}_0\widetilde{V} + \frac{\xi}{2\epsilon}\zeta^2(y)\frac{\widetilde{V}_{xx}\widetilde{V}_y^2}{\widetilde{V}_x^2} + \left[a\lambda + rx + a\eta q^{(0)} + (\mu(y) - r)l^{(0)} \right]\widetilde{V}_x$$

$$+ \frac{\xi+1}{2}\left[b^2 q_t^{(0)2} + \sigma^2(y)l_t^{(0)2} \right]\widetilde{V}_{xx} + \frac{1}{\sqrt{\epsilon}}\rho l^{(0)}\sigma(y)\zeta(y)\left(\widetilde{V}_{xy} + \xi\frac{\widetilde{V}_{xx}\widetilde{V}_y}{\widetilde{V}_x} \right) = 0,$$

with the terminal condition $\widetilde{V}(T,x,y) = U(x)$. Using the operators introduced in (17), the PDE can be rewritten as

$$\widetilde{V}_t + \frac{1}{\epsilon}\mathcal{L}_0\widetilde{V} + \mathcal{L}_{t,x}(\psi_\xi^2(y))\widetilde{V} + \frac{\xi}{2\epsilon}\zeta^2(y)\frac{\widetilde{V}_{xx}\widetilde{V}_y^2}{\widetilde{V}_x^2} + \frac{1}{\sqrt{\epsilon}}\rho l^{(0)}\sigma(y)\zeta(y)\left(\widetilde{V}_{xy} + \xi\frac{\widetilde{V}_{xx}\widetilde{V}_y}{\widetilde{V}_x} \right) = 0,$$

where $\psi_\xi^2(y) = \frac{a^2\eta^2/b^2 + \varpi^2(y)}{\xi+1}$. Then similarly we expand

$$\widetilde{V}(t,x,y) = \widetilde{V}^{(0)} + \sqrt{\epsilon}\widetilde{V}^{(1)}(t,x,y) + \epsilon\widetilde{V}^{(2)}(t,x,y) + \epsilon^{\frac{3}{2}}\widetilde{V}^{(3)}(t,x,y) + \cdots.$$

We attempt to show that $\widetilde{V}^{(i)} \equiv V^{(i)}, i = 0, 1$ and therefore $\widetilde{V}$ coincides with V in (26) up to and including order $\sqrt{\epsilon}$.

Inserting the expansion for $\widetilde{V}$ and collecting the ϵ^{-1} terms in the PDE:

$$\mathcal{L}_0 \widetilde{V}^{(0)} + \frac{\xi}{2} \zeta^2(y) \frac{\widetilde{V}_{xx}^{(0)} \widetilde{V}_y^{(0)2}}{\widetilde{V}_x^{(0)2}} = 0,$$

which takes derivatives of y on $\widetilde{V}^{(0)}$, we choose $\widetilde{V}^{(0)} = \widetilde{V}^{(0)}(t, x)$ independent of y to satisfy this equation. At the order $\epsilon^{-1/2}$: $\mathcal{L}_0 \widetilde{V}^{(1)} = 0$, we again choose $\widetilde{V}^{(1)} = \widetilde{V}^{(1)}(t, x)$ independent of y.

We find the order one equation:

$$\mathcal{L}_0 \widetilde{V}^{(2)} + \mathcal{L}_{t,x}(\psi_\xi^2(y)) \widetilde{V}^{(0)} = 0.$$

This equation coincides with the equation (18). Therefore, similarly we conclude that $\widetilde{V}^{(0)} \equiv V^{(0)}, \widetilde{V}^{(2)} \equiv V^{(2)}$. Then the order $\sqrt{\epsilon}$ equation is given by

$$\mathcal{L}_0 \widetilde{V}^{(3)} + \mathcal{L}_{t,x}(\psi_\xi^2(y)) \widetilde{V}^{(1)} - \frac{\rho \zeta(y) \varpi(y)}{2(\xi + 1)^2} \chi_y(y)(D_1^2 - \xi D_1) \widetilde{V}^{(0)} = 0,$$

which coincides with the equation (24). Therefore, we also prove that $\widetilde{V}^{(1)} \equiv V^{(1)}$.

In summary, the zeroth order strategy pair (27) is sufficient to generate an objective value which approximates the optimal value function up to $\sqrt{\epsilon}$. It provides a convincing explanation for practitioners to just apply the zeroth order strategy, as long as $R^{(0)}$ is obtained and y is observable. The risk-tolerance function $R^{(0)}$ can be solved through the PDE (21). For some specific utility functions, we provide the explicit formulas of $R^{(0)}$ in Table 2. For the general utility function, we may rely on the numerical methods on solving $R^{(0)}$, which is detailed in the Section 5. However, to implement the strategy (27), we still need the volatility factor y,

which complicates the implementation. We overcome this problem by proposing the "practical" strategy in Section 4.

4. Practical solution

In fact, it is difficult to observe the volatility y and the growth rate processes μ. In this section, we attack the HJB problem (10) again by restricting to the admissible strategies of independent of y, i.e. $\pi_p = \pi_p(t, X_t) = (q_p, l_p)'$. Here, we use the subscript "p" to highlight results associated with the practical solution. For implementation purpose, we further assume that $\mu(Y_t) = \mu$ is constant, that is estimated by the historical data. We seek for the asymptotic practical solution by considering the expansions for V_p, q_p and l_p of the forms:

$$V_p(t, x, y) = v^{(0)}(t, x, y) + \sqrt{\epsilon} v^{(1)}(t, x, y) + \epsilon v^{(2)}(t, x, y) + \epsilon^{\frac{3}{2}} v^{(3)}(t, x, y) + \cdots ,$$

$$q_p(t, x) = q_0(t, x) + \sqrt{\epsilon} q_1(t, x) + \cdots , \quad l_p(t, x) = l_0(t, x) + \sqrt{\epsilon} l_1(t, x) + \cdots ,$$

where q_p and l_p are independent of y.

We then collect the order ϵ^{-1} equation in (10):

$$\mathcal{L}_0 v^{(0)} + \frac{\xi}{2} \zeta^2(y) \frac{v_{xx}^{(0)} v_y^{(0)2}}{v_x^{(0)2}} = 0.$$

We observe that $v^{(0)} = v^{(0)}(t, x)$ independent of y satisfies the equation. At order $\epsilon^{-\frac{1}{2}}$, we have $\mathcal{L}_0 v^{(1)} = 0$. Again, $v^{(1)} = v^{(1)}(t, x)$ independent of y satisfies the corresponding equation.

At order one, we have

$$v_t^{(0)} + \sup_{\pi_0} \left\{ \mathcal{L}_0 v^{(2)} + [a\lambda + rx] v_x^{(0)} + [a\eta q_0 + (\mu - r) l_0] v_x^{(0)} \right.$$
$$\left. + \frac{\xi + 1}{2} [b^2 q_0^2 + \sigma^2(y) l_0^2] v_{xx}^{(0)} \right\} = 0. \quad (28)$$

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To make $\pi_0 = (q_0, l_0)'$ independent of y, we enforce

$$\mathcal{L}_0 v^{(2)} = -\frac{\xi+1}{2}(\sigma^2(y) - \langle \sigma^2(y) \rangle) l_0^2 v_{xx}^{(0)},$$

$$\text{and thus } v^{(2)} = -\frac{\xi+1}{2}\kappa(y) l_0^2 v_{xx}^{(0)}, \text{ where } \mathcal{L}_0 \kappa(y) = \sigma^2(y) - \langle \sigma^2(y) \rangle.$$

With this choice, the equation (28) becomes

$$v_t^{(0)} + \sup_{\pi_0} \left\{ [a\lambda + rx]v_x^{(0)} + [a\eta q_0 + (\mu - r)l_0]v_x^{(0)} \right.$$

$$\left. + \frac{\xi+1}{2}[b^2 q_0^2 + \langle \sigma^2(y) \rangle l_0^2]v_{xx}^{(0)} \right\} = 0,$$

and the supremum is attained at

$$q_0^*(t) = -\frac{a\eta}{b^2(\xi+1)}\frac{v_x^{(0)}}{v_{xx}^{(0)}}, \quad l_0^*(t) = -\frac{1}{\xi+1}\frac{\mu - r}{\langle \sigma^2(y) \rangle}\frac{v_x^{(0)}}{v_{xx}^{(0)}}, \tag{29}$$

which are independent of y. Then the order one equation becomes

$$v_t^{(0)} + [a\lambda + rx]v_x^{(0)} - \frac{1}{2(\xi+1)}\left[\frac{a^2\eta^2}{b^2} + \frac{(\mu-r)^2}{\langle \sigma^2(y) \rangle} \right]\frac{v_x^{(0)2}}{v_{xx}^{(0)}} = 0,$$

Introducing the usual notations

$$\widehat{R}^{(0)}(t,x) = -\frac{v_x^{(0)}}{v_{xx}^{(0)}}, \quad \widehat{D}_k = \widehat{R}^{(0)}(t,x)^k \frac{\partial^k}{\partial x^k}, \quad \widehat{\psi}_\xi^2 = \frac{1}{\xi+1}\left[\frac{a^2\eta^2}{b^2} + \frac{(\mu-r)^2}{\langle \sigma^2(y) \rangle} \right],$$

$$\widehat{\mathcal{L}}_{t,x}(\psi^2) = \frac{\partial}{\partial t} + [a\lambda + rx]\frac{\partial}{\partial x} + \frac{1}{2}\psi^2\widehat{D}_2 + \psi^2\widehat{D}_1,$$

where $\widehat{R}^{(0)}$ satisfies (21) with $\bar{\psi}_\xi = \widehat{\psi}_\xi$ by Lemma 3.1, the PDE governing $v^{(0)}$ is given by

$$\widehat{\mathcal{L}}_{t,x}(\widehat{\psi}_\xi^2)v^{(0)} = 0, \quad v^{(0)}(T,x) = U(x).$$

Thus $v^{(0)}(t,x) = V^{(0)}(t,x;\widehat{\psi}_\xi)$ by the same arguments in the previous section.

At order $\sqrt{\epsilon}$, we have

$$v_t^{(1)} + [a\lambda + rx]v_x^{(1)} + \sup_{\pi_1} \Big\{ \mathcal{L}_0 v^{(3)} + [a\eta q_0^* + (\mu - r)l_0^*]v_x^{(1)}$$

$$+ [a\eta q_1 + (\mu - r)l_1]v_x^{(0)} + \frac{\xi + 1}{2}[b^2 q_0^{*2} + \sigma^2(y)l_0^{*2}]v_{xx}^{(1)}$$

$$+ (\xi + 1)[b^2 q_0^* q_1 + \sigma^2(y)l_0^* l_1]v_{xx}^{(0)} + \rho l_0^* \sigma(y)\zeta(y)\left(v_{xy}^{(2)} + \xi\frac{v_{xx}^{(0)}v_y^{(2)}}{v_x^{(0)}}\right)\Big\} = 0$$

Choosing $v^{(3)}$ such that the terms inside the supremum independent of y and substituting q_0^* and l_0^*, we have

$$\widehat{\mathcal{L}}_{t,x}(\widehat{\widetilde{\psi_\xi}^2})v^{(1)} = \frac{\rho}{2(\xi + 1)^2}\left(\frac{\mu - r}{\langle\sigma^2(y)\rangle}\right)^3 \langle\sigma(y)\zeta(y)\kappa_y(y)\rangle(\widehat{D}_1^2 - \xi\widehat{D}_1)v^{(0)}.$$

It is seen that π_1 has no effects on solving $v^{(1)}$. Then similarly, by Lemma 3.2,

$$v^{(1)}(t, x) = \frac{T - t}{(\xi + 1)^2}\left(\frac{\mu - r}{\langle\sigma^2(y)\rangle}\right)^3 \widehat{B}(\widehat{D}_1^2 - \xi\widehat{D}_1)v^{(0)}, \ \widehat{B} = -\frac{\rho}{2}\langle\sigma\zeta\kappa_y\rangle,$$

where $\sqrt{\epsilon}\widehat{B} = \mathcal{V}_3^\epsilon$ can be calibrated from the option implied volatility surface (see Chapter 5 in Fouque et al. (2011)).

In summary, trading with the strategy (29), specifically $\pi_0^* = (q_0^*, l_0^*)'$:

$$q_0^*(t) = \frac{a\eta}{b^2(\xi + 1)}\widehat{R}^{(0)}(t, x), \ l_0^*(t) = \frac{1}{\xi + 1}\frac{\mu - r}{\langle\sigma^2(y)\rangle}\widehat{R}^{(0)}(t, x),$$

we obtain the corrected practical value function

$$V(t, x, y) \simeq \left[1 + \frac{T - t}{(\xi + 1)^2}\left(\frac{\mu - r}{\langle\sigma^2(y)\rangle}\right)^3 \mathcal{V}_3^\epsilon(\widehat{D}_1^2 - \xi\widehat{D}_1)\right]V^{(0)}(t, x; \widehat{\psi_\xi}),$$

which has the similar structure in (26) with different averaged coefficient (ψ). This conclusion is also consistent with what we asserts in Section 3.3. We remark here that the practical strategy is achievable as long as $\widehat{R}^{(0)}$ can be solved and the effective volatility of the underlying asset, $\langle\sigma^2(y)\rangle$, is estimated historically.

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5. Numerical study

This section offers several numerical studies to examine the impacts of the SV factor and ambiguity aversion on the value function. The computations of the optimal strategy pair (essentially the risk-tolerance function $R^{(0)}$) and the value function are also addressed. We first use the common family of power utilities, for which there is an explicit solution provided in the Table 2, to examine the effect of fast mean-reverting SV correction. Then we introduce the mixture of power utilities that is treatable in our formulation. Finally, we investigate the sensitivity of the value function with respect to the model parameters in the cases of different ambiguity aversion coefficients.

5.1. Performance of the optimal strategy pair under power utility

The power (or CRRA) utility is a utility function of the form

$$U(x) = c\frac{x^{1-\gamma}}{1-\gamma}, \ c, \gamma > 0, \ \gamma \neq 1,$$

where γ is the measure of risk aversion, and c is the scale parameter. Then the corresponding Arrow-Pratt measure of relative risk aversion is constant:

$$AP[U] := -x\frac{U''(x)}{U'(x)} = \gamma.$$

With the assumption of $\lambda = 0$, the zeroth order risk-tolerance function $R^{(0)}$ and $V^{(0)}$ is given in the Table 2. Then the corrected value function (26) can be explicitly written out as

$$
\begin{aligned}
V(t, x, y; \psi) \ \simeq \ & c\frac{x^{1-\gamma}}{1-\gamma} \exp\left\{\left[\frac{(2r\gamma + \psi^2)(1-\gamma)}{2\gamma}\right](T-t)\right\} \\
& \cdot \left\{1 + \sqrt{\epsilon}B\frac{T-t}{(\xi+1)^2}\left[\left(\frac{1-\gamma}{\gamma}\right)^2 - \xi\frac{1-\gamma}{\gamma}\right]\right\}.
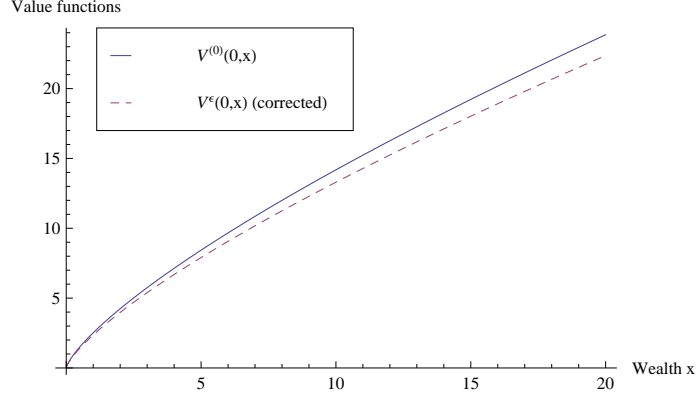\end{aligned}
$$

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Figure 1: Corrected value function under the power utility with $c = 1, \gamma = 0.25$

Figure 1 shows the zeroth order approximated value function and the corrected value function with the parameters $r = 0.05, T = 1, \psi = 0.4, \sqrt{\epsilon}B = -0.01, \xi = 0.2, c = 1, \gamma = 0.25$. Noting that the correction is due to the stochastic volatility, then it is naturally expected that the correction lowers the value function. Moreover, it can be seen that the correction is enlarged for the larger wealth level, because the risk-tolerance (function) is increasing for the wealth.

5.2. *Performance of the optimal strategy pair under mixture of power utilities*

Our formulation with ambiguity aversion applies to a general class of utility functions. We show an example with a mixture of two power utilities:

$$U(x) = c_1 \frac{x^{1-\gamma_1}}{1-\gamma_1} + c_2 \frac{x^{1-\gamma_2}}{1-\gamma_2}, \quad c_1, c_2 \geq 0, \quad \gamma_1 \geq \gamma_2 > 0, \quad \gamma_{1,2} \neq 1.$$

The remarkable advantage of the mixture of power utilities is that it can produce the non-constant relative risk aversion and nonlinear risk tolerance, which are computed as

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- Arrow-Pratt measure of relative risk aversion:

$$AP[U] := -x \frac{U''(x)}{U'(x)} = \frac{c_1 \gamma_1 x^{-(\gamma_1 - \gamma_2)} + c_2 \gamma_2}{c_1 x^{-(\gamma_1 - \gamma_2)} + c_2}.$$

- Risk-tolerance function:

$$R^{(0)}(T, x) = -\frac{U'(x)}{U''(x)} = \frac{c_1 x^{-(\gamma_1 - \gamma_2)} + c_2}{c_1 \gamma_1 x^{-(\gamma_1 - \gamma_2)} + c_2 \gamma_2} x \sim \begin{cases} \frac{1}{\gamma_2} x, & x \to \infty, \\ \frac{1}{\gamma_1} x, & x \to 0. \end{cases}$$

We refer to Brunnermeir and Nagel (2008) for the empirical study of time-varying risk aversion.

We concern with the corrected value function and the risk-tolerance function $R^{(0)}$ with the mixture of power utilities. The relationship between $R^{(0)}$ and $V^{(0)}$ enables us to focus on solving for $R^{(0)}$. This can be achieved by a numerical PDE method for $R^{(0)}$ in (21):

$$R_t^{(0)} + \frac{1}{2} \psi_\xi^2 (R^{(0)})^2 R_{xx}^{(0)} + (a\lambda + rx) R_x^{(0)} - r R^{(0)} = 0, \quad t \in [0, T], x \in [0, x^*],$$

$$R^{(0)}(T, x) = -U'(x)/U''(x),$$

where $\psi = \bar{\psi}_\xi, \widehat{\psi_\xi}$, or $\bar{\psi}_\xi(z)$ is determined according to what kind of solution we seek for, and x^* is set as upper limit of the wealth in order to apply the numerical methods, such as finite difference method. Noting that when the wealth x is large, the insurer behaves as if she has a power utility with γ_2. These deduce the following boundary conditions for solving $R^{(0)}$:

$$R^{(0)}(t, 0) = 0, \ R_x^{(0)}(t, x^*) = \frac{1}{\gamma_2}.$$

Obtained the $R^{(0)}(t, x)$, we can also infer $V_x^{(0)}$ and $V^{(0)}$ as

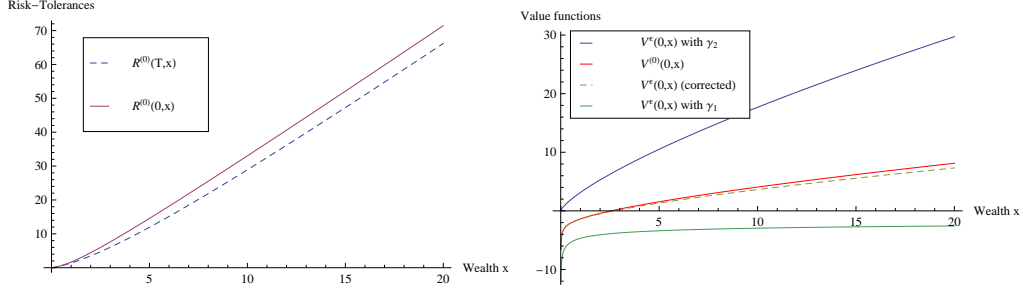$$V_x^{(0)}(t, x) = V_x^{(0)}(t, x^*) \exp\left(\int_x^{x^*} \frac{1}{R^{(0)}(t, \nu)} d\nu\right),$$

Figure 2: Risk-tolerance function (left) and corrected value function under the mixture of power utilities with $c_1 = c_2 = 0.5, \gamma_1 = 1.2,$ and $\gamma_2 = 0.25$

and

$$V^{(0)}(t, x) = V^{(0)}(t, x^*) - \int_x^{x^*} V_x^{(0)}(t, \nu) d\nu.$$

In order to compute $V^{(0)}(t, x^*)$ and $V_x^{(0)}(t, x^*)$, we require x^* is large enough for us to use the large wealth asymptotics, which is given by

$$V^{(0)} \sim c_1 \frac{x^{1-\gamma_1}}{1 - \gamma_1} g_{12}(t) + c_2 \frac{x^{1-\gamma_2}}{1 - \gamma_2} g_2(t), \quad \text{as } x \to \infty,$$

where

$$\begin{aligned}
g_2(t) &= \exp\left(\left[r(1 - \gamma_2) + \frac{\psi^2}{2}\left(\frac{1 - \gamma_2}{\gamma_2}\right)\right](T - t)\right), \\
g_{12}(t) &= \exp\left(\left[r(1 - \gamma_1) + \frac{\psi^2}{\gamma_2^2}(1 - \gamma_1)\left(\gamma_2 - \frac{1}{2}\gamma_1\right)\right](T - t)\right).
\end{aligned}$$

Differentiating $V^{(0)}$ with respect to x yields

$$V_x^{(0)} \sim c_1 x^{-\gamma_1} g_{12}(t) + c_2 x^{-\gamma_2} g_2(t), \quad \text{as } x \to \infty.$$

For the given $V^{(0)}$ and $R^{(0)}$, we can induce the $V^{(1)}$ by using (25).

Figure 2 shows the risk-tolerance function and the corrected value function under the mixture of power utilities with the parameters $x^* = 50, r = 0.05, T =$

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$1.5, \psi = 0.4, \sqrt{\epsilon}B = -0.01, \xi = 0.2, c_1 = c_2 = 0.5, \gamma_1 = 1.2,$ and $\gamma_2 = 0.25$. We also plot the corrected value functions with the individual power utilities (blue line and green line) in the right figure for the comparison purpose. Numerically solving the PDE (21) is efficient. It can be seen that even for a general utility function, we can also solve for the robust problem.

5.3. The impact of ambiguity aversion

We further examine the impact of the ambiguity-aversion coefficient, ξ, on the corrected value functions with respect to the asset's expected return μ, the claim rate a, the effective volatility σ and the volatility of the claim b. To illustrate idea, we use the power utility function with the risk aversion parameter $\gamma = 0.5$ and $c = 1$. We use the following parameters: $r = 0.05, \mu = 0.2, \sigma = 0.3, a = 0.4, b = 0.1, \eta = 0.1, \sqrt{\epsilon}B = -0.01, T = 1$. The plots of the corrected value function against different model parameters are shown in the Figure 3.

It can be seen that the higher the ambiguity aversion the lower the value function, as expected. It reveals the trade-off between optimization and robustness. From the top two graphs in the Figure 3, the value functions are flatter with a larger ambiguity aversion level. In other words, the value function is less sensitive to μ and a when the degree of ambiguity aversion increases. However, the bottom two graphs in the Figure 3 show that only the scale of the value functions are diminished and no robustness on σ and b is observed. Therefore, our robust formulation, which follows the existing literature, is mainly robust with respect to the drift terms of the wealth process.

This result is not surprising. Our formulation is based on the concept of equivalent measures and the change of measure, which essentially manipulates the drift terms only. It could be partially explained by the robust wealth process in the
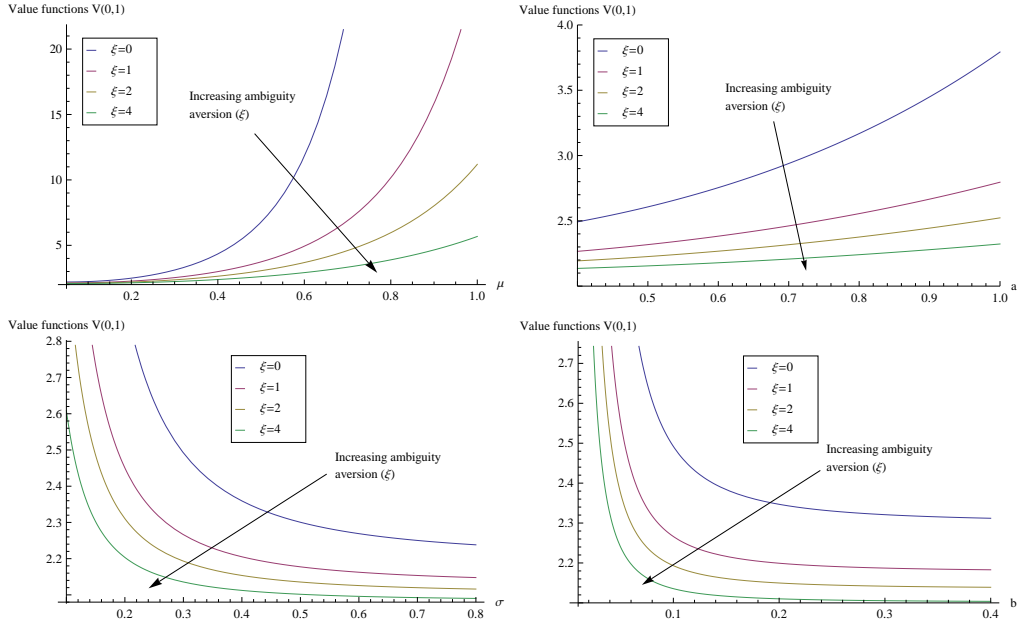
Figure 3: Sensitivity of the corrected value function with different degrees of ambiguity aversion

Theorem 2.1. Such a result is economically useful when the decision maker has less confidence in the estimation of the drift terms. The difficulty of the statistical estimation of the drift terms has been documented in many empirical studies.

6. Conclusion

We present the formulation of the robust investment and reinsurance problem with a general utility function for an ambiguity-averse insurer. With our choice of the preference function to ambiguity aversion, we first transform the robust stochastic control problem to a standard stochastic control problem. The impact of stochastic volatility on this problem is studied through asymptotic approximations. From the practical point of view, we also present a practical strategy, which is suboptimal but does not require the unobservable of the stochastic volatility.

Our numerical studies reveal the fact that our formulation, and the existing literature, mainly concerns with the robustness on the drift terms of the wealth process. There is a trade-off between the optimization and the robustness. Our results provide the robust decision rule for the ambiguity-averse insurer with a general utility function investing into a financial market with stochastic volatility. Future research may investigate the impact of transaction costs and the construction of ambiguity against volatility uncertainty in stead of drift uncertainty.

A. Multiscale stochastic volatility

Here, we consider the multiscale stochastic volatility model, where there is one fast volatility factor and one slow, for the process S:

$$
\begin{cases}
dS_t &= \mu(Y_t, Z_t)S_t dt + \sigma(Y_t, Z_t)S_t dW_t^S, \\
dY_t &= \frac{1}{\epsilon}\theta(Y_t)dt + \frac{1}{\sqrt{\epsilon}}\zeta(Y_t)dW_t^Y, \\
dZ_t &= \delta\omega(Z_t)dt + \sqrt{\delta}\beta(Z_t)dW_t^Z,
\end{cases}
\tag{30}
$$

where $0 < \epsilon, \delta \ll 1$, with $\mathbb{E}[dW_t^S dW_t^Y] = \rho_1 dt$, $\mathbb{E}[dW_t^S dW_t^Z] = \rho_2 dt$, $\mathbb{E}[dW_t^Y dW_t^Z] = \rho_{12}dt$ and $1 + 2\rho_1\rho_2\rho_{12} - \rho_1^2 - \rho_2^2 - \rho_{12}^2 > 0$. Analogous to the problem formulation in the Section 2, $\varphi^{\mathbb{Q}}$ is replaced by $(\varphi_G^{\mathbb{Q}}(t), \varphi_S^{\mathbb{Q}}(t), \varphi_Y^{\mathbb{Q}}(t), \varphi_Z^{\mathbb{Q}}(t))$. Then the wealth process is revised as follows.

$$
\begin{cases}
dX_t &= \Big[a\lambda + [a\eta + b\varphi_G^{\mathbb{Q}}(t)]q + [\mu(Y_t, Z_t) - r + \sigma(Y_t, Z_t)\varphi_S^{\mathbb{Q}}(t)]l + rX_t\Big] dt \\
&\quad + bqd\widetilde{W}_t^G + \sigma(Y_t, Z_t)ld\widetilde{W}_t^S, \\
dY_t &= \Big[\frac{1}{\epsilon}\theta(Y_t) + \frac{1}{\sqrt{\epsilon}}\zeta(Y_t)(\rho_1\varphi_S^{\mathbb{Q}}(t) + \rho_A\varphi_Y^{\mathbb{Q}}(t) + \rho_B\varphi_Z^{\mathbb{Q}}(t))\Big] dt + \frac{1}{\sqrt{\epsilon}}\zeta(Y_t)d\widetilde{W}_t^Y, \\
dZ_t &= \Big[\delta\omega(Z_t) + \sqrt{\delta}\beta(Z_t)(\rho_2\varphi_S^{\mathbb{Q}}(t) + \rho_C\varphi_Y^{\mathbb{Q}}(t) + \rho_D\varphi_Z^{\mathbb{Q}}(t))\Big] dt + \sqrt{\delta}\beta(Z_t)d\widetilde{W}_t^Z,
\end{cases}
$$

with $\mathbb{E}[d\widetilde{W}_t^S d\widetilde{W}_t^Y] = \rho_1 dt$, $\mathbb{E}[d\widetilde{W}_t^S d\widetilde{W}_t^Z] = \rho_2 dt$, $\mathbb{E}[d\widetilde{W}_t^Y d\widetilde{W}_t^Z] = \rho_{12}dt$, where $\rho_A^2 + \rho_B^2 = 1 - \rho_1^2$, $\rho_C^2 + \rho_D^2 = 1 - \rho_2^2$ and $\rho_A\rho_C + \rho_B\rho_D = \rho_{12} - \rho_1\rho_2$. Then the

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value function

$$V(t, x, y, z) = \sup_{\pi \in \Pi} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left[U(X_T) + \int_t^T \frac{P(s)}{\phi(s)} ds \,\middle|\, X_t = x, Y_t = y, Z_t = z \right]$$

has the associated HJBI PDE:

$$V_t + [a\lambda + rx]V_x + \sup_\pi \inf_{\varphi^{\mathbb{Q}}} \Big\{ ([a\eta + b\varphi_G(t)]q + [\mu(y) - r + \sigma(y)\varphi_S(t)]l) \, V_x$$

$$+ \frac{1}{2}[b^2 q_t^2 + \sigma^2(y)l_t^2]V_{xx} + \frac{1}{\sqrt{\epsilon}}\zeta(y)(\rho_1\varphi_S^{\mathbb{Q}}(t) + \rho_A\varphi_Y^{\mathbb{Q}}(t) + \rho_B\varphi_Z^{\mathbb{Q}}(t))V_y + \frac{1}{\sqrt{\epsilon}}\rho l\sigma(y)\zeta(y)V_{xy}$$

$$+ \sqrt{\delta}\beta(z)(\rho_2\varphi_S^{\mathbb{Q}}(t) + \rho_C\varphi_Y^{\mathbb{Q}}(t) + \rho_D\varphi_Z^{\mathbb{Q}}(t))V_z + \sqrt{\delta}\beta(z)l(t)\sigma(y,z)\rho_2 V_{xz}$$

$$+ \sqrt{\frac{\delta}{\epsilon}}\zeta(y)\beta(z)\rho_{12}V_{yz} + \frac{1}{\epsilon}\mathcal{L}_0 V + \delta\mathcal{M}_0 V + \frac{1}{2\xi\phi}\left(\varphi_G^2(t) + \varphi_S^2(t) + \varphi_Y^2(t) + \varphi_Z^2(t)\right) \Big\} = 0,$$

with the terminal condition $V(T, x, y, z) = U(x)$, and where $\mathcal{L}_0$ is the infinitesimal generator of Y defined in (15) and $\mathcal{M}_0$ is the infinitesimal generator of Z:

$$\mathcal{M}_0 = \frac{1}{2}\beta^2(z)\frac{\partial^2}{\partial z^2} + \omega(z)\frac{\partial}{\partial z}$$

Then the infimum is attained at

$$\varphi_G^* = -\xi\phi bq V_x, \;\; \varphi_S^* = -\xi\phi\left[l\sigma(y)V_x + \frac{1}{\sqrt{\epsilon}}\zeta(y)\rho_1 V_y + \sqrt{\delta}\beta(y)\rho_2 V_z\right],$$

$$\varphi_Y^* = -\xi\phi\left[\frac{1}{\sqrt{\epsilon}}\zeta(y)\rho_A V_y + \sqrt{\delta}\beta(y)\rho_C V_z\right], \;\; \varphi_Z^* = -\xi\phi\left[\frac{1}{\sqrt{\epsilon}}\zeta(y)\rho_B V_y + \sqrt{\delta}\beta(y)\rho_D V_z\right].$$

With the choice of ϕ, (7), the HJB equation becomes

$$V_t + [a\lambda + rx]V_x + \frac{1}{\epsilon}\mathcal{L}_0 V + \delta\mathcal{M}_0 V + \sup_\pi \Big\{ (a\eta q + [\mu(y) - r]l) \, V_x$$

$$+ \frac{\xi+1}{2}[b^2 q_t^2 + \sigma^2(y)l_t^2]V_{xx} + \frac{\xi}{2\epsilon}\zeta^2(y)\frac{V_{xx}V_y^2}{V_x^2} + \frac{\xi}{2}\delta\beta^2(z)\frac{V_{xx}V_z^2}{V_x^2}$$

$$+ \frac{1}{\sqrt{\epsilon}}\zeta(y)l\sigma(y,z)\rho_1\left(V_{xy} + \xi\frac{V_{xx}V_y}{V_x}\right) + \sqrt{\delta}\beta(z)l\sigma(y,z)\rho_2\left(V_{xz} + \xi\frac{V_{xx}V_z}{V_x}\right)$$

$$+ \sqrt{\frac{\delta}{\epsilon}}\zeta(y)\beta(z)\rho_{12}\left(V_{yz} + \xi\frac{V_{xx}V_y V_z}{V_x^2}\right) \Big\} = 0,$$

whose supremum is attained at $q^*(t) = -\frac{a\eta}{(\xi+1)b^2}\frac{V_x}{V_{xx}}$ and

$$l^*(t) = -\frac{\mu(y)-r}{(\xi+1)\sigma^2(y,z)}\frac{V_x}{V_{xx}} - \frac{\rho_1\zeta(y)}{\sqrt{\epsilon}(\xi+1)\sigma(y,z)}\left(\frac{V_{xy}}{V_{xx}}+\xi\frac{V_y}{V_x}\right) - \frac{\sqrt{\delta}\rho_2\beta(y)}{(\xi+1)\sigma(y,z)}\left(\frac{V_{xz}}{V_{xx}}+\xi\frac{V_z}{V_x}\right).$$

Then with the substitutions of these controls, the HJB PDE becomes

$$V_t + [a\lambda + rx]V_x + \frac{1}{\epsilon}\left[\mathcal{L}_0 V + \frac{\xi}{2}\zeta^2(y)\frac{V_{xx}V_y^2}{V_x^2}\right] + \delta\left[\mathcal{M}_0 V + \frac{\xi}{2}\delta\beta^2(z)\frac{V_{xx}V_z^2}{V_x^2}\right]$$

$$+\sqrt{\frac{\delta}{\epsilon}}\zeta(y)\beta(z)\rho_{12}\left(V_{yz}+\xi\frac{V_{xx}V_yV_z}{V_x^2}\right) - \frac{a^2\eta^2}{2(\xi+1)b^2}\frac{V_x^2}{V_{xx}} - \frac{1}{2\xi\sigma^2(y,z)V_{xx}}\left[(\mu(y)-r)V_x+\right.$$

$$\left.\frac{1}{\sqrt{\epsilon}}\zeta(y)\sigma(y,z)\rho_1\left(V_{xy}+\xi\frac{V_{xx}V_y}{V_x}\right)+\sqrt{\delta}\beta(z)\sigma(y,z)\rho_2\left(V_{xz}+\xi\frac{V_{xx}V_z}{V_x}\right)\right]^2 = 0.$$

We first expand the value function with respect to $\sqrt{\delta}$:

$$V^{\epsilon,\delta} = V_0^\epsilon + \sqrt{\delta}V_1^\epsilon + \cdots.$$

Then applying the regular perturbation techniques to solve for V_0^ϵ and V_1^ϵ. The equation governing V_0^ϵ is induced by setting $\delta = 0$ in the equation for $V^{\epsilon,\delta}$:

$$V_{0t}^\epsilon + [a\lambda + rx]V_{0x}^\epsilon + \frac{1}{\epsilon}\left[\mathcal{L}_0 V_0^\epsilon + \frac{\xi}{2}\zeta^2(y)\frac{V_{0xx}^\epsilon V_{0y}^{\epsilon 2}}{V_{0x}^{\epsilon 2}}\right] - \frac{a^2\eta^2}{2b^2(\xi+1)}\frac{V_{0x}^{\epsilon 2}}{V_{0xx}^\epsilon}$$

$$-\frac{\left[\frac{1}{\sqrt{\epsilon}}\rho_1\sigma(y,z)\zeta(y)\left(V_{0xy}^\epsilon+\xi\frac{V_{0xx}^\epsilon V_{0y}^\epsilon}{V_{0x}^\epsilon}\right)+(\mu(y,z)-r)V_{0x}^\epsilon\right]^2}{2(\xi+1)\sigma^2(y,z)V_{0xx}^\epsilon} = 0.$$

This PDE problem has been studied in the Section 3. If we write $V_0^\epsilon = V^{(0)} + \sqrt{\epsilon}V^{(1,0)} + \cdots$, then

$$V^{(0)}(t,x,z) = V^{(0)}(t,x,z;\bar{\psi}_\xi(z)), \quad V^{(1,0)}(t,x,z) = \frac{T-t}{(\xi+1)^2}\widetilde{B}(\widetilde{D}_1^2 - \xi\widetilde{D}_1)V^{(0)}(t,x,z),$$

where

$$\bar{\psi}_\xi(z) = \frac{a^2\eta^2}{b^2(\xi+1)} + \left\langle\frac{(\mu(\cdot,z)-r)^2}{(\xi+1)\sigma^2(\cdot,z)}\right\rangle, \quad \widetilde{B} = -\frac{\rho_1}{2}\left\langle\frac{\zeta(\cdot)(\mu(\cdot,z)-r)}{\sigma(\cdot,z)}\widetilde{\chi}_y(\cdot,z)\right\rangle,$$

$$\widetilde{D}_1 = -\frac{V_x^{(0)}}{V_{xx}^{(0)}}\frac{\partial}{\partial x}, \ \mathcal{L}_0\widetilde{\chi}(y,z) = \frac{(\mu(y,z)-r)^2}{\sigma^2(y,z)} - \left\langle\frac{(\mu(\cdot,z)-r)^2}{\sigma^2(\cdot,z)}\right\rangle.$$

At the order $\sqrt{\delta}$, we have

$$V_{1t}^\epsilon + [a\lambda + rx]V_{1x}^\epsilon + \frac{1}{\epsilon}\left[\mathcal{L}_0 V_1^\epsilon + \frac{\xi}{2}\zeta^2(y)\left(\frac{2V_{0xx}^\epsilon V_{0y}^\epsilon V_{1y}^\epsilon + V_{1xx}^\epsilon V_{0y}^{\epsilon2}}{V_{0x}^\epsilon} + \frac{2V_{1x}^\epsilon V_{0xx}^\epsilon V_{0y}^{\epsilon2}}{V_{0x}^{\epsilon2}}\right)\right]$$

$$+\frac{1}{\sqrt{\epsilon}}\zeta(y)\beta(z)\rho_{12}\left(V_{0yz}^\epsilon + \xi\frac{V_{0xx}^\epsilon V_{0y}^\epsilon V_{0z}^\epsilon}{V_{0x}^{\epsilon2}}\right) - \frac{a^2\eta^2}{2b^2(\xi+1)}\left(\frac{2V_{0x}^\epsilon V_{1x}^\epsilon}{V_{0xx}^\epsilon} + \frac{V_{0x}^{\epsilon2}V_{1xx}^\epsilon}{V_{0xx}^{\epsilon2}}\right)$$

$$+\frac{1}{2(\xi+1)\sigma^2(y,z)}\left\{\frac{V_{1xx}^\epsilon}{V_{0xx}^{\epsilon2}}\left[\frac{1}{\sqrt{\epsilon}}\rho_1\sigma(y,z)\zeta(y)\left(V_{0xy}^\epsilon + \xi\frac{V_{0xx}^\epsilon V_{0y}^\epsilon}{V_{0x}^\epsilon}\right) + (\mu(y,z)-r)V_{0x}^\epsilon\right]^2 - \frac{2}{V_{0xx}^\epsilon}\right.$$

$$\cdot\left[\frac{1}{\sqrt{\epsilon}}\rho_1\sigma(y,z)\zeta(y)\left(V_{0xy}^\epsilon + \xi\frac{V_{0xx}^\epsilon V_{0y}^\epsilon}{V_{0x}^\epsilon}\right) + (\mu(y,z)-r)V_{0x}^\epsilon\right]\left[(\mu(y,z)-r)V_{1x}^\epsilon + \rho_2\sigma(y,z)\right.$$

$$\cdot\beta(z)\left(V_{0xz}^\epsilon + \xi\frac{V_{0xx}^\epsilon V_{0z}^\epsilon}{V_{0x}^\epsilon}\right) + \frac{1}{\sqrt{\epsilon}}\zeta(y)\sigma(y,z)\rho_1\left(V_{1xy}^\epsilon + \frac{V_{1xx}^\epsilon V_{0y}^\epsilon + V_{0xx}^\epsilon V_{1y}^\epsilon - V_{0xx}^\epsilon V_{0y}^\epsilon\frac{V_{1x}^\epsilon}{V_{0x}^\epsilon}}{V_{0x}^\epsilon}\right)\right]\right\} = 0.$$

We expand V_1^ϵ as $V_1^\epsilon = V^{(0,1)} + \sqrt{\epsilon}V^{(1,1)} + \epsilon V^{(2,1)} + \cdots$. The order $\frac{1}{\epsilon}$ and $\frac{1}{\sqrt{\epsilon}}$ equations yield $\mathcal{L}_0 V^{(0,1)} = 0$ and $\mathcal{L}_0 V^{(1,1)} = 0$, which are satisfied by taking $V^{(0,1)}$ and $V^{(1,1)}$ independent of y. With these choices, the order one equation is given by

$$\mathcal{L}_0 V^{(2,1)} + V_t^{(0,1)} + [a\lambda + rx]V_x^{(0,1)} + \frac{1}{2}\left[\frac{a^2\eta^2}{(\xi+1)b^2} + \frac{(\mu(y,z)-r)^2}{(\xi+1)\sigma^2(y,z)}\right]\frac{V_x^{(0)2}V_{xx}^{(0,1)}}{V_{xx}^{(0)2}}$$

$$-\left[\frac{a^2\eta^2}{(\xi+1)b^2} + \frac{(\mu(y,z)-r)^2}{(\xi+1)\sigma^2(y,z)}\right]\frac{V_x^{(0)}V_x^{(0,1)}}{V_{xx}^{(0)}} - \frac{\rho_2\beta(z)(\mu(y,z)-r)}{(\xi+1)\sigma(y,z)}\left(\frac{V_x^{(0)}V_{xz}^{(0)}}{V_{xx}^{(0)}} + \xi V_z^{(0)}\right) = 0.$$

The solvability condition for $V^{(2,1)}$ gives that

$$\widetilde{\mathcal{L}}_{t,x}(\bar\psi_\xi(z))V^{(0,1)} + \frac{\rho_2}{\xi+1}\left\langle\frac{\mu(\cdot,z)-r}{\sigma(\cdot,z)}\right\rangle\beta(z)(\widetilde{D}_1 + \xi)V_z^{(0)} = 0.$$

Thus $V^{(0,1)} = \frac{\rho_2}{2(\xi+1)}(T-t)\left\langle\frac{\mu(\cdot,z)-r}{\sigma(\cdot,z)}\right\rangle\beta(z)(\widetilde{D}_1 + \xi)V_z^{(0)}$, which can be verified analogously to the proofs of Lemma 3.1 and Proposition 3.2 in Fouque et al.

(2013). Therefore, the first order corrected value function can be obtained as

$$V(t, x, y, z) \simeq V^{(0)}(t, x, z) + \sqrt{\epsilon}V^{(1,0)}(t, x, z) + \sqrt{\delta}V^{(0,1)}(t, x, z),$$

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